CHAPTER IV

FOURIER TRANSFORMS

The materials of this chapter are drawn from references [5], [8],[9], [13].

4.1 L^1 Theory

4.1.1 Definition: Let $f \in L(R^n)$. The function $\hat{f}$ given by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(y) e^{i(x.y)} dy \tag{1}$$

is called the Fourier transform of f.

We note that $\hat{f}$ is everywhere defined since the integral (1) is absolutely convergent for every $x \in R^n$. Indeed,

$$|\hat{f}(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{R^n} |f(y)| dy = (2\pi)^{-n/2} \|f\|_1,$$

so $\hat{f}$ is bounded.

4.1.2 Lemma. If $f(x) = f_1(x_1) \ldots f_n(x_n)$ where each $f_i(x_i) \in L(R)$, then $\hat{f}(x) = \hat{f}_1(x_1) \ldots \hat{f}_n(x_n)$.

Proof:
$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(y) e^{i(x.y)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} f_1(y_1) \ldots f_n(y_n) e^{i(x_1 y_1 + \ldots + x_n y_n)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(y_1) e^{ix_1 y_1} \cdots f_n(y_n) e^{ix_n y_n} dy_1 \cdots dy_n$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f_1(y_1) e^{ix_1 y_1} dy_1 \cdots \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f_n(y_n) e^{ix_n y_n} dy_n$$

$$= \hat{f}_1(x_1) \cdots \hat{f}_n(x_n).$$

The Fourier transform is a linear transformation, since for any scalar a and b

$$\widehat{(af + bg)}(x) = \frac{1}{(2\pi)^{n/2}} \int (af + bg)(y) e^{i(x \cdot y)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int (af(y) e^{i(x \cdot y)} + bg(y) e^{i(x \cdot y)}) dy$$

$$= \frac{a}{(2\pi)^{n/2}} \int f(y) e^{i(x \cdot y)} dy + \frac{b}{(2\pi)^{n/2}} \int g(y) e^{i(x \cdot y)} dy$$

$$= a\hat{f}(x) + b\hat{g}(x) \qquad (x \in R^n).$$

Thus $\widehat{(af + bg)} = a\hat{f} + b\hat{g}$.

4.1.3 <u>Lemma</u>. Let $a \in R^n$ and let $f_a(x) = f(x + a)$. Then,

$$\hat{f}_a(x) = e^{-i(x \cdot a)} \hat{f}(x) .$$

<u>Proof</u>: $$\hat{f}_a(x) = \frac{1}{(2\pi)^{n/2}} \int f_a(y) e^{i(x \cdot y)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int f(y + a) e^{i(x \cdot y)} dy .$$

Letting $y + a = z$, so $y = z - a$ and $dy = dz$, we obtain

$$= \frac{1}{(2\pi)^{n/2}} \int f(z) e^{i[x.(z - a)]} dz$$

$$= \frac{1}{(2\pi)^{n/2}} \int f(z) e^{-i(x.a)} e^{i(x.z)} dz$$

$$= e^{-i(x.a)} \hat{f}(x) .$$

4.1.4 Lemma. Let $a \in R^n$ and let $g(x) = e^{i(x.a)} f(x)$. Then,

$$\hat{g}(x) = \hat{f}(x + a).$$

Proof: $$\hat{g}(x) = \frac{1}{(2\pi)^{n/2}} \int g(y) e^{i(x.y)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int e^{i(y.a)} f(y) e^{i(x.y)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int f(y) e^{i[(x + a).y]} dy$$

$$= \hat{f}(x + a).$$

4.1.5 Lemma. Let $c \neq 0$ be a real scalar and let $f_c(x) = f(cx)$.

Then, $$\hat{f}_c(x) = \frac{1}{|c|^n} f\left(\frac{x}{c}\right).$$

Proof: $$\hat{f}_c(x) = \frac{1}{(2\pi)^{n/2}} \int f_c(y) e^{i(x.y)} dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int f(cy) e^{i(x.y)} dy.$$

Using the change of variable $z = cy$, so $dz = |c|^n dy$ we have

$$\hat{f}_c(x) = \frac{1}{|c|^n (2\pi)^{n/2}} \int f(z) e^{i(\frac{x}{c}.z)} dz$$

$$= \frac{1}{|c|^n}\hat{f}(\frac{x}{c}).$$

We recall that a function f is said to be <u>even</u> if $f(-x) = f(x)$, and is said to be <u>odd</u> if $f(-x) = -f(x)$. Moreover, any f can be decomposed into its even and odd parts; namely

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

4.1.6 <u>Lemma.</u> The Fourier transform of an even function is even. The Fourier transform of an odd funcfion is odd.

<u>Proof</u>: $\hat{f}(-x) = \frac{1}{(2\pi)^{n/2}}\int f(y)e^{i(-x.y)}dy.$

Let $y = -z$

$$\hat{f}(-x) = \frac{1}{(2\pi)^{n/2}}\int f(-z)e^{i(x.z)}dz.$$

If f is even, then

$$\hat{f}(-x) = \frac{1}{(2\pi)^{n/2}}\int f(z)e^{i(x.z)}dz = \hat{f}(x).$$

If f is odd, then

$$\hat{f}(-x) = \frac{1}{(2\pi)^{n/2}}\int [-f(z)]e^{i(x.z)}dz = -\hat{f}(x).$$

4.1.7 <u>Example.</u> Let χ_k be the characteristic function of the interval $(-k,k)$. Then

$$\hat{\chi}_k(x) = \frac{1}{(2\pi)^{1/2}}\int_{-k}^{+k} e^{ixy}dy = \frac{1}{(2\pi)^{1/2}}\frac{e^{ixy}}{ix}\Bigg|_{-k}^{+k}$$

$$= \frac{1}{(2\pi)^{1/2}}\frac{e^{ixk}-e^{-ixk}}{ix} = \sqrt{\frac{2}{\pi}}\frac{\sin kx}{x}.$$

Thus $\hat{\chi}_k$ is a bounded continuous function which is not integrable on R, though it belongs to $L^p(R)$ for all $p > 1$. Moreover $\hat{\chi}_k(x) \longrightarrow 0$ as $|x| \to \infty$.

4.1.8 Example. Let $\chi_{a,b}$ be the characteristic function of the interval (a,b). Let $c = \frac{a+b}{2}$ and $k = \frac{b-a}{2}$, we have that

$$\chi_k(x-c) = \chi_k\left(x - \frac{a+b}{2}\right).$$

Consider $-k \leqslant y \leqslant k$, where $y = x - \frac{a+b}{2}$, i.e.

$$-\frac{b-a}{2} \leqslant x - \frac{a+b}{2} \leqslant \frac{b-a}{2}$$

$$\frac{a+b}{2} - \frac{b-a}{2} \leqslant x \leqslant \frac{a+b}{2} + \frac{b-a}{2}$$

$$a \leqslant x \leqslant b$$

therefore $\chi_{a,b}(x) = \chi_k(x-c)$.

