CHAPTER I

3-DIMENSIONAL NILPOTENT ALGEBRAS OVER AN ALGEBRICALLY CLOSED FIELD OF CHARACTERISTIC $\neq 2$.

In this chapter we classify the nilpotent algebras of dimension 3 over arbitrary algebraically closed fields of characteristic $\neq 2$ up to isomorphism. The material of this chapter is drawn from reference [1].

Let A be a nilpotent algebra of dimension 3 over a field K. Then there exists a $m > 1$ such that $A^m = \{0\}$. Let k be the smallest such m. We claim that $A \supset A^2 \supset A^3 \supset \ldots \supset A^k = \{0\}$. Suppose that $A^n = A^{n+1}$ for some $n < k$, then we can see that

$$A^{n+2} = A^{n+1}. A = A^{n+1} = A^n$$

$$A^{n+3} = A^{n+2}. A = A^{n+1} = A^n$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$A^k = A^n,$$

which implies that $A^n = \{0\}$. But this contradicts to the smallest of k. Therefore $A \supset A^2 \supset A^3 \supset \ldots \supset A^k = \{0\}$. Thus we see that dimension $A^2 = 2$ or 1 or 0. Dimension $A^2 = 0$ is the trivial case, so we just consider the case where dimension $A^2 = 1$, or dimension $A^2 = 2$. If dimension $A^2 = 1$, then $A^3 = \{0\}$. If dimension A^2 is 2, then dimension A^3 is 1 or 0 and $A^4 = \{0\}$.

The case where the dimension of A^2 is 2 and $A^3 = \{0\}$ is impossible. See proof in [1] page 41.

Next, we shall consider the other cases of a nilpotent algebra of dimension 3.

Remark: The following theorem is true for arbitrary fields. In [1] it was proven only for $\mathbb{R}$.

Theorem: Let A be a nilpotent algebra of dimension 3 over the field K. If dimension of A^2 is 2 and dimension of $A^3 = 1$, $A^4 = \{0\}$, then the multiplication in A is uniquely determined up to isomorphism.

Proof: Since the dimension of A is 3, dimension of $A^2 = 2$, dimension of $A^3 = 1$ and $A^4 = \{0\}$, we let $\{e_1, e_2, e_3\}$ be a basis in A such that $\{e_2, e_3\}$ is a basis of A^2 and e_3 is a basis of A^3. For each x, y in A we have

$$x = \sum_{i=1}^{3} a_i e_i, \quad y = \sum_{j=1}^{3} b_j e_j, \quad \{a_i, b_j\} \subset K, \ i,j = 1,2,3.$$

Hence

$$xy = \sum_{j=1}^{3} \sum_{i=1}^{3} a_i b_j e_i e_j .$$

Since $e_2^2, e_1e_3, e_3e_1 \varepsilon A^4 = \{0\}$, $e_2e_3, e_3e_2 \varepsilon A^5 = \{0\}$ and $e_3^2 \varepsilon A^6 = \{0\}$, we have

$$xy = a_1b_1e_1^2 + a_1b_2e_1e_2 + a_2b_1e_2e_1 .$$

Since $e_1^2 \varepsilon A^2$, we can write $e_1^2 = k_1e_2 + k_2e_3$ for some k_1, k_2 in K and since $e_1e_2, e_2e_1 \varepsilon A^3$ we get $e_1e_2 = k_3e_3$ and $e_2e_1 = k_4e_3$ for some k_3, k_4 in K. Thus xy can be expressed in the form

$$(*) \qquad xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_3a_1b_2 + k_4a_2b_1)e_3 .$$

Now we consider k_1, k_2, k_3, k_4. Since dimension of $A^2 = 2$, the case $k_1 = 0$ and the case $k_2 = k_3 = k_4 = 0$ can not occur. The proof proceeds with 7 cases. The proof of case 1 to case 6 is the same as [1] page 45-47. Now we consider the last step of proof.

