CHAPTER II

SOME FUNCTIONAL EQUATIONS OF THE CAUCHY TYPE

$f(x \circ y) = f(x) * f(y)$; $f(x \circ f(y)) = f(x) * f(f(y))$.

The materials of this chapter are based on the references [1] and [5], and all functions are assumed to map the reals $\mathbb{R}$ into itself.

Cauchy functional equations are given by

(2.1) $f(x + y) = f(x) + f(y)$

(2.2) $f(x + y) = f(x) . f(y)$

(2.3) $f(x . y) = f(x) . f(y)$

(2.4) $f(x . y) = f(x) + f(y)$.

<u>Continuous Solutions of Eq (2.1)</u>. Suppose f is a solution of Eq(2.1). Then putting $x = 0 = y$, Eq (2.1) becomes

$$f(0) = f(0) + f(0)$$

so that

(2.5) $f(0) = 0$.

<u>Lemma 2.1</u>. $f(n) = nf(1)$, for all $n \in \mathbb{Z}$.

<u>Proof</u>. We prove first when $n \geq 0$ by induction on n. Since Eq(2.5) implies that the lemma holds for $n = 0$ and the conclusion is clearly for $n = 1$, assume the lemma holds for lesser values of n. Then

$$f(n) = f(n - 1 + 1) = f(n - 1) + f(1)$$
$$= (n - 1) f(1) + f(1) = nf(1)$$

by induction hypothesis. Thus the lemma holds for $n \geq 0$.

If n is a positive integer, then

$$0 = f(0) = f(n - n) = f(n) + f(-n).$$

Therefore by the first part of the proof,

$$f(-n) = -f(n) = -nf(1).$$

Hence the lemma holds for any integer n /

Lemma 2.2. $f(r) = rf(1)$, for any rational number r.

Proof. For $q \in \mathbb{Z}$ (>0);

$$f(1) = f(q \cdot \frac{1}{q}) = f(\underbrace{\frac{1}{q} + \dots + \frac{1}{q}}_{q \text{ times}})$$

$$= \underbrace{f(\frac{1}{q}) + \dots + f(\frac{1}{q})}_{q \text{ times}}$$

$$= qf(\frac{1}{q}),$$

so that

(2.6) $$f(\frac{1}{q}) = \frac{1}{q}f(1) \quad (q \in \mathbb{Z} \ (>0)).$$

Let r be any positive rational number, $r = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ (>0) and $q \neq 0$.

$$f(r) = f(\frac{p}{q}) = f(p. \frac{1}{q})$$

$$= f(\underbrace{\frac{1}{q} + \dots + \frac{1}{q}}_{p \text{ times}})$$

$$= pf(\frac{1}{q})$$

$$= p \cdot \frac{1}{q} f(1)$$

by Eq (2.6). Hence $f(r) = rf(1)$ for $r \geqslant 0$. Since $f(0) = 0$,

$f(-r) = -f(r)$. Thus $f(r) = rf(1)$ for any rational number r /

Now we will prove the first main results :

<u>Theorem 2.3</u>. If $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

(2.1) $$f(x + y) = f(x) + f(y)$$

and if f is continuous at $x = 0$, then (f is continuous overywhere and) $f(x) = ax$ for some a in $\mathbb{R}$.

<u>Proof</u>. Suppose f is continuous at $x = 0$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x| < \delta \text{ implies } |f(x) - f(0)| = |f(x)| < \varepsilon .$$

Then, if for any x and y in $\mathbb{R}$, $|x - y| < \delta$ then $|f(x - y)| < \varepsilon$.

But $$|f(x - y)| = |f(x) - f(y)| ,$$

hence

$$|f(x) - f(y)| < \varepsilon .$$

Therefore f is everywhere continuous.

Let $x \in \mathbb{R}$. Since the rationals are dense in $\mathbb{R}$, we can find a sequence $\{r_n\}$ of rationals converges to x. Since f is continuous,

$$\lim_{n\to\infty} f(r_n) = f(\lim_{n\to\infty} r_n) = f(x).$$

But by Lemma 2.2,

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n f(1) = xf(1).$$

Therefore,

$$f(x) = xf(1) \qquad (x \in \mathbb{R}).$$

$$f(x) = ax \qquad (x \in \mathbb{R}).$$

for some $a = f(1)$ in $\mathbb{R}$. Moreover, this function satisfies Eq (2.1).