We obtain by Example (4.1.7) and Lemma (4.1.3), that

$$\hat{\chi}_{a,b}(x) = \hat{\chi}_k(x-c) = e^{icx}\hat{\chi}_k(x)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin kx}{x}\,e^{icx}.$$

We note again that $\hat{\chi}_{a,b}(x) \longrightarrow 0$ as $|x| \to \infty$. The next Theorem shows that this is a general property of Fourier transforms of integrable functions.

4.1.9 Theorem. (Riemann - Lebesgue) If $f \in L(R^n)$ then $\hat{f}$ is bounded and uniformly continuous : moreover,

(a) $\hat{f}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Proof: As we saw earlier, $|\hat{f}(x)| \leqslant (2\pi)^{-n}\|f\|_1$, so $\hat{f}$ is a bounded function.

Let $\varepsilon > 0$ and $h = (h_1, \dots, h_n) \in R^n$; then, for any $x \in R^n$,

$$\begin{aligned} |\hat{f}(x+h) - \hat{f}(x)| &= \left|(2\pi)^{-n/2}\int f(y)[e^{i[(x+h).y]} - e^{i(x.y)}]dy\right| \\ &= \left|(2\pi)^{-n/2}\int f(y)e^{i(x.y)}[e^{i(h.y)} - 1]dy\right| \\ &\leqslant \int |f(y)||e^{i(h.y)} - 1|\,dy \\ &= \int_{|y|\leqslant M} + \int_{|y|>M} = I_1 + I_2. \end{aligned}$$

Now, $I_2 \leqslant 2\int_{|y|>M} |f(y)|\,dy < \varepsilon$ if M is large enough (but fixed), since f is integrable.

On the other hand, if $|y| \leqslant M$ then $|e^{i(h.y)} - 1| \to 0$ as $|h| \to 0$, so by Lebesgue Dominated Convergence Theorem (2.35)

$$I_1 < \varepsilon \text{ if } |h| \text{ is small enough.}$$

Thus $\hat{f}$ is uniformly continuous.

Finally we prove (a) in a series of steps.

(1) If $f = g + h$, where g satisfies (a) and $\|h\|_1$ is arbitrarily small, then f satisfies (a), since $\hat{f} = \hat{g} + \hat{h}$, $\hat{g}(x) \to 0$ as $|x| \to \infty$, and $|\hat{h}(x)| \leqslant (2\pi)^{-n}\|h\|_1$ is small.

(2) Characteristic functions of 1-cells (intervals) satisfy (a), as we noted in Example (4.1.8), above.

(3) Characteristic functions of n-cells $a_i \leqslant x_i \leqslant b_i$, $i = 1, \dots, n$, satisfy (a). This follows directly from step (2) and Lemma(4.1.2).

(4) Simple functions, i.e. finite linear combinations of functions in step (3) satisfy (a).

(5) Simple functions are dense in $L(R^n)$ [13 :P.67], so by step(1), we are done.

In view of the presence of the factor $(2\pi)^{-n/2}$ in the definition of Fourier transform, it is convenient to normalize accordingly to the definition of convolution. So if $f,g \in L(R^n)$ we set

$$h(x) = (f * g)(x) \doteq (2\pi)^{-n/2}\int f(y)g(x-y)dy.$$

We know that $h \in L$ also, hence $\hat{h}$ exists.

The next theorem illustrates further the relationship between convolution and multiplication of functions.

4.1.10 Theorem. Let $f,g \in L(R^n)$. If $h = f*g$ then $\hat{h} = \hat{f}\hat{g}$.

Proof:
$$\hat{h}(x) = (2\pi)^{-n/2}\int h(y)e^{i(x.y)}dy$$
$$= (2\pi)^{-n/2}\,[(2\pi)^{-n/2}\int f(z)g(y-z)dz]e^{i(x.y)}dy$$

exists for every x. Since

$$\int[\int|f(z)g(y-z)e^{i(x.y)}|dy]dz = \iint|f(z)||g(y-z)|dydz$$
$$= \int|f(z)|dz\int|g(y-z)|dy = \|f\|_1\|g\|_1 < +\infty,$$

and $e^{i(x.y)} = e^{i[x.(y-z)]}e^{i(x.z)}$. We obtain by the Fubini's theorem that

$$\hat{h}(x) = (2\pi)^{-n/2}\int[(2\pi)^{-n/2}\int f(z)g(y-z)e^{i(x.y)}dz]dy$$
$$= (2\pi)^{-n/2}\int[(2\pi)^{-n/2}\int f(z)g(y-z)e^{i(x.y)}dy]dz$$
$$= (2\pi)^{-n/2}\int[(2\pi)^{-n/2}\int g(y-z)e^{i[x.(y-z)]}dy]f(z)e^{i(x.z)}dz$$
$$= (2\pi)^{-n/2}\int f(z)e^{i(x.z)}dz(2\pi)^{-n/2}\int g(y-z)e^{i[x.(y-z)]}dy$$
$$= \hat{f}(x)\hat{g}(x) \qquad (x \in R^n).$$

4.1.11 Example. Let $x \in \mathbb{R}$ and $0 < k \leqslant h$. Consider the "trapezoidal function" $\chi_{h,k}$ defined as follows: $\chi_{h,k}$ is even, and

$$\chi_{h,k}(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant h-k \\ 0 & \text{if } x > h+k \\ \dfrac{h+k-x}{2k} & \text{if } x \in (h-k\ ,\ h+k) \end{cases}.$$

We shall shows that this function is the convolution of the characteristic functions χ_h and χ_k.

$$(\chi_h * \chi_k)(x) = (2\pi)^{-1/2} \int \chi_h(t)\chi_k(x-t)\,dt$$
$$= (2\pi)^{-1/2} \int_{-h}^{h} \chi_k(x-t)\,dt.$$

Since $-k \leqslant x-t \leqslant k$, $x-k \leqslant t \leqslant x+k$. Therefore

$$(2\pi)^{-1/2} \int_{-h}^{h} \chi_k(x-t)\,dt = (2\pi)^{-1/2} \int_{-h}^{h} \chi_{x-k,x+k}(t)\,dt.$$

If $0 \leqslant x \leqslant h-k$ then $x+k \leqslant h$, $-k \leqslant x-k$. Therefore

$$(2\pi)^{-1/2} \int_{-h}^{h} \chi_{x-k,x+k}(t)\,dt = (2\pi)^{-1/2} \left[\int_{-h}^{x-k} + \int_{x-k}^{x+k} + \int_{x+h}^{h} \right]$$

$$= (2\pi)^{-1/2}(x+k-x+k) = \sqrt{\frac{2}{\pi}}\, k = \sqrt{\frac{2}{\pi}}\, k\, \chi_{h,k}(x),$$

since $\chi_{h,k}(x) = 1$ whenever $0 \leqslant x \leqslant h-k$.

If $h+k \leqslant x$ then $h \leqslant x-k$. Therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-h}^{h} \chi_{x-k,x+k}(t)\,dt = 0 = \sqrt{\frac{2}{\pi}}\, k\chi_{h,k}(x),$$

since $\chi_{h,k}(x) = 0$ whenever $h+k \leqslant x$.

