CHAPTER III

FUNCTIONAL EQUATIONS AND BINARY OPERATIONS

The materials of this chapter are drawn from reference [3].

In this chapter, we want to study the relations between some functional equations to some properties of the binary operations of the domain and range spaces. We begin these discussion by reviewing the following definitions :

Definition 3.1. A semi-group is a pair $(G, *)$ where G is a set and "$*$" is a binary operation on G such that the associative law holds : $x * (y * z) = (x * y) * z$.

Definition 3.2. A monoïd $(G, *)$ is a semi-group with an identity element.

Definition 3.3. A left cancellative semi-group $(G, *)$ is a semi-group such that the left cancellation law holds ;i.e, for any x, z, z', if

$$x * z = x * z'$$

then $z = z'$.

Definition 3.4. A zero element of a semi-group $(G, *)$ is an element 0 such that $0 * x = x * 0 = 0$ for all x in G.

Definition 3.5. Let f be a function defined over $(G, *)$. The right translation of f by a , f_a, is defined by

$$f_a(x) = f(x * a).$$

<u>Convention.</u> Throughout this chapter $(G, *)$ is a semi-group.

<u>Functional Equations Related to Some Properties of Binary Operations.</u>

Let $f : (G, *) \longrightarrow (G, *)$.

We now define a new binary operation on G, depending on f, by

$$x \,\square\, y = f(x * f(y)). \tag{3.1}$$

<u>Lemma 3.6.</u> If $\square$ and $*$ are commutative binary operations, then $(G, \square)$ is a semi-group.

<u>Proof.</u> Since $\square$ is commutative, $x \,\square\, y = y \,\square\, x$. Thus by Eq(3.1),

$$f(x * f(y)) = f(y * f(x))$$

$$= f(f(x) * y) \quad (* \text{ is commutative}). \tag{3.2}$$

Then
$$\begin{aligned}
(x \,\square\, y) \,\square\, z &= f(x \,\square\, y * f(z)) \\
&= f(f(x * f(y)) * f(z)) \\
&= f((x * f(y)) * f^{(2)}(z)) && (\text{Eq}(3.2) \text{ and } f^{(2)}(z) = f(f(z))) \\
&= f(x * f^{(2)}(z) * f(y)) && (* \text{is associative and commutative}) \\
&= f(f(x * f^{(2)}(z)) * y) && (\text{Eq } (3.2)) \\
&= f(f(f(x) * f(z)) * y) && (\text{Eq } (3.2)) \\
&= f(f(x) * f(z) * f(y)) && (\text{Eq } (3.2)) \\
&= f(f(x) * f(y) * f(z)) && (* \text{is commutative}).
\end{aligned}$$

Similary, $x \square (y \square z) = f(x * f(y \square z))$

$$= f(x * f(f(y * f(z))))$$

$$= f(f(x) * f(y * f(z)))$$

$$= f(f(y * f(z)) * f(x))$$

$$= f(f(y) * f(z) * f(x))$$

$$= f(f(x) * f(y) * f(z))$$

$$= (x \square y) \square z.$$

Hence $(G, \square)$ is a semi-group/

Lemma 3.7. If $(G, \square)$ is a monoïd and e is an identity for a commutative $(G, *)$ which is also an idempotent for $\square$, then $\square$ is commutative.

Proof. Since $\square$ is associative, $(x \square y) \square z = x \square (y \square z)$. Therefore

$$f(x * f(y)) \square z = x \square f(y * f(z))$$

$$f(f(x * f(y)) * f(z)) = f(x * f(f(y * f(z))))$$

$$= f(x * f^{(2)}(y * f(z))). \tag{3.3}$$

Suppose e is an identity for $(G, *)$, i.e, $x * e = e * x = x$ for all $x \in G$. Suppose also that e is an idempotent for $\square$, i.e, $e \square e = e$. Then by Eq(3.1),

