CHAPTER II

FACTORIZABLE GENERALIZED TRANSFORMATION SEMIGROUPS

The main purpose of this chapter is to characterize factorizable generalized partial transformation semigroups, factorizable generalized full transformation semigroups and factorizable generalized 1-1 partial transformation semigroups. Moreover, characterizations of the semigroup of almost identical partial transformations, the semigroup of almost identical full transformations and the semigroup of almost identical 1-1 partial transformations on a set which are factorizable are also studied.

Throughout this chapter, if S is a transformation semigroup on a set and $\theta \in S$, the operation on the generalized transformation semigroup (S, θ) will be denoted by $*$.

The following proposition shows that for any set X, if S is T_X, $\mathcal{J}_X$ or I_X and $\theta \in S$, then the generalized transformation semigroup (S, θ) is regular if and only if θ is a permutation on X.

2.1 <u>Proposition</u>. Let X be any set and S be T_X, $\mathcal{J}_X$ or I_X. If $\theta \in S$, then the generalized transformation semigroup (S, θ) is regular if and only if θ is a permutation on X.

<u>Proof</u> : Let $\theta \in S$. Assume that θ is a permutation on X, that is $\theta \in G_X$. To show (S, θ) is regular, let $\alpha \in S$. Because under

the composition of maps, S is a regular semigroup [Introduction, page 4], then there exists $\beta \in S$ such that $\alpha = \alpha\beta\alpha$. Thus $\alpha = \alpha\beta\alpha = \alpha\theta\theta^{-1}\beta\theta^{-1}\theta\alpha = \alpha * \theta^{-1}\beta\theta^{-1} * \alpha$. Since $\theta^{-1} \in G_X \subseteq S$ and $\beta \in S$, we have that $\theta^{-1}\beta\theta^{-1} \in S$, so α is regular.

Conversely, assume that the semigroup (S, θ) is regular. Let $\alpha \in G_X$. Then $\alpha \in S$. Since (S, θ) is regular, there exists $\beta \in S$ such that $\alpha = \alpha * \beta * \alpha = \alpha\theta\beta\theta\alpha$. Then $\alpha^{-1} = \theta\beta\theta$, so $X = \Delta\alpha^{-1} = \Delta\theta\beta\theta = \Delta\theta(\beta\theta) \subseteq \Delta\theta$ and $X = \nabla\alpha^{-1} = \nabla\theta\beta\theta = \nabla(\theta\beta)\theta \subseteq \nabla\theta$. Thus $\Delta\theta = \nabla\theta = X$. Suppose θ is not one-to-one. Then there exist $a, b \in \Delta\theta = X$ such that $a \neq b$ and $a\theta = b\theta$. Therefore $a\alpha^{-1} = a\theta\beta\theta = (a\theta)\beta\theta = (b\theta)\beta\theta = b\theta\beta\theta = b\alpha^{-1}$ which is a contradiction since α^{-1} is one-to-one. Hence θ is one-to-one. Thus θ is a permutation on X. #

Let X be a set and S be T_X, $\mathcal{J}_X$ or I_X. Let θ be a permutation on X. For $\alpha \in S$, $\alpha * \theta^{-1} = \alpha\theta\theta^{-1} = \alpha$ and $\theta^{-1} * \alpha = \theta^{-1}\theta\alpha = \alpha$. Then θ^{-1} is the identity of the semigroup (S, θ). Claim that G_X is the group of units of the semigroup (S, θ). Let $\alpha \in G_X$. Then $\theta\alpha\theta \in G_X$, so $(\theta\alpha\theta)^{-1} \in G_X$. Since $G_X \subseteq S$, $(\theta\alpha\theta)^{-1} \in S$. Because $\alpha * (\theta\alpha\theta)^{-1} = \alpha\theta(\theta\alpha\theta)^{-1} = \alpha\theta\theta^{-1}\alpha^{-1}\theta^{-1} = \theta^{-1}$ and $(\theta\alpha\theta)^{-1} * \alpha = (\theta\alpha\theta)^{-1}\theta\alpha = \theta^{-1}\alpha^{-1}\theta^{-1}\theta\alpha = \theta^{-1}$, it follows that α belongs to the group of units of (S, θ). Thus G_X is a subset of the group of units of (S, θ).

