CHAPTER II

PRELIMINARIES

The materials of this chapter are drawn from references [1], [2], [3], [7], [10], [12], [13], [15].

To make this thesis essentially self contained, we recall some relevant notions and facts from integration theory. However we first recall some properties of upper and lower semi-continuous function.

2.1 <u>Definition</u>. Let X, Y be topological spaces. The function $f : X \longrightarrow Y$ is said to be continuous at $x_o \in X$, if given any open set V in Y with $f(x_o) \in V$, then there exists an open set U in X with $x_o \in U$, such that $f(U) \subset V$.

Furthermore, f is continuous if and only if f is continuous at each point of X.

2.2 <u>Definition</u>. Let f be an extended real-valued function with domain $D \subset R^n$. For each $y \in R^n$ let $\mathcal{N}_y$ be the collection of neighborhoods of y. If $F \subset D$ and x_o is any point of $\overline{F}$, we define

$$\liminf_{\substack{x \longrightarrow x_o \\ x \in F}} f(x) = \sup_{V \in \mathcal{N}_{x_o}} \left[\inf_{x \in V \cap F} f(x)\right]$$

$$\limsup_{\substack{x \longrightarrow x_o \\ x \in F}} f(x) = \inf_{V \in \mathcal{N}_{x_o}} \left[\sup_{x \in V \cap F} f(x)\right]$$

if $F = D$ we simply write $\liminf_{x \to x_0} f(x)$ and $\limsup_{x \to x_0} f(x)$.

2.3 Definition. An extended real-valued function f defined on a topological space is said to be lower semicontinuous at a point $x_0 \in X$, if for each $\alpha \in [-\infty, \infty]$ such that $\alpha < f(x_0)$ then there exists a neighborhood V of x_0 such that $\alpha < f(x)$ for all $x \in V$. An extended real-valued function f defined on a topological space is said to be upper semicontinuous at a point $x_0 \in X$ if $-f$ is lower semicontinuous at a point x_0.

2.4 Theorem. A function f is lower semicontinuous at x_0 if and only if

$$\liminf_{x \to x_0} f(x) = f(x_0).$$

Proof : Assume that f is lower semicontinuous at x_0; for each $\alpha \in [-\infty, \infty]$ such that $\alpha < f(x_0)$, there exists a neighborhood V of x_0 such that $\alpha < f(x)$ for all $x \in V$. Since

$$\inf_{x \in V} f(x) \leq f(x_0), \qquad (V \in \mathcal{N}_{x_0})$$

$$\sup_{V \in \mathcal{N}_{x_0}} \left[\inf_{x \in V} f(x)\right] \leq f(x_0).$$

Suppose that $\sup_{V} \left[\inf_{x \in V} f(x)\right] < f(x_0)$. Then

$$\inf_{x \in V} f(x) < f(x_0) \quad \text{for all } V;$$

there exists $x \in V$ such that $f(x) < f(x_o)$. Then there exists $\beta \in [-\infty, \infty]$ such that $f(x) < \beta < f(x_o)$; i.e. there exists $\beta \in [-\infty, \infty]$ such that $\beta < f(x_o)$ and for all neighborhood V of x_o there exists $x \in V$ such that $f(x) < \beta$. This is the contradiction.

Hence $$\liminf_{x \to x_o} f(x) = f(x_o) .$$

Conversely, assume that $\liminf_{x \to x_o} f(x) = f(x_o)$.

For given any $\alpha \in [-\infty, \infty]$ such that $\alpha < f(x_o)$,

$$\alpha < f(x_o) = \sup_{V_{x_o}} [\inf_{x \in V_{x_o}} f(x)] .$$

Then there exists a neighbothood V of x_o such that

$$\alpha < \inf_{x \in V} f(x), \text{ since otherwise}$$

$\sup_V \left[\inf_{x \in V} f(x)\right] \leqslant \alpha$. Hence f is lower semicontinuous.

Consequently, we can show that f is upper semicontinuous at x_o if and only if $\limsup_{x \to x_o} f(x) = f(x_o)$.

2.5 <u>Theorem</u>. A real-valued function is continuous at a point $x_o \in X$ if and only if it is both upper semicontinuous and lower semicontinuous at a point x_o .

<u>Proof</u> : Let f be any continuous function at a point $x_o \in X$. Then for any (α, β) with $f(x_o) \in (\alpha, \beta)$, there exists a neighborhood U of x_o such that $f(U) \subseteq (\alpha, \beta)$. Thus f is both lower and upper semicontinuous at a point x_o. Conversely, assume that f is both

lower and upper semicontinuous at a point x_o ; for each $\alpha, \beta \in [-\infty, \infty]$ such that $\alpha < f(x_o)$, $\beta > f(x_o)$ then there exists neighborhood V_1, V_2 of x_o such that $\alpha < f(x)$, $\beta > f(x)$ for all $x \in V_1$ and $x \in V_2$ respectively. Therefore for any (α, β) with $f(x_o) \in (\alpha, \beta)$, there exists a neighborhood $U = V_1 \cap V_2$ of x_o such that $f(U) \subset (\alpha, \beta)$. So that f is continuous at a point x_o.

2.6 <u>Theorem</u>. Any continuous mapping f of a compact metric space X into a metric space Y is uniformly continuous.

<u>Proof</u> : Suppose that f is continuous but not uniformly continuous on X, then for some $\varepsilon > 0$, and every positive integer n there exists x_n, $y_n \in X$ such that

$$d(x_n, y_n) < \frac{1}{n}$$

and

$$d'(f(x_n), f(y_n)) \geq \varepsilon \quad .$$

Since X is compact (and hence countable compact), the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $x \in X$, and as $d(x_{n_k}, y_{n_k}) < \frac{1}{n_k}$, it follows from the triangle inequality that the subsequence $\{y_{n_k}\}$ also converges to x. But f is continuous at the point x, hence there is a $\delta > 0$ such that $d'(f(x), f(x_o)) < \varepsilon/2$ for $d(x, x_o) < \delta$. Take k such that $d(x, x_{n_k}) < \delta$, $d(x, y_{n_k}) < \delta$, then $d'(f(x_{n_k}), f(y_{n_k})) < \varepsilon$ contrary to the definition of the sequences $\{x_n\}$ and $\{y_n\}$. Hence f is uniformly continuous on X.

2.7 <u>Definition</u>. A collection $\mathcal{M}$ of subsets of a set X is said to be an <u>algebra</u> in X if $\mathcal{M}$ has the following three properties :

1) $\mathcal{M} \neq \emptyset$

2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$, where A^c is the complement of A relative to X.

3) If $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

If $\mathcal{M}$ is an algebra and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ whenever $A_i \in \mathcal{M}$ then $\mathcal{M}$ is called a <u>σ-algebra</u> in X. If $\mathcal{M}$ is a σ-algebra in X, then X is called a <u>measurable space</u>, and the members of $\mathcal{M}$ are called the <u>measurable sets</u> in X.

2.8 <u>Definition</u>. If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be <u>measurable</u> provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

2.9 <u>Theorem</u>. If $\mathcal{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathcal{M}^*$ in X such that $\mathcal{F} \subset \mathcal{M}^*$.

This $\mathcal{M}^*$ is sometimes called the σ-algebra generated by $\mathcal{F}$.

<u>Proof</u> : Let $\mathcal{C}$ be the family of all σ-algebras $\mathcal{M}$ in X which contain $\mathcal{F}$. Since the collection of all subsets of X is such a σ-algebra, $\mathcal{C}$ is not empty. Let $\mathcal{M}^*$ be the intersection of all $\mathcal{M} \in \mathcal{C}$. It is clear that $\mathcal{F} \subset \mathcal{M}^*$ and that $\mathcal{M}^*$ lies in every σ-algebra in X which contains $\mathcal{F}$. To complete the proof, we have to show that $\mathcal{M}^*$ is itself a σ-algebra.

If $A_n \in \mathcal{M}^*$ for $n = 1, 2, \ldots$, and if $\mathcal{M} \in \mathcal{C}$, then $A_n \in \mathcal{M}$, so $\bigcup A_n \in \mathcal{M}$, since $\mathcal{M}$ is a σ-algebra. Since $\bigcup A_n \in \mathcal{M}$ for every $\mathcal{M} \in \mathcal{C}$, we conclude that $\bigcup A_n \in \mathcal{M}^*$. The other two definiting properties of a σ-algebra are verified in the same manner.

2.10 Definition. Let X be a topological space. By Theorem (2.9), there exists a smallest σ-algebra $\mathcal{B}$ in X such that every open set in X belongs to $\mathcal{B}$. The members of $\mathcal{B}$ are called the Borel sets of X.

Since $\mathcal{B}$ is a σ-algebra, we may now regard X as a measurable space, with the Borel sets playing the role of the measurable sets.

2.11 Definition. If X is a Borel measurable space, Y is a topological space, and f is a mapping of X into Y, Then f is said to be Borel measurable provided that $f^{-1}(V)$ is a Borel set in X for every open set V in Y.

If Y is the real line or the complex plane, the Borel measurable functions will be called Borel functions.

2.12 Theorem. If $f_n : X \longrightarrow [-\infty, \infty]$ is measurable, for $n = 1, 2, \ldots$ and

$$g = \sup_{n \geq 1} f_n \quad , \quad h = \limsup_{n \longrightarrow \infty} f_n \quad ,$$

then g and h are measurable.