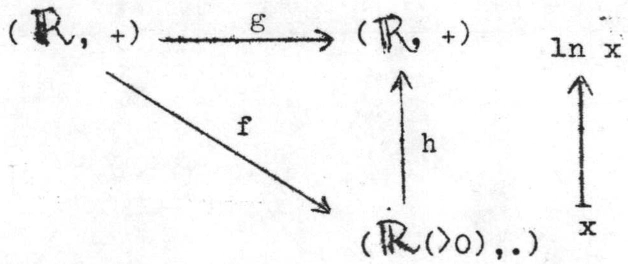
Thus the theorem is proved /

Continuous Solutions of Eq (2.2).

To solve this equation, we will construct a new function, based on f, which satisfies Eq (2.1) whose continuous solutions are readily available.

Theorem 2.4. If $f : \mathbb{R} \to \mathbb{R}(>0)$ is a continuous function satisfies Eq (2.2), then $f(x) = \lambda^x$ for some λ in $\mathbb{R}$.

Proof. Consider the diagram :



where $g = h \circ f$. Since h and f are continuous, g is also continuous and

$$g(x) = \ln (f(x)).$$

Then

$$\begin{aligned} g(x + y) &= \ln (f(x + y)) = \ln (f(x) \cdot f(y)) \\ &= \ln (f(x)) + \ln (f(y)) \\ &= g(x) + g(y). \end{aligned}$$

By applying Theorem 2.3,

$$g(x) = ax \qquad (x \in \mathbb{R})$$

for some a in $\mathbb{R}$. Therefore

$$\ln (f(x)) = ax.$$

Hence $f(x) = e^{ax} = \lambda^{x}$ where $\lambda = e^{a}$ in $\mathbb{R}$.

Moreover, this function satisfies Eq (2.2). Hence the theorem is now proved /

<u>Continuous Solutions of Eq (2.3)</u>.

We will solve Eq (2.3) by the same method as we have used in solving Eq (2.2).

<u>Theorem 2.5</u>. If $f : \mathbb{R} \to \mathbb{R}(>0)$ is a continuous function satisfies Eq (2.3), then $f(x) = x^{a}$ for some a in $\mathbb{R}$.

<u>Proof</u>. Consider the diagram :

$$\begin{array}{ccccccc}
x & & (\mathbb{R}, +) & \xrightarrow{\quad g \quad} & (\mathbb{R}, +) & & \ln x \\
\downarrow & & \downarrow h_1 & & \uparrow h_2 & & \uparrow \\
e^{x} & & (\mathbb{R}, .) & \xrightarrow{\quad f \quad} & (\mathbb{R}(>0), .) & & x
\end{array}$$

where $g = h_2 \circ f \circ h_1$. Since h_1, f and h_2 are continuous, g is continuous and

$$g(x) = \ln (f(e^{x})).$$

Then

$$\begin{aligned}
g(x + y) &= \ln (f(e^{x+y})) = \ln (f(e^{x}. e^{y})) \\
&= \ln (f(e^{x}) . f(e^{y})) = \ln (f(e^{x})) + \ln (f(e^{y})) \\
&= g(x) + g(y).
\end{aligned}$$

Therefore, by applying Theorem 2.3, for some a in $\mathbb{R}$

$$g(x) = ax \quad (x \in \mathbb{R}).$$

Thus $\ln (f(e^x)) = ax$.

That is $f(e^x) = e^{ax}$.

Let $x = \ln t$. Then

$$f(t) = f(e^{\ln t}) = e^{a \ln t} = e^{\ln t^a} = t^a$$

so that

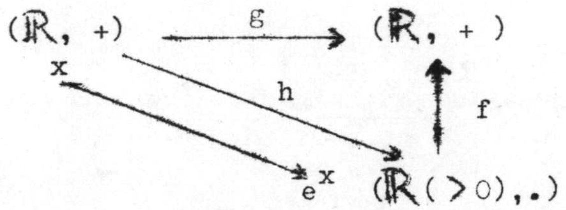
$$f(x) = x^a \qquad (x \in \mathbb{R}).$$

Conversely, this function clearly satisfies Eq (2.3) /

Continuous Solutions of Eq (2.4).

