CHAPTER III

SOLUTION OF $f(x+y) = f(x)f(y)$

In solving trigonometric functional equations treated in this work, it turns out that solutions of the trigonometric functional equations are expressible in terms of homomorphisms from a group G into certain subgroups of the multiplicative group of complex numbers. In this chapter, we shall characterize these homomorphisms for the case $G = R^n$. Our main results are theorem 3.2.6 and its corollary and theorem 3.3.5.

3.1 Vector Space

A non-empty set F with two binary operations +, •, known as addition and multiplication respectively, is said to form a field if

(i) F forms a commutative group under addition.

(ii) $F^* = F - \{0\}$, where 0 is the additive identity, forms a group under multiplication.

(iii) For any $a,b,c \in F$, we have

$$a(b + c) = ab + ac$$

and

$$(b + c)a = ba + ca .$$

(F, +) and $(F^*, \cdot)$ will be refered to as the additive group and the multiplicative group of F, respectively.

Let (F,+,•) be a field and (V,+) be a commutative group with a rule of multiplication which assigns to any $a \in F$, $u \in V$ a product $au \in V$. Then V is called a vector space over F if the following axioms hold :

(1) For any $a \in F$ and any $u, v \in V$, $a(u+v) = au + av$.

(2) For any $a, b \in F$ and any $u \in V$, $(a+b)u = au + bu$.

(3) For any $a, b \in F$ and any $u \in V$, $(ab)u = a(bu)$.

(4) For any $u \in V$, we have $1.u = u$,

where 1 is the multiplicative identity of F.

The elements of F and V will be refered to as scalars and vectors, respectively.

Let V be a vector space over a field F and let $u_1, \ldots, u_m \in V$. If $v = \alpha_1 u_1 + \ldots + \alpha_m u_m$, where $\alpha_i \in F$, $i = 1, \ldots, m$, then we say that v is a linear combination of $u_1, \ldots, u_m$. The vectors $v_1, \ldots, v_m \in V$ are said to be linearly independent if for any scalars $a_1, \ldots, a_m \in F$ $a_1 v_1 + \ldots + a_m v_m = 0$ implies that $a_1 = 0, \ldots, a_m = 0$. An arbitrary set A of vectors is said to be a linearly independent set if every finite subset of A is linearly independent. If B is a linearly independent subset of V such that for every $v \in V$, v can be written as a linear combination of vectors in B, we say that B is a basis of V. It can be shown that every vector in V has a unique representation as a linear combination of elements of B.

Observe that the set $\mathbb{R}$ of real numbers can be considered as a vector space over the field Q of rational numbers. It can be shown that $\mathbb{R}$ has a basis over Q. Such a basis is known as a Hamel basis. A proof of the existence of such a basis will be given in the Appendix.

3.2 Solution of $f(x+y) = f(x)\,f(y)$

3.2.1 Theorem Let V be a vector space over a field F with $\mathcal{B} = \{V_\alpha : \alpha \in I\}$ as a basis. Let f be a function on V into a commutative group G'. Then f satisfies

$$(3.2.1.1) \qquad f(x+y) = f(x)\,f(y) ,$$

iff there exists a family $\{f_\alpha : \alpha \in I\}$ of homomorphisms from the additive group of F into G' such that for any $x = \sum_{i=1}^{n} a_i V_{\alpha_i}$ in V, we have

$$f(x) = f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} f_{\alpha_i}(a_i) .$$

Proof Assume that $f : V \longrightarrow G'$ satisfies (3.2.1.1)

For each $V_\alpha \in \mathcal{B}$, define $f_\alpha(a) = f(aV_\alpha)$.

Observe that for each $\alpha \in I$, $f_\alpha : F \longrightarrow G'$.

And
$$\begin{aligned} f_\alpha(a+b) &= f((a+b)\,V_\alpha) , \\ &= f(aV_\alpha + bV_\alpha) , \\ &= f(aV_\alpha)\,f(bV_\alpha), \\ &= f_\alpha(a)\,f_\alpha(b) . \end{aligned}$$

For any $x \in V$, we have $x = \sum_{i=1}^{n} a_i V_{\alpha_i}$, where $a_i \in F$, $V_{\alpha_i} \in \mathcal{B}$.