If $h-k < x < h+k$ then $h-2k < x-k < h$, $h < x+k < h+2k$.

Since $k \leqslant h$, therefore $-h \leqslant h-2k$.

$$\frac{1}{\sqrt{2\pi}}\int_{-h}^{h}\chi_{x-k,x+k}(t)\,dt = \frac{1}{\sqrt{2\pi}}\left[\int_{-h}^{x-k} + \int_{x-k}^{h} + \int_{h}^{x+k}\right] = \frac{1}{\sqrt{2\pi}}(h-x+k)$$

$$= \sqrt{\frac{2}{\pi}}\, k\, \frac{h+k-x}{2k} = \sqrt{\frac{2}{\pi}}\, k\, \chi_{h,k}(x).$$

Hence $\displaystyle \frac{1}{\sqrt{2\pi}}\int_{-h}^{h}\chi_k(x-t)\,dt = \sqrt{\frac{2}{\pi}}\, k\, \chi_{h,k}(x).$

So, $\displaystyle \chi_{h,k} = \frac{1}{k}\sqrt{\frac{\pi}{2}}\, \chi_h * \chi_k$ and $\displaystyle \hat{\chi}_{h,k} = \frac{1}{k}\sqrt{\frac{\pi}{2}}\, \hat{\chi}_h\, \hat{\chi}_k.$

From Example (4.1.7), it follows that

(b) $\displaystyle \hat{\chi}_{h,k}(x) = \frac{1}{k}\sqrt{\frac{2}{\pi}}\, \frac{\sin hx}{x}\, \frac{\sin kx}{x}, \quad 0 < k \leqslant h.$

In particular, if $k = h$, we have that "<u>triangular function</u>"

$$\chi_{h,h}(x) = \left(1-\frac{|x|}{2h}\right)^+$$

where $f^+(x) = \max\,[f(x), 0]$ is the positive part of $f(x)$, so, from formula (b), we have

$$\hat{\chi}_{h,h}(x) = \frac{1}{h}\sqrt{\frac{2}{\pi}}\, \frac{\sin^2 hx}{x^2}, \quad h > 0.$$

We note that $\hat{\chi}_{h,h}(x)$ is non-negative and integrable on R.

4.1.12 <u>Example</u>. Let $H(x) = e^{-|x|}$, $-\infty < x < \infty$. Then

$$\hat{H}(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-y}e^{ixy}\,dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{y}e^{ixy}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-y(1-ix)}\,dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{y(1+ix)}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-ix} + \frac{1}{1+ix}\right) = \sqrt{\frac{2}{\pi}}\, \frac{1}{1+x^2}.$$

Moreover, if $\varepsilon > 0$ and $H_\varepsilon(x) = H(\varepsilon x) = e^{-\varepsilon|x|}$, then by Lemma(4.1.5),

$$\hat{H}_\varepsilon(x) = \sqrt{\frac{2}{\pi}}\frac{\varepsilon}{\varepsilon^2+x^2}$$ which is the Poisson Kernel in R.

4.1.13 Theorem. Let $f,g \in L(R^n)$ then

$$\int \hat{f}(x)g(x)dx = \int f(x)\hat{g}(x)dx.$$

Proof: Since
$$\iint |f(y)g(x)e^{i(x\cdot y)}|dydx \leq \iint |f(y)||g(x)|dydx \leq \int|g(x)|dx\int|f(y)|dy = \|g\|_1\|f\|_1 < +\infty.$$

We have by the Fubini's theorem that

$$\begin{aligned}\int \hat{f}(x)g(x)dx &= (2\pi)^{-n/2}\iint f(y)g(x)e^{i(x\cdot y)}dydx\\ &= (2\pi)^{-n/2}\iint f(y)g(x)e^{i(x\cdot y)}dxdy\\ &= \int f(y)\left[(2\pi)^{-n/2}\int g(x)e^{i(x\cdot y)}dx\right]dy\\ &= \int f(y)\hat{g}(y)dy.\end{aligned}$$

4.1.14 Definition: f is a radial function if $f(x) = f(|x|), (x \in R^n)$.

Equivalently, a radial function is invariant under all rotations about the origin. For example $x \longmapsto |x|^a$, $x \longmapsto e^{|x|^a}$ are all radial functions of R^n into R.

4.1.15 Definition: A linear transformation of vector space V over the real numbers into itself is called an orthogonal transformation if and only if $|Tx| = |x|$ for all $x \in V$.

4.1.16 Lemma. A function f is radial if and only if for any orthogonal transformations T of R^n into itself, we have $f(Tx) = f(x) \quad (x \in R^n)$.

Proof: Let T be any orthogonal transformation. Then

$$|Tx| = |x| \qquad \text{for all } x.$$

Assume that f is radial; i.e. $f(x) = f(|x|)$, and $f(Tx) = f(|Tx|)$. Then $f(Tx) = f(|Tx|) = f(|x|) = f(x)$. Conversely, assume that $f(Tx) = f(x)$ for all orthogonal transformation T. Since any rotation R about the origin is a linear transformation such that $|Rx| = |x|$. Then R is an orthogonal transformation, and hence f is radial.

4.1.17 Theorem. The Fourier transform of a radial function is radial.

Proof: For any orthogonal transformation T, we have by Lemma(4.1.16) that

$$\begin{aligned} \hat{f}(Tx) &= (2\pi)^{n/2} \int f(Ty)\, e^{i(x\cdot y)} dy \\ &= (2\pi)^{n/2} \int f(y)\, e^{i(x\cdot y)} dy \\ &= \hat{f}(x)\ , \end{aligned}$$

so $\hat{f}$ is also radial.

4.2 The Fourier Inversion Formula

If $f \in L(R^n)$ and $\hat{f}(y) = (2\pi)^{-n/2} \int f(x) e^{i(x.y)} dx$ is it Fourier transform, we seek a representation of f by means of its Fourier transform

(a) $(2\pi)^{-n/2}\int \hat{f}(y)e^{-i(x.y)}dy$ (Inversion Formula).

However $\hat{f}$ need not be integrable (Example 4.1.7) so the integral (a) is not convergent in general. For this reason we introduc now the concept of summability of integrals.

Let $a(u)$, $0 \leqslant u < \infty$, be a locally integrable function so that the "partial integrals"

$$A(w) = (2\pi)^{-n/2}\int_0^w a(u)du < +\infty .$$

But $$(2\pi)^{-n/2}\int_0^\infty a(u)du = \lim_{w\to\infty} A(w) \text{ need not be finite;}$$

in other words, the integral need not converge.

Consider a function $Z(u)$, $0 \leqslant u < \infty$, satisfying the following conditions:

1) Z is of bounded variation on $[0,\infty)$,

2) $Z(u) \longrightarrow 0$ as $u \to \infty$,

3) Z is continuous at $u = 0$ and $Z(0) = 1$.