$$f(e * f(e)) = f^{(2)}(e) = e.$$

Take $x = e$ in Eq(3.3) we get

$$f(f(e * f(y)) * f(z)) = f(e * f^{(2)}(y * f(z)))$$

so that

$$(3.4)\qquad f(f^{(2)}(y) * f(z)) = f^{(3)}(y * f(z)).$$

Let $y = e$ in Eq(3.4) ; we have

$$f(f^{(2)}(e) * f(z)) = f^{(4)}(z).$$

But $f^{(2)}(e) = e$; hence

$$(3.5)\qquad f^{(2)}(z) = f^{(4)}(z).$$

Now applying f to each member of Eq(3.4),

$$f^{(2)}(f^{(2)}(y) * f(z)) = f^{(4)}(y * f(z))$$

$$= f^{(2)}(y * f(z))$$

follows from Eq (3.5). Now from the last equation, replacing $f(z)$ by $f^{(2)}(z)$, we get

$$f^{(2)}(f^{(2)}(y) * f^{(2)}(z)) = f^{(2)}(y * f^{(2)}(z)).$$

By interchanging y and z, we have

$$f^{(2)}(f^{(2)}(z) * f^{(2)}(y)) = f^{(2)}(z * f^{(2)}(y)).$$

Since $*$ is commutative, the last two equations give

$$(3.6)\qquad f^{(2)}(f^{(2)}(y) * f^{(2)}(z)) = f^{(2)}(y * f^{(2)}(z)) = f^{(2)}(z * f^{(2)}(y)).$$

Replacing $f(z)$ by $f^{(2)}(z)$ in Eq (3.3) , we have

$$f(f(x * f(y)) * f^{(2)}(z)) = f(x * f^{(2)}(y * f^{(2)}(z)))$$

and for $z = e$, we obtain

$$f(f(x * f(y)) * f^{(2)}(e)) = f(x * f^{(2)}(y * f^{(2)}(e))).$$

But $f^{(2)}(e) = e$; hence

$$(3.7) \qquad f^{(2)}(x * f(y)) = f(x * f^{(2)}(y))$$

which gives with $y = e$,

$$f^{(2)}(x * f(e)) = f(x * f^{(2)}(e)).$$

Thus

$$(3.8) \qquad f^{(2)}(x * f(e)) = f(x).$$

By applying $f^{(2)}$ to Eq (3.8) we have

$$f^{(4)}(x * f(e)) = f^{(3)}(x)$$

and then from Eq (3.5), $f^{(2)}(x * f(e)) = f^{(3)}(x)$.

Therefore, from the last equation and Eq (3.8), we get

$$(3.9) \qquad f^{(3)}(x) = f(x).$$

Now applying f to Eq (3.7)

$$f^{(3)}(x * f(y)) = f^{(2)}(x * f^{(2)}(y))$$

which gives, by Eq (3.9), $f(x * f(y)) = f^{(2)}(x * f^{(2)}(y))$.

Therefore by Eq (3.1),

$$x \square y = f^{(2)}(x * f^{(2)}(y)).$$

Finally, by interchanging x and y and then from Eq (3.6) we get the conclusion of the lemma, i.e,

$$y \square x = f^{(2)}(y * f^{(2)}(x)) = f^{(2)}(x * f^{(2)}(y)) = x \square y .$$

Hence $\square$ is commutative/

<u>Definition 3.8</u>. A function $f : (G, *) \longrightarrow (G, *)$ for which $\square$ is commutative is called a <u>multiplicative symmetric (MS) function</u>.

In other word, f is multiplicative symmetric function if f satisfies :

$$f(x * f(y)) = f(y * f(x)).$$

<u>Theorem 3.9.</u> Let $(G, *)$ be a commutative monoid and a function $f : (G, *) \longrightarrow (G, *)$ such that $f(e) = e$ where e is the identity of $(G, *)$. Then $(G, \square)$ is a semi-group if and only if f is a multiplicative symmetric function.

<u>Proof.</u> Assume that f is MS. Then by Lemma 3.6, $(G, \square)$ is a semi-group.