Case 7. Assume that all k_i, $i = 1,2,3,4$ are not zero. Then the multiplication (*) is

$$(7.1) \qquad xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_3a_1b_2 + k_4a_2b_1)e_3 .$$

Let $z = \sum_{\ell=1}^{3} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3} \subset K$. Then (7.1) implies that

$$(xy)z = \left[\left(\sum_{i=1}^{3} a_ie_i\right)\left(\sum_{j=1}^{3} b_je_j\right)\right]\left(\sum_{\ell=1}^{3} c_\ell e_\ell\right)$$

$$= \left[k_1a_1b_1e_2 + (k_2a_1b_1 + k_3a_1b_2 + k_4a_2b_1)e_3\right]\left(\sum_{\ell=1}^{3} c_\ell e_\ell\right)$$

$$= k_4(k_1a_1b_1)c_1e_3$$

whereas, $$x(yz) = \left(\sum_{i=1}^{3} a_ie_i\right)\left[\left(\sum_{j=1}^{3} b_je_j\right)\left(\sum_{\ell=1}^{3} c_\ell e_\ell\right)\right]$$

$$= \left(\sum_{i=1}^{3} a_ie_i\right)\left[k_1b_1c_1e_2 + (k_2b_1c_1 + k_3b_1c_2 + k_4b_2c_1)e_3\right]$$

$$= k_3a_1(k_1b_1c_1)e_3 .$$

Since A is an associative algebra, we must have that $k_1k_4a_1b_1c_1 = k_1k_3a_1b_1c_1$. That is $k_3 = k_4$. Hence the multiplication (7.1) becomes

$$(7.2) \qquad xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_3a_1b_2 + k_3a_2b_1)e_3.$$

We claim that this multiplication is isomorphic to the multiplication in case 4. In case 4 we have that

$$(4.1) \qquad x \circ y = a_1'b_1'e_2' + (a_1'b_2' + a_2'b_1')e_3'$$

where $x = \sum_{i=1}^{3} a_i'e_i'$, $y = \sum_{j=1}^{3} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, $i,j = 1,2,3.$

To prove this, let $f : A \to A$ be the linear map defined by

$$f(e_1') = e_1 ,$$
$$f(e_2') = k_1 e_2 + k_2 e_3 ,$$
$$f(e_3') = k_1 k_3 e_3 , \quad k_1, k_2, k_3 \in K.$$

We have that

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & k_1 & k_2 \\ 0 & 0 & k_1 k_3 \end{bmatrix} = k_1^2 k_3 \neq 0 .$$

Therefore, f is 1-1 and onto. (4.1) implies that

$$f(x o y) = f\left[\left(\sum_{i=1}^{3} a_i' e_i'\right) o \left(\sum_{j=1}^{3} b_j' e_j'\right)\right]$$

$$= f\left[a_1' b_1' e_2' + (a_1' b_2' + a_2' b_1') e_3'\right]$$

$$= k_1 a_1' b_1' e_2 + (k_2 a_1' b_1' + k_1 k_3 a_1' b_2' + k_1 k_3 a_2' b_1') e_3 ,$$

on the other hand, (7.2) gives

$$f(x) f(y) = f\left(\sum_{i=1}^{3} a_i' e_i'\right) f\left(\sum_{j=1}^{3} b_j' e_j'\right)$$

$$= \left[a_1' e_1 + k_1 a_2' e_2 + (k_2 a_2' + k_1 k_3 a_3') e_3\right]\left[b_1' e_1 + k_1 b_2' e_2 + (k_2 b_2' + k_1 k_3 b_3') e_3\right]$$

$$= k_1 a_1' b_1' e_2 + (k_2 a_1' b_1' + k_1 k_3 a_1' b_2' + k_1 k_3 a_2' b_1') e_3 .$$

That is $f(x o y) = f(x) f(y)$. Therefore, this two multiplications are isomorphic.

