CHAPTER III

ON THE RAMSEY NUMBERS $N(q_1,q_2,\ldots, q_\ell ; 2)$

3.0 Introduction

This chapter deals with the determination of the values of some Ramsey numbers $N(q_1,q_2,\ldots, q_\ell ; 2)$ where $q_i \geq 2$ for all $i = 1,2,\ldots, \ell$. In showing that of the Ramsey numbers $N(q_1,q_2,\ldots, q_\ell ; 2)$, we must find a positive integer n and show that $N(q_1,q_2,\ldots, q_\ell ; 2) > n$ and $N(q_1,q_2,\ldots,q_\ell ; 2) \leq n + 1$. When this is done, we can conclude that $N(q_1,q_2,\ldots,q_\ell ; 2) = n + 1$. Basic concepts of graph theory are introduced in section 3.1. In section 3.2 we shall discuss a general method for constructing chromatic graphs that will be helpful in showing the inequality $N(q_1,q_2,\ldots, q_\ell ; 2) > n$. In section 3.3 we derive some inequalities on $N(q_1,q_2,\ldots, q_\ell ; 2)$. These inequalities give us the upper bounds of certain Ramsey numbers. Section 3.4 deals with the determination of the values of some Ramsey numbers $N(q_1,q_2; 2)$. Section 3.5 deals with the determination of the values of some Ramsey numbers $N(3, 3,\ldots,3 ; 2)$.

3.1 Basic Concepts

By a graph G we mean an ordered pair (S,E), where S is a finite set and E is a set of 2-subsets of S. Any graph (S,E) can be represented geometrically by representing elements of S by points and each 2-subset $\{x, y\}$ in E by a line segment joining the points x, y. For example, G = (S, E) is a graph where

$S = \{1, 2, 3, 4, 5, 6\}$,

$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}, \{2,5\}\}$.

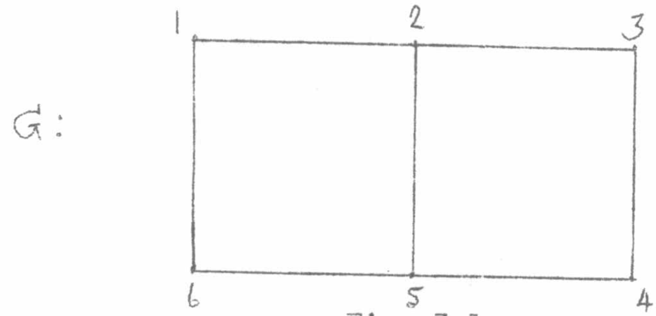

Fig. 3.1

Note that a complete graph is a graph (S,E) in which E consists of all 2-subsets of S.

By a <u>subgraph</u> of a graph $G = (S,E)$ we mean any graph $G_1 = (S_1, E_1)$ such that S_1 and E_1 are subsets of S and E, respectively. For example, G_1, G_2 are subgraphs of G.

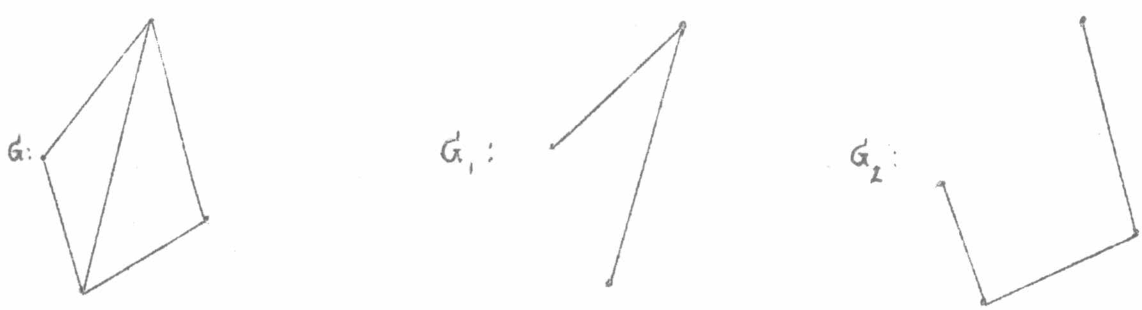

Fig. 3.2

If S_1 is a set of points of a graph (S,E), the subgraph (S_1,E_1), where $E_1 = P_2(S_1) \cap E$, is called the <u>subgraph induced by S_1</u>.