Conversely, the hypothesis $e \square x = x \square e$ implies

$$f(e * f(x)) = f(x * f(e))$$

by virtue of Eq (3.1). Since $f(e) = e$, it follows that $f^{(2)}(x) = f(x)$. Therefore

$$f^{(2)}(e) = f(e) = e.$$

Thus e is an idempotent for $\square$. By Lemma 3.7, $\square$ is commutative. Hence f is a MS function/

Let us now define :

$$(3.10) \qquad x \triangle y = x * f(y).$$

<u>Definition 3.10.</u> A function $f : (G, *) \longrightarrow (G, *)$ is called a <u>semi-multiplicative symmetric (SMS) function</u> if $f(x * f(y)) = f(x) * f(y)$.

<u>Theorem 3.11.</u> Let $(G, *)$ be a left cancellative semi-group, and a function $f : (G, *) \longrightarrow (G, *)$. Then $(G, \triangle)$ is a semi-group if and only if f is a semi-multiplicative symmetric function.

Proof. Assume (G, Δ) is a semi-group. Then by Eq (3.10)

$$(x \Delta y) \Delta z = (x * f(y)) * f(z) = x * (f(y) * f(z))$$
$$= x \Delta (y \Delta z) \qquad = x * f(y * f(z)).$$

Since $(G, *)$ is a left cancellative semi-group.

$$f(y * f(z)) = f(y) * f(z).$$

Hence f is SMS.

Conversely assume that f is SMS. Then

$$f(y * f(z)) = f(y) * f(z)$$

so that. $x * f(y * f(z)) = x * (f(y) * f(z)) = (x * f(y)) * f(z).$

Hence $\quad x \Delta (y \Delta z) = (x \Delta y) \Delta z.$

Now the proposition is completely proved/

Let us define now :

$$(3.11) \qquad x \perp y = f(y * x).$$

Definition 3.12. A function $f : (G, *) \longrightarrow (G, *)$ is called an interpolating (I) function if $f(x * f(y)) = f(x * y)$.

Theorem 3.13. Let $(G, *)$ be a commutative monoïd and e the identity element of $(G, *)$ which is an idempotent for $\perp$. A necessary and sufficient condition for $(G, \perp)$ to be a commutative semi-group is that f is an interpolating function.

Proof. Suppose first that $(G, \perp)$ is a commutative semi-group. Then by Eq (3.11),

$$(x \perp y) \perp z = f(y * x) \perp z = f(z * f(y * x))$$
$$= x \perp (y \perp z)$$
$$= f((y \perp z) * x) = f(f(z * y) * x).$$

Therefore, $f(z* f(y* x)) = f(f(z*y)*x)$.

Take $x = e$ in the last equation, we get

$$f(z* f(y)) = f(f(z*y)) = f^{(2)}(z*y). \tag{3.12}$$

Put $y = e$ in Eq (3.12) to obtain

$$f(z* f(e)) = f^{(2)}(z).$$

But from the hypothesis, $e \perp e = e$ and hence $f(e* e) = f(e) = e$; so that

$$f(z* e) = f(z) = f^{(2)}(z).$$

Thus Eq (3.12) becomes

$$f(z* f(y)) = f(z* y)$$

and f is an interpolating function.

On the other hand, assume f is an interpolating function, i.e,

$$f(x* f(y)) = f(x*y).$$

Then
$$\begin{aligned}(x \perp y) \perp z &= f(z* f(y* x)) = f(z*(y*x)) \\ &= f(x*(z* y)) = f(x*f(z*y)) \\ &= f(f(z* y)* x) = x \perp (y \perp z).\end{aligned}$$

Since $*$ is commutative, $x \perp y = f(y* x) = f(x*y) = y \perp x$.

Thus $(G, \perp)$ is a commutative semi-group/

Some Classes of Functional Equations.