Hence, we have already proved that the multiplication in a nilpotent algebra A of dimension 3 over the field K with dimension $A^2 = 2$, dimension $A^3 = 1$ and $A^4 = \{0\}$ is uniquely determined up to isomorphism. #

Suppose A is a nilpotent algebra of dimension 3 with dimension $A^2 = 1$ and $A^3 = \{0\}$. Let $\{e_1, e_2, e_3\}$ and $\{e_1', e_2', e_3'\}$ be bases in A such that e_3 and e_3' are in A^2. If $f : A \to A$ is an isomorphism, then $f : A^2 \to A^2$. Therefore, $f(e_3) \in A^2$. Consequently, we may write

$$f(e_1) = m_1e_1' + m_2e_2' + m_3e_3' ,$$

$$f(e_2) = p_1e_1' + p_2e_2' + p_3e_3' ,$$

$$f(e_3) = qe_3', \qquad \{m_i, p_j, q\} \subset K,\ i,j = 1,2,3,$$

$$q \neq 0 \text{ in } K.$$

The classification of 3-dimensional nilpotent algebras A over $\mathbb{R}$ with dimension $A^2 = 1$ has already done in [1]. Now we begin to discuss the classification of 3-dimensional nilpotent algebras A over arbitrary algebraically closed fields K of characteristic $\neq 2$ such that dimension $A^2 = 1$, by choosing a basis e_1, e_2, e_3 in A such that $e_3 \in A^2$. First, note that it isn't necessary to check associativity in this case since $A^3 = \{0\}$. For each x, y in A we have

$$x = \sum_{i=1}^{3} a_ie_i ,$$

$$y = \sum_{j=1}^{3} b_je_j, \quad \{a_i, b_j\} \subset K,\ i,j = 1,2,3.$$

It follows that

$$xy = \sum_{j=1}^{3} \sum_{i=1}^{3} a_i b_j e_i e_j .$$

Since $e_1e_3, e_3e_1, e_2e_3, e_3e_2 \in A^3 = \{0\}$ and $e_3^2 \in A^4 = \{0\}$, we have that

$$xy = a_1b_1e_1^2 + a_1b_2e_1e_2 + a_2b_1e_2e_1 + a_2b_2e_2^2.$$

Since $e_1^2, e_1e_2, e_2e_1, e_2^2 \in A^2$, we can write

$$e_1^2 = k_1e_3 ,$$

$$e_1e_2 = k_2e_3 ,$$

$$e_2e_1 = k_3e_3 ,$$

$$e_2^2 = k_4e_3 , \text{ for some } k_i \in K,\ i = 1,2,3,4.$$

Therefore,

$$(**) \qquad xy = (k_1a_1b_1 + k_2a_1b_2 + k_3a_2b_1 + k_4a_2b_2)e_3 .$$

Now we begin to classify the multiplications xy by studying k_i in K, $i = 1,2,3,4$. Since dimension of $A^2 = 1$, the case $k_1 = k_2 = k_3 = k_4 = 0$ cannot occur. Therefore, we consider the following cases.

Case 1. If $k_1 \neq 0$ and $k_2 = k_3 = k_4 = 0$, then the multiplication (**) becomes

$$xy = k_1a_1b_1e_3$$

As in [1], we choose a new basis $e_1' = e_1$, $e_2' = e_2$, $e_3' = k_1e_3$.

Therefore,

$$xy = a_1'b_1'(e_1')^2+a_1'b_2'e_1'e_2'+a_2'b_1'e_2'e_1'+a_2'b_2'(e_2')^2,$$

for $x = \sum_{i=1}^{3} a_i'e_i'$, $y = \sum_{j=1}^{3} b_j'e_j'$, $\{a_i',b_j'\} \subset K$, i, j = 1,2,3.

Since $(e_1')^2 = e_1^2 == k_1e_3 = e_3'$.

$e_1'e_2' = k_2e_3 = 0$, $e_2'e_1' = k_3e_3 = 0$, $(e_2')^2 = k_4e_3 = 0$,

we have

(1.1) $xy = a_1'b_1'e_3'$.

Case 2. If $k_4 \neq 0$ and $k_1 = k_2 = k_3 = 0$, then the multiplication (**) can be written as

(2.1) $xoy = k_4a_2b_2e_3$.

