CHAPTER IV

GENERALIZATIONS OF NON-CREATIVITY

In this chapter, we want to define a generalized criterion of non-creativity and show that explicit definitions also satisfy this criterion.

4.1 Definition. Let L and L' be two first-order languages such that $L \subset L'$. A sentence ϕ in L' is non-creative with respect to L if and only if : for all sentences ψ in L; if $\phi \vdash \psi$ then $\vdash \psi$.

We want to show that non-creativity with respect to the first-order language of theory T (L(T)) is more general than non-creativity with respect to T.

4.2 Theorem. If a sentence ϕ in $L' \supset L(T)$ is non-creative with respect to L(T), then ϕ is non-creative with respect to T.

proof. Assume ϕ is non-creative with respect to L(T), i.e. for all sentences ψ in L(T); if $\phi \vdash \psi$ then $\vdash \psi$.

Want to show that ϕ is non-creative with respect to T, i.e. show that, for all sentences ψ in L(T) ; if $T \vdash \phi \rightarrow \psi$ then $T \vdash \psi$.

Let ψ be any sentence in L(T). Assume $T \vdash \phi \rightarrow \psi$. Then there exists a finite sequence of formulas $\theta_1, \ldots, \theta_n$ such that $\theta_n = \phi \longrightarrow \psi$ and for each i, θ_i is a logical axiom, or $\theta_i \varepsilon T$, or θ_i comes from θ_j,

θ_k $(j,k < i)$ by MP, or θ_i comes from θ_j $(j < i)$ by generalization. Let T_1 be a set of sentences used in this deduction, so T_1 is finite. Let $T_1 = \{\sigma_1,\ldots,\sigma_n\}$ where each σ_i, $1 \leq i \leq n$, is a sentence in T used in this deduction, hence we get $T_1 \vdash \phi \to \psi$, i.e. $\{\sigma_1,\ldots,\sigma_n\} \vdash \phi \to \psi$. By Deduction Theorem, we get $\{\phi\} \cup \{\sigma_1,\ldots,\sigma_n\} \vdash \psi$, i.e. $\phi \wedge \sigma_1 \wedge \ldots \wedge \sigma_n \vdash \psi$. By Deduction Theorem; $\phi \wedge \sigma_1 \wedge \ldots \wedge \sigma_{n-1} \vdash \sigma_n \to \psi$. Use Deduction Theorem again and again until we get $\phi \vdash \sigma_1 \to (\sigma_2 \to (\ldots \to (\sigma_n \to \psi))\ldots)$. Since $\sigma_1 \to (\sigma_2 \to (\ldots \to (\sigma_n \to \psi))\ldots)$ is sentence in L(T), by first assumption, we get $\sigma_1 \to (\sigma_2 \to (\ldots \to (\sigma_n \to \psi))\ldots)$. Use Deduction Theorem again and again, so we get $\{\sigma_1,\ldots,\sigma_n\} \vdash \psi$, i.e. $T_1 \vdash \psi$. Hence $T \vdash \psi$.

4.3 Theorem. (Interpolation Theorem). Let ϕ and ψ be sentences in first-order language without identity such that $\vdash \phi \longrightarrow \psi$. Then

(i) if ϕ and ψ contain common symbols, then there is a sentence θ such $\vdash \phi \to \theta$ and $\vdash \theta \to \psi$ and the symbols of θ are common to ϕ and ψ,

and (ii) if ϕ and ψ contain no common symbols, then either $\vdash \sim \phi$ or $\vdash \psi$.

proof. We can find this proof in [2].

4.4 Theorem. The converse of Theorem 4.2 is not necessarily true.

proof. To prove this theorem, we must show that there exists a sentence ϕ in $L' \supset L(T)$ which is non-creative with respect to theory T but not non-creative with respect to L(T). So we must find a sentence

ϕ in $L' \supset L(T)$ and a sentence ψ in $L(T)$ such that : for all formulas t in $L(T)$; if $T \vdash \phi \longrightarrow t$ then $T \vdash t$ and $\phi \vdash \psi$ but $\nvdash \psi$.