Conversely, let α be a unit in (S, θ). Then there exists $\alpha' \in S$ such that $\alpha * \alpha' = \alpha' * \alpha = \theta^{-1}$. Thus $\alpha\theta\alpha' = \alpha'\theta\alpha = \theta^{-1}$ which implies $\alpha(\theta\alpha'\theta) = (\theta\alpha'\theta)\alpha = 1$. Since G_X is the group of units of T_X, $\mathcal{J}_X$ and I_X under the composition of maps. Then $\alpha \in G_X$. This proves that the group of units of (S, θ) is a subset of G_X.

Therefore G_X is the group of units of the semigroup (S, θ).

Let X be a set. If $\theta \in \mathcal{J}_X$, then $E((\mathcal{J}_X, \theta)) = E((T_X, \theta)) \cap \mathcal{J}_X$ since $(\mathcal{J}_X, \theta)$ is a subsemigroup of (T_X, θ). If $\theta \in I_X$, then $E((I_X, \theta)) = E((T_X, \theta)) \cap I_X$ because (I_X, θ) is a subsemigroup of (T_X, θ).

Let X be a set. The next proposition gives the set of all idempotents of the generalized transformation semigroup (T_X, θ) where $\theta \in G_X$.

2.2 <u>Proposition</u>. Let X be a set and θ be a permutation on X. Then $E((T_X, \theta)) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\theta\alpha \text{ and } x\theta\alpha = x \text{ for all } x \in \nabla\alpha\}$.

<u>Proof</u> : Let $\beta \in E((T_X, \theta))$. Then $\beta*\beta = \beta$, so $\beta\theta\beta = \beta$. To show $\nabla\beta \subseteq \Delta\theta\beta$, let $x \in \nabla\beta$. Then there exists $y \in \Delta\beta$ such that $x = y\beta$. Because $\beta\theta\beta = \beta$ and $x = y\beta$, we have that $x = y\beta = y\beta\theta\beta = x\theta\beta$. This shows that $\nabla\beta \subseteq \Delta\theta\beta$ and $x\theta\beta = x$ for all $x \in \nabla\beta$. Therefore $E((T_X, \theta)) \subseteq \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\theta\alpha \text{ and } x\theta\alpha = x \text{ for all } x \in \nabla\alpha\}$.

Let $\alpha \in T_X$ be such that $\nabla\alpha \subseteq \Delta\theta\alpha$ and $x\theta\alpha = x$ for all $x \in \nabla\alpha$. Because $\nabla\alpha \subseteq \Delta\theta\alpha$, $\Delta\alpha*\alpha = \Delta\alpha\theta\alpha = (\nabla\alpha \cap \Delta\theta\alpha)\alpha^{-1} = (\nabla\alpha)\alpha^{-1} = \Delta\alpha$. Let $x \in \Delta\alpha$. Then $x\alpha \in \nabla\alpha$, so $(x\alpha)\theta\alpha = x\alpha$. Hence $x(\alpha*\alpha) = x\alpha\theta\alpha = (x\alpha)\theta\alpha = x\alpha$. Thus $\alpha*\alpha = \alpha$. Therefore $\alpha \in E((T_X, \theta))$. This proves that $\{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\theta\alpha \text{ and } x\theta\alpha = x \text{ for all } x \in \nabla\alpha\} \subseteq E((T_X, \theta))$.

Hence, $E((T_X, \theta)) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\theta\alpha \text{ and } x\theta\alpha = x \text{ for all } x \in \nabla\alpha\}$. #

Let X be a set. If $\alpha \in I_X$, then $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are identity maps on $\Delta\alpha$ and $\nabla\alpha$, respectively, that is, $\Delta\alpha\alpha^{-1} = \Delta\alpha$, $\Delta\alpha^{-1}\alpha = \nabla\alpha$, $x\alpha\alpha^{-1} = x$ for all $x \in \Delta\alpha$ and $x\alpha^{-1}\alpha = x$ for all $x \in \nabla\alpha$.

From Proposition 2.2, we have the following corollary.

