CHAPTER II

PRELIMINARIES

2.1 Quasi-groups and Groups

A quasi-group is an ordered pair (Q, o), where Q is non-empty set and o is a binary operation on Q such that for every p, q in Q, there exists unique elements x and y such that $q \circ x = p$ and $y \circ q = p$. In what follows we shall consider only finite quasi-groups, i.e. a quasi-group (Q, o) such that Q is finite set. The number of elements of Q will be called the order of Q. For any subset A of Q and any element q of Q the set $\{q \circ a \mid a \in A\}$ will be denoted by $q \circ A$. A fact that will be oftenly used in our arguments in this chapter and chapter III is that $|A| = |q \circ A|$, where the symbol $|S|$ denotes the cardinality of the set S.

A mapping Θ from a quasi-group (Q, o) to a quasi-group $(Q^*, *)$ is said to be a homomorphism if for every p, q in Q, $\Theta(p \circ q) = \Theta(p) * \Theta(q)$. If a homomorphism Θ is a one-to-one correspondence, then Θ is called an isomorphism. If there exists an isomorphism from (Q, o) to $(Q^*, *)$, then we say that (Q, o) is isomorphic to $(Q^*, *)$ and will be denoted by $Q \cong Q^*$. If Q is an isomorphism from (Q, o) onto itself, then Θ is called an automorphism of (Q, o).

Let (Q, o) be a quasi-group. If the quasi-group (Q, o) is associative, i.e. for every p, q, r in Q, $(p \circ q) \circ r = p \circ (q \circ r)$, then (Q, o) is said to be a group. If (Q, o) is a group, then there exists unique element e in Q such that for each q in Q, $q \circ e = e \circ q = q$. Such an element e will be called the identity of Q. Also, we have that for each q in Q, there exists unique element p in Q such that $p \circ q = q \circ p = e$. Such an element p will be called the inverse of q and will be denoted by q^{-1}.

A non-empty subset P of a group Q is said to be a subgroup of Q if P itself is a group under the same operation as Q. For each subgroup P of a group Q and each element q in Q. $q \circ P$ is called a left coset of P in Q. For each subgroup P of a group Q the following hold:

(2.1.1) Any two left cosets of P in Q are either disjoint or identical set of elements,

(2.1.2) Q is the disjoint union of the left cosets of P in Q; and

(2.1.3) $$[Q : P] = \frac{|Q|}{|P|},$$

where $[Q:P]$ is the number of left cosets of P in Q and it is called the index of P in Q.

Let S be a non-empty set. Then a one-to-one function of S onto itself will be called permutation. It can be shown that the set of all permutation of S forms a group. This group is know as the symmetric group of S. If Q is a subgroup of the symmetric group, then we say that Q is a permutation group. A permutation group Q is transitive on a subset V of S if

(1) for each σ in Q and each v in V, $\sigma(v)$ belongs to V;

(2) for every v, v' in V, there exists σ in Q such that $\sigma(v) = v'$.

2.2 Graphs

A graph G is an ordered pair $(V,\mathcal{E})$, where V is a finite non-empty set and $\mathcal{E}$ is a set of 2-subsets of V. Elements of V and $\mathcal{E}$ are called vertices and edges of G. If $\bar{\mathcal{E}} = S_2(V)-\mathcal{E}$, where $S_2(V)$ is the set of all 2-subsets of V, then $\bar{G} = (V,\bar{\mathcal{E}})$ is called the complementary graph of G.

Let $G = (V,\mathcal{E})$ and $G_1 = (V_1,\mathcal{E}_1)$ be graphs. A one-to-one mapping ψ from V onto V_1 is call an isomorphism from G onto G_1 if for every u, v in V,

$$\{u,v\} \text{ belongs to } \mathcal{E} \text{ if and only if } \{\psi(u),\psi(v)\} \text{ belongs to } \mathcal{E}_1.$$

If there exists an isomorphism from G onto G_1, then we say that G is isomorphic to G_1 and write $G \cong G_1$. If ψ is an isomorphism from G onto itself, then ψ is called an automorphism of G.

Let $G = (V,\mathcal{E})$ be a graph. For each v in V, we define

$$N_G(v) = \{u \mid \{u,v\} \in \mathcal{E}\}$$

For each v in V the degree of v in G, denoted by $d_G(v)$, is defined by

$$d_G(v) = |N_G(v)|$$

If for every u, v in V, $d_G(u) = d_G(v)$, then G is said to be regular.