Hence
$$f(x) = f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right).$$

By (3.2.1.1), we have

$$f(x) = \prod_{i=1}^{n} f(a_i V_{\alpha_i}).$$

Hence $$f(x) = \prod_{i=1}^{n} f_{\alpha_i}(a_i).$$

To prove the converse, assume that $\{f_\alpha : \alpha \in I\}$ is a family of homomorphisms on the additive group of F into G' and f is given by $f(\sum_{i=1}^{n} a_i V_{\alpha_i}) = \prod_{i=1}^{n} f_{\alpha_i}(a_i)$.

Then for any $x, y \in V$, we may write

$$x = \sum_{i=1}^{n} a_i V_{\alpha_i}, \quad y = \sum_{i=1}^{n} b_i V_{\alpha_i},$$

where $a_i, b_i \in F$ and $V_{\alpha_i} \in \mathcal{B}$.

Hence,

$$\begin{aligned} f(x+y) &= f(\sum_{i=1}^{n} a_i V_{\alpha_i} + \sum_{i=1}^{n} b_i V_{\alpha_i}), \\ &= f(\sum_{i=1}^{n} (a_i + b_i) V_{\alpha_i}), \\ &= \prod_{i=1}^{n} f_{\alpha_i}(a_i + b_i), \\ &= \prod_{i=1}^{n} (f_{\alpha_i}(a_i)\, f_{\alpha_i}(b_i)), \\ &= \prod_{i=1}^{n} f_{\alpha_i}(a_i) \prod_{i=1}^{n} f_{\alpha_i}(b_i), \\ &= f(\sum_{i=1}^{n} a_i V_{\alpha_i})\; f(\sum_{i=1}^{n} b_i V_{\alpha_i}), \\ &= f(x)\, f(y). \end{aligned}$$

3.2.2 <u>Lemma</u> Let h be a homomorphism from the additive group Q of rational numbers into a commutative group G'. Then $h(na) = (h(a))^n$, for all $a \in Q$ and all $n \in Z$, where Z is the set of all integers.

Proof Let $a \in Q$.

Since h is a homomorphism, hence $h(0) = 1$.

Therefore $h(0.a) = h(0) = 1 = (h(a))^0$.

Assume that k is a non-negative integer such that

$$h(k.a) = (h(a))^k.$$

Then,

$$\begin{aligned} h((k+1)a) &= h(ka+a), \\ &= h(ka)\, h(a), \\ &= (h(a))^k\, h(a), \\ &= (h(a))^{k+1}. \end{aligned}$$

Hence $h(na) = (h(a))^n$ for all non-negative integers n.

For any negative integer m, -m is a positive integer.

Hence,

$$\begin{aligned} 1 = h(0) &= h(ma + (-m)a), \\ &= h(ma)\, h((-m)a), \\ &= h(ma)(h(a))^{-m}. \end{aligned}$$

Therefore $h(ma) = (h(a))^m$.

Thus $h(na) = (h(a))^n$ for all $n \in Z$.

3.2.3 Theorem h is a homomorphism from Q into G', where G' is $\mathbb{R}^+$ or Δ, iff there exists $r \in G'$ such that $h(a) = r^a$, for $a \in Q$.

Proof Assume that h is a homomorphism from Q into G'.

Let $a \in Q$.

Then $a = \frac{p}{q}$, where p, q are integers, $q \neq 0$.

We have

$$(h(\tfrac{p}{q}))^q = h(q.\tfrac{p}{q}) ,$$
$$= h(p) ,$$
$$= h(p.1) ,$$
$$= (h(1))^p .$$

Hence $h(\frac{p}{q}) = (h(1))^{\frac{p}{q}}$.

i.e. we have $h(a) = r^a$, where $r = h(1) \in G'$.

Conversely, assume that there exists $r \in G'$ such that

$$h(a) = r^a , \quad \text{for } r \in G' .$$

Then,

$$h(a+b) = r^{a+b} = r^a . r^b ,$$
$$= h(a)\, h(b).$$

Hence h is a homomorphism.

3.2.4 <u>Theorem</u> Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. A function $f : \mathbb{R} \longrightarrow G'$, where G' is $\mathbb{R}^+$ or Δ , satisfies

$$(3.2.4.1) \qquad f(x+y) = f(x)\, f(y)$$

iff there exists a function b on H into G' such that for each $x = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$, we have

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i} \right) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} .$$

<u>Proof</u> Assume that $f : \mathbb{R} \longrightarrow G'$, where G' is $\mathbb{R}^+$ or Δ, satisfies (3.2.4.1).