From the first section, we have seen some classes of functions, namely, the multiplicative symmetric, the semi-multiplicative symmetric and the interpolating functions. In this section, we will define a new class of functions —— the demi-multiplicative symmetric function, and then we shall consider some elementary properties of all these functions.

Definition 3.14. A function $f : (G, *) \longrightarrow (G, *)$ is called a demi-multiplicative symmetric (DMS) function if $f(x * f(y)) = f(f(x) * f(y))$.

Lemma 3.15. Let $(G, *)$ be a commutative semi-group and $f : G \longrightarrow G$ be a function of the MS or SMS type. Then for any $a \in G$, f_a is of the same type.

Proof. Suppose f is a MS function. We prove that $f_a : x \longmapsto x * a$ is also a MS function.

$$\begin{aligned} f_a(x * f_a(y)) &= f((x * f_a(y)) * a) = f((x * f(y * a)) * a) \\ &= f((x * a) * f(y * a)) \\ &= f((y * a) * f(x * a)) \qquad (f \text{ is MS}) \\ &= f((y * a) * f_a(x)) \\ &= f_a(y * f_a(x)). \end{aligned}$$

Therefore f_a is MS.

Similarly, if f is SMS we can show that f_a is of the same type.

Hence the lemma is proved/

This lemma will be useful to change " scales " on $(G, *)$. We shall say that a semi-group $(G_o, *)$ is a group with zero if $(G = G_o \setminus \{0\}, *)$ is a group and 0 is the zero element of G_o.

Lemma 3.16. Let $(G_o, *)$ be a group with a zero and f be a MS function on $(G_o, *)$. Then $f(0) = 0$ or f is constant on G_o.

Proof. Since f is MS on $(G_0, *)$, if $(x * f(y)) = f(y * f(x))$. Take $x = 0$, then

$$f(0 * f(y)) = f(0) = f(y * f(0)).$$

If $f(0) \neq 0$, then for any x in $G_0 \setminus \{0\} = G$,

$$x = x * e = x * (f(0))^{-1} * f(0)$$

so that we can write x as

$$x = y * f(0) \quad \text{where } y = x * (f(0))^{-1}.$$

Thus

$$f(x) = f(y * f(0))$$

$$= f(0 * f(y))$$

by the MS-property of f. Therefore

$$f(x) = f(0) \qquad (x \in G).$$

That is f is constant on G_0.

Hence the lemma is proved/

Remark. This lemma can be used for semi-multiplicative symmetric function or interpolating function without commutativity assumption.

The proof of this remark is similar to that of Lemma 3.16.

Lemma 3.17. Let $(G, *)$ be a semi-group. Let $G_0 = G \cup \{0\}$ and define $0 * x = x * 0 = 0$. Let $f : G \longrightarrow G$. We can extend f to

$$f_{(0)} : G_0 \longrightarrow G_0$$

by defining $f_{(0)}(0) = 0$. If f is MS, SMS, DMS or I on $(G, *)$, $f_{(0)}$ is of the same type on $(G_0, *)$.

Proof. Assume f is MS on $(G, *)$. Then $f_{(0)}$ is MS on $(G, *)$. To show that $f_{(0)}$ is MS on $(G_0, *)$, it suffices to show that

$$f_{(0)}(x * f_{(0)}(y)) = f_{(0)}(y * f_{(0)}(x)) \quad \text{if } x = 0.$$

Suppose $x = 0$; then

$$f_{(0)}(0 * f_{(0)}(y)) = f_{(0)}(0) = 0$$

by definitions. Now consider

$$f_{(0)}(y * f_{(0)}(0)) = f_{(0)}(y * 0) = f_{(0)}(0) = 0.$$

Therefore $f_{(0)}(0 * f_{(0)}(y)) = f_{(0)}(y * f_{(0)}(0))$, and $f_{(0)}$ is MS on $(G_0, *)$.

Similarly, if f is SMS, DMS or I on $(G, *)$, we can show that $f_{(0)}$ is of the same type on $(G_0, *)$/