Any subgraph (S_1, E_1) of a graph (S,E) such that $S_1 = S$ will be called a <u>spanning graph</u>. In Fig. 3.2, G_2 is a spanning subgraph of G.

By a <u>union</u> of the graphs (S_1, E_1), (S_2, E_2),..., (S_ℓ, E_ℓ)

we mean a graph (S,E) where $S = \bigcup_{i=1}^{\ell} S_i$ and $E = \bigcup_{i=1}^{\ell} E_i$. In Fig. 3.2, G is the union of G_1 and G_2.

Two graphs (S_1,E_1), (S_2,E_2) with $S_1 = S_2$ are said to be line-disjoint if $E_1 \cap E_2 = \emptyset$.

By a decomposition of the complete graph $(S, P_2(S))$ we mean an ℓ-tuple of subgraphs $((S,E_1), (S, E_2),\ldots,(S,E_\ell))$ such that

(1) each (S,E_i) is a spanning subgraph of the complete graph $(S, P_2(S))$, $i = 1, 2, \ldots, \ell$,

(2) $P_2(S) = \bigcup_{i=1}^{\ell} E_i$,

(3) any two graphs (S, E_i), (S, E_j) where $i \neq j$ are line-disjoint.

If each line of the complete graph $(S, P_2(S))$ is colored by one and only one of the ℓ given colors, we obtain what is known as a chromatic graph. For such a graph, if we let E_i consist of all lines colored by the i^{th} color, $i = 1,2,\ldots, \ell$, then $((S,E_1), (S,E_2),\ldots, (S,E_\ell))$ is a decomposition of the complete graph $(S, P_2(S))$. We shall refer to any decomposition of the complete graph $(S, P_2(S))$ as a coloring of $(S, P_2(S))$. Hence a chromatic graph can be view as an ordered pair $(G; C)$, where G is a complete graph $(S, P_2(S))$ and C is a coloring of $(S, P_2(S))$. For convenience, we shall denote any chromatic graph $((S, P_2(S)); ((S, E_1), (S, E_2),\ldots, (S, E_\ell)))$ simply by $((S,E_1), (S,E_2),\ldots, (S,E_\ell))$. Observe that colorings are in one-to-one correspondence with partitions of $P_2(S)$ under the correspondence

$$(E_1, E_2, \ldots, E_\ell) \longleftrightarrow ((S,E_1), (S,E_2), \ldots, (S,E_\ell)).$$

We shall refer to $((S,E_1, (S, E_2), \ldots, (S, E_\ell))$ as the coloring induced by the partition $(E_1, E_2, \ldots, E_\ell)$. If $((S,E_1), (S,E_2), \ldots, (S,E_\ell))$ is a coloring of the complete graph $(S, P_2(S))$ such that for each $i = 1,2,\ldots, \ell$ there does not exist any q_i-subset of S which forms a complete subgraph of (S,E_i), we say that $((S,E_1), (S,E_2), \ldots, (S,E_\ell))$ is a $(q_1, q_2, \ldots, q_\ell ; 2)$-coloring of the complete graph $(S, P_2(S))$. When $\ell = 2$, the first and the second colors will be assumed to be red and blue, respectively. Hence a coloring $((S,E_1), (S,E_2))$ is a $(p, q; 2)$-coloring of the complete graph $(S_2 P_2(S))$ if no red K_p or blue K_q occurs in the chromatic graph $((S,E_1),(S,E_2))$. By a $(q_1, q_2, \ldots, q_\ell ; 2)$-chromatic graph we mean a chromatic graph in which its coloring is a $(q_1, q_2, \ldots, q_\ell ; 2)$-coloring.

Let S_1 be a set of points of a chromatic graph $((S,E_1), (S,E_2), \ldots, (S,E_\ell))$, the chromatic graph $((S_1,E_1'), (S_1,E_2'), \ldots, (S_1,E_\ell'))$, where $E_1' = P_2(S_1) \cap E_i$, $i = 1, 2, \ldots, \ell$, will be called the chromatic subgraph of $((S, E_1), (S, E_2), \ldots, (S,E_\ell))$ induced by S_1.