2.3 Corollary. For any set X and θ is a permutation on X,

$$E((\mathcal{J}_X, \theta)) = \{\alpha \in \mathcal{J}_X \mid x\theta\alpha = x \text{ for all } x \in \nabla\alpha\}$$

and

$$E((I_X, \theta)) = \{\theta^{-1}|_A \mid A \subseteq X\}$$

where for a map f from X and for $A \subseteq X$, $f|_A$ denotes the restriction of f to the set A.

Proof : Let X be a set and θ be a permutation on X. It is obvious from Proposition 2.2 that $E((\mathcal{J}_X, \theta)) = \{\alpha \in \mathcal{J}_X \mid x\theta\alpha = x$ for all $x \in \nabla\alpha\}$ since $\mathcal{J}_X \subseteq T_X$ and for all $\alpha \in \mathcal{J}_X$, $\Delta\alpha = X$.

Let $\alpha \in E((I_X, \theta))$. Then $\alpha \in E((T_X, \theta))$, so $\nabla\alpha \subseteq \Delta\theta\alpha$ and $x\theta\alpha = x$ for all $x \in \nabla\alpha$. Since $\alpha^{-1}\alpha$ is the identity map on $\nabla\alpha$, $(x\theta)\alpha = x = (x\alpha^{-1})\alpha$ for all $x \in \nabla\alpha$. Because α is one-to-one, $x\theta = x\alpha^{-1}$ for all $x \in \nabla\alpha$. Hence $\alpha^{-1} = \theta|_{\nabla\alpha}$, so $\alpha = (\theta|_{\nabla\alpha})^{-1} = \theta^{-1}|_{(\nabla\alpha)\theta}$. Thus $E((I_X, \theta)) \subseteq \{\theta^{-1}|_A \mid A \subseteq X\}$. Let $\beta = \theta^{-1}|_A$ for some $A \subseteq X$. Then $\Delta\beta = A$ and $x\beta = x\theta^{-1}$ for all $x \in A$. Thus $A\beta = A\theta^{-1}$ which implies $(\Delta\beta)\theta^{-1} = (\Delta\beta)\beta = \nabla\beta$. Because $\Delta\theta\beta = (\nabla\theta \cap \Delta\beta)\theta^{-1} = (\Delta\beta)\theta^{-1} = \nabla\beta$, $\Delta\beta*\beta = \Delta\beta\theta\beta = (\nabla\beta \cap \Delta\theta\beta)\beta^{-1} = (\nabla\beta \cap \nabla\beta)\beta^{-1} = (\nabla\beta)\beta^{-1} = \Delta\beta$. Let $x \in \Delta\beta$. Then $x\beta = x\theta^{-1}$ which implies $x(\beta*\beta) = x\beta\theta\beta = (x\beta)\theta\beta = x\theta^{-1}\theta\beta = x\beta$. Thus $\beta*\beta = \beta$, so $\beta \in E((I_X, \theta))$. Therefore $\{\theta^{-1}|_A \mid A \subseteq X\} \subseteq E((I_X, \theta))$. Hence $E((I_X, \theta)) = \{\theta^{-1}|_A \mid A \subseteq X\}$. #

It has been showed in [4] that for any set X, the partial transformation semigroup on X, T_X, is factorizable if and only if X is finite. Using this fact, the following theorem is obtained.

2.4 <u>Theorem</u>. For any set X and $\theta \in T_X$, the generalized partial transformation semigroup on X is factorizable if and only if θ is a permutation on X and X is a finite set.

