CHAPTER V

SOME RESULTS ON CONTRACTIVE MAPPINGS

We first give the definition of contractive mappings.

5.1 Definition.* A self-mapping T of a metric space (X,d) is said to be contractive mapping provided that

$$d(T(x),T(y)) < d(x,y)$$

for all $x, y \in X$, $x \neq y$.

In this chapter, we will observe the relationship between a contraction mapping and a contractive mapping on a compact subset of a metric space. We recall that a self-mapping T on a metric space X is a contraction mapping if, there exists $0 \leq k < 1$ such that

$$d(T(x),T(y)) \leq k\, d(x,y)$$

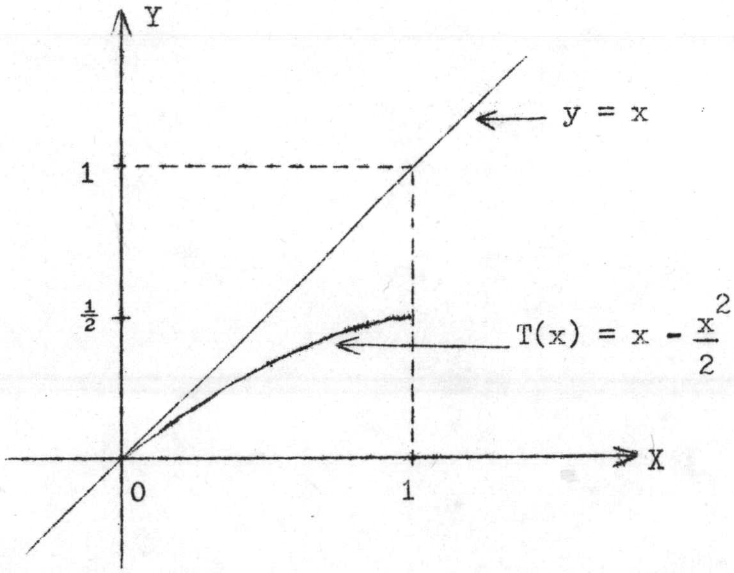
for all x, y in X. Then, it is clear that every contraction mapping is also a contractive mapping.

However, our question is that, if T is a contractive self-mapping of a compact subset of a metric space, is T a contraction mapping? The following example will show that the answer is in the negative.

5.2 Example.* Let C be a compact subset of a metric space X. Then there exists a contractive mapping $T : C \longrightarrow C$ which is not a contraction mapping.

Let $C = [0,1]$, and define the mapping $T : C \longrightarrow C$ by

$$T(x) = x - \frac{x^2}{2} \quad , \quad (x \in C)$$



It is clear that C is a compact subset of a metric space $\mathbb{R}^1$, and T is a contractive mapping of C into itself. In fact, the slope of T in the interval (0,1) is less than 1 and greater than 0. Then the mean valued theorem gives that, for any $x \neq y \in C$,

$$\frac{|T(x) - T(y)|}{|x - y|} = |T'(\phi_{x,y})| \quad \text{.... (1)}$$

for some $x < \phi_{x,y} < y$. Since $0 < T'(t) < 1$ for all $t \in (0,1)$,

$$\frac{|T(x) - T(y)|}{|x - y|} < 1 \quad ,$$

i.e., $|T(x) - T(y)| < |x - y|$.

Since $T'(x) \longrightarrow 1$ as $x \longrightarrow 0$, we have by (1) that we can not find a number $0 \leq k < 1$ such that $|T(x) - T(y)| \leq k\,|x - y|$. That is T is not a contraction mapping.

Hence, we conclude that the class of all contraction mappings on a compact subset of a metric space is a proper subclass of contractive mappings.

In the following theorem, we can see that the star-shaped hypothesis in Theorem 3.8 can be relaxed whenever the mapping T is a contractive mapping.

5.3 Theorem.* Suppose X is a compact metric space, and suppose T is a contractive mapping of X into itself, i.e.,

$$d(T(x),T(y)) < d(x,y)$$

holds for all $x,y \in X$, $x \neq y$. Then T has a unique fixed point in X.