This multiplication is isomorphic to(1.1)incase 1. See proof in [1] page 52.

Case 3. If $k_3 \neq 0$ and $k_1 = k_2 = k_4 = 0$, then from (**) we have that

$$xy = k_3a_2b_1e_3$$

Like the other cases we choose a new basis $e_1' = e_1, e_2' = e_2$, $e_3' = k_3e_3$ and get the result,

(3.1) $xy = a_2'b_1'e_3'$,

where $x = \sum_{i=1}^{3} a_i'e_i'$, $y = \sum_{j=1}^{3} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, i,j = 1,2,3.

Notice that A is not a commutative algebra over K with respect to this multiplication, but A is commutative with respect to the multiplication (1.1) in case 1. Therefore, the multiplication in this case is not isomorphic to the one in case 1 (and hence in case 2.).

Case 4. Assume that $k_2 \neq 0$ and $k_1 = k_3 = k_4 = 0$. The multiplication (**) becomes

$$xoy = k_2 a_1 b_2 e_3 . \tag{4.1}$$

This multiplication is isomorphic to (3.1) in case 3. The proof is the same as [1] page 53.

Case 5. Suppose that $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 = k_4 = 0$. Then the multiplication (**) is

$$xoy = (k_1 a_1 b_1 + k_2 a_1 b_2) e_3 . \tag{5.1}$$

This multiplication is isomorphic to (3.1) in case 3. The proof is the same as [1] page 54.

Case 6. Let $k_3 \neq 0$, $k_4 \neq 0$, $k_1 = k_2 = 0$. Then from (**) we have that

$$xoy = (k_3 a_2 b_1 + k_4 a_2 b_2) e_3 . \tag{6.1}$$

By the same proof as [1] page 55. we have that (6.1) is isomorphic to (3.1) in case 3.

Case 7. Assume that $k_1 \neq 0$, $k_3 \neq 0$ and $k_2 = k_4 = 0$. Then the multiplication (**) is

$$(7.1) \qquad xoy = (k_1 a_1 b_1 + k_3 a_2 b_1) e_3 .$$

As in the above case, this multiplication is isomorphic to (3.1) in case 3. The proof is the same as [1] page 56

Case 8. In this case we take $k_2 \neq 0$, $k_4 \neq 0$ and $k_1 = k_3 = 0$ in (**). Then from (**) we have that

$$(8.1) \qquad xoy = (k_2 a_1 b_2 + k_4 a_2 b_2) e_3 .$$

This multiplication is isomorphic to (3.1) in case 3. The proof is the same as [1] page 57.

Case 9. Suppose that $k_2 \neq 0$, $k_3 \neq 0$ and $k_1 = k_4 = 0$, Then we have from (**) that

$$xoy = (k_2 a_1 b_2 + k_3 a_2 b_1) e_3 .$$

Like the previous cases, we choose a new basis $e''_1 = e_1$, $e''_2 = e_2$, $e''_3 = k_2 e_3$ such that

$$xoy = (a''_1 b''_2 + \frac{k_3}{k_2} a''_2 b''_1) e''_3 ,$$

for $x = \sum_{i=1}^{3} a''_i e''_i$, $y = \sum_{j=1}^{3} b''_j e''_j$, $\{a''_i, b''_j\} \subset K$, $i,j = 1,2,3$.

Let $k'' = \frac{k_3}{k_2}$, then we have

$$(9.1) \qquad xoy = (a''_1 b''_2 + k'' a''_2 b''_1) e''_3 , \quad k'' \neq 0 \text{ in } K.$$

We can prove that (9.1) is not isomorphic to the multiplications in case 1 and case 3. The proof is the same as [1] page 60.

Suppose that e_1', e_2', e_3' is another basis of A such that

$$(9.2) \qquad x * y = (a_1'b_2' + k'a_2'b_1')e_3' \,, \quad k' \neq 0 \text{ in } K,$$

for $x = \sum_{i=1}^{3} a_i'e_i'$, $y = \sum_{j=1}^{3} b_j'e_j'$, $\{a_i', b_j'\} \subset K$.