Let $T = \{\sigma\}$ where σ is a sentence in first-order language without identity such that $\nvdash \sigma$. Let $L' = L(T) \cup \{P\}$ where P is a new 1-placed relation symbol. Let ϕ be the sentence $(\sigma \wedge \exists v P v)$ in L'. Want to show that, for all formulas t in L(T); if $T \vdash \phi \rightarrow t$ then $T \vdash t$. To show this, let t be any formula in L(T). Assume $T \vdash \phi \rightarrow t$, i.e. $\{\sigma\} \vdash (\sigma \wedge \exists v P v) \rightarrow t$. By Deduction Theorem; we get $\sigma \wedge \exists v P v \vdash t$ and so $\exists v P v \vdash \sigma \rightarrow t$. Since $\exists v P v$ and $\sigma \rightarrow t$ are sentences contain no common symbols, by Interpolation Theorem, we get either $\vdash \sim (\exists v P v)$ or $\vdash \sigma \rightarrow t$. By Gödel's Completeness Theorem, we get $\models \sim (\exists v P v)$ which is impossible, hence $\nvdash \sim (\exists v P v)$. So $\vdash \sigma \rightarrow t$ and hence $\sigma \vdash t$, i.e. $\{\sigma\} \vdash t$. Then ϕ is non-creative with respect to T.

Let ψ be a sentence σ in T, we see that $(\sigma \wedge \exists v P v) \vdash \sigma$ but $\nvdash \sigma$. Thus ϕ is not non-creative with respect to L(T).

Hence, from Theorem 4.2 and 4.4, we see that non-creativity with respect L(T) is more general than non-creativity with respect to T.

4.5 Theorem. Let L and L' be two first-order languages such that $L \subset L'$, and ϕ be a sentence in L'. If for all models M of L, there exist a model M^* of L such that $M \equiv M^*$ and M^* can be expanded to a model M' of L' in which $M' \models \phi$, then ϕ is non-creative with respect to L.

proof. Assume for all models M of L, there exist a model M^* of

L such that $M \equiv M^*$ and M^* can be expanded to a model M' of L' in which $M' \models \phi$. Want to show that ϕ is non-creative with respect to L, i.e. show that for all sentences ψ in L; if $\phi \vdash \psi$ then $\vdash \psi$. To show this, let ψ be any sentence in L. Assume $\phi \vdash \psi$. Suppose $\nvdash \psi$. By Gödel's Completeness Theorem, we can suppose $\not\models \psi$, therefore there exists a model M of L such that $M \not\models \psi$. From first assumption, there exists model $M^* \equiv M$ and a model expansion of M^*, say M', such that $M' \models \phi$. Since $M \not\models \psi$, we have $M^* \not\models \psi$ and $M' \not\models \psi$, and from $\phi \models \psi$ (i.e. $M' \models \phi \Rightarrow M' \models \psi$), we get $M' \not\models \phi$ which is a contradiction. Thus $\vdash \psi$.

4.6 <u>Definition</u>. Let L and L' be two first-order languages such that $L \subset L'$. A sentence ϕ in L' is said to be semantically non-creative with respect to L if and only if : for all models M of L, there exist a model M' of L' such that M' is an expansion of M and $M' \models \phi$.

4.7 <u>Theorem</u>. Let L and L' be two first-order languages such that $L \subset L'$ and ϕ be a sentence in L'. If ϕ is semantically non-creative with respect to L, then for all models M of L there exist a model M^* of L such that $M \equiv M^*$ and M^* can be expanded to a model M' of L' in which $M' \models \phi$.

<u>proof</u>. Assume ϕ is semantically non-creative with respect to L, i.e. for all models M of L, there exist a model M' of L' such that M' is an expansion of M and $M' \models \phi$. Let M^* be M, so we get for all models M of L, there exist a model M^* of L such that $M \equiv M^*$ and M^* can be expanded to a model M' of L' in which $M' \models \phi$.

4.8 <u>Theorem</u>. Let L and L' be two first-order languages such that $L \subset L'$ and ϕ be a sentence in L'. If ϕ is semantically non-creative with respect to L, then ϕ is non-creative with respect to L.

proof. From Theorem 4.5 and 4.7.

Next, we describe two first-order languages L and L' such that $L \subset L'$, and a sentence ϕ in L' such that for all models M of L, there exist a model M^* of L such that $M \equiv M^*$ and M^* can be expanded to a model M' of L' in which $M' \models \phi$.

Let $L = \{ P, R \}$ and $L' = \{P, R, F \}$; where P is a 1-placed relation symbol, R is a 2-placed relation symbol and F is a 1-placed function symbol. Thus $L \subset L'$.

Let ψ in L be the sentence : $\exists x\, P x \wedge \forall x\, (\sim R(x, x)) \wedge \forall x\, \exists y\, (P y \wedge R(x, y)) \wedge \forall x\, \forall y\, \forall z\, (R(x, y) \wedge R(y, z) \longrightarrow R(x,z))$.