By Theorem 3.2.1, we see that f must be of the form

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} f_{\alpha_i}(a_i) ,$$

where f_{α_i} is a homomorphism from $\mathbb{Q}$ into G'.

By Theorem 3.2.3, each f_{α_i} must be of the form

$$f_{\alpha_i}(a) = b_{\alpha_i}^{a} , \quad \text{for some } b_{\alpha_i} \in G'.$$

Let $b : H \longrightarrow G'$ be defined by $b(V_{\alpha_i}) = b_{\alpha_i}$.

Then we have,

$$\begin{aligned} f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) &= \prod_{i=1}^{n} f_{\alpha_i}(a_i) , \\ &= \prod_{i=1}^{n} b_{\alpha_i}^{a_i} , \\ &= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} . \end{aligned}$$

On the other hand, if b is any function on H into G', and f is defined by

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} ,$$

then, for any $x = \sum_{i=1}^{n} a_i V_{\alpha_i}$, $y = \sum_{i=1}^{n} a_i' V_{\alpha_i}$ in $\mathbb{R}$, we have

$$f(x+y) = f(\sum_{i=1}^{n} a_i V_{\alpha_i} + \sum_{i=1}^{n} a_i' V_{\alpha_i}) ,$$

$$= f(\sum_{i=1}^{n} (a_i + a_i') V_{\alpha_i}) ,$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i + a_i'} ,$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i'} ,$$

$$= f(\sum_{i=1}^{n} a_i V_{\alpha_i}) f(\sum_{i=1}^{n} a_i' V_{\alpha_i}) ,$$

$$= f(x) f(y).$$

3.2.5 Corollary Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. A function $f : \mathbb{R} \longrightarrow \mathbb{C}^*$ satisfies

$$(3.2.5.1) \qquad f(x+y) = f(x) f(y)$$

iff there exists a function c on H into $\mathbb{C}^*$ such that for each

$x = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, we have

$$f(\sum_{i=1}^{n} a_i V_{\alpha_i}) = \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} .$$

Proof Assume that $f : \mathbb{R} \longrightarrow \mathbb{C}^*$ satisfies (3.2.5.1).

Let $f(x) = \emptyset(x) . \frac{f}{\emptyset}(x)$,

where $\emptyset(x) = |f(x)|$ and $\frac{f}{\emptyset}(x) = \frac{f(x)}{\emptyset(x)}$.

Observe that $\emptyset : \mathbb{R} \longrightarrow \mathbb{R}^+$,

and $\frac{f}{\emptyset} : \mathbb{R} \longrightarrow \Delta$.

Hence,

$$\begin{aligned}
\emptyset(x+y) &= |f(x+y)| , \\
&= |f(x)\ f(y)| , \\
&= |f(x)|\ |f(y)| , \\
&= \emptyset(x)\ \emptyset(y) .
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{f}{\emptyset}(x+y) &= \frac{f(x+y)}{\emptyset(x+y)} , \\
&= \frac{f(x)\ f(y)}{\emptyset(x)\ \emptyset(y)} , \\
&= \frac{f(x)}{\emptyset(x)} \cdot \frac{f(y)}{\emptyset(y)} , \\
&= \frac{f}{\emptyset}(x)\ \frac{f}{\emptyset}(y) .
\end{aligned}$$

Therefore, by using Theorem 3.2.4, there exists a function b_1 on H into $\mathbb{R}^+$ and a function b_2 on H into Δ such that for each

$x = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$,

we have $\emptyset(x) = \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i}$,

and $\frac{f}{\emptyset}(x) = \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i}$. 000621

Let $c : H \longrightarrow \mathbb{C}^*$ be defined by

$$c(V_{\alpha_i}) = b_1(V_{\alpha_i})\ b_2(V_{\alpha_i}).$$

So we have,

$$f(x) = \emptyset(x) \cdot \frac{f}{\emptyset}(x) ,$$
$$= \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i} \cdot \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} ,$$
$$= \prod_{i=1}^{n} (b_1(V_{\alpha_i}) \, b_2(V_{\alpha_i}))^{a_i} ,$$
$$= \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} .$$

Conversely, if c is a function on H into $\mathbb{C}^*$, and f is defined by

$$f(\sum_{i=1}^{n} a_i V_{\alpha_i}) = \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} ,$$

then it can be verified in the same way as in theorem 3.2.4, that $f(x+y) = f(x)\, f(y)$.