2.3 Hall's Representation Theorem

To prove our main result of this chapter we need Hall's Representation Theorem. First we introduce some terminologies.

Let $(N_v)_{v \in V}$ be a system of sets, i.e. for each v in V, N_v is a set. If $(u_v)_{v \in V}$ is a system of elements such that u_v belongs to N_v

for all v belongs to V, then we say that $(u_v)_{v \in V}$ is a system of representative of $(N_v)_{v \in V}$. Furthermore, if the u_v's are distinct we call $(u_v)_{v \in V}$ a system of distinct representative, to be abbreviated SDR, of $(N_v)_{v \in V}$. For each system $(N_v)_{v \in V}$ of sets and each subset S of V we shall denote $\bigcup_{v \in S} N_v$ by N(S). Now, Hall's Representation Theorem can be stated as follows.

2.3.1 Theorem Let $(N_v)_{v \in V}$ be any finite system of subset of a set X, i.e. V is finite and for each v in V, N_v is a subset of X. Then $(N_v)_{v \in V}$ has SDR if and only if $|N(S)| \geq |S|$ for all subset S of V. This theorem is due to Hall [7].

2.4 A Property of Regular Graphs.

The following result on regular graph is essential to our study on quasi-group hypergraphs.

2.4.1 Theorem Let $G = (V,\mathcal{E})$ be a regular graph and W be any set such that $|W| = |N_G(v)|$. Then for each v in V we can associate a one-to-one function π_v from W onto $N_G(v)$ such that

$$\forall u, v \in V \ (u \neq v \rightarrow \forall w \in W \ (\pi_u(w) \neq \pi_v(w))).$$

Proof Let $(V,\mathcal{E})$ be a regular graph of degree k. If k = 0 we have $W = \emptyset$. In this case we can take $\pi_v = \emptyset$ for all v in V. So, we are left to consider the case where k > 0. In this case Hall's Representation Theorem (Theorem 2.3.1) will be used. For convenience, in the remaining of this proof we shall denote $N_G(v)$ by N_v. First we shall show that $(N_v)_{v \in V}$ has an SDR. Let S be any subset of V. To verify that $|N(S)| \geq |S|$, let

$$O_u = \{(u,v) \mid v \in V \text{ and } u \in N_v\},$$

$$K_u = \{(u,v) \mid v \in S \text{ and } u \in N_v\},$$

$$L_v = \{(u,v) \mid u \in N_v\}.$$

Observe that

$$\bigcup_{u \in N(S)} K_u = \bigcup_{v \in S} L_v$$

Note that each side of the above equation is disjoint union, hence we have

$$\sum_{u \in N(S)} |K_u| = \sum_{v \in S} |L_v| .$$

Clearly $K_u \subseteq O_u$. Therefore we have

$$\sum_{u \in N(S)} |O_u| \geqslant \sum_{v \in S} |L_v| .$$

Observe that

$$|O_u| = |N_u| = k$$

and

$$|L_v| = |N_v| = k ,$$

hence we have

$$\sum_{u \in N(S)} k \geqslant \sum_{v \in S} k ,$$

i.e.

$$|N(S)|\, k \geqslant |S|\, k.$$

Therefore

$$|N(S)| \geqslant |S|$$

Hence, by theorem 2.3.1, $(N_v)_{v \varepsilon V}$ has an SDR. Let $(u_v)_{v \varepsilon V}$ be an SDR of $(N_v)_{v \varepsilon V}$.

For each v in V, we define $u_v^{(1)} = u_v$ and $N_v^{(1)} = N_v$. Observe that we have define $u_v^{(1)}$ and $N_v^{(1)}$ such that

(1) for each u in V, $|N_u^{(1)}| = k = k-(1-1)$, and

(2) $(N_v^{(1)})_{v \varepsilon V}$ has $(u_v^{(1)})_{v \varepsilon V}$ as an SDR.

Let ℓ be any positive integer less than or equal to k. Assume that $u_v^{(j)}$ and $N_v^{(j)}$ have been defined for all v in V and for all positive integer $j < \ell$ such that

(3) for each $j < \ell$ and each u in V, $|N_u^{(j)}| = k-(j-1)$, and

(4) for each $j < \ell$, $(N_v^{(j)})_{v \varepsilon V}$ has $(u_v^{(j)})_{v \varepsilon V}$ as an SDR.