If $\{u, v\}$ is a line in a graph G, we say that u and v are adjacent points. Any set S of points in which no two are adjacent is said to be an independent set. If u is a point and x is a line such that $u \in x$, we say that u and x are incident. Any two lines x, y are said to be adjacent lines if they are incident with a common point. By a degree of a point v of a graph G we mean the number of lines which are incident with the point v. Two graphs (S_1,E_1), (S_2,E_2) is said to be isomorphic if there exists a one-to-one correspondence f from S_1 to S_2 such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$, i.e. adjacency is preserved under f. Hence isomorphic graphs must have the same number of points and the same number of lines. As an example, G_1 and G_2 in Fig. 3.3 are isomorphic under the correspondence $v_i \longleftrightarrow u_i$, i = 1, 2, 3, 4, 5, 6.

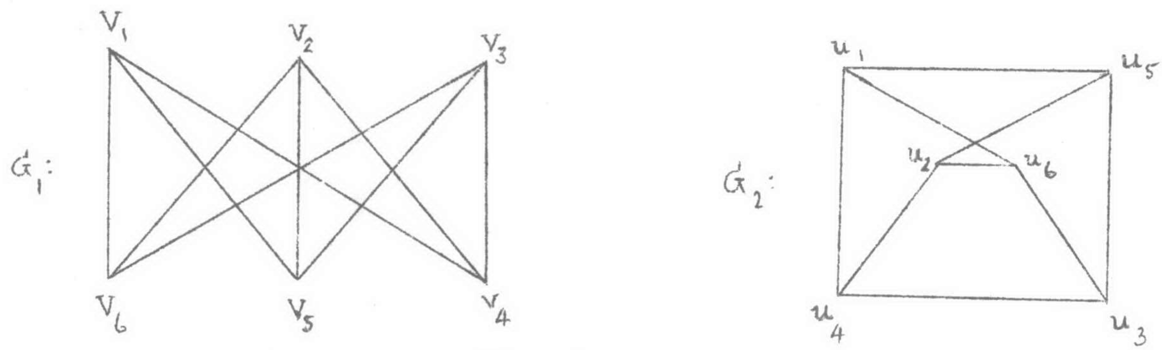

Fig. 3.3

3.1.1 Remark Let f be an isomorphism from a graph G to another graph G'. We have the following.

(1) For any point v of G, the degree of v and the degree of f(v) must be the same.

(2) For any set S_1 of points of G, the subgraph of G induced by S_1 is isomorphic to the subgraph of G' induced by $f(S_1)$.

Two chromatic graphs $((S,E_1), (S,E_2),\ldots, (S,E_\ell))$, $((S,E_1'),(S,E_2'),\ldots, (S, E_\ell'))$ are said to be isomorphic if the graphs (S, E_i) and (S, E_i') are isomorphic for each $i = 1,2,\ldots, \ell$.

By a complement of a graph (S,E) we mean a graph $(S, \overline{E})$ such that two points are adjacent in $(S, \overline{E})$ if and only if they are not adjacent in (S,E). Fig. 3.4 shown below is example of a graph G and its complement $\overline{G}$.

Fig. 3.4

Note that the union of the graph (S,E) and its complement (S, $\overline{E}$) form a complete graph (S, P_2(S)). In any chromatic graph ((S,E_1), (S,E_2)), the graphs (S,E_1) and (S,E_2) are complement of each other.

3.2 Construction of Chromatic Graphs.

3.2.1 Theorem Let H be a multiplicative subgroup of a finite field (F, +, ·) with n elements such that $h \in H$ implies that $-h \in H$. Let the set of all cosets of H be partitioned into ℓ sets $S_1, S_2, \ldots, S_\ell$. Let us label the points of K_n be elements of F. If for each pair of points, say x, y, the line xy is colored by the i^{th} color if and only if x - y belongs to some coset of S_i, $i \in \{1, 2, \ldots, \ell\}$. Then the coloring of lines of K_n is well-defined.

Proof: Let x, y be any two points. Therefore, x - y belongs to some coset, say x_oH, in S_i for some $i \in \{1,2,\ldots, \ell\}$. Then $x - y = x_oh$ for some $h \in H$. Since $y - x = x_o(-h)$. Hence y - x belongs to x_oH. Therefore, y - x belongs to coset x_oH of H in S_i, that is the line xy and yx have the same color. Hence the coloring of lines of K_n is well-defined. Q.E.D.

3.2.2 Definition The coloring of lines of K_n as given in Theorem 3.2.1 will be called an ℓ-coloring of K_n induced by $(S_1, S_2, \ldots, S_\ell)$.