Theorem 2.6. If $f : \mathbb{R}(>0) \to \mathbb{R}$ is a continuous function satisfies Eq (2.4), then $f(x) = a \ln x$ for some a in $\mathbb{R}$.

Proof. Consider the diagram :



where $g = f \circ h$. Since f and h are continuous, g is continuous and

$$g(x) = f(e^x).$$

Therefore,

$$\begin{aligned} g(x + y) &= f(e^{x+y}) \\ &= f(e^x \cdot e^y) = f(e^x) + f(e^y) \\ &= g(x) + g(y). \end{aligned}$$

Apply Theorem 2.3, we have for some a in $\mathbb{R}$

$$g(x) = ax \qquad (x \in \mathbb{R}).$$

Thus $f(e^x) = ax \quad (x \in \mathbb{R})$

so that $f(t) = a \ln t \quad (t \in \mathbb{R})$.

Moreover, this function satisfies Eq (2.4).

Hence the theorem is proved /

Functional Equations of Cauchy Type.

In this section, we shall consider some variations of Cauchy's functional equations :

$$(2.7) \qquad f(x + f(y)) = f(x) + f(f(y))$$

$$(2.8) \qquad f(x + f(y)) = f(x) \cdot f(f(y))$$

$$(2.9) \qquad f(x \cdot f(y)) = f(x) \cdot f(f(y))$$

$$(2.10) \qquad f(x \cdot f(y)) = f(x) + f(f(y)).$$

Eq (2.7). Let f be a continuous solution of Eq (2.7). We then immediately get

$$(2.11) \qquad f(x + z) = f(x) + f(z) \quad (x \in \mathbb{R},\ z \in f(\mathbb{R})).$$

If $f \equiv c$, then

$$c = c + c$$

which implies that $c = 0$ so that we get a proposition :

Proposition 2.7. If f is identically constant, then $f \equiv 0$.

Now further assume that f is non-constant and let $a = f(x_0) \neq 0$. Then

$$f(f(y)) = f(0 + f(y)) = f(0) + f(f(y)).$$

Thus $f(0) = 0$.

Lemma 2.8. $f(na) = nf(a) \quad (n \in \mathbb{Z})$.

Proof. We will show that $f(na) = nf(a)$ for all $n \in \mathbb{Z}\ (\geq 0)$. The case $n = 0$ is true because $f(0) = 0$. For $n = 1$,

$$f(1.a) = 1.f(a).$$

Assume the lemma holds for lesser values of n. Then from induction hypothesis,

$$\begin{aligned} f(n \cdot a) = f(na - a + a) &= f(na - a) + f(a) \\ &= (n - 1)\, f(a) + f(a) \\ &= nf(a). \end{aligned}$$

Hence by induction, the lemma holds for all non-negative integers.

If n is a non-negative integer, then

$$0 = f(0) = f(na - na) = f(na) + f(-na)$$

by another inductive argument so that

$$f(-na) = -f(na).$$

Therefore the lemma holds for all integers n /

Lemma 2.9. $f(ra) = rf(a)$, for any rational number r.

Proof. Since $0, a \in f(\mathbb{R})$ and f is continuous, it follows from the Intermediate Value Theorem that $\frac{a}{q} \in f(\mathbb{R})$ for all $q \in \mathbb{Z}\ (>0)$.