We now define $N_v^{(\ell)}$ as follows. For each v in V, let

$$N_v^{(\ell)} = N_v^{(\ell-1)} - \{u_v^{(\ell-1)}\} .$$

We shall show that $(N_v^{(\ell)})_{v \varepsilon V}$ has an SDR. Let S be any subset of V. To verify that $|N^{(\ell)}(S)| \geqslant |S|$ where $N^{(\ell)}(S) = \bigcup_{v \varepsilon S} N_v^{(\ell)}$, let

$$O_u^{(\ell)} = \{(u,v) \mid v \varepsilon V \text{ and } u \varepsilon N_v^{(\ell)} \},$$

$$K_u^{(\ell)} = \{(u,v) \mid v \varepsilon S \text{ and } u \varepsilon N_v^{(\ell)} \},$$

$$L_v^{(\ell)} = \{(u,v) \mid u \varepsilon N_v^{(\ell)}\} .$$

Observe that

$$\bigcup_{u \in N^{(\ell)}(S)} K_u^{(\ell)} = \bigcup_{v \in S} L_v^{(\ell)}$$

Note that each side of the above equation is a disjoint union, hence we have

$$\sum_{u \in N^{(\ell)}(S)} |K_u^{(\ell)}| = \sum_{v \in S} |L_v^{(\ell)}|.$$

Clearly $K_u^{(\ell)} \subseteq O_u^{(\ell)}$. Therefore we have

$$\sum_{u \in N^{(\ell)}(S)} |O_u^{(\ell)}| \geq \sum_{v \in S} |L_v^{(\ell)}| .$$

For each element u in V and for each $i < \ell$, u is among $(u_v^{(i)})_{v \in V}$, i.e. $u = u_{v_i}^{(i)}$ for some v_i in V. Since $N_{v_i}^{(\ell)} = N_{v_i} - \{u_{v_i}^{(1)}, u_{v_i}^{(2)}, \ldots, u_{v_i}^{(\ell-1)}\}$, hence $u \notin N_{v_i}^{(\ell)}$, Therefore $(u, v_i) \notin O_u^{(\ell)}$. Hence $O_u^{(\ell)} \subseteq O_u - \{(u, v_i) | i = 1,2,\ldots,\ell-1\}$. If $u \in N_v$ where $v \neq v_i$ for any $i < \ell$, then we must have $u \neq u_v^{(i)}$ for any $i < \ell$. For otherwise $u_v^{(i)} = u = u_{v_i}^{(i)}$ for some i. This is contary to the fact that $(u_v^{(i)})_{v \in V}$ is an SDR. Hence $u \in N_v^{(\ell)}$. Therefore $O_u - \{(u, v_i) | i = 1,2,\ldots,\ell-1\} \subseteq O_u^{(\ell)}$. Hence

$$O_u^{(\ell)} = O_u - \{(u, v_i) \mid i = 1,2,\ldots, \ell-1\}$$

If $1 \leq i < j < \ell$, then $u_{v_i}^{(i)} \notin N_{v_i}^{(j)}$, but $u_{v_i}^{(j)} \in N_{v_i}^{(j)}$. Hence $u_{v_i}^{(i)} \neq u_{v_i}^{(j)}$.

Therefore, if $i \neq j$ we must have $v_i \neq v_j$. For otherwise we would have $u_{v_i}^{(i)} = u_{v_i}^{(j)}$. Hence

$$|O_u^{(\ell)}| = |O_u - \{(u,v_i) \mid i = 1,2,\ldots, \ell-1\}|$$

$$= |O_u| - |\{(u,v_i) \mid i = 1,2,\ldots, \ell-1\}|$$

$$= k - (\ell-1).$$

Clearly,

$$|L_v^{(\ell)}| = |N_v^{(\ell)}| = k - (\ell-1).$$

Therefore we have

$$\sum_{u \in N^{(\ell)}(S)} k-(\ell-1) \geq \sum_{v \in S} k-(\ell-1),$$

i.e.

$$|N^{(\ell)}(S)|\,(k-(\ell-1)) \geq |S|(k-(\ell-1))$$

Therefore

$$|N^{(\ell)}(S)| \geq |S| .$$

Hence, by theorem 2.3.1, $(N_v^{(\ell)})_{v \in V}$ has an SDR. Let $(u_v^{(\ell)})_{v \in V}$ be an SDR of $(N_v^{(\ell)})_{v \in V}$.