If $\ell = 2$, $S_1 = \{H\}$, the coloring of lines of K_n will be called a 2-coloring induced by H. In this case the first and the second colors will be assumed to be red and blue, respectively.

If H has ℓ cosets and each S_i contains exactly one coset, the coloring of lines of K_n will be called a coloring induced by H.

3.2.3 Theorem Let H be a multiplicative subgroup of a field F with n elements such that

(1) $h \in H$ implies that $-h \in H$,

(2) for any $h_1, h_2 \in H$ $h_1 - h_2 \neq 1$.

Let K_n have a coloring induced by H. Then there exists no 3-subset of F with all its lines colored by the same color.

Proof : Assume that there exists 3-subset $\{a, b, c\}$ of F with all its lines colored by the same color. Then all $a - b$, $a - c$, $b - c$ belong to some coset, say $x_0 H$. Hence all $0 - (a - b)$, $(a - b) - (c - b)$, $0 - (c - b)$ belong to coset $x_0 H$. Therefore, the 3-subset $\{0, (a - b), (c - b)\}$ has all its lines colored by the same color. Since $(a - b)$ is not zero element in the field F, multiplication by $(a - b)^{-1}$ is permissible. Let $A = (c - b)(a - b)^{-1}$, then all $0 - 1$, $1 - A$, $0 - A$ belong to the coset $(a - b)^{-1} x_0 H$. Hence the 3-subset $\{0, 1, A\}$ has all its lines colored by the same color. Since $-1 \in H$, hence the coset $(a - b)^{-1} x_0 H$ must be H. Therefore, all 1, A, $1 - A$ belong to H. This contradicts to the hypothesis of the theorem. Hence there exists no 3-subset of F with all its lines colored by the same color. Q.E.D.

3.2.4 Corollary Let H be as in Theorem 3.2.3 . Let K_n have a 2-coloring induced by H. Then there exists no 3-subset of F with all its lines colored red.

Proof : Let us recolor the lines of K_n by a coloring induced by H. Observe that if in the original 2-coloring of K_n we have a 3-subset $\{a,b,c\}$ such that all the lines joining them are red, then we have all of $a - b$, $b - c$, $c - a$ belong to H. This implies that all the lines joining these three points are colored by the same color in the coloring induced by H, which is contrary to Theorem 3.2.3. Hence we can not have any 3-subset of F with all its lines colored red in the original 2-coloring of K_n.

Q.E.D.

3.3 Inequalities for $N(q_1,q_2,\ldots,q_\ell ; 2)$

3.3.1 Theorem

$$N(q_1,q_2,\ldots, q_\ell ; 2) \leq N(q_1-1,q_2,\ldots, q_\ell ; 2) + N(q_1,q_2-1,q_3,\ldots,q_\ell ; 2) + \ldots + N(q_1,\ldots,q_{\ell-1},q_\ell-1;2)-(\ell-2)$$

for all $q_i \geq 2$, $i = 1, 2,\ldots, \ell$.

Proof : Let S be any set of $N(q_1-1,q_2,\ldots,q_\ell ; 2) + N(q_1,q_2-1,q_3,\ldots,q_\ell ; 2) + \ldots + N(q_1,\ldots,q_{\ell-1},q_\ell-1 ; 2) - (\ell - 2)$ points. We shall show that $P_2(S)$ has no $(q_1,q_2,\ldots,q_\ell ; 2)$-partition. Let $(E_1,E_2,\ldots, E_\ell)$ be any partition of $P_2(S)$. This partition induces a chromatic graph $((S,E_1),(S,E_2),\ldots,(S,E_\ell))$. In this chromatic graph, points of S are taken to be the points of our complete graph and any line xy is colored by the i^{th} color if and only if $\{x , y\} \in E_i$. Select a point, say v, from S. For $i = 1,2,\ldots, \ell$ let S_i denote the set of all points such that the lines joining them to v are of the i^{th} color, n_i denote number of points in S_i. Thus

$$n_1 + n_2 + \dots + n_\ell + 1 = N(q_1-1, q_2, \dots, q_\ell ; 2) + N(q_1, q_2-1, q_3, \dots, q_\ell ; 2) + \dots + N(q_1, \dots, q_{\ell-1}, q_\ell - 1; 2) - (\ell - 2).$$