4.2.1 Definition. The integral $\frac{1}{\sqrt{2\pi}}\int_0^\infty a(u)du$ is said to be summable-Z to a value I if the integrals

$$I_\varepsilon = \frac{1}{\sqrt{2\pi}}\int_0^\infty a(u)Z(\varepsilon u)du \text{ converge for all } \varepsilon > 0$$

and $\lim_{\varepsilon\to 0} I_\varepsilon = I$.

4.2.2 Example. Let $Z(u) = \begin{cases} 1 & \text{if } 0 \leqslant u \leqslant 1 \\ 0 & \text{if } u > 1 \end{cases}$.

Clearly Z satisfies the previous three conditions: moreover

$$I_\varepsilon = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} a(u) Z(\varepsilon u)\, du = \frac{1}{\sqrt{2\pi}} \int_0^{1/\varepsilon} a(u)\, du.$$

Thus, in this case, summability corresponds to ordinary convergence.

4.2.3 Theorem. Let a be locally integrable function on $[0,\infty)$. If the integral $(2\pi)^{-1/2} \int_0^{\infty} a(u)\, du$ converges to a value I, then it is summable-Z to the value I.

Proof: We must show that $I_\varepsilon = (2\pi)^{-1/2} \int_0^{\infty} a(u) Z(\varepsilon u)\, du$ exists,

and that $I_\varepsilon \longrightarrow I$, as $\varepsilon \to 0$.

By our assumption, the function $A(u) = \frac{1}{\sqrt{2\pi}} \int_0^{u} a(v)\, dv < +\infty$

and $A(u) \longrightarrow I$ as $u \to \infty$. Moreover,

$$A'(u) = (2\pi)^{-1/2} a(u) \quad \text{almost everywhere.}$$

Then $$\frac{1}{\sqrt{2\pi}} \int_0^{w} a(u) Z(\varepsilon u)\, du = \int_0^{w} Z(\varepsilon u) A'(u)\, du = \int_0^{w} Z(\varepsilon u)\, dA(u).$$

Integrating by parts, we have that

$$\int_0^{w} Z(\varepsilon u)\, dA(u) = Z(\varepsilon w) A(w) - \int_0^{w} A(u)\, dZ(\varepsilon u).$$

We denote by V the total variation of $Z(u)$ on $[0,\infty)$.

Let $|A(u)| \leqslant M$,

$$\left| \int_0^{w} A(u)\, dZ(\varepsilon u) \right| \leqslant \int_0^{w} |A(u)|\, |dZ(\varepsilon u)| \leqslant M \int_0^{w} |dZ(\varepsilon u)|$$

$$[illegible] \leqslant MV < +\infty ,$$

since $Z(u)$ and $Z(\varepsilon u)$ have the same total variation on $[0,\infty)$. Letting $w \to \infty$, we obtain

$$I_\varepsilon = \frac{1}{\sqrt{2\pi}} \int_0^\infty a(u) Z(\varepsilon u) du = -\int_0^\infty A(u) dZ(\varepsilon u),$$

since $Z(\varepsilon w) \to 0$ and $A(w) \to I$. Hence I_ε exists. If we set $A(u) = I + h(u)$, so $h(u) \to 0$ as $u \to \infty$, then

$$\begin{aligned} I_\varepsilon &= -I \int_0^\infty dZ(\varepsilon u) - \int_0^\infty h(u) dZ(\varepsilon u) \\ &= -I[Z(\infty) - Z(0)] - \int_0^\infty h(u) dZ(\varepsilon u) \\ &= I - \int_0^\infty h(u) dZ(\varepsilon u) \end{aligned}$$

and it remains to show that this last integral tends to zero as $\varepsilon \to 0$.

By our assumption $|h(u)| = |A(u) - I| \leqslant |A(u)| + |I| = N$ and for any given $\eta > 0$, there exists u_0 such that $|h(u)| < \eta$ whenever $u \geqslant u_0$. Let

$$\int_0^\infty h(u) dZ(\varepsilon u) = \int_0^{u_0} + \int_{u_0}^\infty = I_1 + I_2.$$

Now, $|I_2| < \eta V$ is arbitrarily small, for all ε. On the other hand,

$$\begin{aligned} |I_1| &\leqslant N(\text{variation of } Z(\varepsilon u) \text{ on } [0, u_0]) \\ &= N(\text{variation of } Z(u) \text{ on } [0, \varepsilon u_0]) \to 0 \text{ as } \varepsilon \to 0, \end{aligned}$$

since $Z(u)$ is of bounded variation and continuous at $u = 0$, hence its variation is also continuous.

We now extend the concept of summability to multiple integrals. Let $a(u)$ be a locally integrable function of $u \in R^n$, and $\int a(u)du = \int \cdots \int a(u_1, \ldots, u_n)du_1 \ldots du_n$.

Here it is most natural to consider partial integrals taken over spheres with center at the origin; namely,

$$S(R) = \frac{1}{(2\pi)^{n/2}} \int_{|u| \leqslant R} a(u)du.$$

Then, we say that $\frac{1}{(2\pi)^{n/2}} \int a(u)du$ converges spherically to a value I if $S(R) \to I$ as $R \to \infty$.

4.2.4 Example. Let $x \in R^n$. Consider the function

$$f(x) = \frac{1}{(1+|x|^2)^{(n+1)/2}}.$$

Then $\frac{1}{(2\pi)^{n/2}} \int f(x)dx$ converges spherically to $\frac{1}{(2\pi)^{n/2}} \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$.

For details see (5.1).

4.2.5 Example. Consider the function $f(x) = c$ for all $x \in R^n$. For fixed k, $\int_{|x| \leqslant k} f(x)dx = c \int_{|x| \leqslant k} dx = ck^n V_n$ is finite, where V_n is the volume of unit ball. Then f is a locally integrable function. But

$$\frac{1}{(2\pi)^{n/2}} \int f(x)dx \text{ is not converges spherically.}$$

The corresponding notion of spherical summability is defined as follows: we consider integrals

$$I_\varepsilon = \frac{1}{(2\pi)^{n/2}}\int a(u)Z(\varepsilon|u|)du$$

where Z satisfies conditions 1), 2) and 3) listed earlier. If I_ε exists for every $\varepsilon > 0$ and $I_\varepsilon \to I$ as $\varepsilon \to 0$, then we say that $(2\pi)^{-n/2}\int a(u)du$ is summable-Z to the value I.

4.2.6 Theorem. Let a be a locally integrable function on R^n. If the integral $(2\pi)^{-n/2}\int a(u)du$ converges spherically to a value I, then it is also summable-Z to the same value I.

Proof: By hypothesis, $S(R) = (2\pi)^{-n/2}\int_{|u|\leqslant R} a(u)du \to I$ as $R \to \infty$.

$$\frac{1}{(2\pi)^{n/2}}\int_{|u|\leqslant w} a(u)Z(\varepsilon|u|)du = \int_0^w Z(\varepsilon R)dS(R).$$

Integrating by parts,

$$\int_0^w Z(\varepsilon R)dS(R) = Z(\varepsilon w)S(w) - \int_0^w S(R)dZ(\varepsilon R) .$$

Letting $w \to \infty$, we have that

$$I_\varepsilon = \frac{1}{(2\pi)^{n/2}}\int a(u)Z(\varepsilon|u|)du = - \int_0^\infty S(R)dZ(\varepsilon R).$$

The rest of the proof is identical to the proof of Theorem(4.2.3).