Proof : We claim that $g^{-1}((\alpha, +\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, +\infty])$.

if $g^{-1}((\alpha, +\infty]) = \emptyset$, then $g^{-1}((\alpha, +\infty]) \subset \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, +\infty])$.

if $\quad g^{-1}((\alpha,+\infty]) \neq \emptyset$, then let $x \in g^{-1}((\alpha,+\infty])$;

$$g(x) > \alpha .$$

Since $g(x) = \sup_{n \geq 1} f_n(x)$, there exists n_o such that

$$f_{n_o}(x) > \alpha \quad , \quad x \in f_{n_o}^{-1}((\alpha,+\infty]) \text{ for some } n_o ,$$

$x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty])$.

Thus $\quad g^{-1}((\alpha,+\infty]) \subset \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty])$.

Conversely, if $\bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty]) = \emptyset$ then $\bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty]) \subset g^{-1}((\alpha,+\infty])$.

If $\bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty]) \neq \emptyset$, then let $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty])$, there exists n_o such that $f_{n_o}(x) > \alpha$. Since $g(x) = \sup_{n \geq 1} f_n(x)$,

$$g(x) > \alpha \quad , \quad \text{so that } x \in g^{-1}((\alpha,+\infty]).$$

Then $\quad g^{-1}((\alpha,+\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,+\infty])$, and hence g is measurable, since for each f_n is measurable.

The same result holds of course with inf in place of sup, and since $h = \inf_{k \geq 1} \{ \sup_{i \geq k} f_i \}$, it follows that h is measurable.

2.13 Definition : A positive measure is a function μ, defined on a σ-algebra $\mathcal{M}$, whose range is in $[0,\infty]$ and which is countably additive. This means that if $\{A_i\}$ is a disjoint countable collection of members of $\mathcal{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A <u>measure space</u> is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

2.14 <u>Definition</u> : Let R^n denote n-dimensional Euclidean space. By n-cell in R^n we mean the set of points $x = (x_1, \ldots, x_n)$ such that

$$(1) \qquad a_i \leq x_i \leq b_i \qquad (i = 1, \ldots, n),$$

or the set of points which is charactorized by (1) with any or all of the $\leq$ signs replaced by $<$.

For any n-cell I in R^n, we define

$$\text{Vol } I = \prod_{i=1}^{n} (b_i - a_i) ,$$

no matter whether equality is included or excluded in any of the inequalities (1).

Note that, if I and J are n-cells then $I \cap J$ is n-cell, and if I is n-cell then I^c is a finite union of disjoint n-cells.

2.15 <u>Definition</u> : The set $E \subset R^n$ is said to be <u>elementary set</u> if E is the union of a finite number of disjoint n-cells. Let $\mathcal{E}$ be the class of all elementary sets. Note that $\mathcal{E}$ is an algebra.

$m : \mathcal{E} \longrightarrow \tilde{R}^+$ is defined by

$$m(E) = \sum_{n=1}^{N} \text{vol } I_n \quad \text{where } E = \bigcup_{n=1}^{N} I_n \in \mathcal{E} .$$

2.16 Lemma. To prove that m in (2.15) is increasing, finite additive and $m(\emptyset) = 0$. Moreover m is well-defined.

Proof : 1) To prove that $m(\emptyset) = 0$

Since $\emptyset = \{ x \in R^n : a < x_i \leq a \}$, $m(\emptyset) = \prod_{i=1}^{n} (a-a) = 0$.

2) To prove that m is increasing.

If $E, F \in \mathcal{E}$ and $E \subset F$, then we must show that $m(E) \leq m(F)$.

Let $E = \bigcup_{n=1}^{N} I_n$, $F = \bigcup_{n=1}^{M} J_m$ where $\{I_1, \ldots . I_N\}$, $\{J_1, \ldots, J_M\}$

are two systems of disjoint n-cells.

Since

$$\bigcup_{I_n \subset J_m} I_n \subset J_m ,$$

$$m(\bigcup_{I_n \subset J_m} I_n) \leq \text{Vol } J_m .$$

By addition,

$$m(E) \leq m(F) .$$

3) To prove that m is finite additive.

If $E, F \in \mathcal{E}$ and $E \cap F = \emptyset$, then we must show that

$m(E \cup F) = m(E) + m(F)$.

Let $E = \bigcup_{n=1}^{N} I_n$, $F = \bigcup_{m=1}^{M} J_m$.

$E \cup F = \bigcup_{n=1}^{N} I_n \cup \bigcup_{m=1}^{M} J_m$ is disjoint union of n-cells.

By definition.

$$m(E \cup F) = \sum_{n=1}^{N} \text{vol } I_n + \sum_{m=1}^{M} \text{vol } J_m = m(E) + m(F) .$$

4) To prove that m is well-defined.

Let $\{I_1,\dots,I_N\}$, $\{J_1,\dots,J_M\}$ be two systems of disjoint n-cells

such that $E = \bigcup_{n=1}^{N} I_n = \bigcup_{m=1}^{M} J_m$.

For each n, $I_n = \bigcup_{m=1}^{M} I_n \cap J_m$. By (3)

$$\mathrm{Vol}\ I_n = \mathrm{Vol}\left(\bigcup_{m=1}^{M} I_n \cap J_m\right) = \sum_{m=1}^{M} \mathrm{vol}\ (I_n \cap J_m) .$$

Therefore

$$\sum_{n=1}^{N} \mathrm{Vol}\ I_n = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathrm{vol}\ (I_n \cap J_m) = \sum_{m=1}^{M} \sum_{n=1}^{N} \mathrm{vol}\ (I_n \cap J_m)$$

$$= \sum_{m=1}^{M} \mathrm{vol} \left\{ \bigcup_{n=1}^{N} I_n \cap J_m \right\} = \sum_{m=1}^{M} \mathrm{vol}\ \{ J_m \} .$$

Hence m is well-defined.

2.17 <u>Lemma</u>. If $E \in \mathcal{E}$ and $\{E_i\}$ is a sequence in $\mathcal{E}$ such that

$E \subset \bigcup_{i=1}^{\infty} E_i$ then $m(E) \leqslant \sum_{i=1}^{\infty} m(E_i)$.

<u>Proof</u> : We may assume that $m(E_i) < +\infty$ for all i.

Given any $\varepsilon > 0$ there exists a closed set $F \in \mathcal{E}$ such that

$F \subset E$ and $m(F) \geqslant m(E) - \frac{\varepsilon}{2}$.

For each E_i there exists an open elementary set $\hat{E}_i$

such that $E_i \subset \hat{E}_i$ with

$$m(\hat{E}_i) \leqslant m(E_i) + \frac{\varepsilon}{2^{i+1}} .$$

Then $F \subset \bigcup_{i=1}^{\infty} \hat{E}_i$. By Heine-Borel theorem there exists finite system $\hat{E}_{i_1}, \hat{E}_{i_2}, \dots \hat{E}_{i_M}$ from $\{\hat{E}_i\}$ such that

$$F \subset \bigcup_{m=1}^{M} \hat{E}_{i_m}$$

Therefore

$$m(F) \leq \sum_{m=1}^{M} m(\hat{E}_{i_m}) .$$

Then

$$\begin{aligned} m(E) &\leq m(F) + \frac{\varepsilon}{2} \\ &\leq \sum_{m=1}^{M} m(\hat{E}_{i_m}) + \frac{\varepsilon}{2} \leq \sum_{m=1}^{\infty} m(\hat{E}_{i_m}) + \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^{\infty} m(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2} = \sum_{i=1}^{\infty} m(E_i) + \varepsilon \end{aligned}$$

it is true for all $\varepsilon > 0$.

Hence $m(E) \leq \sum_{i=1}^{\infty} m(E_i)$.

2.18 Theorem. A set function m in (2.15) is measure on $\mathcal{E}$.

m is called the Lebesgue measure.

Proof : Obviously $m(\emptyset) = 0$. Let $\{E_n\}$ be any disjoint sequence in $\mathcal{E}$ such that $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$.

By Lemma (2.17) , we have

$$m(E) \leq \sum_{n=1}^{\infty} m(E_n) .$$

But $$E \supset \bigcup_{n=1}^{N} E_n \quad \text{for all } N.$$

Then $$m(E) \geq \sum_{n=1}^{N} m(E_n) \quad \text{for all } N,$$

so that $$m(E) \geq \sum_{n=1}^{\infty} m(E_n) . \quad \text{Thus } m(E) = \sum_{n=1}^{\infty} (E_n).$$

Hence m is a measure on $\mathcal{E}$.

Moreover, there exists the unique measure $\bar{m}$ on $\mathcal{M}(\mathcal{E})$ such that $\bar{m}(E) = m(E)$ for all $E \in \mathcal{E}$, and $\bar{m}$ is called a Lebesgue measure on $\mathcal{M}(\mathcal{E})$.

2.19 Definition. A function s on a measurable space X whose range consists of only finitely many points in $[0,\infty)$ will be called a simple function.

Let $E \subset X$, and put

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

χ_E is called the characteristic function of E.

Suppose the range of s consists of the distinct numbers $c_1, \ldots, c_m$. Let

$$E_i = \{x : s(x) = c_i\} \qquad (i = 1, \ldots, m).$$

Then clearly

$$s = \sum_{i=1}^{m} c_i \chi_{E_i} ,$$

that is, every simple function is a finite linear combination of characteristic functions. If is also clear that s is measurable if and only if each of the sets E_i is measurable.