By the same proof as [1] page 61-64, we conclude that the multiplications (9.1) and (9.2) are isomorphic iff $k' = k''$ or $k' = \frac{1}{k''}$.

<u>Case 10.</u> Let $k_1 \neq 0$, $k_4 \neq 0$ and $k_2 = k_3 = 0$. Then (**) becomes

$$x * y = (k_1a_1b_1 + k_4a_2b_2)e_3$$

Let k', k'' be the roots of the polynomial $x^2 - \frac{k_1}{k_4}$. Now choose one of these numbers. Let k' denote the choice. Choose a new basis e_1', e_2', e_3' such that $e_1' = e_1$, $e_2' = k'e_2$, $e_3' = k_1e_3$ and get

$$(10.1) \qquad x * y = (a_1'b_1' + a_2'b_2')e_3' \,,$$

for $x = \sum_{i=1}^{3} a_i'e_i'$, $y = \sum_{j=1}^{3} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, $i, j = 1,2,3$.

This multiplication is not isomorphic to the multiplication in case 1. Since the center C of A under the multiplication in case 1 is $C = [e_2, e_3]$ and dimension of C is 2, whereas the center C' of A under the multiplication (10.1) is $C' = [e_3]$ and dimension of C' is 1. Moreover, the algebra A is not commutative under the multiplication (3.1) of case 3, but A is commutative under the multiplication (10.1). Therefore, the multiplications (10.1) and (3.1) cannot be isomorphic.

Recall that the multiplication (9.1) of case 9 is

$$(9.1) \qquad x \circ y = (a''_1 b''_2 + k'' a''_2 b''_1) e''_3 \ , \quad k'' \neq 0 \ \text{ in } K,$$

for $x = \sum_{i=1}^{3} a''_i e''_i$, $y = \sum_{j=1}^{3} b''_j e''_j$, $\{a''_i, b''_j\} \subset K$, $i,j = 1,2,3$.

We claim that the multiplications (10.1) and (9.1) are isomorphic iff $k'' = 1$. First we assume that the multiplications (10.1) and (9.1) are isomorphic. Therefore, we can find a linear, 1-1, onto function $f\colon A \to A$ such that

$$f(x * y) = f(x) \circ f(y) \ .$$

This function f is in the form

$$f(e'_1) = m_1 e''_1 + m_2 e''_2 + m_3 e''_3 \ ,$$

$$f(e'_2) = p_1 e''_1 + p_2 e''_3 + p_3 e''_3 \ ,$$

$$f(e'_3) = q e''_3 \ , \quad \{ m_i, p_j, q \} \subset K, \ i,j = 1,2,3, \quad q \neq 0 \ \text{ in } K.$$

Therefore, (10.1),(9.1) and the fact that $f(x * y) = f(x) \circ f(y)$ imply that, for $x = e_1'$, $y = e_1'$

(1) $$m_1 m_2(1+k'') = q .$$

If $x = e_1'$, $y = e_2'$, then

(2) $$m_1 p_2 + k'' m_2 p_1 = 0 .$$

If $x = e_2'$, $y = e_1'$, then

(3) $$m_2 p_1 + k'' m_1 p_2 = 0 .$$

If $x = e_2'$, $y = e_2'$, then

(4) $$p_1 p_2(1+k'') = q .$$

Since $q \neq 0$, equation (1) implies that $k'' \neq -1$. From (2) and (3) we have that

(5) $$m_1 p_2(k''^2-1) = 0$$

Since $m_1 \neq 0$, $p_2 \neq 0$ and $k'' \neq -1$, (5) implies that

$$k''- 1 = 0$$

$$k'' = 1 .$$

Conversely, suppose that $k'' = 1$. We let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1''+ e_2'' ,$$

$$f(e_2') = ie_1''- ie_2'' ,$$

$$f(e_3') = 2e_3'', \qquad i = \sqrt{-1} \text{ in } K.$$

Then

$$\det [f] = \det \begin{bmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 2 \end{bmatrix} = -2i-2i = -4i .$$