Let θ in L' be the sentence : $\forall x\, (\sim Px \longrightarrow \exists y\, (P y \wedge F(y) = x))$.

Let ϕ in L' be the sentence : $\psi \longrightarrow \theta$. (Intuitively, ϕ says that "If { x/x is P } is infinite then there is a function F that maps {x/x is P } onto {x/x is not P} ".).

Before we show the above, we need some lemmas.

4.9 Lemma. If $M \models \psi$, then I_p is infinite, where I_p is the interpretation of P in M.

proof. Assume $M \models \psi$ and let $M = < A, I_p, I_R >$ where A is the universe and I_p, I_R are interpretations of P and R, respectively, in M.

Suppose I_p is finite, let $I_p = \{ a_1, \ldots, a_n / a_i \in A, 1 \leq i \leq n \}$. Since $M \models \psi$, we can define $B = \{ b_1, \ldots, b_n / b_1 = a_1$ and $b_{i+1} = a_j$ where $(b_i, a_j) \in I_R \}$. From this set, we see that $(b_i, b_{i+1}) \in I_R$ and if $(b_i, b_{i+1}) \in I_R$ and $(b_{i+1}, b_{i+2}) \in I_R$, then $(b_i, b_{i+2}) \in I_R$; $1 \leq i \leq n$. At last, we get $(b_i, b_n) \in I_R$ $\forall i$, $1 \leq i \leq n$, but $(b_n, b_k) \in I_R$ for some k, $1 \leq k \leq n$, therefore $(b_n, b_n) \in I_R$ contradiction. Hence I_p is infinite.

4.10 Lemma. For all infinite models M of L, there exist a countable (infinite) model M^* of L such that $M \equiv M^*$.

proof. Let M be any infinite model of L. Let $T = \{\phi / \phi$ is a sentence in which $M \models \phi\}$, then T is consistent. By Theorem 2.57, T has a countable model, say M^*. Now we want to show that $M \equiv M^*$. Let ψ be any sentence in L. Suppose $M \models \psi$, then $\psi \in T$, and so $M^* \models \psi$. Suppose $M^* \models \psi$. If $\psi \in T$, then we get $M \models \psi$. If $\psi \notin T$, and suppose that $M \not\models \psi$ then $M \models \sim \psi$, therefore $\sim \psi \in T$, so $M^* \models \sim \psi$, i.e. $M^* \not\models \psi$ which is a contradiction. Thus $M \models \psi$.

4.11 Theorem. The converse of Theorem 4.7 is not necessarily true.

proof. To prove this theorem, we must show that there exist two first-order languages L and L' such that $L \subset L'$ and a sentence ϕ in L' such that for all models M of L, there exist a model M^* of L such that $M \equiv M^*$ and M^* can be expanded to a model M' of L' in which $M' \models \phi$, but ϕ is not semantically non-creative with respect to L.

Let $L = \{P, R\}$, $L' = \{P, R, F\}$ where P is an 1-placed relation symbol, R is a 2-placed relation symbol and F is an 1-placed function symbol. Thus $L \subset L'$.

Let ψ in L be the sentence : $\exists x\, Px \wedge \forall x\, (\sim R(x, x)) \wedge \forall x\, \exists y\, (Py \wedge R(x, y)) \wedge \forall x\, \forall y\, \forall z\, (R(x, y) \wedge R(y, z) \longrightarrow R(x, z))$,

θ in L' be the sentence : $\forall x\, (\sim Px \longrightarrow \exists y\, (Py \wedge F(y) = x))$.

and ϕ in L' be the sentence : $\psi \longrightarrow \theta$.

Let $M = < A, I_p, I_R >$ where A is the universe, I_p and I_R are interpretations of P and R, respectively, in M; be any model of L.

<u>case 1</u> : M is finite. Let $M^* = M = <A, I_p, I_R>$ where A is finite and $I_p \subseteq A$ is also finite. Let $M' = < M^*, I_F >$ where I_F is any interpretation of F. By Lemma 4.8, we get $M' \not\models \psi$. Hence M' is an expansion of M^* such that $M' \models \psi \rightarrow \theta$.

<u>case 2</u> : M is countably infinite. Suppose $M = <A, I_p, I_R >$ where A is countably infinite and I_p is finite. Let $M^* = M$ and $M' = < M^*, I_F >$ where I_F is any interpretation of F, M' is an expansion of M^*. By Lemma 4.8, $M' \not\models \psi$, so we get $M' \models \psi \rightarrow \theta$.