We now return to the Fourier Inversion Formula

$$\text{(b)} \qquad \frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x.y)}dy, \qquad f \in L(R^n).$$

We will show that, under some additional assumptions on the function $Z(|x|)$, the integral in (b) is summable-Z to $f(x)$ for almost every x.

We recall that , with $r = |x|$, $Z(r)$ satisfies conditions

1) $Z(r)$ is of bounded variation on $[0,\infty)$;

2) $Z(r) \longrightarrow 0$ as $r \longrightarrow \infty$,

3) $Z(r)$ is continuous at $r = 0$ and $Z(0) = 1$.

Assume, in addition, that

4) $Z(|x|) \in L(R^n)$.

If we denote $H(x)$ the Fourier transform of $Z(|x|)$, then $H(x)$ is a bounded radial function . We have by Lemma(4.1.3) and Theorem(4.1.13), that

$$\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x\cdot y)}Z(\varepsilon|y|)dy = \frac{1}{(2\pi)^{n/2}}\int \hat{f}(y+x)Z(\varepsilon|y|)dy$$

$$= \frac{1}{(2\pi)^{n/2}}\int f(y+x)\hat{Z}(\varepsilon|y|)dy$$

$$= \frac{1}{(2\pi)^{n/2}}\int f(x-y)\hat{Z}(\varepsilon|y|)dy.$$

But, by Lemma(4.1.5),

$$\hat{Z}(\varepsilon|y|) = \varepsilon^{-n}\hat{Z}\left(\frac{|y|}{\varepsilon}\right) = \varepsilon^{-n}H\left(\frac{y}{\varepsilon}\right) = H_{\varepsilon}(y) \text{ ; hence,}$$

(4.2.7) $$\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x\cdot y)}Z(\varepsilon|y|)dy$$

$$= \frac{1}{(2\pi)^{n/2}}\int f(x-y)H_{\varepsilon}(y)dy = (f*H_{\varepsilon})(x).$$

Recalling that H is bounded, we assume finally that

5) $H(x) = \hat{Z}(|x|) = O(|x|^{-n-1})$ for $|x| \geq 1$,

(hence $H(x)$ is integrable).

6) $\int H(x)dx = 1$.

<u>Note</u> We can show that $Z(|x|) = e^{-(x_1^2+\dots+x_n^2)}$ satisfies conditions 1) through 6).

In view of Theorem(3.12) and (3.15), we obtain the following conclusion:

4.2.8 Theorem. If $f \in L(R^n)$ and $Z(|x|)$ satisfies conditions 1) through 6) above, then, as $\varepsilon \to 0$,

$$\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x.y)}Z(\varepsilon|y|)dy \longrightarrow f(x)$$

almost everywhere and in L^1norm.

Proof: From (4.2.7) we have

$$\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x.y)}Z(\varepsilon|y|)dy = (f*H_\varepsilon)(x).$$

Hence we must show that as $\varepsilon \to 0$, $(f*H_\varepsilon)(x) \longrightarrow f(x)$ almost everywhere and in L^1norm. This result follows from Theorem(3.12) and (3.15) where $p = 1$.

4.2.9 Corollary.(Uniqueness of the Fourier transform) If $f \in L(R^n)$ and $\hat{f}(x) \equiv 0$, then $f(x) = 0$ a.e..

4.2.10 Theorem. If the integral $\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x.y)}dy$ converges(spherically). Then it converges a.e. to the value $f(x)$.

Proof: By hypothesis, let $g(x)$ be such that

$$\frac{1}{(2\pi)^{n/2}}\int_{|y|\leq R} \hat{f}(y)e^{-i(x.y)}dy \longrightarrow g(x) \quad \text{as} \quad R \to \infty.$$

It is enough to show that $g(x) = f(x)$ a.e..

By Theorem(4.2.6) with $Z(|x|)$ satisfies conditions 1) though 6) above $(2\pi)^{-n/2}\int \hat{f}(y)e^{-i(x.y)}dy$ is summable-Z to $g(x)$, i.e.

$$\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x.y)}Z(\varepsilon|y|)dy \longrightarrow g(x), \text{ as } \varepsilon \longrightarrow 0.$$

By Theorem(4.2.8), as $\varepsilon \to 0$,

$$\frac{1}{(2\pi)^{n/2}}\int \hat{f}(y)e^{-i(x.y)}Z(\varepsilon|y|)dy \longrightarrow f(x) \quad \text{a.e.}.$$

Hence $\qquad g(x) = f(x) \quad$ a.e..

We next investigate the behaviour of

$$(2\pi)^{-n/2}\int \hat{f}(y)e^{-i(x.y)}dy \quad \text{at a particular point } x \text{ in } R^n.$$

4.2.11 <u>Theorem</u>. If $f \in L^{\infty}(R^n)$, then for $x_0 \in R^n$

$$\liminf_{x \to x_0} f(x) \leqslant (2\pi)^{-n/2}\int \hat{f}(y)e^{-i(x_0.y)}dy \leqslant \limsup_{x \to x_0} f(x).$$

<u>Proof</u>: Suppose that $g(x_0) = (2\pi)^{-n/2}\int \hat{f}(y)e^{-i(x_0.y)}dy.$

Consider the function $Z(|x|) \geqslant 0$ satisfies conditions 1) through 6), then its Fourier transform $H(x) \geqslant 0$ and from(4.2.7)

$$\text{(c)} \qquad (f*H_\varepsilon)(x_0) = (2\pi)^{-n/2}\int \hat{f}(y)e^{-i(x_0.y)}Z(\varepsilon|y|)dy.$$

The integrands on the right side of (c) are bounded by $(2\pi)^{-n/2}M|\hat{f}(t)|$ where $|Z(\varepsilon|y|)| \leqslant M$, and since $Z(\varepsilon|y|) \longrightarrow 1$ as $\varepsilon \to 0$, the right side of (c) converges to $g(x_0)$, by the dominated convergence theorem. We get

$$\lim_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) = g(x_0).$$

On the other hand, we obtain by Theorem(3.7) that

$$\limsup_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \leqslant \limsup_{x \to x_0} f(x)$$

and

$$\liminf_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \geqslant \liminf_{x \to x_0} f(x).$$

Hence

(d) $$\liminf_{x \to x_0} f(x) \leqslant \liminf_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \leqslant \lim_{\varepsilon \to 0} (f*H_\varepsilon)(x) = g(x_0),$$

and

(e) $$g(x_0) = \lim_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \leqslant \limsup_{\varepsilon \to 0} (f*H_\varepsilon)(x_0) \leqslant \limsup_{x \to x_0} f(x)$$

i.e.,

$$\liminf_{x \to x_0} f(x) \leqslant (2\pi)^{-n/2} \int \hat{f}(y) e^{-i(x_0 \cdot y)} dy \leqslant \limsup_{x \to x_0} f(x).$$

4.2.12 Corollary. Let $f \in L(R^n)$.

1) If f is lower semicontinuous at a point x_0, then

$$(2\pi)^{-n/2} \int \hat{f}(y) e^{-i(x_0 \cdot y)} dy \geqslant f(x_0).$$

2) If f is upper semicontinuous at a point x_0, then

$$(2\pi)^{-n/2} \int \hat{f}(y) e^{-i(x_0 \cdot y)} dy \leqslant f(x_0).$$

3) If f is continuous at a point x_0, then

$$(2\pi)^{-n/2} \int \hat{f}(y) e^{-i(x_0 \cdot y)} dy = f(x_0).$$

Proof: The proof of 1) and 2) follow from (d) and (e) respectively, and 3) follows from 1) and 2).