<u>Proof</u> : Let X be a set and $\theta \in T_X$. Assume the generalized partial transformation semigroup (T_X, θ) is factorizable. Then (T_X, θ) is regular since every factorizable semigroup is regular [4, Proposition 2.2]. By Proposition 2.1, θ is a permutation on X. To show X is a finite set, suppose not. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$, so there exists a one-to-one and onto map $\alpha : X \setminus \{a\} \to X$. Thus $\alpha \in T_X$ with $\Delta\alpha = X \setminus \{a\}$ and $\nabla\alpha = X$. Since (T_X, θ) is factorizable, $T_X = G_X * E((T_X, \theta))$. Then there exist $\beta \in G_X$, $\gamma \in E((T_X, \theta))$ such that $\alpha = \beta * \gamma$. It then follows that $\alpha = \beta\theta\gamma$ which implies $\nabla\alpha = \nabla(\beta\theta)\gamma \subseteq \nabla\gamma$. But $\nabla\alpha = X$, then $\nabla\gamma = X$. Since $\gamma \in E((T_X, \theta))$, $\nabla\gamma \subseteq \Delta\theta\gamma$ and $x\theta\gamma = x$ for all $x \in \nabla\gamma$. Then $\Delta\theta\gamma = X$ which implies $X = (\nabla\theta \cap \Delta\gamma)\theta^{-1} = (\Delta\gamma)\theta^{-1}$. Hence $\Delta\gamma = X\theta = X$. Since $\Delta\beta = \Delta\theta = \Delta\gamma = X$, $\Delta\alpha = \Delta\beta\theta\gamma = X$ which is a contradiction. Therefore X is a finite set.

Conversely, assume that θ is a permutation on X and X is a finite set. Then $T_X = G_X E(T_X)$ [4, Theorem 3.1]. Let $\alpha \in T_X$. Since $T_X = G_X E(T_X)$, there exist $\beta \in G_X$ and $\gamma \in E(T_X)$ such that

$\alpha = \beta\gamma$. It then follows that $\alpha = \beta\gamma = \beta\theta\theta^{-1}\gamma = \beta*(\theta^{-1}\gamma)$. Claim that $\theta^{-1}\gamma \in E((T_X, \theta))$. Since $\gamma \in E(T_X)$, $\nabla\gamma \subseteq \Delta\gamma$ and $x\gamma = x$ for all $x \in \nabla\gamma$. Then $(\nabla\gamma)\theta \subseteq (\Delta\gamma)\theta$. Because $\nabla\theta^{-1}\gamma = (\nabla\theta^{-1} \cap \Delta\gamma)\gamma = (\Delta\gamma)\gamma = \nabla\gamma \subseteq \Delta\gamma$ and $\Delta\theta(\theta^{-1}\gamma) = \Delta\gamma$, it follows that $\nabla\theta^{-1}\gamma \subseteq \Delta\theta(\theta^{-1}\gamma)$. Let $x \in \nabla\theta^{-1}\gamma = \nabla\gamma$. Then $x\gamma = x$. Thus $x\theta(\theta^{-1}\gamma) = x\gamma = x$. Therefore $\theta^{-1}\gamma \in E((T_X, \theta))$ by Proposition 2.2. This proves that the semigroup (T_X, θ) is factorizable. #

2.5 <u>Theorem</u>. For any set X and $\theta \in \mathcal{T}_X$, the generalized full transformation semigroup on X is factorizable if and only if θ is a permutation on X and X is a finite set.

<u>Proof</u> : Let X be a set and $\theta \in \mathcal{T}_X$. Assume the generalized full transformation semigroup $(\mathcal{T}_X, \theta)$ is factorizable. Then $(\mathcal{T}_X, \theta)$ is regular since every factorizable semigroup is regular [4, Proposition 2.2]. By Proposition 2.1, θ is a permutation on X. To show X is a finite set, suppose not. Let $a \in X$. Then $|X| = |X \setminus \{a\}|$, so there exists a one-to-one map α with $\Delta\alpha = X$ and $\nabla\alpha = X \setminus \{a\}$. Thus $\alpha \in \mathcal{T}_X$. Since $(\mathcal{T}_X, \theta)$ is factorizable, $\mathcal{T}_X = G_X * E((\mathcal{T}_X, \theta))$. Then there exist $\beta \in G_X$, $\gamma \in E((\mathcal{T}_X, \theta))$ such that $\alpha = \beta*\gamma = \beta\theta\gamma$. Thus $\gamma = (\beta\theta)^{-1}\alpha$, so $\nabla\gamma = \nabla(\beta\theta)^{-1}\alpha = (\nabla(\beta\theta)^{-1} \cap \Delta\alpha)\alpha = (\Delta\alpha)\alpha = \nabla\alpha = X \setminus \{a\}$. Claim that $x\gamma = x\theta^{-1}$ for all $x \in X$. Let $x \in X$. Then there exist $b, c \in X$ such that $b\beta\theta = x$ (since $\nabla\beta\theta = X$) and $c\beta = x\gamma$ (since $\nabla\beta = X$). Therefore $b\alpha = b\beta\theta\gamma = x\gamma = x(\gamma*\gamma) = x\gamma\theta\gamma = c\beta\theta\gamma = c\alpha$. Since α is one-to-one, $b = c$. Then $b\beta = c\beta = x\gamma$. Because $b\beta\theta = x$, $b\beta = x\theta^{-1}$. Then $x\theta^{-1} = b\beta = x\gamma$. This proves that $x\gamma = x\theta^{-1}$ for all