3.2.6 Theorem Let $f : \mathbb{R}^n \longrightarrow G'$, where G' is $\mathbb{C}^*$ or Δ. f satisfies

(3.2.6.1) $\quad f(x+y) = f(x)\, f(y)$

iff for each i = 1,....,n, there exists a function f_i on $\mathbb{R}$ to G' satisfying

$$f_i(x+y) = f_i(x)\, f_i(y)$$

such that for each $x = (x_1,....,x_n) \in \mathbb{R}^n$, we have

$$f(x) = \prod_{i=1}^{n} (f_i \circ p_i)(x) ,$$

where the p_i's are given by $p_i(x_1,....,x_n) = x_i$, i = 1,....,n.

<u>Proof</u> Assume that f satisfies (3.2.6.1).

For each $i = 1,\dots,n$, let $\pi_i : \mathbb{R} \longrightarrow \mathbb{R}^n$ be defined by

$$\pi_i(x) = xe_i ,$$

where $e_i = (\delta_{i1},\dots, \delta_{in})$, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Set $f_i = f \circ \pi_i$,

hence $f_i : \mathbb{R} \longrightarrow G'$ and

$$\begin{aligned} f_i(x+y) &= (f \circ \pi_i)(x+y) , \\ &= f(\pi_i(x+y)) , \\ &= f((x+y)e_i), \\ &= f(xe_i + ye_i) , \\ &= f(xe_i)\, f(ye_i), \\ &= f(\pi_i(x))\, f(\pi_i(y)), \\ &= f_i(x)\, f_i(y) . \end{aligned}$$

Also, from $f_i = f \circ \pi_i$, we have

$$f_i \circ p_i = (f \circ \pi_i) \circ p_i ,$$

where p_i is defined by $p_i(x_1,\dots, x_n) = x_i$.

Hence, for any $x = (x_1,\dots, x_n)$, we have

$$\begin{aligned} f_i \circ p_i(x) &= f(\pi_i(p_i(x_1,\dots,x_n))), \\ &= f(\pi_i(x_i)), \\ &= f(x_i e_i). \end{aligned}$$

Therefore,

$$\prod_{i=1}^{n} f_i \circ p_i(x) = \prod_{i=1}^{n} f(x_i e_i) ,$$

$$= f(x_1 e_1) \dots . f(x_n e_n) ,$$

$$= f(x_1 e_1 + \dots . + x_n e_n) ,$$

$$= f(x_1, \dots , x_n) ,$$

$$= f(x).$$

Conversely, assume that $f(x) = \prod_{i=1}^{n} f_i \circ p_i(x)$, where each f_i , $i = 1, \dots, n$, satisfies $f_i(x+y) = f_i(x) f_i(y)$ for all $x, y \in \mathbb{R}$.

We have

$$f(x+y) = \prod_{i=1}^{n} (f_i(p_i(x+y))),$$

$$= \prod_{i=1}^{n} f_i(x_i + y_i) ,$$

$$= \prod_{i=1}^{n} (f_i(x_i) f_i(y_i)) ,$$

$$= \prod_{i=1}^{n} f_i(x_i) \prod_{i=1}^{n} f_i(y_i) ,$$

$$= \prod_{i=1}^{n} f_i(p_i(x)) \prod_{i=1}^{n} f_i(p_i(y)) ,$$

$$= f(x) f(y).$$

3.2.7 Corollary By using corollary 3.2.5, we see that $f : \mathbb{R}^n \longrightarrow \mathbb{C}^*$ satisfies $f(x+y) = f(x)\, f(y)$ if and only if for $j = 1, \ldots, n$, there exist functions c_j on H, where H is a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, into $\mathbb{C}^*$ such that for each $x = (\sum_{i=1}^{m} a_{1i} V_{\alpha_i}, \ldots, \sum_{i=1}^{m} a_{ni} V_{\alpha_i})$ we have

$$f(x) = \prod_{j=1}^{n} \prod_{i=1}^{m} c_j(V_{\alpha_i})^{a_{ji}} .$$

3.3 Continuous Solution of $f(x+y) = f(x)\, f(y)$.

In this section, we shall determine all the continuous solutions of $f(x+y) = f(x)\, f(y)$, where f is a function from $\mathbb{R}^n$ into $\mathbb{C}^*$.