Suppose $M = < A, I_p, I_R >$ where A is countably infinite and I_p is also countably infinite. Let $M^* = M$. Since A and I_p are also countably infinite, there exists a function I_F maps from I_p onto $A-I_p$. So let $M' = < M^*, I_F >$, thus $M' \models \theta$. Hence M' is an expansion of M^* such that $M' \models \psi \rightarrow \theta$.

<u>case 3</u> : M is uncountable. By Lemma 4.9, there exists a model M^* such that M^* is countable and $M \equiv M^*$. As in case 2, there exists an

expansion $M' = \langle M^*, I_F \rangle$ of M^* such that $M' \models \psi \longrightarrow \theta$.

Finally, we must show that ϕ is not semantically non-creative with respect to L, i.e. there exists a model M of L such that for all model M' of L' which M' are expansions of M, $M' \not\models \phi$ (i.e. $M' \models \psi$ and $M' \not\models \theta$).

Let $M = \langle \mathbb{R}, \mathbb{Q}, < \rangle$, we see that $M \models \psi$. Let $M' = \langle \mathbb{R}, \mathbb{Q}, <, I_F \rangle$ is any model of L', we get M' is an expansion of M, therefore $M' \models \psi$. If I_F is any function maps from $\mathbb{Q}$ to $\mathbb{R} - \mathbb{Q}$, then I_F is not onto, so $M' \not\models \theta$.

4.12 Theorem. The converse of Theorem 4.8 is not necessarily true.

proof. To proof this theorem, we must show that there exist two first-order languages L and L' such that $L \subset L'$ and a sentence ϕ in L' such that ϕ is non-creative with respect to L but is not semantically non-creative with respect to L.

Let L, L' and ϕ as in Theorem 4.10. By Theorem 4.4, ϕ is non-creative with respect to L.

We see that semantical non-creativity with respect to L $\Longrightarrow$ non-creativity with respect to L(T) $\Longrightarrow$ non-creativity with respect to T, but the converses are not true. Hence semantical non-creativity with respect to L is the most general criterion of non-creativity among these three.

4.13 Theorem. Explicit definitions are semantically non-creative.

<u>proof</u>. Let L and L' be two first-order languages such that $L \subset L' = L \cup \{P\}$ where P is a new n-placed relation symbol. Let ϕ be an explicit definition, therefore ϕ is of the form $(\forall v_1) \ldots (\forall v_n) (P(v_1 \ldots v_n) \longleftrightarrow S)$, where S is a formula in L. We must show that ϕ is semantically non-creative with respect to L.

Let $M = < A, \mathcal{I} >$ be any model of L. Want to show that there exists a model M' of L' which is an expansion of M and such that $M' \models (\forall v_1) \ldots (\forall v_n) (P(v_1 \ldots v_n) \longleftrightarrow S)$.

Let $B = \{ (a_1, \ldots, a_n)/a_i \varepsilon A$ and there exists $(b_1, \ldots, b_n, \ldots)$ satisfies S in M such that $b_1 = a_1, \ldots, b_n = a_n \}$.

Let interpretation of $P = I_p = B$. Let $M' = < M, I_p >$, hence M' is an expansion of M.

Next, we want to show that $M' \models P(v_1 \ldots v_n) \longleftrightarrow S$, i.e. to show all sequence of elements of A satisfy $P(v_1 \ldots v_n) \longleftrightarrow S$ in M'.

Let $s = (c_1, \ldots, c_n, \ldots)$ be any sequence of elements of A which satisfies $P(v_1 \ldots v_n)$ in M', therefore $(c_1, \ldots, c_n) \varepsilon I_p$. Then there exists sequence of elements of A : $s' = (b_1, \ldots b_n, \ldots)$ such that $b_1 = c_1, \ldots, b_n = c_n$ satisfies S in M'. By Lemma 2.38, s satisfies S in M'.

Let $s = (d_1, \ldots, d_n, \ldots)$ be any sequence of elements of A does not satisfy $P(v_1 \ldots v_n)$ in M', therefore $(d_1, \ldots, d_n) \notin I_p$. Then for all sequence of elements of A : $(b_1, \ldots, b_n, \ldots)$ such that $b_1 = d_1, \ldots, b_n = d_n$ does not satisfy S in M'. So s does not satisfy S in M'.

Thus, all sequences of elements of A satisfy $P(v_1 \ldots v_n) \longleftrightarrow S$ in M' and so $M' \models P(v_1 \ldots v_n) \longleftrightarrow S$. Hence $M' \models (\forall v_1) \ldots (\forall v_n)$ (P $(v_1 \ldots v_n) \longleftrightarrow S)$.