Suppose that $n_i < N(q_1, q_2, \dots, q_i-1, q_{i+1}, \dots, q_\ell ; 2)$ for all $i = 1, 2, \dots, \ell$. Then

$$n_1 + n_2 + n_3 + \dots + n_\ell + (\ell - 1) < N(q_1-1, q_2, \dots, q_\ell ; 2) + N(q_1, q_2-1, q_3, \dots, q_\ell ; 2) + \dots + N(q_1, \dots, q_{\ell-1}, q_\ell - 1 ; 2).$$

So that we have

$$n_1 + n_2 + \dots, n_\ell + 1 < N(q_1-1, q_2, \dots, q_\ell ; 2) + N(q_1, q_2-1, q_3, \dots, q_\ell ; 2) + \dots + N(q_1, \dots, q_{\ell-1}, q_\ell - 1 ; 2) - (\ell - 2).$$

This is a contradiction. Hence

$n_{i_o} \geqslant N(q_1, \dots, q_{i_o}-1, q_{i_o+1}, \dots, q_\ell ; 2)$ for some $i_o \in \{1, 2, \dots, \ell\}$.

Therefore, S_{i_o} contains either

(1) a $(q_{i_o} - 1)$-subset with all its lines colored by the i_o^{th} color, or

(2) some q_i-subset, where $i \neq i_o$, with all its lines colored by the i^{th} color.

If (1) holds, then this $(q_{i_o} - 1)$-subset together with V forms a q_{i_o}-subset of S with all its lines colored by the i_o^{th} color. Hence S contains some q_i-subset with all its lines colored by the i^{th} color for some $i \in \{1, 2, \dots, \ell\}$. This shows that $P_2(S)$ has no $(q_1, q_2, \dots, q_\ell ; 2)$-partition. Hence

$$N(q_1,q_2,\dots,q_\ell \; ; \; 2) \leq N(q_1-1,q_2,\dots,q_\ell \; ; \; 2) + N(q_1,q_2-1,q_3,\dots,q_\ell ; 2) + \dots + N(q_1,\dots,q_{\ell-1},q_\ell-1 \; ; \; 2) - (\ell - 2).$$

Q.E.D.

3.3.2 Corollary $N(q_1,q_2 \; ; \; 2) \leq N(q_1-1,q_2; \; 2) + N(q_1,q_2-1 \; ; \; 2)$ for all q_1, q_2 such that $q_1 > 2$, $q_2 > 2$.

Proof : Let S be any set of $N(q_1-1,q_2; \; 2) + N(q_1,q_2-1 \; ; \; 2)$ points. We shall show that $P_2(S)$ has no $(q_1,q_2; \; 2)$-partition. Let (E_1,E_2) be any partition of $P_2(S)$. This partition induces a chromatic graph $((S,E_1), (S,E_2))$. Select one point , say v, from S. Let S_1 denote the set of all points such that the lines joining them to v are red, S_2 denote the set of all points such that the lines joining them to v are blue. Let n_i denote the number of points in S_i. Since all points other than v belong to either S_1 or S_2, hence $n_1 + n_2 + 1 = N(q_1-1,q_2 \; ; \; 2) + N(q_1,q_2-1 \; ; \; 2)$.

Case I : If $n_1 < N(q_1-1,q_2; \; 2)$, then $n_2 \geq N(q_1,q_2-1; \; 2)$. Hence either

(I-a) there exists a q_1-subset of S_2 with all its lines colored red, or

(I-b) there exists a (q_2-1)-subset of S_2 with all its lines colored blue.

If (I-b) holds, then this (q_2-1)-subset together with v forms a q_2-subset of S with all its lines colored blue. Hence S contains either a q_1-subset with all its lines colored red or a q_2-subset with all its lines colored blue. This shows that $P_2(S)$ has no $(q_1,q_2 \; ; \; 2)$-partition.

Case II : If $n_1 \geq N(q_1-1, q_2 \; ; \; 2)$, then either

(II-a) there exists a $(q_1 - 1)$-subset of S_1 with all its lines colored red, or

(II-b) there exists a q_2-subset of S_1 with all its lines colored blue.

If (II-a) holds, then this $(q_1 - 1)$-subset together with v forms a q_1-subset of S with all its lines colored red. Hence S contains either a q_1-subset with all its lines colored red or a q_2-subset with all its lines colored blue.