Since characteristic $K \neq 2$, $\det [f] = -4i \neq 0$. Hence f is 1-1 and onto. The multiplication (10.1) implies that

$$f(x * y) = f[(a_1'b_1' + a_2'b_2')e_3']$$

$$= 2(a_1'b_1' + a_2'b_2')e_3'' ,$$

whereas, (9.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{3} a_i'e_i')o\ f(\sum_{j=1}^{3} b_j'e_j')$$

$$= [(a_1'+ia_2')e_1''+(a_1'-ia_2')e_2''+2a_3'e_3'']o[(b_1'+ib_2')e_1''$$

$$+(b_1'-ib_2')e_2''+2b_3'e_3'']$$

$$= [(a_1'+ia_2')(b_1'-ib_2')+(a_1'-ia_2')(b_1'+ib_2')]\ e_3''$$

$$= 2(a_1'b_1' + a_2'b_2')e_3'' .$$

That is $f(x * y) = f(x) \circ f(y)$ for $k'' = 1$.

<u>Case 11.</u> Assume that $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$ and $k_1 = 0$. Then from (**) we have that

$$x * y = (k_2a_1b_2 + k_3a_2b_1 + k_4a_2b_2)e_3$$

Choose a new basis $e_1' = \frac{k_4}{k_2} e_1$, $e_2' = e_2$, $e_3' = k_4 e_3$,

then it is immediate that

$$x * y = (a_1'b_2' + \frac{k_3}{k_2} a_2'b_1' + a_2'b_2')e_3' ,$$

for $x = \sum_{i=1}^{3} a_i'e_i'$, $y = \sum_{j=1}^{3} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, $i,j = 1,2,3.$

Let $k' = \frac{k_3}{k_2}$, then

$$(11.1) \qquad x * y = (a_1'b_2' + k'a_2'b_1' + a_2'b_2')e_3' , \text{ for } k' \neq 0 \text{ in } K.$$

By using the same proof as [1] page 67 we conclude that if $k' \neq -1$, then this multiplication is isomorphic to (9.1) of case 9 whenever $k' = k''$.

If $k' = -1$, then (11.1) becomes

$$(11.2) \qquad x * y = (a_1'b_2' - a_2'b_1' + a_2'b_2')e_3' .$$

We can easily see that the algebra A is not commutative under the multiplication (11.2) while A is commutative under the multiplication in case 1. Therefore, the multiplication (11.2) cannot be isomorphic to the multiplication in case 1. Moreover, the left center C_L of A under the multiplication (11.2) is $[e_3]$ and hence C_L has dimension 1. Therefore, the multiplication (11.2) cannot be isomorphic to the multiplication (3.1) where the left center $C_L' = [e_1, e_3]$ and has dimension 2. Furthermore, the multiplication (11.2) is not isomorphic to the multiplication (9.1) in case 9. The proof is the same as [1] page 68.

Case 12. Suppose that $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ and $k_4 = 0$. Then from (**) we have that

$$xoy = (k_1a_1b_1 + k_2a_1b_2 + k_3a_2b_1)e_3$$

Like the other cases, we choose a new basis $e_1'' = e_1$, $e_2'' = \frac{k_1}{k_2} e_2$, $e_3'' = k_1e_3$ and get

$$xoy = (a_1''b_1'' + a_1''b_2'' + \frac{k_3}{k_2} a_2''b_1'')e_3'' ,$$

for $x = \sum_{i=1}^{3} a_i''e_i''$, $y = \sum_{j=1}^{3} b_j''e_j''$, $\{a_i'', b_j''\} \subset K$, $i,j = 1,2,3$.

Let $k'' = \frac{k_3}{k_2}$, then

$$\text{(12.1)} \quad xoy = (a_1''b_1'' + a_1''b_2'' + k''a_2''b_1'')e_3'' , \quad k'' \neq 0 \text{ in } K.$$

By the same proof as [1] page 69, we can prove that this multiplication is isomorphic to the multiplication (11.1) in case 11 whenever $k' = \frac{1}{k''}$.