3.3.1 Lemma Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$(3.3.1.1) \qquad g(x+y) = g(x) + g(y) ,$$

then $g(x) = bx$ for some b in $\mathbb{R}$.

Proof We first claim that $g(na) = ng(a)$ for all integer n and all $a \in \mathbb{R}$.

Since g is a homomorphism, hence $g(0) = 0$.

Therefore $g(0.a) = g(0) = 0 = 0.g(a)$.

Assume that k is a non-negative integer such that

$$g(ka) = kg(a).$$

Then,

$$\begin{aligned} g((k+1)a) &= g(ka+a), \\ &= g(ka) + g(a), \\ &= kg(a) + g(a), \\ &= (k+1)\, g(a). \end{aligned}$$

Hence $g(na) = ng(a)$ for all non-negative integer n.

For any negative integer m, -m is a positive integer.

Hence,

$$0 = g(0) = g(ma + (-m)a),$$
$$= g(ma) + g((-m)a),$$
$$= g(ma) + (-m)\,g(a),$$

Thus $g(ma) = mg(a)$.

Therefore $g(na) = ng(a)$ for all integer n.

For $r = \frac{p}{q}$, where p, q are integers and $q \neq 0$,

we have

$$qg(r) = qg(\tfrac{p}{q}),$$
$$= g(q.\tfrac{p}{q}),$$
$$= g(p),$$
$$= g(p.1),$$
$$= pg(1).$$

Thus $g(r) = \frac{p}{q}\,g(1) = rg(1)$.

Let $x \in \mathbb{R}$. Since the set of rational numbers is dense in $\mathbb{R}$, we can find a sequence $\{r_n\}$ of rational numbers converging to x.

Since g is continuous, hence $\lim_{n \to \infty} g(r_n) = g(x)$.

But $\lim_{n \to \infty} g(r_n) = \lim_{n \to \infty} r_n g(1) = xg(1)$.

Therefore $g(x) = xg(1)$, $x \in \mathbb{R}$.

Thus $g(x) = bx$, where $b = g(1) \in \mathbb{R}$.

3.3.2 <u>Theorem</u> Let g be a continuous function on the set of real numbers into the set of positive real numbers. g satisfies

$$(3.3.2.1) \qquad g(x+y) = g(x)\, g(y)$$

iff $g(x) = e^{ax}$ for some a in $\mathbb{R}$.

Proof Assume that g satisfies (3.3.2.1).

Let $h(x) = \ln x , \quad x > 0.$

Put $f = h \circ g.$

Since both h and g are continuous, hence f is also continuous.

We also have

$$\begin{aligned} f(x+y) &= h(g(x+y)) , \\ &= \ln(g(x+y)) , \\ &= \ln(g(x)\, g(y)) , \\ &= \ln g(x) + \ln g(y) , \\ &= h(g(x)) + h(g(y)) , \\ &= f(x) + f(y). \end{aligned}$$

Therefore, by lemma 3.3.1, there exists $a \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ $f(x) = ax.$

Then,

$$\begin{aligned} \ln g(x) &= h(g(x)) , \\ &= f(x) , \\ &= ax . \end{aligned}$$

Therefore $g(x) = e^{ax}$, where $a \in \mathbb{R}$.

Conversely, let $g(x) = e^{ax}$ for some a in $\mathbb{R}$.

Thus
$$\begin{aligned} g(x+y) &= e^{a(x+y)} , \\ &= e^{ax+ay} , \\ &= e^{ax} e^{ay} , \\ &= g(x)\, g(y). \end{aligned}$$

3.3.3 Theorem Let $I : (\mathbb{R}, +) \longrightarrow \Delta$ be a continuous function. I satisfies

$$(3.3.3.1) \qquad I(x+y) = I(x)I(y)$$

iff there exists a real number k such that $I(x) = e^{ikx}$.

Proof Assume that I satisfies (3.3.3.1).

Since $I : \mathbb{R} \longrightarrow \Delta$, hence $|I(x)| = 1$ for all x.

Thus $|I(1)| = 1$.

Therefore $\exists\, k \in \mathbb{R}$ such that $I(1) = e^{ik}$.