It is of interest that every function can be approximated by simple functions. In the next theorem we consider only the case of measurable function.

2.20 Theorem. Let $f : X \longrightarrow [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that

1) $0 \leqslant s_1 \leqslant s_2 \leqslant \dots \leqslant f$.

2) $s_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, for every $x \in X$.

Proof : For $n = 1, 2, 3, \dots$, and for $1 \leqslant i \leqslant n2^n$, define

$$E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \quad \text{and} \quad F_n = f^{-1}([n, \infty])$$

and put

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n} .$$

Since f is measurable function, $E_{n,i}$ and F_n are measurable sets. Then s_n are measurable functions.

To prove 1), for any $x \in E_{n,i}$;

$$\frac{i-1}{2^n} \leqslant f(x) < \frac{i}{2^n} ,$$

$$\frac{2(i-1)}{2^{n+1}} \leqslant f(x) < \frac{2i}{2^{n+1}} , \quad \text{implies that}$$

either $\frac{2i-2}{2^{n+1}} \leqslant f(x) < \frac{2i-1}{2^{n+1}}$ or $\frac{2i-1}{2^{n+1}} \leqslant f(x) < \frac{2i}{2^{n+1}}$.

Then either $x \in E_{n+1,2i-1}$ or $x \in E_{n+1,2i}$, so that

$$s_{n+1}(x) = \frac{2i-2}{2^{n+1}} = \frac{i-1}{2^n} = s_n(x) \quad \text{or}$$

$$s_{n+1}(x) = \frac{2i-1}{2^{n+1}} > \frac{2i-2}{2^{n+1}} = \frac{i-1}{2^n} = s_n(x).$$

Therefore $s_{n+1}(x) \geq s_n(x)$ for $x \in E_{n,i}$.

If $x \in F_n$ then $f(x) \geq n$;

either $n \leq f(x) < n+1$ or $n+1 \leq f(x)$,

either $\frac{n2^{n+1}}{2^{n+1}} \leq f(x) < \frac{(n+1)2^{n+1}}{2^{n+1}}$ or $x \in F_{n+1}$,

either $s_{n+1}(x) \geq \frac{n2^{n+1}}{2^{n+1}} = n = s_n(x)$ or

$$s_{n+1}(x) = n+1 > n = s_n(x) .$$

For any $x \in X$; either $x \in E_{n,i}$ for some i or $x \in F_n$.

If $x \in E_{n,i}$ then $f(x) \geq \frac{i-1}{2^n}$ and $s_n(x) = \frac{i-1}{2^n}$.

If $x \in F_n$ then $f(x) \geq n$ and $s_n(x) = n$.

Hence, we conclude that

$$f(x) \geq s_{n+1}(x) \geq s_n(x) \qquad (x \in X).$$

To prove 2) If x is such that $f(x) = +\infty$ then $x \in F_n$, $s_n(x) = n$. Then $\lim_{n\to\infty} s_n(x) = f(x)$. If x is such that $f(x) < +\infty$ then $x \in E_{n,i}$ for some n (if n is large enough). Then

$$\frac{i-1}{2^n} \leqslant f(x) < \frac{i}{2^n} ,$$

$$s_n(x) = \frac{i-1}{2^n} = \frac{i}{2^n} - 2^{-n} > f(x) - 2^{-n} ,$$

$s_n(x) > f(x) - 2^{-n}$ when n is large enough.

Therefore $f(x) \geqslant s_n(x) > f(x) - 2^{-n}$ when n is large enough.

$f(x) \geqslant \lim_{n\to\infty} s_n(x) \geqslant f(x)$, since $s_n(x)$ is increasing sequence.

Thus $\lim_{n\to\infty} s_n(x) = f(x)$.

2.21 Definition. If s is a measurable simple function on X, of the form

$$s = \sum_{i=1}^{m} c_i \chi_{E_i} ,$$

where c_i and E_i as in Definition (2.19), and if $E \in \mathcal{M}$, we define

$$\int_E s \, d\mu = \sum_{i=1}^{m} c_i \mu(E_i \cap E).$$

The convention $0.\infty = 0$ is used here; it may happen that $c_i = 0$ for some i and that $\mu(E_i \cap E) = \infty$.

If $f : X \longrightarrow [0,\infty]$ is measurable, and $E \in \mathcal{M}$, we define

(*) $$\int_E f \, d\mu = \sup_{0 \leqslant s \leqslant f} \int_E s \, d\mu .$$

The left member of (*) is called the Lebesgue integral of f over E, with respect to the measure μ .

2.22 Theorem. Let f and g : $X \longrightarrow \tilde{R}^+$ be measurable functions.

1) If $f \leq g$, then $\int_E f\, d\mu \leq \int_E g\, d\mu$.

2) If $A \subset B$, then $\int_A f\, d\mu \leq \int_B f\, d\mu$.

3) If c is a constant, $0 \leq c < \infty$, then

$$\int_E c f\, d\mu = c \int_E f\, d\mu .$$

4) If $f(x) = 0$ for all $x \in E$, then $\int_E f\, d\mu = 0$, even if $\mu(E) = \infty$.

5) If $\mu(E) = 0$, then $\int_E f\, d\mu = 0$, even if $f(x) = \infty$ for every $x \in E$.

6) $\int_E f\, d\mu = \int \chi_E f\, d\mu$.

Proof : 1) Let $\mathcal{S} = \{ s : s$ is simple measurable and $0 \leq s \leq f \}$, $\mathcal{S}' = \{ s : s$ is simple measurable and $0 \leq s \leq g \}$. Since $f \leq g$. Then $\mathcal{S} \subset \mathcal{S}'$ which will give the result.

2) Let $s_n = \sum_{i=1}^{m} c_{n_i} \chi_{E_{n_i}}$.

$$\int_A f\, d\mu = \sup_{0 \leq s_n \leq f} \int_A s_n\, d\mu = \sup_{0 \leq s_n \leq f} \{ \sum_{i=1}^{m} c_{n_i} \mu(E_{n_i} \cap A)\}.$$

Since $A \subset B$, $\mu(E_{n_i} \cap A) \leq \mu(E_{n_i} \cap B)$.

$$\sup_{0 \leq s_n \leq f} \{ \sum_{i=1}^{m} c_{n_i} \mu(E_{n_i} \cap A)\} \leq \sup_{0 \leq s_n \leq f} \{ \sum_{i=1}^{m} c_{n_i} \mu(E_{n_i} \cap B)\} = \sup_{0 \leq s_n \leq f} \int_B s_n\, d\mu = \int_B f\, d\mu .$$

Hence $$\int_A f\,d\mu \leqslant \int_B f\,d\mu .$$

3) $$\int_E c\,f\,d\mu = \sup_{0 \leqslant s_n \leqslant f} \int_E c\,s_n\,d\mu = c \sup_{0 \leqslant s_n \leqslant f} \int_E s_n\,d\mu = c\int_E f\,d\mu ,$$

since $\int_E cs_n\,d\mu = c\sum_{i=1}^{m} c_{n_i}\mu(E_{n_i}\cap E) = c\int_E s_n\,d\mu$, where s_n as in 2).

4) Let s_n be as in 2),

$$(*) \quad \int_E f\,d\mu = \sup_{0 \leqslant s_n \leqslant f} \int_E s_n\,d\mu = \sup_{0 \leqslant s_n \leqslant f} \left\{ \sum_{i=1}^{m} c_{n_i}\mu(E_{n_i}\cap E)\right\} .$$

If $f(x) = 0$ for all $x \in E$, then $s_n(x) = 0$ for all $x \in E$, where $0 \leqslant s_n \leqslant f$. Then $c_{n_i} = 0 \;\forall\, n_i$ and hence $\int_E f\,d\mu = 0$.

5) From (*) in 4), if $\mu(E) = 0$ then

$\mu(E_{n_i}\cap E) = 0$ for all n_i, since $E_{n_i}\cap E \subset E$. Hence

$$\int_E f\,d\mu = 0.$$

6) Let $s = \sum_{i=1}^{m} c_i \chi_{E_i}$ be any simple function.

We claim that $\int_E s\,d\mu = \int \chi_E\, s\,d\mu$.

$$\int_E s\,d\mu = \int_E \sum_{i=1}^{m} c_i \chi_{E_i}\,d\mu = \sum_{i=1}^{m} c_i \int_E \chi_{E_i}\,d\mu$$

$$= \sum_{i=1}^{m} c_i \int_X \chi_E \cdot \chi_{E_i}\,d\mu = \int \chi_E \sum_{i=1}^{m} c_i \chi_{E_i}\,d\mu = \int \chi_E\, s\,d\mu .$$

Then $\int_E f\,d\mu = \sup_{0 \leq s_n \leq f} \int_E s_n d\mu = \sup_{0 \leq s_n \leq f} \int \chi_E\, s_n d\mu$

$$= \sup_{0 \leq \chi_E s_n \leq \chi_E f} \int \chi_E\, s_n d\mu = \int \chi_E\, f\, d\mu .$$

2.23 <u>Theorem</u>. Let s and t be measurable simple functions on X. We define

$$\varphi(E) = \int_E s\, d\mu \qquad (E \in \mathcal{M}).$$

Then φ is a measure on $\mathcal{M}$ and

$$(*) \qquad \int (s+t) d\mu = \int s\, d\mu + \int t\, d\mu .$$

<u>Proof</u> : Let $s = \sum_{i=1}^{m} a_i \chi_{A_i}$.