4.3 L^2 Theory

We recall that $L^2 = L^2(R^n)$ is a Hilbert space, with inner product $(f,g) = \int f(x)\overline{g(x)}dx$ which is finite by Höder's Inequality, and with norm

$$\|f\|_2 = (f,f)^{1/2} = (\int |f(x)|^2 dx)^{1/2}.$$

We denote by $\mathcal{S}$ the class of all simple functions (i.e. finite linear combinations of characteristic functions of n-cells) and recall that $\mathcal{S}$ is a dense subspace of L^2 [13 :p.67] .

If $f \in L^2$ then, by Schwarz's inequality, f is locally integrable and hence the following integrals exists:

$$\hat{f}_R(x) = (2\pi)^{-n/2} \int_{|y| \leq R} f(y) e^{i(x.y)} dy.$$

4.3.1 Theorem. (Parseval-Plancherel) Let $f \in L^2(R^n)$, then the Fourier transform

$$\hat{f}(x) = (2\pi)^{-n/2} \int f(y) e^{i(x.y)} dy$$

exists as a limit in L^2 norm of the $\hat{f}_R(x)$, $R \to \infty$. Also,

$$\|\hat{f}\|_2 = \|f\|_2 \quad \text{(Parseval Formula).}$$

Moreover the Inversion Formula

$$f(x) = (2\pi)^{-n/2} \int \hat{f}(y) e^{-i(x.y)} dy$$

holds in the sense that

$$f(x) = \underset{R \to \infty}{\text{limit in } L^2} (2\pi)^{-n/2} \int_{|y| \leq R} \hat{f}(y) e^{-i(x.y)} dy.$$

Proof: We shall first prove the theorem for simple functions and then pass to the limit. We do so in the following series of steps.

step. 1 Characteristic function on R^1.

We recall from Example(4.1.11) that the triangular function

$$\chi_{h,h}(x) = (1 - \frac{|x|}{2h})^+, \quad h > 0,$$

is continuous and integrable on R^1, and that the same is true of its Fourier transform

$$\hat{\chi}_{h,h}(x) = \frac{1}{h}\sqrt{\frac{2}{\pi}}\,\frac{\sin^2 hx}{x^2}.$$

Hence, by the Inversion Formula, we have that for every x

$$\frac{1}{\pi h}\int_{-\infty}^{\infty}\frac{\sin^2 hy}{y^2}\,e^{-ixy}dy = \chi_{h,h}(x).$$

In particular, at $x = 0$

$$\frac{1}{\pi h}\int_{-\infty}^{\infty}\frac{\sin^2 hy}{y^2}\,dy = 1$$

and letting $h = 1$, we obtain

$$\int_{-\infty}^{\infty}(\frac{\sin x}{x})^2 dx = \pi .$$

If χ_h is the characteristic function of the interval $(-h,h)$, then its Fourier transform

$$\hat{\chi}_h(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin hx}{x}$$

exists pointwise and also in the L^2 sense, since with $f(x) = \chi_h(x)$, the integrals

$$\hat{f}_R(x_o) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(y) e^{ix_o y} dy$$

are all the same for $R \geqslant h$. Moreover,

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{\chi}_h^2(x)\,dx &= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 hx}{x^2}\,dx \\ &= \frac{2h}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2}\,dy \qquad (y = hx) \\ &= 2h = \int_{-\infty}^{\infty} \chi_h^2(x)\,dx\end{aligned}$$

whence, taking square roots, we see that Parseval's Formula holds.

If χ is the characteristic function of any interval $(a,b) = (c-h, c+h)$, then we know that

$$\hat{\chi}(x) = \hat{\chi}_h(x) e^{icx}$$

so $|\hat{\chi}(x)|^2 = |\hat{\chi}_h(x)|^2 = \hat{\chi}_h^2(x)$ and the conclusion follows from the previous argument.

<u>step</u>. 2 Characteristic function on R^n.

If χ is the characteristic function of an n-cell $a_i \leqslant x_i \leqslant b_i$, $i = 1, \ldots, n$, then

$$\chi(x) = \chi_1(x_1) \ldots \chi_n(x_n)$$

where χ_i is the characteristic function of the 1-cell (interval) (a_i, b_i); also $\hat{\chi}(x)$ is defined in the usual way and

$$\hat{\chi}(x) = \hat{\chi}_1(x_1) \ldots \hat{\chi}_n(x_n).$$

Then

$$\begin{aligned}\int_{R^n} |\hat{\chi}(x)|^2 dx &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\hat{\chi}_1(x_1)|^2 \ldots |\hat{\chi}_n(x_n)|^2 dx_1 \ldots dx_n \\ &= \int_{-\infty}^{\infty} |\hat{\chi}_1(x_1)|^2 dx_1 \ldots \int_{-\infty}^{\infty} |\hat{\chi}_n(x_n)|^2 dx_n\end{aligned}$$

$$= \int_{-\infty}^{\infty}|\chi_1(x_1)|^2dx_1\cdots\int_{-\infty}^{\infty}|\chi_n(x_n)|^2dx_n$$
$$= \int_{R^n}|\chi(x)|^2dx$$

whence, taking square roots, we see the Parseval's Formula holds.

step. 3 Simple function on R^1.

Suppose that $f(x)$ takes values $c_1,\ldots,c_m$ on m non-overlapping 1-cells(intervals) $I_1,\ldots,I_m$ with characteristic functions $\chi_1,\ldots,\chi_m$, and $f(x) = 0$ elsewhere. So

$$f(x) = \sum_1^m c_k\chi_k(x)$$

and

$$\hat{f}(x) = \sum_1^m c_k\hat{\chi}_k(x).$$

Then,

$$\int_{-\infty}^{\infty}|\hat{f}(x)|^2dx = \int_{-\infty}^{\infty}\hat{f}(x)\overline{\hat{f}(x)}dx$$
$$= \int_{-\infty}^{\infty}\left\{\sum_1^m c_k\hat{\chi}_k(x)\right\}\left\{\sum_1^m \bar{c}_j\overline{\hat{\chi}_j(x)}\right\}dx$$
$$= \sum_1^m |c_k|^2\int_{-\infty}^{\infty}|\hat{\chi}_k(x)|^2dx + \sum_{k\neq j} c_k\bar{c}_j\int\hat{\chi}_k(x)\overline{\hat{\chi}_j(x)}dx = A + B.$$

We claim that $B = 0$.