By using the same argument as in the proof of lemma 3.2.2, it can be shown that $I(na) = (I(a))^n$ for all integer n and $a \in \mathbb{R}$.

Thus, for any rational number $r = \frac{n}{m}$, where n, m are integers and $m \neq 0$, we get

$$\begin{aligned}(I(r))^m &= (I(\tfrac{n}{m}))^m, \\ &= I(m.\tfrac{n}{m}), \\ &= I(n), \\ &= I(n.1), \\ &= (I(1))^n. \\ &= e^{ikn}.\end{aligned}$$

Hence $I(r) = I(\frac{n}{m}) = e^{\frac{ikn}{m} + \frac{2\pi li}{m}}$,

for some $l = 0,1,\ldots, m-1$.

For any integer $t \neq 0$, we have

$$e^{\frac{ikn}{m} + \frac{2\pi li}{m}} = I(\tfrac{n}{m}) = I(\tfrac{nt}{mt}) = e^{\frac{iknt}{mt} + \frac{2\pi li}{mt}}.$$

Therefore $e^{\frac{2\pi li}{m}} = e^{\frac{2\pi li}{mt}}$ for all integer $t \neq 0$.

Hence,

$$1 = e^{2\pi li} = (e^{\frac{2\pi li}{m}})^m = (e^{\frac{2\pi li}{mt}})^m = e^{\frac{2\pi li}{t}},$$

for all integer $t \neq 0$.

If $l > 0$, then for some $t_o \neq 0$, we have $0 < \frac{l}{t_o} < 1$.

Hence $e^{2\pi(\frac{l}{t_o})i} \neq 1$, which is a contradiction.

Therefore, we have $l = 0$.

Thus $I(r) = I(\frac{n}{m}) = e^{ik\frac{n}{m}} = e^{ikr}$ for all rational r.

For any $x \in \mathbb{R}$, there exists a sequence $\{r_n\}$ such that $r_n \in \mathbb{Q}$ and $\lim_{n\to\infty} r_n = x$.

By continuity of I, $I(x) = \lim_{n\to\infty} I(r_n) = \lim_{n\to\infty} e^{ikr_n} = e^{ikx}$.

Conversely, let $I(x) = e^{ikx}$ for some real number k.

Then we have

$$I(x+y) = e^{ik(x+y)} = e^{ikx+iky} = e^{ikx} \cdot e^{iky} = I(x)\, I(y).$$

3.3.4 <u>Theorem</u> Let $h : (\mathbb{R}, +) \longrightarrow (\mathbb{C}^*, \cdot)$ be a continuous function. h satisfies

(3.3.4.1) $$h(x+y) = h(x)\, h(y)$$

iff there exists $r \in \mathbb{C}$ such that $h(x) = e^{rx}$.

<u>Proof</u> Assume that h satisfies (3.3.4.1).

Let $g(x) = |h(x)|$ and $I(x) = \frac{h(x)}{g(x)}$.

Observe that $g : \mathbb{R} \longrightarrow \mathbb{R}^+$,

and $I : \mathbb{R} \longrightarrow \Delta$.

Since h is continuous, so are g and I.

Also, $g(x+y) = |h(x+y)| = |h(x)\, h(y)|,$

$= |h(x)|\,|h(y)| = g(x)\, g(y).$

By using Theorem 3.3.2, we get $g(x) = e^{cx}$ for some $c \in \mathbb{R}$.

Observe that $I(x+y) = \dfrac{h(x+y)}{g(x+y)} = \dfrac{h(x)\, h(y)}{g(x)\, g(y)},$

$= \dfrac{h(x)}{g(x)} \cdot \dfrac{h(y)}{g(y)} = I(x)\, I(y).$

By using Theorem 3.3.3, we get $I(x) = e^{ikx}$ for some $k \in \mathbb{R}$.

Thus $h(x) = I(x)\, g(x),$

$= e^{ikx} \cdot e^{cx},$

$= e^{(c+ik)x}$

$= e^{rx}$, where $r = (c+ik) \in \mathbb{C}$.

Conversely, let $h(x) = e^{rx}$ where $r \in \mathbb{C}$.