It is clear that $\varphi(E)$ is set function.

$$\varphi(\emptyset) = \int_{\emptyset} s\, d\mu = \sum_{i=1}^{m} a_i \mu(\emptyset) = 0 .$$

Let $\{E_k\}$ be any disjoint sequence in $\mathcal{M}$ and let $E = \bigcup_k E_k$.

The countable additivity of μ shows that

$$\varphi(E) = \sum_{i=1}^{m} a_i \mu(A_i \cap E) = \sum_{i=1}^{m} a_i \sum_{k=1}^{\infty} \mu(E_k \cap A_i)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{i=1}^{m} a_i \mu(E_k \cap A_i)\right)$$

$$= \sum_{k=1}^{\infty} \int_{E_k} s\, d\mu = \sum_{k=1}^{\infty} \varphi(E_k).$$

Hence φ is measure on $\mathcal{M}$.

Next, let s be as before, let $b_1, \ldots, b_m$ be the distinct values of t, and let $B_j = \{x : t(x) = b_j\}$. If $E_{ij} = A_i \cap B_j$, then

$$\int_{E_{ij}} (s+t)\,d\mu = (a_i + b_j)\mu(E_{ij})$$

$$= a_i \mu(E_{ij}) + b_j \mu(E_{ij}).$$

Since X is the disjoint union of the sets E_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m)$, the first half implies that (*) holds.

2.24 Lemma. Let μ be a positive on a σ-algebra $\mathcal{M}$. If $\{E_n\}$ is an increasing sequence in $\mathcal{M}$ such that $E = \bigcup_{n=1}^{\infty} E_n$ then $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.

Proof : Put $F_1 = E_1$, $F_n = E_n - E_{n-1}$ for $n = 2, 3, 4, \ldots$.
Then $F_n \in \mathcal{M}$, $F_i \cap F_j = \emptyset$ if $i \neq j$, $E_n = F_1 \cup F_2 \cup \ldots \cup F_n$, and $E = \bigcup_{i=1}^{\infty} F_i$. Hence

$$\mu(E) = \mu\Big(\bigcup_{i=1}^{\infty} F_i\Big) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i)$$

$$= \lim_{n\to\infty} \mu\Big(\bigcup_{i=1}^{n} F_i\Big) = \lim_{n\to\infty} \mu(E_n).$$

2.25 <u>Lebesgue's Monotone Convergence Theorem</u>. Let $\{f_n\}$ be a sequence of measurable functions on X and suppose that

a) $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq \infty$ for every $x \in X$,

b) $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, for every $x \in X$.

Then f is measurable, and

$$\int f_n d\mu \longrightarrow \int f \, d\mu \qquad \text{as } n \longrightarrow \infty .$$

<u>Proof</u>. Since $\int f_n d\mu \leq \int f_{n+1} d\mu$, there exists $c \in [0,\infty]$ such that

(1) $$\lim_{n \to \infty} \int f_n d\mu = c .$$

By Theorem (2.12), f is measurable. Since $f_n \leq f$, we have

$$\int f_n d\mu \leq \int f \, d\mu$$ for all n, so (1) implies

(2) $$c \leq \int f \, d\mu .$$

Let s be any simple measurable function such that $0 \leq s \leq f$, let k be a constant, $0 < k < 1$, and define

$$E_n = \{x: f_n(x) \geq k\, s(x)\} \qquad (n = 1, 2, 3, \ldots).$$

Each E_n is measurable, since $f_n - ks$ is measurable and $E_n = (f_n - ks)^{-1}([0,+\infty])$. $E_1 \subset E_2 \subset, \ldots,$ and $X = \bigcup E_n$. If $f(x) = 0$, then $x \in E_1$.

If $f(x) > 0$, then $ks(x) < f(x)$, since $k < 1$; hence $x \in E_n$ for some n. Also

(3) $$\int f_n d\mu \geq \int_{E_n} f_n d\mu \geq k \int_{E_n} s \, d\mu \qquad \text{for all } n.$$

Let $n \longrightarrow \infty$, applying Theorem (2.23) and Lemma (2.24) to the last integral in (3). The result is

(4) $$c \geq k \int s \, d\mu \, .$$

Since (4) holds for every $0 < k < 1$, we have

(5) $$c \geq \int s \, d\mu$$

for every simple measurable s satisfying $0 \leq s \leq f$, so that

(6) $$c \geq \int f \, d\mu \, .$$

The theorem follows from (1), (2) and (6).

2.26 Theorem. If $f_n : X \longrightarrow \widetilde{R}^{+}$ is measurable, for n = 1, 2,... and $f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X)$, then

$$\int f(x) d\mu = \sum_{n=1}^{\infty} \int f_n(x) d\mu \, .$$

Proof : By Theorem (2.20) there are increasing sequences $\{s_i'\}$ $\{s_i''\}$ of measurable simple functions such that

$s_i' \longrightarrow f_1$ and $s_i'' \longrightarrow f_2$, as $i \longrightarrow \infty$.

Let $s_i(x) = s_i'(x) + s_i''(x) \qquad (x \in X)$. Then

$$s_i \longrightarrow f_1 + f_2 \quad \text{as} \quad i \longrightarrow \infty \, .$$

By the Lebesgue's monotone convergence theorem (2.25),

$$\begin{aligned}\int (f_1 + f_2)d\mu &= \lim_{i\to\infty} \int s_i d\mu \\ &= \lim_{i\to\infty} \int (s_i' + s_i'')d\mu \\ &= \lim_{i\to\infty} \left[\int s_i' d\mu + \int s_i'' d\mu\right] \\ &= \lim_{i\to\infty} \int s_i' d\mu + \lim_{i\to\infty} \int s_i'' d\mu \\ &= \int f_1 d\mu + \int f_2 d\mu .\end{aligned}$$

Let $F_N = f_1 + f_2 + \ldots + f_N$, by induction,

$$\int F_N \, d\mu = \sum_{n=1}^{N} \int f_n d\mu .$$

Since $F_N \uparrow f$, by Theorem (2.25)

$$\begin{aligned}\int f \, d\mu &= \lim_{N\to\infty} \int F_N \, d\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu .\end{aligned}$$

2.27 Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that an extened real-valued measurable function f is integrable on X if and only if $\int |f| \, d\mu$ is finite. The class of all integrable functions on X will be denoted by $L^1(\mu)$.

2.28 Definition. A property is said to holds a.e., if it holds every where on X except on a measurable set of μ-measure zero.

2.29 <u>Theorem</u>. If $f \geq 0$ is measurable, and $\int f \, d\mu = 0$, then $f = 0$ a.e.

<u>Proof</u> : For all integer n, let

$$E_n = f^{-1}([\tfrac{1}{n}, +\infty]) \text{ , and let}$$

$$E = f^{-1}((0, +\infty]).$$

Since f is measurable, E_n, E are measurable.

Since $\{E_n\}$ is an increasing sequence and $E = \bigcup_{n=1}^{\infty} E_n$,

$$\mu(E) = \lim_{n \to \infty} \mu(E_n).$$

For each n,

$$f \geq \frac{1}{n} \chi_{E_n} \quad \text{on } X. \text{ Therefore}$$

$$0 \leq \frac{1}{n} \mu(E_n) = \int \frac{1}{n} \chi_{E_n} d\mu \leq \int f \, d\mu = 0 ,$$

$$\mu(E_n) = 0 \qquad \text{for all } n.$$

Then $\mu(E) = 0$, and hence $f = 0$ a.e. .

2.30 <u>Theorem</u>. Let $(X, \mathcal{M})$ be a measurable space, $E \in \mathcal{M}$ and $f : E \longrightarrow \widetilde{R}$ be measurable on E. Then the function $[f]_E$ given by

$$[f]_E = \begin{cases} f(x) & (x \in E) \\ 0 & (x \in E^c) \end{cases}$$

is measurable on X.

Proof : Since f is measurable on E, for any real number r, we have that

$$\{x \in E : f(x) > r\} \in \mathcal{M} .$$

On the other hand

$$[f]_E^{-1}((r,+\infty]) = f^{-1}((r,+\infty]) \qquad \text{if } r > 0.$$

$$[f]_E^{-1}((r,+\infty]) = f^{-1}((r,+\infty]) \cup E^c \text{ if } r \leqslant 0.$$

Hence $[f]_E$ is measurable on X.

2.31 Theorem. If $f \in L^1(\mu)$ then f is finite a.e. on X.

Proof : By taking $f = [f]_{X-Z}$ where Z is the set of measure zero, we may assume that f is measurable.

Since $f \in L^1(\mu)$, $f^+ f^- \in L^1(\mu)$. Then it is enough to prove the theorem for non-negative $f \in L^1(\mu)$.

Let $f \geqslant 0$, $f \in L^1(\mu)$, and let

$$E_n = \{x \in X : f(x) \geqslant n\} \qquad \text{for all } n > 0. \text{ Then}$$

E_n is measurable. So that χ_{E_n} is measurable.

Since $$n\chi_{E_n} \leqslant f \qquad \text{on } X.$$

$$n\,\mu(E_n) \leqslant \int f\,d\mu \quad .$$

If $E_\infty = \{x \in X : f(x) = +\infty\}$, then $E_\infty = \bigcap_{n=1}^{\infty} E_n$, which implies that E_∞ is measurable and $E_\infty \subset E_n$ for all n.

$$0 \leqslant \mu(E_\infty) \leqslant \mu(E_n) \leqslant \frac{1}{n}\int f\,d\mu \qquad \text{which is true for all n.}$$

Then $\mu(E_\infty) = 0$, and hence f is finite a.e.