In any case we see that $P_2(S)$ has no $(q_1, q_2; 2)$-partition. Hence $N(q_1,q_2;2) \leq N(q_1-1,q_2;2) + N(q_1,q_2-1; 2)$.

Q.E.D.

3.3.3 Corollary $N(q_1,q_2;2) \leq \binom{q_1+q_2-2}{q_1-1}$ for all q_1, q_2 such that $q_1 \geq 2$, $q_2 \geq 2$.

Proof : Let $S = \{ (q_1,q_2) / q_1 \geq 2, q_2 \geq 2 \}$, and

$$T = \left\{ (q_1,q_2) / N(q_1,q_2;2) \leq \binom{q_1+q_2-2}{q_1-1} \right\}.$$ Thus

$T \subseteq S$. We shall apply Theorem A-1 of the appendix to show that $T = S$. By Lemma 2.5.2 and Theorem 2.4.2, we have $N(q_1,2;2) = q_1$ and $N(2,q_2; 2) = q_2$. Since $\binom{q_1}{q_1-1} = q_1$ and $\binom{q_2}{1} = q_2$. Hence we see that

(1) if $q_i = 2$ for some i, then (q_1, q_2) belongs to T. To verify that T has the property (2) of the hypothesis of Theorem A-1, we assume that $q_1 \geq 2$, $q_2 \geq 2$ are any positive integers such that $(q_1 - 1, q_2)$ and $(q_1, q_2 - 1)$ belong to T, i.e.

$$N(q_1-1,q_2;\ 2) \leq \binom{(q_1-1)+q_2-2}{(q_1-1)-1} \quad ,$$

and $N(q_1,q_2+1;2) \leq \binom{q_1+(q_2-1)-2}{q_1-1}$.

Thus $N(q_1-1,q_2;2) + N(q_1,q_2-1;2) \leq \binom{(q_1-1)+q_2-2}{(q_1-1)-1} + \binom{q_1+(q_2-1)-2}{q_1-1}$.

But $\binom{(q_1-1)+q_2-2}{(q_1-1)-1} + \binom{q_1+(q_2-1)-2}{q_1-1}$

$$= \frac{(q_1+q_2-3)!}{(q_1-2)!(q_2-1)!} + \frac{(q_1+q_2-3)!}{(q_1-1)!(q_2-2)!} \quad ,$$

$$= \frac{(q_1+q_2-3)!\left[(q_1-1)+(q_2-1)\right]}{(q_1-1)!\ (q_2-1)!}$$

$$= \frac{(q_1+q_2-2)!}{(q_1-1)!(q_2-1)!}$$

$$= \binom{q_1+q_2-2}{q_1-1} .$$

From Corollary 3.3.2 we have

$$N(q_1,q_2;2) \leq N(q_1-1,\ q_2;\ 2) + N(q_1,q_2-1;\ 2).$$

Hence $N(q_1,q_2;2) \leq \binom{q_1+q_2-2}{q_1-1}$. Therefore, (q_1,q_2) belongs to T. Thus T = S. Hence

$$N(q_1,q_2;2) \leq \binom{q_1+q_2-2}{q_1-1} \qquad \text{for all } q_1 \geq 2,\ q_2 \geq 2.$$

Q.E.D.

3.3.4 Remark If we let $P(\ell)$ denote the statement

$$"N(q_1,q_2,\ldots,q_\ell;2) \leqslant \frac{(q_1+q_2+\ldots+q_\ell-\ell)!}{(q_1-1)!(q_2-1)!\ldots(q_\ell-1)!}"$$

. Then the above Corollary 3.3.3 says that P(2) holds. By using Theorem A-1 of the appendix, it can be shown that P(k) implies P(k + 1). The proof of this implication is similar to that used in the proof of Theorem 2.5.3. Hence $P(\ell)$ holds for all $\ell \geq 2$, i.e. we have

$$N(q_1,q_2,\ldots,q_\ell;2) \leqslant \frac{(q_1+q_2+\ldots+q_\ell-\ell)!}{(q_1-1)!(q_2-1)!\ldots(q_\ell-1)!} \quad \text{for all}$$

$\ell \geq 2$.