$x \in X$. Thus $X\gamma = X\theta^{-1} = X$, it is a contradiction because $X\gamma = \nabla\gamma = X \setminus \{a\}$.

Conversely, assume θ is a permutation on X and X is a finite set. By Theorem 2.4, (T_X, θ) is factorizable. To show the semigroup $(\mathcal{J}_X, \theta)$ is factorizable, let $\alpha \in \mathcal{J}_X$. Then $\alpha \in T_X$, so there exist $\beta \in G_X$, $\gamma \in E((T_X, \theta))$ such that $\alpha = \beta * \gamma = \beta\theta\gamma$. Then $\gamma = (\beta\theta)^{-1}\alpha$. Since $\Delta\alpha = \Delta(\beta\theta)^{-1} = X$, $\Delta\gamma = X$. Then $\gamma \in \mathcal{J}_X \cap E((T_X, \theta)) = E((\mathcal{J}_X, \theta))$. Therefore, the semigroup $(\mathcal{J}_X, \theta)$ is factorizable. #

2.6 <u>Theorem.</u> For any set X and $\theta \in I_X$, the generalized 1-1 partial transformation semigroup on X is factorizable if and only if θ is a permutation on X and X is a finite set.

<u>Proof</u> : Let X be a set and $\theta \in I_X$. Assume the generalized 1-1 partial transformation semigroup (I_X, θ) is factorizable. Then (I_X, θ) is regular by Proposition 2.2 [4]. By Proposition 2.1, θ is a permutation on X. To show X is a finite set, suppose not. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$, so there exists a one-to-one map α with $\Delta\alpha = X \setminus \{a\}$ and $\nabla\alpha = X$. Thus $\alpha \in I_X$ but $\alpha \notin G_X$. Since (I_X, θ) is factorizable, $I_X = G_X * E((I_X, \theta))$. Then there exist $\beta \in G_X$, $\gamma \in E((I_X, \theta))$ such that $\alpha = \beta * \gamma$. Thus $\alpha = \beta\theta\gamma$ which implies $\nabla\alpha = \nabla(\beta\theta)\gamma \subseteq \nabla\gamma$. But $\nabla\alpha = X$, then $\nabla\gamma = X$. Because $\gamma \in E((I_X, \theta))$, $\gamma = \theta^{-1}|_{\Delta\gamma}$. Then $X = \nabla\gamma = \nabla\theta^{-1}|_{\Delta\gamma} = (\Delta\gamma)\theta^{-1}$ which implies $X\theta = \Delta\gamma$. But $X\theta = X$, so $\Delta\gamma = X$. Therefore $\gamma = \theta^{-1}|_X = \theta^{-1}$. Since $\alpha = \beta\theta\gamma$, $\alpha = \beta\theta\theta^{-1} = \beta \in G_X$ which is a contradiction. Then X is a finite set.

Conversely, assume θ is a permutation on X and X is a finite set. By Theorem 2.4, (T_X, θ) is factorizable. To show the semigroup (I_X, θ) is factorizable, let $\alpha \in I_X$. Then $\alpha \in T_X$, so $\alpha = \beta * \gamma = \beta\theta\gamma$ for some $\beta \in G_X$ and $\gamma \in E((T_X, \theta))$. It then follows that $\gamma = (\beta\theta)^{-1}\alpha$. Since $(\beta\theta)^{-1}$ and α are one-to-one, we have that γ is one-to-one. Hence $\gamma \in E((T_X, \theta)) \cap I_X = E((I_X, \theta))$. This shows that the semigroup (I_X, θ) is factorizable, as required. #

The following corollary follows directly from Theorem 2.4, Theorem 2.5 and Theorem 2.6.

