#### CHAPTER II

#### PRELIMINARIES

In this chapter we collect the relevant definitions and results from topology and group theory. The material is standard and can be found in [2]. We shall assume that the reader is familiar with common terms used in set theory.

# 2.1 Topological Concepts.

By a topological space we mean an ordered paired  $(X, \mathcal{I})$ , where X is a set and  $\mathcal{I}$  is a family of subsets of X satisfying the following conditions:

- (a) X and Ø are in J.
- (b) The intersection of any finite number of members of is in J.
  - (c) The arbitrary union of members of J is in J.

Any family I satisfying these three conditions will be called a topology for X. Each member of I will be called an open set, or more precisely a I - open set. Occasionally, we shall denote any topological space (X,I) simply by X. It can be shown that if Y is any subset of a topological space (X,I), then the family  $I_Y = I \cap Y : I \cap I$  forms a topology for Y. The topological space  $(Y,I_Y)$  obtained in this way will be called a subspace of (X,I) and the topology  $I_Y$  will be called a relative topology.

Let (X, ) be a topological space. A subfamily B of J

is said to be a base of T if for each T and T there exists T is a union of members of T. Or, equivalently, each T in T is a union of members of T. It can be shown that if a family T of subsets of T has the properties,

- (i) the union of sets in B is X,
- (ii) for each  $B_1$ ,  $B_2 \in \mathbb{R}$  ,  $B_1 \cap B_2$  is the union of members of  $\mathbb{R}$ ,

then  $\mathcal B$  is a base for some topology for X. This topology consists of all sets that can be written as unions of sets in  $\mathcal B$ . Observe that the family of all open intervals form a base for a topology on the set  $\mathcal R$  of real numbers. This topology is known as the usual topology of  $\mathcal R$  and will be denoted by  $\mathcal U$ .

Let  $(X_{\omega}, J_{\omega})$ ,  $\omega = 1, \ldots, n$ , be any n topological spaces. The family of all sets of the form  $T_1 \times T_2 \times \cdots \times T_n$ , where  $T_{\omega} \in J_{\omega}$ ,  $\omega = 1, \ldots, n$ , form a base of a topology  $J_{\omega}$  on the Cartesian product  $X = X_1 \times \cdots \times X_n$ . This topology  $J_{\omega}$  will be called the product topology. The set X endowed with the product topology is called a topological product space. If we let each  $(X_{\omega}, J_{\omega})$  be (R, U),  $\omega = 1, \ldots, n$ , then the product topology will be denoted by  $U^n$  and will be called the usual topology for  $R^n$ .

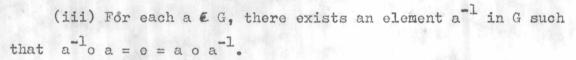
Let X and Y be two topological spaces. A mapping f of X into Y is said to be continuous if for each open set V in Y,  $f^{-1}[V] = \left\{x \in X : f(x) \in V\right\} \text{ is an open set of X.}$ 

### 2.2 Groups

A non-empty set G is said to form a group under a binary

operation o if the following conditions hold:

- (i) For all a, b, c ∈ G, ao (b o c) = (a o b) oc
- (ii) There exists an element e in G such that  $e \circ a = a = a \circ e$  for all  $a \notin G$ .



Let G be a group. If for all a, b & G we have a o b = b o a, then G is said to be a commutative group. Usually, we shall denote any group G under a binary operation o by (G, o) or simply by G.

If H is any non-empty subset of a group (G, o) such that H form a group under the restriction of o to H \* H, we say that (H, o) is a subgroup of (G, o). It can be shown that any non-empty set H forms a subgroup of (G, o) if and only if x o y -1 & H for any x,y in H.

A mapping h on a group (G, o) into a group (G, \*) is said to be a homomorphism if h satisfies the condition  $h(x \circ y) = h(x) * h(y) , \text{ for } x, y \text{ in } G.$ 

### 2.3 Topological Groups

If (G, o) is a group and I is a topology on G such that

- (i) the binary operation o is continuous,
- (ii) the mapping  $i: G \longrightarrow G$  defined by  $i(x) = x^{-1}$  is continuous, we say that  $(G, \mathcal{I}, o)$  is a topological group.

## 2.4 Examples of Topological Groups.

The followings are examples of topological groups :

- (a) The set R of real numbers with addition as the group operation and the usual topology form a topological group.
- (b) The set  $\mathbb{R}$  of nonzero real numbers with multiplication as the group operation and the relative topology of the usual topology of  $\mathbb{R}$  form a topological group.
- (c) The set  $\mathbb{R}^*$  of positive real numbers with multiplication as the group operation and the relative topology of the usual topology of  $\mathbb{R}$  form a topological group.
- (d) The set  $\mathbb{R}^n$  of all real n-tuples with an addition + defined by  $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ , as a group operation and the usual topology of  $\mathbb{R}^n$  form a topological group.
- (e) The set f of nonzero complex numbers with complex multiplication. defined by (x, y). (z, w) = (xz yw, yz + xw), where (x, y),  $(z, w) \in f$ , as a group operation, and the relative topology of the usual topology of  $R^2$  form a topological group.
- (f) The unit circle  $\Delta = \{z \in \mathbb{C} : |z| = 1\}$  with complex multiplication as the group operation and the relative topology of the usual topology of  $\mathbb{R}^2$  form a topological group.