Let $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}(>0)$.

$$f(a) = f(\tfrac{q}{q}.\, a) = f(\underbrace{\tfrac{a}{q} + \ldots + \tfrac{a}{q}}_{q \text{ times}})$$

$$= q\, f\left(\frac{a}{q}\right)$$

by Eq (2.11) so that

$$(2.12) \qquad f\left(\frac{a}{q}\right) = \frac{1}{q} f(a).$$

Now $f(ra) = f(\frac{p}{q} \cdot a) = f(\underbrace{\frac{a}{q} + \dots + \frac{a}{q}}_{p \text{ times}})$

$$= p\, f\left(\frac{a}{q}\right) = p \cdot \frac{1}{q} f(a)$$

by Eq (2.11) and Eq (2.12). Hence for any non-negative rational number r,

$$f(ra) = rf(a) .$$

But $0 = f(0) = f(ra - ra) = f(ra) + f(-ra)$, hence $f(-ra) = -f(ra)$.

Therefore the lemma holds for all rational number /

Now we will prove :

<u>Theorem 2.10.</u> If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying

(2.7) $\qquad f(x + f(y)) = f(x) + f(f(y)),$

then $f(x) = kx$ for some k in $\mathbb{R}$.

<u>Proof</u>. We may assume that $f \not\equiv 0$.

For any x in $\mathbb{R}$, there exists a sequence $r_1, r_2, \dots$ of rational number such that $r_n \cdot a$ converging to x, where $a = f(x_o) \neq 0$. Since f is continuous,

$$\lim_{n\to\infty} f(r_n a) = f(\lim_{n\to\infty} r_n a) = f(x) .$$

But from Lemma 2.9,

$$\lim_{n\to\infty} f(r_n \cdot a) = \lim_{n\to\infty} (r_n \cdot f(a)) = \frac{x}{a} \cdot f(a).$$

Hence $\qquad f(x) = \frac{x}{a} f(a)$

$$= kx$$

where $k = \frac{f(a)}{a}$ is in $\mathbb{R}$. Moreover, this function clearly satisfies Eq (2.7).

Hence the theorem is completely proved /

Eq (2.8). Suppose that f is a continuous solution of Eq (2.8).

First, suppose that $f(f(y)) = 0$ for all y, then $f(x + f(y)) = 0$ for all x , y in $\mathbb{R}$. If further $f \not\equiv 0$, there exists y_0 in $\mathbb{R}$ such that $f(y_0) \neq 0$. Therefore

$$\begin{aligned} f(x + f(y_0)) &= f(x) \cdot f(f(y_0)) \\ &= 0 \end{aligned}$$

which contradicts to the assumption that $f \not\equiv 0$. Hence

Proposition 2.11. If $f(f(y)) = 0$ for all y, then f is identically 0.

Suppose now that there exists y_0 such that $f(f(y_0)) \neq 0$. Then from Eq (2.8),

$$f(f(y_0)) = f(0 + f(y_0)) = f(0) \cdot f(f(y_0)).$$

Therefore $f(0) = 1$.

Proposition 2.12. If there exists y_0 such that $f(f(y_0)) \neq 0$. Then $f(0) = 1$.

Henceforth we shall assume that $f(0) = 1$, and $f(1) \neq 0$. Note that $f(1) = 0$ implies $f(x + 1) = f(x) \cdot f(1) = 0$ for all x which is impossible.

Lemma 2.13. For $n \in \mathbb{Z}$, $f(x + n) = f(x)\, f(1)^n$ so that $f(1)^n = (f(1))^n$.

Proof. By induction on n, the lemma is obviously true for $n = 0$. (Here we need $f(1) \neq 0$). Now Eq (2.8) gives $f(x + 1) = f(x+f(0)) = f(x) \cdot f(1)$ so that the lemma holds for $n = 1$.

Assume the lemma holds for lesser values of n. Therefore

$$f(x+n) = f(x+n-1+1) = f(x+n-1) \cdot f(1)$$
$$= f(x)\, f(1)^{n-1} f(1) = f(x)\cdot f(1)^n,$$

by induction hypothesis.

Hence the lemma holds for any non-negative integer.

Since f(0) = 1 and for any positive integer n

$$f(x) = f(x-n+n) = f(x-n)\, f(1)^n ,$$
$$f(x-n) = f(x)\, f(1)^{-n} .$$

Thus $f(x+n) = f(x)\, f(1)^n$ for all $n \in \mathbb{Z}$.

Take now x = 0. Then $f(n) = f(0)\, f(1)^n = f(1)^n$ $(n \in \mathbb{Z})$ /

<u>Lemma 2.14</u>. $f(r) = f(1)^r$, for any rational number r.