2.32 <u>Theorem</u>. If $f, g \in L^1(\mu)$, a, b are real numbers, then $af + bg \in L^1(\mu)$ and

$$\int (af+bg)\, d\mu = a\int f\, d\mu + b\int g\, d\mu .$$

<u>Proof</u> : By Theorem (2.31) f and g are finite a.e., so that af+bg is defined and finite a.e. on X. Furthermore af+bg is measurable on X-Z, where Z is the set of measure zero, and $|af+bg| \leqslant |a||f| + |b||g| \in L^1(\mu)$. Then $af+bg \in L^1(\mu)$.

To prove the last past of the theorem, it is suffieient to show that

$$(1) \qquad \int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$$

$$(2) \qquad \int a\, f\, d\mu = a\int f\, d\mu .$$

Take $h = f+g$. Then $h^+ - h^- = f^+ - f^- + g^+ - g^-$.

On X-Z, $h^+ + f^- + g^- = h^- + f^+ + g^+$.

By Theorem (2.26)

$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- d\mu + \int f^+ d\mu + \int g^+ d\mu \quad , \text{ and}$$

since each three integrals are finite, we have that

$$\int h^+ d\mu - \int h^- d\mu = \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu .$$

$$\int h\, d\mu = \int f\, d\mu + \int g\, d\mu .$$

Thus (1) holds.

If $a \geqslant 0$ then
$$\int af\,d\mu = \int (af)^+ d\mu - \int (af)^- d\mu$$
$$= a\int f^+ d\mu - a\int f^- d\mu$$
$$= a\int (f^+ - f^-)d\mu = a\int f\,d\mu .$$

If $a < 0$ then
$$\int af\,d\mu = \int (af)^+ d\mu - \int (af)^- d\mu$$
$$= (-a)\int f^- d\mu - (-a)\int f^+ d\mu$$
$$= a\int (f^+ - f^-)d\mu = a\int f\,d\mu .$$

Hence (2) holds.

2.33 <u>Theorem</u>. If $f \in L'(\mu)$, then

$$\left|\int f\,d\mu\right| \leqslant \int |f|\,d\mu .$$

<u>Proof</u> :

$$\left|\int f\,d\mu\right| = \left|\int f^+ d\mu - \int f^- d\mu\right|$$
$$\leqslant \left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right|$$
$$= \int f^+ d\mu + \int f^- d\mu$$
$$= \int (f^+ + f^-)d\mu = \int |f|\,d\mu .$$

2.34 <u>Fatou's Lemma</u>. If $\{f_n\}$ is a sequence of measurable functions on X and non-negative a.e. on X, then

$$\int (\liminf_{n\to\infty} f_n)d\mu \leqslant \liminf_{n\to\infty} \int f_n d\mu .$$

Proof : Let Z be the set of measure zero such that $f_n \geq 0$ on X-Z, for all n. Then by taking $[f_n]_{X-Z}$, we may assume that $f_n \geq 0$ on X for all n. Let

$$g_n = \inf_{m \geq n} f_m(x) \; , \quad \text{by the Theorem (2.12)}$$

g_n is measurable, for each n, and $g_n \leq f_n$, so that

$$\int g_n \, d\mu \leq \int f_n \, d\mu \qquad (n = 1, 2, 3, \ldots).$$

Since $0 \leq g_1 \leq g_2 \leq \cdots$,

$$\liminf_{n \to \infty} \int f_n d\mu \geq \liminf_{n \to \infty} \int g_n d\mu$$

$$= \lim_{n \to \infty} \int g_n d\mu \; .$$

By the Lebegue's Monotone Convergence Theorem (2.25).

$$\lim_{n\to\infty} \int g_n d\mu = \int \lim_{n\to\infty} g_n d\mu = \int \lim_{n\to\infty} \inf_{m \geq n} f_m d\mu$$

$$= \int (\liminf_{n\to\infty} f_n) d\mu \; .$$

2.35 Lebesgue's Dominated Convergence Theorem. Suppose $\{f_n\}$ is a sequence of measurable functions on X such that $f_n \longrightarrow f$ a.e. on X, and there exists $g \in L^1(\mu)$ such that for all n $|f_n| \leq g$ a.e. on X. Then f_n and $f \in L^1(\mu)$,

(1) $$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

and

(2) $$\int f \, d\mu = \lim_{n\to\infty} \int f_n d\mu \; .$$

Proof : Let Z be the set of measure zero such that $f_n \longrightarrow f$ and $|f_n| \leqslant g$ on X-Z. Then by taking $[f_n]_{X-Z}$ and $[f]_{X-Z}$, we may assume that $f_n \longrightarrow f$ and $|f_n| \leqslant g$ on X. Since $|f| \leqslant g$ and f is measurable, $f \in L^1(\mu)$.

Since $|f_n - f| \leqslant 2g$,

$$\begin{aligned}\liminf_{n \longrightarrow \infty} (2g - |f_n - f|) &= 2g + \liminf_{n \longrightarrow \infty} (-|f_n - f|) \\ &= 2g - \lim_{n \longrightarrow \infty} |f_n - f| \\ &= 2g ,\end{aligned}$$

Fatou's lemma (2.34) applies to $2g - |f_n - f|$ that

$$\begin{aligned}\int 2g \, d\mu &\leqslant \liminf_{n \longrightarrow \infty} \int (2g - |f_n - f|) d\mu \\ &= \int 2g \, d\mu + \liminf_{n \longrightarrow \infty} \left(- \int |f_n - f| d\mu\right) \\ &= \int 2g \, d\mu - \limsup_{n \longrightarrow \infty} \int |f_n - f| \, d\mu .\end{aligned}$$

Since $\int 2g \, d\mu$ is finite, we may subtract it and obtain

$$(3) \qquad \limsup_{n \longrightarrow \infty} \int |f_n - f| \, d\mu \leqslant 0 .$$

If a sequence of nonnegative real numbers fails to converge to 0, then its upper limit is positive. Thus (3) implies (1), and (2) follows from Theorem (2.33).

2.35.1 <u>Corollary</u>. Let A be a σ-compact subset of R^n, B be an open subset of R, and f be continuous on $A \times B$. Assume that there is a function g integrable over A that $|f(x,t)| \leq g(x)$ for every $x \in A$, $t \in B$. Let

$$\phi(t) = \int_A f(x,t)\,d\mu(x), \quad t \in B.$$

Then ϕ is continuous on B. Assume further that $\frac{\partial f}{\partial t}$ is continuous on $A \times B$ and satisfy

$$|f_2(x,t)| = \left|\frac{\partial f}{\partial t}(x,t)\right| \leq h(x) \quad \text{for every} \quad x \in A,\ t \in B,$$

where h is integrable over A. Then

$$\frac{d\phi(t)}{dt} = \int_A f_2(x,t)\,d\mu(x) \qquad t \in B.$$

<u>Proof</u> : The first part is immediate consequently from the Theorem(2.35).

Since $$\frac{\phi(t+h) - \phi(t)}{h} = \int \frac{f(x,t+h) - f(x,t)}{h}\,d\mu(x).$$

Then by the mean value theorem

$$\frac{\phi(t+h) - \phi(t)}{h} = \int f_2(x,t+\theta h)\,d\mu(x) \quad \text{for some} \quad 0 < \theta < 1.$$

Hence by the first part and Theorem (2.35)

$$\lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h} = \frac{d\phi(t)}{dt} = \int_A f_2(x,t)\,d\mu(x) \qquad t \in B.$$

Integration on Product spaces.

2.36 Definition : If X, Y are two sets, we define the set $X \times Y = \{(x,y) : x \in X, y \in Y\}$. If $A \subset X$, $B \subset Y$ then $A \times B$ is called the rectangle of sides A and B.

Suppose $(X, \mathcal{M}')$ and $(Y, \mathcal{M}'')$ are measurable spaces. A measurable rectangle is any set of the form $A \times B$, where $A \in \mathcal{M}'$, $B \in \mathcal{M}''$.

If $E = R_1 \cup \dots \cup R_n$, where R_i is a measurable rectangle, and $R_i \cap R_j = \emptyset$ if $i \neq j$, we say that E is an elementary set, and the class of such sets will be denoted by $\mathcal{E}$.

$\mathcal{M}' \times \mathcal{M}''$ is defined to be the smallest σ-algebra which contains every measurable rectangle.

If $E \subset X \times Y$, $x \in X$, $y \in Y$, we define

$$E_x = \{y : (x,y) \in E\}$$

$$E^y = \{x : (x,y) \in E\}.$$

we call E_x and E^y the x-section and y-section of E respectively.

A monotone class $\mathfrak{M}$ is a collection of sets with the following properties : If $A_i \in \mathfrak{M}$, $B_i \in \mathfrak{M}$, $A_i \subset A_{i+1}$, $B_i \supset B_{i+1}$ for $i = 1, 2, \dots$, and if

$$A = \bigcup_{i=1}^{\infty} A_i, \qquad B = \bigcap_{i=1}^{\infty} B_i,$$

then $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

2.37 Theorem. If $E \in \mathcal{M}' \times \mathcal{M}''$ then $E_x \in \mathcal{M}''$ and $E^y \in \mathcal{M}'$ for every $x \in X$, $y \in Y$.