Proof. Existence :

Consider the mapping $g : X \longrightarrow \mathbb{R}^1$ defined by

$$g(x) = d(x,T(x))$$

for all $x \in X$. We first show that g is continuous. Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$. Then whenever $d(x,y) < \delta$, we have

$$\begin{aligned} |g(x) - g(y)| &= |d(x,T(x)) - d(y,T(y))| \\ &= |d(x,T(x)) - d(y,T(x))| + \\ &\quad |d(y,T(x)) - d(y,T(y))|, \qquad \text{.... (1)} \end{aligned}$$

Next, we have to show that, if (X,d) is a metric space, then

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

for all $x,y,z \in X$.

Since $d(x,z) \leq d(x,y) + d(y,z)$ and

$d(y,z) \leq d(y,x) + d(x,z)$,

$d(x,z) - d(y,z) \leq d(x,y)$ and

$d(y,z) - d(x,z) \leq d(x,y)$.

Then,

$$|d(x,z) - d(y,z)| \leq d(x,y).$$

Hence, from the inequality (1) we have

$$\begin{aligned} |g(x) - g(y)| &\leq d(x,y) + d(T(x),T(y)) \\ &< d(x,y) + d(x,y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \end{aligned}$$

Then g is a continuous mapping from X into $\mathbb{R}^1$.

Since X is compact, g contains its infimum in X, i.e., there exists $x_o \in X$ such that

$$g(x_o) = \inf_{x \in X} \{g(x)\} . \qquad \ldots . (*)$$

We claim that $g(x_o) = 0$, i.e., $d(x_o,T(x_o)) = 0$.

Suppose by contradiction that $d(x_o,T(x_o)) > 0$. Then

$$T(x_o) \neq x_o .$$

Since both x_o and $T(x_o)$ are in X, by definition of g and by the hypothesis we get

$$\begin{aligned} g(T(x_o)) &= d(T(x_o),T(T(x_o))) \\ &< d(x_o,T(x_o)) = g(x_o) . \end{aligned}$$

Then there exists $T(x_o) \in X$ such that $g(T(x_o)) \angle g(x_o)$ which is contradictory to (*).

Hence, $T(x_o) = x_o$, i.e., x_o is a fixed point of T.

<u>Uniqueness</u> :

Assume that there exist $x_1 \neq x_2$ in X such that

$$T(x_1) = x_1 \text{ and } T(x_2) = x_2 .$$

Then

$$d(x_1,x_2) = d(T(x_1),T(x_2)) \angle d(x_1,x_2)$$

which is a contradiction.

Hence, $x_1 = x_2$, and proves the theorem.

From the above theorem, if we omit the compactness, the condition

$$d(T(x),T(y)) \angle d(x,y)$$

is insufficient for the existence of a fixed point of T.

5.4 <u>Example.</u>* Consider the mapping $T : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ defined by

$$T(x) = (1+x^2)^{\frac{1}{2}}.$$

We first show that the mapping T satisfies the condition that

$$|T(x_1) - T(x_2)| \angle |x_1 - x_2|$$

where $x_1 \neq x_2 \in \mathbb{R}^1$.

For this prove we may assume that $x_1 \geq x_2$. Then

$$|(1+x_1^2)^{\frac{1}{2}} - (1+x_2^2)^{\frac{1}{2}}| = (1+x_1^2)^{\frac{1}{2}} - (1+x_2^2)^{\frac{1}{2}}.$$

Suppose that $(1+x_1^2)^{\frac{1}{2}} - (1+x_2^2)^{\frac{1}{2}} \geq (x_1 - x_2)$. Then

$$(1+x_1^2)^{\frac{1}{2}} \geq (1+x_2^2)^{\frac{1}{2}} + (x_1 - x_2)$$

$$(1+x_1^2) \geq (1+x_2^2) + 2(x_1 - x_2)(1+x_2^2)^{\frac{1}{2}} + x_1^2 - 2x_1x_2 + x_2^2$$

$$x_2(x_1 - x_2) \geq (x_1 - x_2)(1+x_2^2)^{\frac{1}{2}}$$

$$x_2 \geq (1+x_2^2)^{\frac{1}{2}}.$$

Then

$$x_2^2 \geq 1 + x_2^2$$

which is a contradiction.

Next, to show that T has no fixed point. Suppose that there exists $x \in \mathbb{R}^1$ such that $T(x) = x$. Then

$$x = (1+x^2)^{\frac{1}{2}},$$

and then

$$x^2 = 1 + x^2$$

which is impossible.