If χ_1 is the characteristic function of $I_1=(x_1-h_1,x_1+h_1)$ and χ_2 is the characteristic function of $I_2=(x_2-h_2,x_2+h_2)$, where I_1 and I_2 do not overlap (say $x_1+h_1 \leq x_2-h_2$, and $h_1 \geq h_2 > 0$), then

$$\hat{\chi}_1 = \sqrt{\frac{2}{\pi}}\,\frac{\sin h_1x}{x}\,e^{ix_1x}$$
$$\hat{\chi}_2 = \sqrt{\frac{2}{\pi}}\,\frac{\sin h_2x}{x}\,e^{ix_2x} \quad .$$

Hence, $\int_{-\infty}^{\infty} \hat{\chi}_1(x)\overline{\hat{\chi}_2(x)}\,dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin h_1 x}{x}\,\frac{\sin h_2 x}{x}\, e^{i(x_1-x_2)x}\,dx.$

If we set

$f(x) = \frac{2}{\pi}\,\frac{\sin h_1 x}{x}\,\frac{\sin h_2 x}{x}$ then f is an integrable function on R^1 and is equal (up to a constant factor) to the Fourier transform of the trapezoidal function χ_{h_1,h_2} (Example4.1.11) which is continuous and integrable. Since the Inversion Formula holds everywhere in this case, we have that, for some constant $c \neq 0$,

$$\int_{-\infty}^{\infty} \hat{\chi}_1(x)\overline{\hat{\chi}_2(x)}\,dx = \int_{-\infty}^{\infty} f(x)e^{i(x_1-x_2)x}\,dx = c\chi_{h_1,h_2}(x_1-x_2)$$

$$= 0 \quad \text{becase } |x_1-x_2| \geq h_1+h_2 \text{ and}$$

therefore χ_{h_1,h_2} vanishes at the point x_1-x_2.

Hence the claim is proved. Finally by Step 1 and using the fact that χ_k have non-overlapping supports, we obtain

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx = \sum_1^m |c_k|^2 \int_{-\infty}^{\infty} |\hat{\chi}_k(x)|^2 dx = \sum_1^m |c_k|^2 \int_{-\infty}^{\infty} \chi_k^2(x)\,dx$$

$$= \int_{-\infty}^{\infty} \Big|\sum_1^m c_k\chi_k(x)\Big|^2 dx$$

$$= \int_{-\infty}^{\infty} |f(x)|^2 dx$$

whence, taking square roots, we see that Parseval's Formula holds.

step 4 Simple functions on R^n.

Suppose that $f(x)$ takes values $c_1,\ldots,c_m$ on m non-overlapping n-cells $I_1,\ldots,I_m$, with characteristic functions $\chi_1,\ldots,\chi_m$, and $f(x) = 0$ elsewhere. So

$$f(x) = \sum_1^m c_k \chi_k(x) \text{ , where}$$

$$\chi_k(x) = \chi_{1_k}(x_1)\ldots\chi_{n_k}(x_n), \text{ and}$$

$$\hat{f}(x) = \sum_1^m c_k \hat{\chi}_k(x).$$

Then

$$\int_{R^n} |\hat{f}(x)|^2 dx = \int_{R^n} \hat{f}(x)\overline{\hat{f}(x)}dx$$

$$= \int_{R^n} \left\{ \sum_1^m c_k \hat{\chi}_k(x) \right\} \left\{ \sum_1^m \bar{c}_j \overline{\hat{\chi}_j(x)} \right\} dx$$

$$= \sum_1^m |c_k|^2 \int_{R^n} |\hat{\chi}_k(x)|^2 dx + \sum_{k \neq j} c_k \bar{c}_j \int_{R^n} \hat{\chi}_k(x)\overline{\hat{\chi}_j(x)}dx = A + B.$$

We claim that $B = 0$. for $k \neq j$,

$$\int_{R^n} \hat{\chi}_k(x)\overline{\hat{\chi}_j(x)}dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{\chi}_{1_k}(x_1)\ldots\hat{\chi}_{n_k}(x_n)\overline{\hat{\chi}_{1_j}(x_1)\ldots}$$

$$\overline{\ldots\hat{\chi}_{n_j}(x_n)}dx_1\ldots dx_n$$

$$= \int_{-\infty}^{\infty} \hat{\chi}_{1_k}(x_1)\overline{\hat{\chi}_{1_j}(x_1)}dx_1 \ldots \int_{-\infty}^{\infty} \hat{\chi}_{n_k}(x_n)\overline{\hat{\chi}_{n_j}(x_n)}dx_n.$$

By step 3 $\int_{R^n} \hat{\chi}_k(x)\overline{\hat{\chi}_j(x)}dx = 0$, and hence the claim is proved.

Finally by step 2 and using the fact that χ_k have non-overlapping supports, we obtain

$$\int_{R^n} |\hat{f}(x)|^2 dx = \sum_1^m |c_k|^2 \int_{R^n} |\hat{\chi}_k(x)|^2 dx = \sum_1^m |c_k|^2 \int_{R^n} |\chi_k(x)|^2 dx$$

$$= \int_{R^n} \left|\sum_1^m c_k \chi_k(x)\right|^2 dx = \int_{R^n} |f(x)|^2 dx$$

whence, taking square roots, we see that Parseval's Formula holds.

step 5 Extension by continuity.

We have seen that the Fourier transform $F(f) = \hat{f}$ is defined on the class $\mathcal{S}$ of simple function, and satisfies Parseval's Formula

(1) $$\|\hat{f}\|_2 = \|f\|_2 \quad \text{for all } f \in \mathcal{S}.$$

Hence F is a bounded (so, continuous) linear operator in L^2. We claim that the operator F can be extended to the closure $\overline{\mathcal{S}} = L^2$, with preservation of norm, that is, in such a way that formula (1) will hold for all $f \in L^2$.

If $f \in L^2$, choose a sequence $f_n \in \mathcal{S}$ such that

$\|f_n - f\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$;

in particular $\{f_n\}$ is a cauchy sequence so

$\|f_n - f_m\|_2 \longrightarrow 0$ as $m,n \longrightarrow \infty$, thus, by continuity of F,

$\|F(f_n) - F(f_m)\|_2 \longrightarrow 0$ as $m,n \to \infty$, i.e.