Case 13. Assume that $k_1 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$ and $k_2 = 0$. Then the multiplication (**) can be written as

$$x * y = (k_1a_1b_1 + k_3a_2b_1 + k_4a_2b_2)e_3 .$$

We can choose a new basis $e_1''' = e_1$, $e_2''' = \frac{k_1}{k_3} e_2$, $e_3''' = k_1e_3$ such that

$$x * y = (a_1'''b_1''' + a_2'''b_1''' + \frac{k_1k_4}{k_3^2} a_2'''b_2''')e_3''' ,$$

for $x = \sum_{i=1}^{3} a_i''' e_i'''$, $y = \sum_{j=1}^{3} b_j''' e_j'''$, $\{a_i''', b_j'''\} \subset K$, $i,j = 1,2,3$.

Let $k''' = \frac{k_1 k_4}{k_3^2}$, then we have that

$$(13.1) \qquad x * y = (a_1''' b_1''' + a_2''' b_1''' + k''' a_2''' b_2''') e_3''', \quad k''' \neq 0 \text{ in } K.$$

We can prove that the multiplication (13.1) and (9.1) are isomorphic iff $k''' = \frac{-k''}{(1-k'')^2}$, $k'' \neq \pm 1$. See proof in [1] page 71-73.

Under the assumption above the $k''' = \frac{-k''}{(1-k'')^2}$ we can see that for a given number k''' we can find k'' to make (13.1) isomorphic to (9.1) only if $k''' \neq 0$ and $k''' \neq \frac{1}{4}$. Therefore we have to consider (13.1) when $k''' = 1/4$.

By the same proof as . [1], we can show that the multiplications (13.1) and (11.2) are isomorphic iff $k''' = 1/4$.

<u>Case 14</u>. Suppose that $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$ and $k_3 = 0$. Then the multiplication (**) is

$$x o y = (k_1 a_1 b_1 + k_2 a_1 b_2 + k_4 a_2 b_2) e_3.$$

As in the other cases, we may choose a new basis $e_1' = e_1$, $e_2' = \frac{k_1 e}{k_2} e_2$, $e_3' = k_1 e_3$ and obtain

$$x o y = (a_1' b_1' + a_1' b_2' + \frac{k_1 k_4}{k^2} a_2' b_2') e_3', \text{ for}$$

$$x = \sum_{i=1}^{3} a_i' e_i', \ y = \sum_{j=1}^{3} b_j' e_j' \ , \ \{a_i', b_j'\} \subset K, \ i,j = 1,2,3.$$

Let $\dfrac{k_1 k_4}{k_2^2} = k'$, then

$$(14.1) \qquad x o y = (a_1' b_1' + a_1' b_2' + k' a_2' b_2') e_3' \ , \ k' \neq 0 \text{ in } K.$$

We claim that (14.1) is isomorphic to (13.1) in case 13 iff $k' = k'''$. To prove this, we first assume that these two multiplications are isomorphic. Therefore, there exists a linear mapping $f: A \to A$ defined by

$$f(e_1') = m_1 e_1''' + m_2 e_2''' + m_3 e_3''' \ ,$$

$$f(e_2') = p_1 e_1''' + p_2 e_2''' + p_3 e_3''' \ ,$$

$$f(e_3') = q e_3''', \ q \neq 0 \text{ in } K \ , \ \{m_i, p_i\}_{i,j=1,2,3} \subset K.. \ ,$$

such that $f(x o y) = f(x) * f(y)$.