Proof : Let $\Omega = \{ E \in \mathcal{M}' \times \mathcal{M}'' : E_x \in \mathcal{M}''$ for every $x \in X \}$.

$$\text{If } E = A \times B, \text{ then } E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases},$$

so that $E_x \in \mathcal{M}''$. Hence every measurable rectangle belongs to Ω ; Ω contains the class of all measurable rectangles. We claim that Ω is a σ-algebra. Since $\mathcal{M}''$ is a σ-algebra, we have

1) $X \times Y \in \Omega$, since $(X \times Y)_x = Y \in \mathcal{M}''$,

2) If $E \in \Omega$, then $(E^c)_x = (E_x)^c \in \mathcal{M}''$. so that $E^c \in \Omega$.

3) If $E_i \in \Omega$ $(i = 1,2,\ldots)$ and $E = \bigcup_i E_i$ then $E_x = \bigcup_i (E_i)_x \in \mathcal{M}''$, since $(E_i)_x \in \mathcal{M}''$ for all i. So that $E \in \Omega$.

Then Ω is a σ-algebra contains the class of all measurable rectangle. Hence $\Omega = \mathcal{M}' \times \mathcal{M}''$.

Similarly, if $\Omega' = \{ E \in \mathcal{M}' \times \mathcal{M}'' : E^y \in \mathcal{M}'$ for every $y \in Y \}$, then $\Omega' = \mathcal{M}' \times \mathcal{M}''$.

Hence $E_x \in \mathcal{M}''$ and $E^y \in \mathcal{M}'$ for every $x \in X$, $y \in Y$.

2.38 Theorem. $\mathcal{M}' \times \mathcal{M}''$ is the smallest monotone class which contains the class of all elementary sets.

Proof : Let $\mathfrak{M}$ be the smallest monotone class which contains $\mathcal{E}$. Since $\mathcal{M}' \times \mathcal{M}''$ is a monotone class, we have

$$\mathfrak{M} \subset \mathcal{M}' \times \mathcal{M}''.$$

We claim that $\mathfrak{M}$ is a σ-algebra.

The identities

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

and

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 - A_2) \times B_1] \cup [(A_1 \cap A_2) \times (B_1 - B_2)]$$

show that the intersection of two measurable rectangles is a measurable rectangle and the difference of two measurable rectangles is the union of two disjoint measurable rectangles, hence is an elementary set.

If $E \in \mathcal{E}$, $F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$ and $E - F \in \mathcal{E}$. Since $E \cup F = (E-F) \cup F$ and $(E-F) \cap F = \emptyset$, we have that $E \cup F \in \mathcal{E}$. Hence $\mathfrak{M}$ is an algebra. Since $\mathfrak{M}$ is monotone class, $\mathfrak{M}$ is a σ-algebra.

Then $$\mathfrak{M} = \mathcal{M}' \times \mathcal{M}''.$$

2.39 Definition : Let f be an extended real valued function defined on $X \times Y$. For $x \in X$, the section of f by x is given by $f_x : Y \longrightarrow \tilde{R}$ such that $f_x(y) = f(x,y)$. Similarly, for $y \in Y$, the section of f by y is the function given by $f^y : X \longrightarrow \tilde{R}$ such that $f^y(x) = f(x,y)$.

2.40 <u>Theorem</u>. Let f be a $\mathcal{M}' \times \mathcal{M}''$ - measurable function on $X \times Y$. Then

a) For each $x \in X$, f^x is a $\mathcal{M}''$ - measurable function on Y.

b) For each $y \in Y$, f^y is a $\mathcal{M}'$ - measurable function on X.

<u>Proof</u>: For any open set V , put

$$E = \{ (x,y) : f(x,y) \in V \} ,$$

Then $E \in \mathcal{M}' \times \mathcal{M}''$, and

$$E_x = \{ y : f_x(y) \in V \} , \quad E^y = \{ x : f^y(x) \in V \}$$

are $\mathcal{M}''$, and $\mathcal{M}'$ - measurable respectively.

2.41 <u>Theorem</u>. Let $(X, \mathcal{M}', \mu)$ and $(Y, \mathcal{M}'', \nu)$ be σ-finite measure spaces. Suppose $E \in \mathcal{M}' \times \mathcal{M}''$. If

(1) $$\varphi(x) = \nu(E_x) , \quad \psi(y) = \mu(E^y)$$

for every $x \in X$ and $y \in Y$, then φ is $\mathcal{M}'$-measurable, ψ is $\mathcal{M}''$-measurable, and

(2) $$\int_X \varphi \, d\mu = \int_Y \psi \, d\nu .$$

<u>Notes</u> : Since

$$\nu(E_x) = \int_Y \chi_E(x,y) d\, \nu(y)$$
$$\mu(E^y) = \int_X \chi_E(x,y) d\, \mu(x),$$

the formula (2) can be written.

(3) $$\int_X d\mu(x) \int_Y \chi_E(x,y)\, d\nu(y) = \int_Y d\nu(y) \int_X \chi_E(x,y)\, d\mu(x) .$$

Proof : Let Ω be the class of all $E \in \mathcal{M}' \times \mathcal{M}''$ for which (2) (or equivalently (3)) holds. We claim that Ω has the following four properties :

a) Every measurable rectangle belongs to Ω .

b) If $E_1 \subset E_2 \subset E_3 \subset \ldots$, if each $E_i \in \Omega$, and if $E = \bigcup_i E_i$, then $E \in \Omega$.

c) If $\{E_i\}$ is a disjoint sequence in Ω , and $E = \bigcup_i E_i$, then $E \in \Omega$.

d) If $\mu(A) < +\infty$ and $\nu(B) < +\infty$, if $A \times B \supset E_1 \supset E_2 \supset \ldots$, if $E = \bigcap_i E_i$ and $E_i \in \Omega$ for $i = 1, 2, \ldots$, then $E \in \Omega$.

To prove (a) we let $E = A \times B$, where $A \in \mathcal{M}'$, $B \in \mathcal{M}''$, then

$$\nu(E_x) = \nu(B)\, \chi_A(x) , \quad \mu(E^y) = \mu(A)\, \chi_B(y).$$

Therefore

$$\int_X d\mu(x) \int_Y \chi_E(x,y)\, d\nu(y) = \int_X \nu(E_x)\, d\mu(x)$$

$$= \int_X \nu(B)\, \chi_A(x)\, d\mu(x) = \mu(A)\, \nu(B),$$

and

$$\int_Y d\nu(y) \int_X \chi_E(x,y)\, d\mu(x) = \int_X \mu(E^y)\, d\nu(y)$$

$$= \int_X \mu(A)\, \chi_B(y)\, d\nu(y) = \mu(A)\, \nu(B) ;$$

i.e.

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$$\int_X d\mu(x) \int_Y \chi_E(x,y)\,d\nu(y) = \int_Y d\nu(y) \int_X \chi_E(x,y)\,d\mu(x).$$

This proves (a).

To prove (b), let φ_i and ψ_i be associated with E_i in the way in which (1) associates φ and ψ with E; i.e.

$$\varphi_i(x) = \nu((E_i)_x) \quad , \quad \psi_i(y) = \mu((E_i)^y).$$

By Lemma (2.24) applied to ν and μ respectively, we get $\varphi_i(x) \longrightarrow \varphi(x)$, $\psi_i(y) \longrightarrow \psi(y)$ as $i \longrightarrow +\infty$, the convergence being monotone increasing at every point. Since for each i, $E_i \in \Omega$; i.e.

$$\int_X \varphi_i(x)\,d\mu(x) = \int_Y \psi_i(y)\,d\nu(y).$$

By the Lebesque Monotone Convergence Theorem (2.25) we have

$$\int_X \varphi(x)\,d\mu(x) = \int_Y \psi(y)\,d\nu(y) \text{ ; so that } E \in \Omega .$$

To prove (c), we set $F_n = E_1 \cup E_2 \cup \dots \cup E_n$. Since Since $E_1, E_2, \dots, E_n$ are disjoint, we have $\chi_{F_n} = \sum_{i=1}^{n} \chi_{E_i}$.

Since for each i

$$\int_X \varphi_i(x)\,d\mu(x) = \int_X \nu((E_i)_x)\,d\mu(x) = \int_Y \mu((E_i)^y)\,d\nu(y)$$

$$= \int_Y \psi_i(y)\,d\nu(y) \quad ,$$

$$\sum_{i=1}^{n} \int_X \nu((E_i)_x) d\mu(x) = \sum_{i=1}^{n} \int_Y \mu((E_i)^y) d\nu(y).$$

Consider, $\displaystyle\sum_{i=1}^{n} \int_X \nu((E_i)_x) d\mu(x) = \int_X \sum_{i=1}^{n} \nu((E_i)_x) d\mu(x)$

$$= \int_X \nu(\bigcup_{i=1}^{n} E_i)_x \, d\mu(x)$$

$$= \int_X \nu(F_n)_x \, d\mu(x).$$

Similarly,

$$\sum_{i=1}^{n} \int_Y \mu((E_i)^y) d\nu(y) = \int_Y \mu(F_n)^y d\nu(y).$$

Therefore,

$$\int_X \nu(F_n)_x d\mu(x) = \int_Y \mu(F_n)^y d\nu(y),$$

that is $F_n \in \Omega$. Now, $\{F_n\}$ is an increasing sequence in Ω. By (b), $E = \bigcup_n F_n \in \Omega$. This proves (c).

