CHAPTER II

PRELIMINARIES

In this chapter we collect the relevant definitions and results from topology and group theory. The material is standard and can be found in [2] . We shall assume that the reader is familiar with common terms used in set theory.

2.1 Topological Concepts.

By a topological space we mean an ordered paired $(X, \mathfrak{T})$, where X is a set and $\mathfrak{T}$ is a family of subsets of X satisfying the following conditions :

(a) X and $\emptyset$ are in $\mathfrak{T}$.

(b) The intersection of any finite number of members of $\mathfrak{T}$ is in $\mathfrak{T}$.

(c) The arbitrary union of members of $\mathfrak{T}$ is in $\mathfrak{T}$.

Any family $\mathfrak{T}$ satisfying these three conditions will be called a topology for X . Each member of $\mathfrak{T}$ will be called an open set, or more precisely a $\mathfrak{T}$ - open set. Occasionally, we shall denote any topological space $(X, \mathfrak{T})$ simply by X. It can be shown that if Y is any subset of a topological space $(X, \mathfrak{T})$, then the family $\mathfrak{T}_y = \{ T \cap Y : T \in \mathfrak{T} \}$ forms a topology for Y. The topological space $(Y, \mathfrak{T}_y)$ obtained in this way will be called a subspace of $(X, \mathfrak{T})$ and the topology $\mathfrak{T}_y$ will be called a relative topology.

Let $(X, \mathfrak{T})$ be a topological space. A subfamily $\mathcal{B}$ of $\mathfrak{T}$

is said to be a <u>base</u> of $\mathcal{T}$ if for each $T \in \mathcal{T}$ and $x \in T$ there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset T$. Or, equivalently, each T in $\mathcal{T}$ is a union of members of $\mathcal{B}$. It can be shown that if a family $\mathcal{B}$ of subsets of X has the properties,

(i) the union of sets in $\mathcal{B}$ is X,

(ii) for each $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is the union of members of $\mathcal{B}$,

then $\mathcal{B}$ is a base for some topology for X. This topology consists of all sets that can be written as unions of sets in $\mathcal{B}$. Observe that the family of all open intervals form a base for a topology on the set $\mathbb{R}$ of real numbers. This topology is known as the <u>usual topology of $\mathbb{R}$</u> and will be denoted by U.

Let $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha = 1,\ldots,n$, be any n topological spaces. The family of all sets of the form $T_1 \times T_2 \times \ldots \times T_n$, where $T_\alpha \in \mathcal{T}_\alpha$, $\alpha = 1,\ldots,n$, form a base of a topology $\mathcal{T}$ on the Cartesian product $X = X_1 \times \ldots \times X_n$. This topology $\mathcal{T}$ will be called the <u>product topology</u>. The set X endowed with the product topology is called a <u>topological product space</u>. If we let each $(X_\alpha, \mathcal{T}_\alpha)$ be $(\mathbb{R}, U)$, $\alpha = 1,\ldots,n$, then the product topology will be denoted by U^n and will be called the <u>usual topology for $\mathbb{R}^n$</u>.

Let X and Y be two topological spaces. A mapping f of X into Y is said to be <u>continuous</u> if for each open set V in Y, $f^{-1}[V] = \{x \in X : f(x) \in V\}$ is an open set of X.

2.2 <u>Groups</u>

A non-empty set G is said to form a <u>group</u> under a binary operation o if the following conditions hold :

(i) For all a, b, c $\in$ G, ao (b o c) = (a o b) oc .

(ii) There exists an element e in G such that

e o a = a = a o e for all a $\in$ G.

(iii) For each a $\in$ G, there exists an element a^{-1} in G such that a^{-1}o a = e = a o a^{-1}.

Let G be a group. If for all a, b $\in$ G we have a o b = b o a, then G is said to be a <u>commutative group</u>. Usually, we shall denote any group G under a binary operation o by (G, o) or simply by G. If H is any non-empty subset of a group (G, o) such that H form a group under the restriction of o to H $\times$ H, we say that (H, o)is a <u>subgroup</u> of (G, o). It can be shown that any non-empty set H forms a subgroup of (G, o) if and only if x o $y^{-1} \in$ H for any x,y in H.

A mapping h on a group (G, o) into a group (G', $*$) is said to be a <u>homomorphism</u> if h satisfies the condition

$$h(x \circ y) = h(x) * h(y) \quad , \text{ for } x, y \text{ in } G.$$

2.3 <u>Topological Groups</u>

If (G, o) is a group and $\mathcal{T}$ is a topology on G such that

(i) the binary operation o is continuous,

(ii) the mapping i : G $\longrightarrow$ G defined by i(x) = x^{-1} is continuous, we say that (G, $\mathcal{T}$, o) is a <u>topological group</u>.

2.4 <u>Examples of Topological Groups.</u>

The followings are examples of topological groups :

(a) The set $\mathbb{R}$ of real numbers with addition as the group operation and the usual topology form a topological group.

(b) The set $\mathbb{R}^*$ of nonzero real numbers with multiplication as the group operation and the relative topology of the usual topology of $\mathbb{R}$ form a topological group.

(c) The set $\mathbb{R}^+$ of positive real numbers with multiplication as the group operation and the relative topology of the usual topology of $\mathbb{R}$ form a topological group.

(d) The set $\mathbb{R}^n$ of all real n-tuples with an addition $+$ defined by $(x_1,\ldots,x_n) + (y_1,\ldots,y_n) = (x_1+ y_1,\ldots,x_n+ y_n)$, as a group operation and the usual topology of $\mathbb{R}^n$ form a topological group.

(e) The set $\mathbb{C}^*$ of nonzero complex numbers with complex multiplication. defined by $(x, y) . (z, w) = (xz - yw, yz + x w)$, where $(x , y), (z, w) \in \mathbb{C}^*$, as a group operation, and the relative topology of the usual topology of $\mathbb{R}^2$ form a topological group.

(f) The unit circle $\Delta = \{ z \in \mathbb{C} : |z| = 1\}$ with complex multiplication as the group operation and the relative topology of the usual topology of $\mathbb{R}^2$ form a topological group.