Hence, for $x = e_1'$, $y = e_1'$, we have that

$$(1) \qquad m_1^2 + m_2 m_1 + k''' m_2^2 = q$$

For $x = e_1'$, $y = e_2'$, we have that

$$(2) \qquad m_1 p_1 + m_2 p_1 + k''' m_2 p_2 = q \ .$$

For $x = e_2'$, $y = e_1'$, we have that

$$(3) \qquad m_1 p_1 + m_1 p_2 + k''' m_2 p_2 = 0 \ .$$

For $x = e_2'$, $y = e_2'$, we have that

$$(4) \qquad p_1^2 + p_1 p_2 + k''' p_2^2 = k' q \ .$$

Take (2)-(3), we get that

$$(5) \qquad m_2p_1 - m_1p_2 = q .$$

Take $p_1 \times (1) - m_1 \times (3)$, we get that

$$p_1q = (m_2p_1 - m_1p_2)(m_1 + k'''m_2).$$

This and (5) imply that

$$(6) \qquad p_1 = m_1 + k'''m_2 .$$

Take $m_1 \times (4) - p_1 \times (3)$, we get that

$$m_1k'q = k'''p_2(m_1p_2 - m_2p_1).$$

This, together with (5), gives us the result that

$$(7) \qquad m_1k' = - p_2k'''.$$

Take $m_2 \times (3) - p_2 \times (1)$, we get that

$$- p_2q = m_1(m_2p_1 - m_1p_2).$$

This and (5) imply that

$$(8) \qquad - p_2 = m_1$$

Take $m_2 \times (4) - p_2 \times (2)$, we get that

$$q(m_2k' - p_2) = p_1(m_2p_1 - m_1p_2).$$

Thus we have that

$$(9) \qquad m_2k' - p_2 = p_1 .$$

If $m_1 = 0$, then $p_2 = 0$ from (8). Therefore, (6) and (9) imply that

$$k' = k''' .$$

If $m_1 \neq 0$, then (7) and (8) imply that

$$k' = k''' .$$

Conversely, if $k' = k'''$, let $f: A \to A$ be the linear map defined by

$$f(e_1''') = e_1' ,$$

$$f(e_2''') = e_1' - e_2' ,$$

$$f(e_3''') = e_3' ,$$

Then [1] page 78 proves that (14.1) and (13.1) are isomorphic.

Case 15. In this final case we assume that all k_1, k_2, k_3, k_4 are not zero. Then the multiplication (**) is

$$x * y = (k_1 a_1 b_1 + k_2 a_1 b_2 + k_3 a_2 b_1 + k_4 a_2 b_2) e_3 .$$

As in case 15 [1] page 79 we choose a new basis e_1', e_2', e_3' such that $e_1' = e_1$, $e_2' = k_2 e_1 - k_1 e_2$, $e_3' = e_3$ and get that

$$(15.1) \qquad x * y = [k_1 a_1' b_1' + k_1 (k_2 - k_3) a_2' b_1' + k_1 (k_1 k_4 - k_2 k_3) a_2' b_2'] e_3' ,$$

for $x = \sum_{i=1}^{3} a_i' e_i'$, $y = \sum_{j=1}^{3} b_j' e_j'$, $\{a_i', b_j'\}_{i,j=1,2,3} \subset K$.

We have no term of the form $a_1' b_2'$ so we are back to case 13.

In conclusion, we see that the multiplications in a 3-dimensional nilpotent algebra A with dimension $A^2 = 1$ and $A^3 = \{0\}$ over an algebraically closed field K of characteristic $K \neq 2$ can be divided into 4 classes. Let M,N be any subsets of $K - \{0,1,-1\}$ such that

$$M \cap N = \emptyset$$

$$M \cup N = K - \{0,1,-1\}$$

and $k \in M$ iff $k^{-1} \notin M$. For each $x = \sum_{i=1}^{3} a_i e_i$, $y = \sum_{j=1}^{3} b_j e_j$,

$\{a_i, b_i\} \subset K$, $i,j = 1,2,3$, we have that

1) $xy = a_1 b_1 e_3$,

2) $xy = a_2 b_1 e_3$,

3) $xy = (a_1 b_2 + k a_2 b_1) e_3$, $k = 1,-1$ or $k \in M$,

4) $xy = (a_1 b_2 - a_2 b_1 + a_2 b_2) e_3$,

are non-isomorphic and every nilpotent algebra is isomorphic to one of the above.