Then $h(x+y) = e^{r(x+y)} = e^{rx+ry},$

$= e^{rx} \cdot e^{ry} = h(x)\, h(y).$

3.3.5 Theorem Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}^*$ be a continuous function. f satisfies

(3.3.5.1) $f(x+y) = f(x)\, f(y)$

iff there exist $r_i \in \mathbb{C}$, $i = 1, \ldots, n$, such that for each $x = (x_1, \ldots, x_n)$ we have $f(x) = e^{r_1 x_1 + \cdots + r_n x_n}$.

Proof Assume that f satisfies (3.3.5.1).

Using Theorem 3.2.6, there exist $f_i : \mathbb{R} \longrightarrow \mathbb{C}^*$ satisfying

$$f_i(x+y) = f_i(x)\, f_i(y), \quad i = 1, \ldots, n,$$

such that for each $x \in \mathbb{R}^n$, we have

$$f(x) = \prod_{i=1}^{n} (f_i \circ p_i)(x) ,$$

where each p_i, $i = 1,\ldots, n$, is given by $p_i(x_1,\ldots,x_n) = x_i$.
Such an f_i is given by $f_i = f \circ \pi_i$, where π_i is defined as in the proof of Theorem 3.2.6.
Since f and π_i are continuous, hence each f_i is continuous.
By using Theorem 3.3.4, we have

$$f_i(x_i) = e^{r_i x_i} \quad \text{for each } i = 1,\ldots,n \text{ and } r_i \in \mathbb{C}.$$

Hence,

$$\begin{aligned} f(x) &= \prod_{i=1}^{n} (f_i \circ p_i)(x) , \\ &= f_1(x_1)\ldots\ldots f_n(x_n) , \\ &= e^{r_1 x_1}\ldots\ldots e^{r_n x_n} , \\ &= e^{r_1 x_1+\ldots\ldots+ r_n x_n} . \end{aligned}$$

Conversely, assume that there exist $r_i \in \mathbb{C}$, $i = 1,\ldots,n$, such that $f(x) = e^{r_1 x_1+\ldots\ldots+ r_n x_n}$, for each $x = (x_1,\ldots\ldots, x_n) \in \mathbb{R}^n$.
Then we have

$$\begin{aligned} f(x+y) &= e^{r_1(x_1+y_1)+\ldots\ldots+r_n(x_n+ y_n)} , \\ &= e^{(r_1x_1+\ldots\ldots+r_nx_n)+(r_1y_1+\ldots\ldots+r_ny_n)} , \\ &= e^{r_1x_1+\ldots\ldots+r_nx_n} . e^{r_1y_1+\ldots\ldots+ r_ny_n} , \\ &= f(x)\, f(y). \end{aligned}$$

3.4 Existence of Discontinuous Solution of $f(x+y) = f(x)\, f(y)$.

The purpose of this section is to provide some examples of a discontinuous solution of $f(x+y) = f(x)f(y)$, where f is a function from $(\mathbb{R}^n,+)$ into $(\mathbb{C}^*,\cdot)$. For simplicity, we give examples of discontinuous solutions from $\mathbb{R}^3$ to $\mathbb{C}^*$.

Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. By using remark 3.2.7, any function $f : \mathbb{R}^3 \longrightarrow \mathbb{C}^*$ satisfying $f(x+y) = f(x)f(y)$ must be of the form

$$f\left(\sum_{i=1}^{m} a_{1i}V_{\alpha_i}, \sum_{i=1}^{m} a_{2i}V_{\alpha_i}, \sum_{i=1}^{m} a_{3i}V_{\alpha_i}\right) = \prod_{j=1}^{3}\prod_{i=1}^{m} c_j(V_{\alpha_i})^{a_{ji}} ,$$

where c_1, c_2, c_3 are functions on H into $\mathbb{C}^*$.

Let us denote such function f by f_{c_1,c_2,c_3}. Hence each triple $c = (c_1,c_2,c_3)$, where $c_j : H \longrightarrow \mathbb{C}^*$, $j = 1,2,3$, defines a function f_c satisfying $f_c(x+y) = f_c(x)f_c(y)$. Discontinuous function f_c satisfying this equation can be obtained by choosing suitable functions c_1, c_2 and c_3. We shall first constructed $c_j : H \longrightarrow \mathbb{C}^*$, $j = 1,2,3$, which will make f_c a discontinuous solution of $f(x+y) = f(x)f(y)$.