$\{F(f_n)\}$ is a Cauchy sequence. Therefore, by the completeness of L^2, there is some $g \in L^2$ such that $F(f_n) \longrightarrow g$ in L^2, as $n \to \infty$. We define $F(f)$ by setting $F(f) = g$. Since if there is $\{f_n'\}$ in $\mathcal{S}$ such that $f_n' \longrightarrow f$ in L^2 and $F(f_n') \longrightarrow g'$ in L^2, then the sequence $f_1, f_1', f_2, f_2' \ldots$ converges to f, and hence the sequence $F(f_1), F(f_1'), F(f_2), F(f_2'), \ldots$ also converges in L^2 therefore we must have that $g = g'$. Then F is well-defined. Finally, from the formula

$\|F(f_m)\|_2 = \|f_m\|_2$, $f_m \in \mathcal{S}$ and $f_m \longrightarrow f$, we obtain, passing to the limit (since the norm is a continuous function), that

$$\|\hat{f}\|_2 = \|F(f)\|_2 = \|f\|_2 \text{ , for every } f \in L^2.$$

So the claim is proved.

step 6 $F(f) = \hat{f}$ is the limit, in L^2 norm, of the Fourier transform $\hat{f}_R$.

We first prove that if $f \in L^2$ with compact support then $F(f)$ is of the form

$$\hat{f}_R(x) = (2\pi)^{-n/2} \int_{|y| \leqslant R} f(y) e^{i(x.y)} dy.$$

Suppose that the support of f is contained in a sphere $|x| \leqslant R$. Choose a sequence of functions $f_k \in \mathcal{S}$ with support in $|x| \leqslant R$ such that, as $k \to \infty$,

$$f_k \longrightarrow f \quad \text{in } L^2.$$

Then, by continuity of F, $F(f_k) \longrightarrow F(f)$ in L^2 and hence there exists a subsequence of $\{F(f_k)\}$ which converges to $F(f)$ pointwise a.e.. Moreover,

$$F(f_k(x)) = (2\pi)^{-n/2} \int_{|y| \leqslant R} f_k(y) e^{i(x.y)} dy$$

and

$$\hat{f}_k(x) = (2\pi)^{-n/2} \int_{|y| \leqslant R} f_k(y) e^{i(x.y)} dy$$

therefore,

$$|F(f_k(x)) - \hat{f}_R(x)| \leqslant (2\pi)^{-n/2} \int_{|y| \leqslant R} |f_k(y) - f(y)| dy$$

by Hölder inequality

$$|F(f_k(x)) - \hat{f}_R(x)| \leqslant (2\pi)^{-n/2} \left\{ \int_{|y| \leqslant R} |f_k(y) - f(y)|^2 dy \right\}^{\frac{1}{2}} \left\{ \int_{|y| \leqslant R} 1^2 dy \right\}^{\frac{1}{2}}$$

$$\leqslant C\|f_k(y)-f(y)\|_2 \longrightarrow 0, \text{ as } k \longrightarrow \infty,$$

so the sequence $\{F(f_k)\}$ converges pointwise to $\hat{f}_R$. Thus $F(f) = \hat{f}_R$ a.e..

Now, for any $f \in L^2$, we consider the functions

$$f_R(x) = \begin{cases} f(x) & \text{if } |x| \leqslant R \\ 0 & \text{if } |x| > R \end{cases}.$$

As $R \longrightarrow \infty$, $f_R \longrightarrow f$ in L^2, so $F(f_R) \longrightarrow F(f)$ in L^2. But, by the previous argument $F(f_R) = \hat{f}_R$ a.e., therefore as $R \to \infty$

$$\hat{f}_R \longrightarrow F(f) = \hat{f} \quad \text{in } L^2.$$

step 7 The Inversion Formula.

Finally, we must verify the Inversion Formula

$$\text{(b)} \qquad f(x) = \lim_{R \to \infty} \text{ in } L^2 (2\pi)^{-n/2} \int_{|y| \leqslant R} \hat{f}(y) e^{-i(x \cdot y)} dy.$$

It should be clear from our previous discussion that it is enough to prove (b) for functions $f \in \mathcal{S}$. Moreover, by linearity, it suffices to prove (b) for characteristic functions of n-cells, but this case reduces easily to the case of characteristic functions of intervals in R^1.

From the formula

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = 1$$

we claim that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 1.$$

Since, $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2} dx$

$$= \lim_{N \to \infty} \left\{ -\frac{1}{2} \frac{1-\cos 2x}{x}\Big|_{-N}^{N} + \int_{-N}^{N} \frac{\sin 2x}{x} dx \right\}$$

$$= \int_{-\infty}^{\infty} \frac{\sin y}{y} dy.$$

Moreover, for any real number r

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin rx}{x} dx = \operatorname{sgn} r$$

where

$$\operatorname{sgn} r = \begin{cases} r/|r| & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

Now, if $f = \chi_{a,b}$ is the characteristic function of the interval $(a,b) = (c-h, c+h)$, then

$$\hat{f}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin hx}{x} e^{icx}, \text{ and}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(y) e^{-ixy} dy = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i(x-c)y} \frac{\sin hy}{y} dy$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(x-c)y \sin hy}{y} dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(x-c+h)y - \sin(x-c-h)y}{y} dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{\sin(x-a)y}{y} - \frac{\sin(x-b)y}{y}\right] dy$$

$$= \frac{1}{2}[\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b)]$$

$$= \begin{cases} 0 & \text{if } x \notin [a,b] \\ 1 & \text{if } x \in (a,b) \\ 1/2 & \text{if } x = a \text{ or } x = b \end{cases}.$$

Therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{-ixy} dy = f(x) \quad \text{a.e.}.$$

This completes the proof of Theorem (4.3.1).

Remark: If we set

$$F^{\sim} g(y) = (2\pi)^{-n/2} \int g(y) e^{-i(x.y)} dy,$$

then the Inversion Formula (b) can be written in the form $f = F^{\sim}F(f)$ which shows that $F^{\sim} = F^{-1}$. We conclude from Theorem (4.3.1) that the Fourier Transform F is a continuous linear isomorphism of L^2.

4.3.2 Corollary: (Plancherel's Formula) If $f, g \in L^2(R^n)$ then

$$\int \hat{f}(x)\overline{\hat{g}(x)}dx = \int f(x)\overline{g(x)}dx.$$

Proof: Since $\widehat{(f + g)} = \hat{f} + \hat{g}$ then, by Parseval's Formula,

$$\int |\hat{f} + \hat{g}|^2 dx = \int |f + g|^2 dx.$$

Now,

$$|f + g|^2 = (f + g)(\bar{f} + \bar{g}) = |f|^2 + |g|^2 + f\bar{g} + \bar{f}g$$
$$= |f|^2 + |g|^2 + 2\,\text{Re}(f\bar{g})$$

and likewise

$$|\hat{f} + \hat{g}|^2 = |\hat{f}|^2 + |\hat{g}|^2 + 2\,\text{Re}(\hat{f}\bar{\hat{g}}).$$

Hence, using Parseval's formula again, we obtain that

$$\text{Re}\left\{\int \hat{f}\bar{\hat{g}}dx\right\} = \text{Re}\left\{\int f\bar{g}dx\right\}.$$

Replacing f by if, the above formula gives the corresponding equality for the imaginary parts of our integrals, and hence Plancherel's formula is proved.