<u>Proof</u>. Since $0, 1 \in f(\mathbb{R})$, $\frac{1}{q} \in f(\mathbb{R})$ for all $q \in \mathbb{Z}(>0)$ by the Intermediate Value Theorem.

For $q \in \mathbb{Z}(>0)$, $f(1) = f(\frac{1}{q} \cdot q)$

$$= f(\underbrace{\tfrac{1}{q} + \dots + \tfrac{1}{q}}_{q \text{ times}})$$

$$= f(\tfrac{1}{q})^q$$

by applying Eq (2.8) repeatedly q times and since $\frac{1}{q} \in f(\mathbb{R})$, then

$$f(\tfrac{1}{q}) = f(1)^{1/q} . \qquad (2.13)$$

Let $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}(>0)$.

$$f(r) = f(\tfrac{p}{q}) = f(\underbrace{\tfrac{1}{q} + \dots + \tfrac{1}{q}}_{p \text{ times}})$$

$$= f(\tfrac{1}{q})^p = f(1)^{\frac{1}{q}\cdot p} = f(1)^r$$

by Eq (2.8) and Eq (2.13). But for any positive rational r,

$$1 = f(-r + r) = f(-r) \,.\, f(r)$$

hence $$f(-r) = f(r)^{-1}.$$

Therefore the lemma holds for all rational numbers /

Now we will prove a main result :

Theorem 2.15. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying

(2.8) $$f(x + f(y)) = f(x) \,.\, f(f(y))$$

then $f \equiv 0$ or $f(x) = \lambda^x$ for some non-zero λ in $\mathbb{R}$.

Proof. Suppose $f \not\equiv 0$. Then $f(0) = 1$ by Proposition 2.11 and 2.12. For any x in $\mathbb{R}$, there exists a sequence $r_1, r_2, \ldots$ of rational numbers converging to x. Since f is continuous

$$\lim_{n\to\infty} f(r_n) = f(\lim_{n\to\infty} r_n) = f(x).$$

But by Lemma 2.14,

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} f(1)^{r_n} = f(1)^x.$$

Hence $f(x) = f(1)^x = \lambda^x$ for some non-zero λ in $\mathbb{R}$.

Moreover, this function satisfies Eq (2.8).

Hence the theorem is proved /

Eq (2.9). Let f be a solution of Eq (2.9).

Note first that if $f \equiv c$, then from Eq (2.9) we have

$$c = c.c$$

Therefore $c = 0$ or 1.

Proposition 2.16. If f is a constant function, then $f \equiv 0$ or 1.

Now assume that <u>f is non-constant</u>.

<u>Theorem 2.17</u>. If $f : \mathbb{R}(>0) \longrightarrow \mathbb{R}(>0)$ is a continuous function satisfying

(2.9) $$f(x\,f(y)) = f(x)\,f(f(y)).$$

Then $f(x) = x^c$ for some c in $\mathbb{R}$.

<u>Proof</u>. Consider the diagram :

$$\begin{array}{ccccc} x & (\mathbb{R}, +) & \xrightarrow{\quad g \quad} & (\mathbb{R}, +) & \ln x \\ \downarrow & \Big\downarrow h_2 & & \Big\uparrow h_1 & \uparrow \\ e^x & (\mathbb{R}(>0), .) & \xrightarrow{\quad f \quad} & (\mathbb{R}(>0), .) & x \end{array}$$

where $g = h_1 \circ f \circ h_2$. Since h_1, f and h_2 are continuous, g is continuous and

$$g(x) = \ln (f(e^x)).$$

Then
$$\begin{aligned} g(x + g(y)) &= \ln (f(e^{x+\ln f(e^y)})) \\ &= \ln (f(e^x . f(e^y))) \\ &= \ln (f(e^x) . f(f(e^y))) \\ &= \ln (f(e^x)) + \ln (f(f(e^y))) \\ &= \ln (f(e^x)) + \ln (f(e^{\ln f(e^y)})) \\ &= g(x) + g(g(y)). \end{aligned}$$

By applying Theorem 2.10,

$$g(x) = cx \quad (x \in \mathbb{R})$$

for some c in $\mathbb{R}$. Thus

$$\ln (f(e^x)) = cx$$

$$f(e^x) = e^{cx}.$$

Therefore $$f(x) = x^c \quad (x \in \mathbb{R}).$$

Moreover, this function satisfies Eq (2.9).