2.7 Corollary. For any set X and θ is a permutation on X, the following are equivalent :

(i) X is a finite set.

(ii) (T_X, θ) is factorizable.

(iii) $(\mathcal{T}_X, \theta)$ is factorizable.

(iv) (I_X, θ) is factorizable.

Let X be a set, and let U_X, V_X and W_X be defined as in Chapter I, that is,

$$U_X = \{\alpha \in T_X \mid |S(\alpha)| < \infty\},$$

$$V_X = \{\alpha \in \mathcal{T}_X \mid |S(\alpha)| < \infty\} = U_X \cap \mathcal{T}_X$$

and

$$W_X = \{\alpha \in I_X \mid |S(\alpha)| < \infty\} = U_X \cap I_X.$$

Recall that U_X, V_X and W_X are regular semigroups under the composition of maps, and $G_X \cap U_X = G_X \cap V_X = G_X \cap W_X$ is the group of units of U_X, V_X and W_X.

Let S be U_X, V_X or W_X and $\theta \in S$.

Assume that $\theta \in G_X$. Then $S(\theta) = S(\theta^{-1})$. Thus $\theta^{-1} \in S$. Let $\alpha \in S$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in S$, so $\alpha = \alpha*(\theta^{-1}\beta\theta^{-1})*\alpha$. Since θ^{-1} and $\beta \in S$, $\theta^{-1}\beta\theta^{-1} \in S$. This shows that if $\theta \in G_X$, then the transformation semigroup (S, θ) is regular.

Suppose that the semigroup (S, θ) is regular. Let $\alpha \in G_X \cap S$. Then $\alpha \in S$, so $\alpha = \alpha*\beta*\alpha = \alpha(\theta\beta\theta)\alpha$ for some $\beta \in S$. Thus $\alpha^{-1} = \theta\beta\theta$. Hence $X = \Delta\alpha^{-1} = \Delta\theta\beta\theta \subseteq \Delta\theta$ and $X = \nabla\alpha^{-1} = \nabla\theta\beta\theta \subseteq \nabla\theta$, so $\Delta\theta = \nabla\theta = X$. Since $\Delta\alpha^{-1} = X$ and α^{-1} is one-to-one, θ is one-to-one. Therefore $\theta \in G_X$.

Hence, we have

2.8 <u>Proposition</u>. For any set X, if S is U_X, V_X or W_X and $\theta \in S$, then (S, θ) is regular if and only if θ is a permutation on X.

Let X be a set, S be U_X, V_X or W_X, and $\theta \in S$. Assume that $\theta \in G_X$. Then $S(\theta^{-1}) = S(\theta)$, so $\theta^{-1} \in G_X \cap S$. Clearly, θ^{-1} is the identity of the semigroup (S, θ). Let $\alpha \in G_X \cap U_X = G_X \cap S$. Then $\alpha^{-1} \in G_X \cap S$. Thus $\theta^{-1}\alpha^{-1}\theta^{-1} \in S$ and $\alpha*(\theta^{-1}\alpha^{-1}\theta^{-1}) = \theta^{-1} = (\theta^{-1}\alpha^{-1}\theta^{-1})*\alpha$. Suppose that $\beta \in S$ such that $\beta*\gamma = \theta^{-1} = \gamma*\beta$ for some $\gamma \in S$. Then $\beta\theta\gamma = \theta^{-1} = \gamma\theta\beta$, so $\beta(\theta\gamma\theta) = 1 = (\theta\gamma\theta)\beta$. Thus $\beta \in G_X$ since G_X is the unit group of T_X, $\mathcal{J}_X$ and I_X under the composition of maps. Hence $\beta \in G_X \cap S = G_X \cap U_X$. Therefore, we have that $G_X \cap U_X = G_X \cap V_X = G_X \cap W_X = G_X \cap S$ is the unit group of the semigroup (S, θ). Because (U_X, θ) is a subsemigroup of (T_X, θ), it immediate from Proposition 2.2 that