To prove (d), since $(E_i)_x \subset (A \times B)_x$,

$$0 \leq \varphi_i(x) = \nu(E_i)_x \leq \nu(A \times B)_x = \nu(B) \chi_A(x).$$

Similarly,

$$0 \leq \psi_i(y) = \mu(E_i)^y \leq (A \times B)^y = \mu(A) \chi_B(y).$$

Since $\mu(A) < +\infty$ and $\nu(B) < +\infty$, we have $\nu(E_i)_x < +\infty$ and $\mu(E_i)^y < +\infty$. Since $\{E_i\}$ is an increasing sequence with converges to E,

$\lim_{i \to \infty} \varphi_i(x) = \lim_{i \to \infty} \nu(E_i)_x = \nu(E_x) = \varphi(x)$; and

$\lim_{i \to \infty} \psi_i(y) = \psi(y)$. By the Lebesgue Dominated Convergence Theorem (2.35), we have

$$\int_X \varphi(x) d\mu(x) = \int_Y \psi(y) d\nu(y) ,$$

That is $\int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$. This proves (d). Since μ and ν are σ-finite, there exists disjoint sequences $\{X_m\}$, $\{Y_n\}$, $m = 1,2,\ldots$, $n = 1,2,\ldots$, such that $X_m \in \mathcal{M}'$, $Y_n \in \mathcal{M}''$, $\mu(X_m) < +\infty$, $\nu(Y_n) < +\infty$ for all m, n, and $X = \bigcup_m X_m$, $Y = \bigcup_n Y_n$

$$E_{m,n} = E \cap (X_m \times Y_n) \qquad (m, n = 1, 2, \ldots)$$

and note that $E_{m,n}$ are disjoint and $\bigcup_{m=n} E_{m,n} = E$. Let $\mathfrak{M}$ be the class of $E \in \mathcal{M}' \times \mathcal{M}''$ such that $E_{m,n} \in \Omega$ for all choices of m and n, then by (b) and (d) show that $\mathfrak{M}$ is a monotone class. (a) and (c) show that $\mathcal{E} \subset \mathfrak{M}$. Since $\mathcal{M}' \times \mathcal{M}''$ is the smallest monotone class which contains $\mathcal{E}$, $\mathcal{M}' \times \mathcal{M}'' \subset \mathfrak{M}$. From Definition of $\mathfrak{M}$ we have $\mathfrak{M} \subset \mathcal{M}' \times \mathcal{M}''$. Then $\mathfrak{M} = \mathcal{M}' \times \mathcal{M}''$, that is for any $E \in \mathcal{M}' \times \mathcal{M}''$, $E_{m,m} \in \Omega$ for all choices of m and n.

Since $E = \bigcup_{m,n} (E_{m,n})$, we have from (c) that $E \in \Omega$. That is

$$\int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y) .$$

2.42 Definition. If $(X, \mathcal{M}', \mu)$ and $(X, \mathcal{M}'', \nu)$ are σ-finite measure spaces and if $E \in \mathcal{M}' \times \mathcal{M}''$, we define

$$(1) \qquad (\mu \times \nu)(E) = \int_X \nu(E_x)\,d\mu(x) = \int_Y \mu(E^y)\,d\nu(y).$$

The equality of the integrals in (1) is the content of Theorem (2.41). We call $\mu \times \nu$ the product of the measures μ and ν. We claim that $\mu \times \nu$ is a measure on $\mathcal{M}' \times \mathcal{M}''$.

Let $\{E_n\}$ be a disjoint sequence in $\mathcal{M}' \times \mathcal{M}''$, and let $E = \bigcup_n E_n$.

By Theorem (2.26)

$$(\mu \times \nu)(E) = \int \nu(E_x)\,d\mu(x) = \int \sum_{n=1}^{\infty} \nu((E_n)_x)\,d\mu(x)$$

$$= \sum_{n=1}^{\infty} \int \nu((E_n)_x)\,d\mu(x) = \sum_{n=1}^{\infty} (\mu \times \nu)(E_n).$$

Clearly, $(\mu \times \nu)(\emptyset) = 0$ and $(\mu \times \nu)(E) \geqslant 0 \quad \forall\ E \in \mathcal{M}' \times \mathcal{M}''$.

Hence $\mu \times \nu$ is a measure on $\mathcal{M}' \times \mathcal{M}''$.

2.43 The Fubini Theorem. Let $(X, \mathcal{M}', \mu)$ and $(Y, \mathcal{M}'', \nu)$ be σ-finite measure spaces, and let f be an $(\mathcal{M}' \times \mathcal{M}'')$- measurable function on $X \times Y$.

a) If $0 \leqslant f \leqslant \infty$, and if

1) $\varphi(x) = \int_Y f_x\,d\nu$, $\psi(y) = \int_X f^y\,d\mu \quad (x \in X,\ y \in Y)$,

then φ is $\mathcal{M}'$- measurable, ψ is $\mathcal{M}''$- measurable, and

2) $\int_X \varphi\,d\mu = \int_{X \times Y} f\,d(\mu \times \nu) = \int_Y \psi\,d\nu$.

b) If f is an extended real valued function and if

3) $$\varphi^*(x) = \int_Y |f|_x d\nu \quad \text{and} \quad \int_X \varphi^* d\mu < \infty ,$$

then $f \in L^1(\mu \times \nu)$.

c) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$; the functions φ and ψ defined by (1) a.e. are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and (2) holds.

Notes : The first and last integrals in (2) can also be written in the more usual form

(4) $$\int_X d\mu(x) \int_Y f(x,y) d\nu(y) = \int_Y d\nu(y) \int_X f(x,y) d\mu(x).$$

These are the so-called " iterated integrals " of f. The middle integral in (2) is often referred to as a double integral.

The combination of (b) and (c) gives the following useful result : If f is ($\mathcal{M}' \times \mathcal{M}''$) - measurable and if

(5) $$\int_X d\mu(x) \int_Y |f(x,y)| d\nu(y) < \infty ,$$

then the two iterated integrals (4) are finite and equal.

Proof : We first consider (a). By Theorem (2.40), the definitions of φ and ψ make sense. Suppose $E \in \mathcal{M}' \times \mathcal{M}''$ and $f = \chi_E$. By definition (2.42),(2) is then the conclusion of Theorem (2.41). In fact that,

$$\int \varphi \, d\mu = \int \left\{ \int f_x d\nu \right\} d\mu = \int \left\{ \int \chi_{E_x} d\nu \right\} d\mu$$

$$= \int \nu(E_x) d\mu$$

$$= \int \mu(E_y) d\nu = \int \left\{ \int \chi_{E_y} d\mu \right\} d\nu$$

$$= \int \left\{ \int f_y d\mu \right\} d\nu = \int \psi \, d\nu .$$

Therefore

$$\int \varphi d\mu = \int f \, d(\mu \times \nu) = \int \psi \, d\nu , \quad \text{since}$$

$$\int \mu(E^y) d\nu = (\mu \times \nu)(E) = \int \chi_E d(\mu \times \nu) = \int f \, d(\mu \times \nu).$$

So (2) holds for all characteristic functions of ($\mathcal{M}' \times \mathcal{M}''$)- measurable set. Hence (2) holds for all non-negative ($\mathcal{M}' \times \mathcal{M}''$)- measurable simple functions. Now let f be any non-negative ($\mathcal{M}' \times \mathcal{M}''$)- measurable function. By Theorem (2.20) there exists an increasing sequence of non-negative ($\mathcal{M}' \times \mathcal{M}''$)- measurable simple functions s_n such that $s_n(x,y) \longrightarrow f(x,y)$ at every point of $X \times Y$. If φ_n is associated with s_n in the same way in which φ was associated to f, we have

$$\int_X \varphi_n \, d\mu = \int s_n \, d\,(\mu \times \nu) \qquad (n = 1,2,\ldots). \tag{6}$$

The monotone convergence theorem (2.25), applied on ($Y, \mathcal{M}'', \nu$),

$$\varphi_n(x) = \int (s_n)_x d\nu \longrightarrow \int f_x d\nu = \varphi(x), \text{ for every } x \in X,$$

as $n \longrightarrow \infty$. Hence the monotone convergence theorem applies again, to the integrals in (6), and the first equality (2) is obtained. The second half of (2) follows by interchanging the roles of x and y. This completes (a).

If we apply (a) to $|f|$, we see that (b) is true.

i.e.

$$\int |f| \, d(\mu \times \nu) = \int_X \varphi \, d\mu = \int_X \left\{ \int_Y |f|_x \, d\nu \right\} d\mu = \int_X \varphi^* \, d\mu < \infty .$$

To prove c), we let

$$\varphi_1(x) = \int_Y (f^+)_x \, d\nu$$

$$\varphi_2(x) = \int_Y (f^-)_x \, d\nu .$$

From (a) we obtain $\int \varphi_1 \, d\mu = \int_{X \times Y} f^+ \, d(\mu \times \nu)$, and

$$\int \varphi_2 \, d\mu = \int_{X \times Y} f^- \, d(\mu \times \nu).$$

Since $f \in L^1(\mu \times \nu)$, we have $f^+, f^- \in L^1(\mu \times \nu)$,

$\Longrightarrow$ $\int \varphi_1 d\mu$ and $\int \varphi_2 d\mu$ are finite

$\Longrightarrow$ φ_1 and φ_2 are finite a.e.

$\Longrightarrow$ $(f^+)_x , (f^-)_x \in L^1(\nu)$ for almost all $x \in X$.