Hence the theorem is now completely proved /

Eq (2.10). Let f be a solution of Eq (2.10).

Let $x = 1$ in Eq (2.10). Then

$$f(1 \cdot f(y)) = f(1) + f(f(y))$$

for any y in $\mathbb{R}$; therefore

$$f(1) = 0$$

Thus $f(x\, f(1)) = f(x) + f(f(1))$

$= f(x) + f(0).$

But $f(x\, f(1)) = f(x \cdot 0) = f(0)$; hence

$$f(0) = f(x) + f(0)$$

which implies that $f(x) = 0$ for all x.

Hence the function which satisfies Eq (2.10) is the zero function. Then we have proved :

Theorem 2.18. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function which satisfies Eq (2.10), then f is identically zero.

Jensen's Functional Equations.

In this section, we shall consider the functional equation of the form :

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

which is known as Jensen's equation (see [1]). This functional equations can be reduced to the Cauchy's functional equation :

$$g(x + y) = g(x) + g(y).$$

The particular functional equation we shall solve is given by :

$$(2.14) \qquad f\left(\frac{x + f(y)}{2}\right) = \frac{f(x) + f(y)}{2}.$$

To find the continuous solutions of this equation, we will reduce it to a semi-multiplicative symmetric equation :

$$g(x + g(y)) = g(x) + g(y).$$

By assuming the validity of Theorem 6.6 in chapter VI, we will find the continuous solutions of Eq (2.14).

For convenience, let us state that theorem first.

Theorem 6.6. If $g : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +)$ is a continuous function satisfying

$$(2.15) \qquad g(x + g(y)) = g(x) + g(y)$$

then g identically 0 or $g(x) = x + c$ for some c in $\mathbb{R}$.

Lemma 2.19. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Eq (2.14), and if there exists an x_o in $\mathbb{R}$ with $f(x_o) = 0$, then $f(0) = 0$.

Proof. Set $x = f(y)$ in Eq (2.14). Then

$$f(f(y)) = f\left(\frac{f(y) + f(y)}{2}\right) = \frac{f(f(y)) + f(y)}{2}$$

so that

$$(2.16) \qquad f(f(y)) = f(y).$$

If $f(x_o) = 0$, then it follows from Eq (2.16) that

$$f(f(x_o)) = f(0) = f(x_o) = 0.$$

Hence the lemma is proved /

Theorem 2.20. The continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

(2.14) $$f\left(\frac{x + f(y)}{2}\right) = \frac{f(x) + f(y)}{2}$$

are either constant or $f(x) = x$.

Proof. It follows from Eq (2.14) that

$$\frac{f(x) + f(y)}{2} = f\left(\frac{x + f(y)}{2}\right)$$

$$= f\left(\frac{x + f(y) - f(0) + f(0)}{2}\right)$$

$$= \frac{f(x + f(y) - f(0)) + f(0)}{2}$$

so that

(2.17) $$f(x + f(y) - a) = f(x) + f(y) - a,$$

where $a = f(0)$. Let

(2.18) $$g(x) = f(x) - a.$$

Then from Eq (2.17) and (2.18), we get

$$g(x + g(y)) = g(x) + g(y).$$

Now, it follows from the Theorem 6.6 that

$$g = 0$$

or $$g(x) = x \quad (x \in \mathbb{R})$$

and by virtue of Eq (2.18),

$$f(x) = a$$

or $$f(x) = x + a \quad (x \in \mathbb{R}).$$

In the latter case, we have $f(-a) = 0$ so that, by Lemma 2.19, $f(0) = 0 = a$. Hence $f \equiv a$ or $f(x) = x$ $(x \in \mathbb{R})$. Moreover, these functions satisfy Eq (2.14).

The theorem is now proved /