$$E((U_X, \theta)) = \{\alpha \in U_X \mid \nabla\alpha \subseteq \Delta\theta\alpha \text{ and } x\theta\alpha = x \text{ for all } x \in \nabla\alpha\}$$

$$= E((T_X, \theta)) \cap U_X.$$

Hence,

$$E((V_X, \theta)) = \{\alpha \in V_X \mid x\theta\alpha = x \text{ for all } x \in \nabla\alpha\}$$

$$= E((\mathcal{I}_X, \theta)) \cap V_X$$

$$= E((U_X, \theta)) \cap V_X$$

and $E((W_X, \theta)) = E((I_X, \theta)) \cap W_X$

$$= E((U_X, \theta)) \cap W_X$$

$$= \{\theta^{-1}|_A \mid A \subseteq X\} \cap W_X$$

$$= \{\theta^{-1}|_A \mid A \subseteq X \text{ and } a\theta^{-1} \neq a \text{ at most a finite number of } a \in A\}$$

$$= \{\theta^{-1}|_A \mid A \subseteq X \text{ and } a\theta \neq a \text{ at most a finite number of } a \in A\}.$$

2.9 Theorem. Let X be a set and S be U_X, V_X or W_X. Assume $\theta \in S$. Then the generalized transformation semigroup (S, θ) is factorizable if and only if θ is a permutation on X.

Proof : Assume that the semigroup (S, θ) is factorizable. By Proposition 2.2 [4], the semigroup (S, θ) is regular. Hence, by Proposition 2.8, θ is a permutation on X.

Conversely, assume $\theta \in G_X$. Then $\theta \in G_X \cap S$, so $\theta^{-1} \in G_X \cap S$.

Case $S = U_X$. To show the semigroup (U_X, θ) is factorizable, let $\alpha \in U_X$. Since $U_X = (G_X \cap U_X)E(U_X)$, $\alpha = \beta\gamma$ for some $\beta \in G_X \cap U_X$

and $\gamma \in E(U_X)$. Then $\alpha = \beta*(\theta^{-1}\gamma)$ and $\gamma \in E(T_X)$. As the proof in Theorem 2.4, we have that $\theta^{-1}\gamma \in E((T_X, \theta))$. Since $\theta^{-1}\gamma \in U_X$, $\theta^{-1}\gamma \in E((T_X, \theta)) \cap U_X = E((U_X, \theta))$. Hence $U_X = (G_X \cap U_X)*E((U_X, \theta))$.

Case $S = V_X$. Let $\alpha \in V_X$. Then $\alpha \in U_X$. From the first case, $\alpha = \beta*\gamma = \beta\theta\gamma$ for some $\beta \in G_X \cap U_X$ and $\gamma \in E((U_X, \theta))$. Thus $\gamma = (\beta\theta)^{-1}\alpha$. Since $\beta\theta \in G_X \cap U_X = G_X \cap V_X$, $(\beta\theta)^{-1} \in G_X \cap V_X \subseteq V_X$. Then $(\beta\theta)^{-1}\alpha \in V_X$, so $\gamma \in V_X$. Thus $\gamma \in E((U_X, \theta)) \cap V_X = E((V_X, \theta))$. This shows that $V_X = (G_X \cap U_X)*E((V_X, \theta))$.

Case $S = W_X$. Let $\alpha \in W_X$. Then $\alpha \in U_X$. From the first case, $\alpha = \beta*\gamma = \beta\theta\gamma$ for some $\beta \in G_X \cap U_X$ and $\gamma \in E((U_X, \theta))$. Thus $\gamma = (\beta\theta)^{-1}\alpha \in W_X$ because $(\beta\theta)^{-1} \in G_X \cap W_X \subseteq W_X$ and $\alpha \in W_X$. Hence $\gamma \in E((U_X, \theta)) \cap W_X = E((W_X, \theta))$. This proves that $W_X = (G_X \cap U_X) * E((W_X, \theta))$.

Hence, the theorem is completely proved. #