$\Longrightarrow$ $f_x \in L^1(\nu)$ for almost all $x \in X$.

Since for all x for which φ_1, φ_2 are finite and at any such x, we have

$$\varphi(x) = \varphi_1(x) - \varphi_2(x) \quad \text{for almost all } x \in X,$$

$$\int \varphi \, d\mu = \int \varphi_1 d\mu - \int \varphi_2 d\mu < +\infty .$$

Hence $\qquad \varphi \in L^1(\mu)$.

Similarly, we can show that $f^y \in L^1(\mu)$ for almost all $y \in Y$ and $\psi \in L^1(\nu)$.

Now (2) holds. In fact that,

$$\int_Y \varphi \, d\mu = \int_X \varphi_1 \, d\mu - \int_X \varphi_2 \, d\mu$$

$$= \int_X \left\{ \int_Y (f^+)_x d\nu \right\} d\mu - \int_X \left\{ \int_Y (f^-)_x d\nu \right\} d\mu$$

$$= \int_{X \times Y} f^+ d(\mu \times \nu) - \int_{X \times Y} f^- d(\mu \times \nu) = \int_{X \times Y} f \, d(\mu \times \nu),$$

and

$$\int_{X \times Y} f^+ d(\mu \times \nu) - \int_{X \times Y} f^- d(\mu \times \nu) = \int_Y \left\{ \int_X (f^+)^y d\mu \right\} d\nu - \int_Y \left\{ \int_X (f^-)^y d\mu \right\} d\nu$$

$$= \int_Y \psi_1 \, d\nu - \int_Y \psi_2 \, d\nu = \int_Y \psi \, d\nu .$$

Hence $\displaystyle \int_X \varphi \, d\nu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \psi \, d\nu$.

The L^p-spaces.

Let $x = (x_1, \ldots, x_n) \in R^n$, where R^n is Euclidean n-dimensional space, with Lebesgue measure $dx = dx_1 \ldots dx_n$. If f(x) is a measurable function defined a.e. on a set $S \subset R^n$, we consider the integral

$$\int_S f(x)dx = \int_S f(x_1, \ldots x_n)\, dx_1 \ldots dx_n .$$

If we set f(x) = 0 outside S, we may write the integral as

$$\int_{R^n} f(x)dx = \int f(x)dx$$

where the domain of integration is understood to be the entire space R^n.

2.44 Definition. If $0 < p < \infty$ and if f is a measurable function on R^n, define

$$\|f\|_p = \left\{ \int |f|^p\, dx \right\}^{1/p}$$

and let $L^p = L^p(R^n)$ consist of all (equivalence classes of) measurable functions for which $\|f\|_p < \infty$. We call $\|f\|_p$ The L^p- norm of f.

2.45 Definition. Suppose $g : X \longrightarrow [0, \infty]$ is a measurable function. Let S be the set of all real γ such that

$$\mu(g^{-1}((\gamma, \infty])) = 0 .$$

If $S = \emptyset$, put $b = \infty$. If $S \neq \emptyset$, put $b = \inf S$.

Since $g^{-1}((b,\infty]) = \bigcup_{n=1}^{\infty} g^{-1}((b+\frac{1}{n},\infty])$, and since the union of a countable collection of sets of measure zero has measure zero, we see that $b \in S$. We call b the <u>essential</u> <u>suppremum</u> of g.

If f is a measurable function on R^n, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and we let $L^\infty(R^n)$ consist of all f for which $\|f\|_\infty < \infty$.

2.46 <u>Theorem</u>. Let p and q be conjugate exponents, $1 < p < \infty$. Let f and g be measurable functions on R^n, with range in $[0,\infty]$. Then

$$(1) \qquad \int fg\,dx \leqslant \left\{\int f^p dx\right\}^{1/p} \left\{\int g^q dx\right\}^{1/q} ,$$

and

$$(2) \qquad \left\{\int (f+g)^p dx\right\}^{1/p} \leqslant \left\{\int f^p dx\right\}^{1/p} + \left\{\int g^p dx\right\}^{1/p} .$$

The inequality (1) is Hölder's, (2) is Minkowski's. If $p = q = 2$, (1) is known as the Schwarz inequality.

<u>Proof</u> : Let A and B be the two factors on the right of (1). If $A = 0$, then $f = 0$ a.e.; hence $fg = 0$ a.e., so (1) holds. If $A > 0$ and $B = \infty$, (1) is again trivial. So we need consider only the case $0 < A < \infty$, $0 < B < \infty$. Put

$$(3) \qquad F = \frac{f}{A} , \qquad G = \frac{g}{B} .$$

This gives

$$(4) \qquad \int F^p\,dx = \int G^q\,dx = 1.$$

If $x \in X$ is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$, there are real numbers s and t such that

$$F(x) = e^{s/p} , \quad G(x) = e^{t/q} , \quad \text{Since } \frac{1}{p} + \frac{1}{q} = 1 ,$$

the convexity of the exponential function implies that

$$e^{s/p + t/q} \leq p^{-1} e^{s} + q^{-1} e^{t} .$$

It follows that

$$(5) \qquad F(x)\, G(x) \leq p^{-1} F(x)^{p} + q^{-1} G(x)^{q} \qquad \text{for every } x \in X.$$

Integration of (5) yields

$$(6) \qquad \int F\, G\, dx \leq p^{-1} + q^{-1} = 1 ,$$

by (4); inserting (3) into (6), we obtain (1).

To prove (2), we write

$$(f+g)^{p} = f\,(f+g)^{p-1} + g\,(f+g)^{p-1} .$$

By Hölder's inequality,

$$(7) \qquad \int f\,(f+g)^{p-1}\, dx \leq \left\{ \int f^{p} dx \right\}^{1/p} \left\{ \int (f+g)^{(p-1)q} dx \right\}^{1/q} .$$

Then we can write

$$(8) \qquad \int g\,(f+g)^{p-1}\, dx \leq \left\{ \int g^{p} dx \right\}^{1/p} \left\{ \int (f+g)^{(p-1)q}\, dx \right\}^{1/q} .$$

Since $(p-1)q = p$, addition of (7) and (8) gives

$$(9) \qquad \int (f+g)^{p} dx \leq \left\{ \int (f+g)^{p} dx \right\}^{1/q} \left[\left\{ \int f^{p} dx \right\}^{1/p} + \left\{ \int g^{p} dx \right\}^{1/p} \right] .$$

Clearly, it is enough to prove (2) in the case that the left side is greater than 0 and the right side is less than ∞. The convexity of the function t^p for $0 < t < \infty$ shows that

$$\left(\frac{f+g}{2}\right)^p \leqslant \frac{1}{2}(f^p + g^p) .$$

Hence the left side of (2) is less than ∞, and (2) is less than , and (2) follows from (9) if we divide by $\left\{\int (f+g)^p dx\right\}^{1/q}$, and use the fact that $1 - \frac{1}{q} = \frac{1}{p}$. This completes the proof.

2.47 Theorem. If p and q are conjugate exponents, $1 \leqslant p \leqslant \infty$, and if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and

(1) $$\|fg\|_1 \leqslant \|f\|_p \|g\|_q .$$

Proof : For $1 < p < \infty$, (1) is simply Hölder's inequality applied to $|f|$ and $|g|$. In fact

$$\|fg\|_1 = \int |fg|\, dx \leqslant \left\{\int |f|^p dx\right\}^{1/p} \left\{\int |g|^q dx\right\}^{1/q}$$

$$= \|f\|_p \|g\|_q < +\infty .$$

If $p = \infty$, then

(2) $$|f(x)\, g(x)| \leqslant \|f\|_\infty\, |g(x)| \quad \text{for almost all } x ;$$

integrating (2), we obtain

$$\|fg\|_1 \leqslant \|f\|_\infty \|g\|_1 < \infty .$$

If $p = 1$, then $q = \infty$, and the same argument applies.

2.48 <u>Definition</u>; $\mathcal{L}^p = \mathcal{L}^p(R^n)$ $(1 \leqslant p < \infty)$ is the space of all vertor functions g are defined by

$$g(x) = (g_1(x), \ldots, g_n(x)) \quad \text{where} \quad g_i \in L^p(R^n),$$

$\forall i = 1, 2, \ldots, n$, and the norm is

$$\| g \|_p = \left(\int (|g_1(x)|^p + \ldots + |g_n(x)|^p) dx \right)^{\frac{1}{p}} .$$

2.49 <u>Definition</u>; Let f and g be two functions on R^n defined for $|x| > c$, and let $g(x) \neq 0$. The symbols

$$f(x) = o(g(x)) , \quad f(x) = O(g(x))$$

mean respectively that $f(x)/g(x) \longrightarrow 0$ as $|x| \to \infty$, and that $f(x)/g(x)$ is bounded for $|x|$ large enough. The same notation is used when $|x|$ tends to a finite limit or to $-\infty$. In particular, an expression is $o(1)$ or $O(1)$ if it tends to 0 or is bounded, respectively.

2.50 <u>Definition</u>; A function f on $[a,b)$ is said to be of bounded variation on $[a,b)$ if the supremum

$$V_{ab} = \sup \left\{ \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| : a \leqslant a_0 \leqslant \ldots \leqslant a_n < b \right\} ,$$

which is taken over all possible finite sequence $a_0, \ldots, a_n$, is finite. V_{ab} is called the total variation of f on $[a,b)$.