Choose three distinct elements $V_{\alpha_1}, V_{\alpha_2}, V_{\alpha_3}$ of H and three nonzero complex numbers z_1, z_2, z_3 such that not all z_i's are 1. Define $c_j : H \longrightarrow \mathbb{C}^*$, $j = 1,2,3$, by putting

$$c_1(V_{\alpha_1}) = z_1 , \quad c_1(V_\alpha) = 1 \quad \text{for all} \quad \alpha \neq \alpha_1 ,$$

$$c_2(V_{\alpha_2}) = z_2 , \quad c_2(V_\alpha) = 1 \quad \text{for all} \quad \alpha \neq \alpha_2 ,$$

and

$$c_3(V_{\alpha_3}) = z_3 , \quad c_3(V_\alpha) = 1 \quad \text{for all} \quad \alpha \neq \alpha_3 .$$

By Remark 3.2.7, f_c satisfies $f_c(x+y) = f_c(x)f_c(y)$. Next, we show that f_c is not continuous.

Suppose that f_c is continuous. By Theorem 3.3.5, f_c must be of the form $f_c(x_1, x_2, x_3) = e^{r_1x_1 + r_2x_2 + r_3x_3}$, where $r_i \in \mathbb{C}$, $i = 1,2,3$.

Observe that $f_c(V_{\alpha_1}, 0, 0) = c_1(V_{\alpha_1})^1 = z_1$,

and $f_c(V_{\alpha_1} + V_{\alpha_2}, 0, 0) = c_1(V_{\alpha_1})^1 \cdot c_1(V_{\alpha_2})^1 = z_1 \cdot 1 = z_1$.

Therefore $f_c(V_{\alpha_1},0,0) = f_c(V_{\alpha_1} + V_{\alpha_2},0,0)$.

Since $f(x_1, x_2, x_3) = e^{r_1x_1 + r_2x_2 + r_3x_3}$.

$$
\begin{aligned}
\text{Hence} \quad e^{r_1V_{\alpha_1}} &= f_c(V_{\alpha_1}, 0, 0), \\
&= f_c(V_{\alpha_1} + V_{\alpha_2},0,0), \\
&= e^{r_1(V_{\alpha_1} + V_{\alpha_2})}.
\end{aligned}
$$

Therefore, $e^{r_1V_{\alpha_2}} = 1.$

Thus $r_1V_{\alpha_2} = 0.$

Since $V_{\alpha_2} \in H$, we have $V_{\alpha_2} \neq 0$.

Therefore $r_1 = 0$.

Similarly, we have

$$e^{r_2V_{\alpha_2}} = f_c(0,V_{\alpha_2},0) = z_2 = f_c(0,V_{\alpha_1} + V_{\alpha_2}, 0) = e^{r_2(V_{\alpha_1} + V_{\alpha_2})},$$

and

$$e^{r_3V_{\alpha_3}} = f_c(0,0,V_{\alpha_3}) = z_3 = f_c(0,0,V_{\alpha_1} + V_{\alpha_3}) = e^{r_3(V_{\alpha_1} + V_{\alpha_3})}.$$

It follows that $r_2 = r_3 = 0$.

Hence we have $f_c(x) \equiv 1$ for all $x = (x_1, x_2, x_3)$.

Since not all z_i's are 1. We may assume that $z_1 \neq 1$.

Hence $f_c(V_{\alpha_1},0,0) = z_1 \neq 1$, which is a contradiction.

Therefore $f_c(x)$ cannot be of the form $e^{r_1x_1+ r_2x_2+ r_3x_3}$. i.e. f_c is not continuous. Hence there exists a discontinuous solution of $f(x+y) = f(x)f(y)$.

It can be seen that if we choose n distinct elements $V_{\alpha_1}, \ldots, V_{\alpha_n}$ in H and any n non-zero complex numbers $z_1, \ldots, z_n$ such that not all z_i's are 1 and define $c_j : H \longrightarrow \mathbb{C}^*$ by

$$c_j(V_{\alpha_i}) = \begin{cases} z_j & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases}$$

then $f_c : \mathbb{R}^n \longrightarrow \mathbb{C}^*$, defined by

$$f_c\left(\sum_i a_{1i}V_{\alpha_i}, \ldots\ldots, \sum_i a_{ni}V_{\alpha_i}\right) = \prod_j \prod_i c_j(V_{\alpha_i})^{a_{ji}},$$

is a discontinuous solution of $f(x+y) = f(x)f(y)$.