Hence for each $i = 1,2,\ldots,k$, we can define $(N_v^{(i)})_{v \in V}$ and $(u_v^{(i)})_{v \in V}$ such that

(5) for each $i = 1,2,\ldots,k$, $(u_v^{(i)})_{v \in V}$ is an SDR of $(N_v^{(i)})_{v \in V}$,

and

(6) for each $i = 1,2,\ldots,k-1$, $N_v^{(i+1)} = N_v^{(i)} - \{u_v^{(i)}\}$.

From (6) it follows that

(7) for each v in V, $u_v^{(1)}, u_v^{(2)}, \ldots, u_v^{(k)}$ are distinct.

Let W be any set such that $|W| = k$. Then there exists a one-to-one function ψ from W onto $\{1,2,\ldots,k\}$. For each v in V, define $\pi_v : W \to N_v$ by

$$\pi_v(w) = u_v^{(\psi(w))} \quad \text{for all } w \text{ in } W.$$

It follows from (7) that π_v is one-to-one function from W onto N_v, and follows from (5) that for every u, v in V if $u \neq v$ then $\pi_u(w) \neq \pi_v(w)$ for all w in W.

Hence for each v in V we can associate a one-to-one function π_v from W onto $N_G(v)$ such that

$$\forall u, v \in V \; (u \neq v \to \forall w \in W \; (\pi_u(w) \neq \pi_v(w))).$$

#

2.5 Hypergraphs

A hypergraph H is an ordered pair $(V, \mathcal{E})$, where V is a finite non-empty set and $\mathcal{E}$ is a set of non-empty subsets of V. The sets in $\mathcal{E}$ are called hyperedges or simply edges while the elements of V are called vertices. By rank of a hypergraph we mean the maximum cardinality of the edges in the hypergraph. A hypergraph in which every edge has the same cardinality is called a uniform hypergraph. In this thesis we shall consider only uniform hypergraphs. In the sequel, by a hypergraph we mean a uniform hypergraph.

Let $H = (V,\mathcal{E})$ and $H_1 = (V_1,\mathcal{E}_1)$ be hypergraphs. A one-to-one mapping ψ from V onto V_1 is called an isomorphism from H onto H_1 if for each subset E of V,

E belongs to $\mathcal{E}$ if and only if $\psi(E)$ belongs to $\mathcal{E}_1$.

Here and in the sequel, $\psi(E)$ denotes the set $\{\psi(v) \mid v \in E\}$. If there is an isomorphism from H onto H_1, then we say that H is <u>isomorphic</u> to H_1 and write $H \cong H_1$. If ψ is isomorphism from H onto itself, then ψ is called an <u>automorphism</u> of H. It can be shown that the set of all automorphisms of any hypergraph H forms a group under composition. This group is known as the <u>automorphism group</u> of H. It will be denoted by $\Gamma(H)$ or $\Gamma(V,\mathcal{E})$.

To each vertex v of a hypergraph $H = (V,\mathcal{E})$ we associate a hypergraph $H_v = (V_v,\mathcal{E}_v)$, where

$$\mathcal{E}_v = \{E - \{v\} \mid E \in \mathcal{E} \text{ and } v \in E\}.$$

and

$$V_v = \cup\, \mathcal{E}_v \,.$$

Following Berge [3], we associate a graph $(H)_2 = (V,(\mathcal{E})_2)$ to each hypergraph $H = (V,\mathcal{E})$, where

$$(\mathcal{E})_2 = \{e \mid e \text{ is a 2-subset of some } E \text{ in } \mathcal{E}\}.$$

2.5.1 <u>Remark</u> Let $H = (V,\mathcal{E})$ be a hypergraph. Then for each v in V, $N_{\overline{(H)}_2}(v) = V - (V_v \cup \{v\})$.

2.5.2 <u>Proposition</u> Let $H = (V,\mathcal{E})$ be a hypergraph. If for every u, v in V, $H_u \cong H_v$, then $\overline{(H)}_2$ is regular.

<u>Proof</u> Let $H = (V,\mathcal{E})$ be a hypergraph such that for every u, v in V, $H_u \cong H_v$. Then $|V_u| = |V_v|$. Hence $|V| - |V_u| - 1 = |V| - |V_v| - 1$. By remark 2.5.1, we have $|N_{\overline{(H)}_2}(u)| = |N_{\overline{(H)}_2}(v)|$.

Hence $\overline{(H)}_2$ is regular. #