3.3.5 Theorem If $N(q_1-1,q_2;2) = 2m$ and $N(q_1,q_2-1;2) = 2n$, then $N(q_1,q_2;2) < 2m + 2n$.

Proof : Let S be a set of 2m + 2n - 1 points. We shall show that $P_2(S)$ has no $(q_1,q_2;2)$-partition. Let (E_1,E_2) be any partition of $P_2(S)$. This partition induces a chromatic graph $((S,E_1),(S,E_2))$. For each point A let r_A, b_A denote the numbers of red lines and blue lines which are incident with A.

First, we shall show that there exists a point A such that $r_A \neq 2m - 1$. Assume the contrary. Hence all points A are such that $r_A = 2m - 1$. If we cut each red line into two half lines, then each point is incident with 2m - 1 red half lines. Hence there are (2m+2n-1)(2m-1) red half lines. This calls for an odd number of red half lines. But since each line has two half lines, the number of red half lines is required to be even. Hence the case $r_A = 2m - 1$

can not hold for each points of $K_{2m+2n-1}$. Observe that $r_A + b_A = 2m+2n-2$. It follows that if $r_A < 2m-1$, then $b_A > 2n-1$. Thus there exists a point A of $K_{2m+2n-1}$ such that $r_A > 2m-1$ or $b_A > 2n-1$.

Case I : Assume that $r_A > 2m-1$. Hence $r_A \geq 2m$. Let S_1 be the set of points that are joined to A by red lines. Therefore, the number of points in S_1 is greater than or equal to $2m = N(q_1-1, q_2 ;2)$. Hence either

(I-a) there exists a (q_1-1)- subset of S_1 with all its lines colored red , or

(I-b) there exists a q_2- subset of S_1 with all its lines colored blue.

If (I-a) holds, then this (q_1-1)-subset together with A forms a q_1- subset of S with all its lines colored red. Hence S contains a q_1- subset with all its lines colored red or a q_2- subset with all its lines colored blue.

Case II : Assume that $b_A > 2n-1$. Hence $b_A \geq 2n$. Let S_2 be the set of points that are joined to A by blue lines. Therefore, the number of points in S_2 is greater than or equal to $2n = N(q_1,q_2-1;2)$. Hence either

(II-a) there exists a q_1- subset of S_2 with all its lines colored red, or

(II-b) there exists a (q_2-1)- subset of S_2 with all its lines colored blue.

If (II-b) holds, then this $(q_2 - 1)$-subset together with A forms a q_2-subset of S with all its lines colored blue. Hence S contains either a q_1-subset with all its lines colored red or a q_2-subset with all its lines colored blue.

In any case we see that $P_2(S)$ has no $(q_1,q_2;2)$-partition. Hence $N(q_1, q_2; 2) \leq 2m + 2n - 1$. Therefore , $N(q_1, q_2; 2) < 2m + 2n$.

Q.E.D.

3.4 Special Values for $N(q_1, q_2; 2)$

In this section we shall determine the values of the Ramsey numbers N(3,3;2), N(3,4;2), N(3,5;) and N(4,4;2).

3.4.1 Theorem N(3, 3; 2) = 6.

Proof : This is already done in Chapter I.

Q.E.D.

3.4.2 Theorem N(3, 4; 2) = 9 and N(3, 5; 2) = 14.

Proof : By Lemma 2.5.2 and Theorem 2.4.2 we have N(2, 4; 2) = 4, N(2, 5; 2) = 5. By Theorem 3.4.1, we have N(3,3;2) = 6. It follows from Theorem 3.3.5 that N(3,4;2) < 10, i.e. $N(3,4;2) \leq 9$. Hence, by Corollary 3.3.2, $N(3,5;2) \leq 14$. First, we shall show that $N(3,5;2) > 13$. Let (F, +, .) be the field of residue classes modulo 13, i.e. F = GF(13). Observe that $H = \{1,5,8,12\}$ is a multiplicative subgroup of F such that $h \in H$ implies that $-h \in H$. Let us label the points of K_{13} by the elements of F and its lines by the 2 - coloring induced by H.

Since $h_1 - h_2 \neq 1$ for all $h_1, h_2 \in H$. It follows from Corollary 3.2.4 that there exists no 3-subset of F with all its lines colored red.

Next, we shall show that there exists no 5-subset of F with all its lines colored blue. Suppose that there exists some 5-subset $\{a,b,c,d,e\}$ of F with all its lines colored blue. Then all a-b, a-c, a-d, a-e, b-c, b-d, b-e, c-d, c-e and d-e are not in H. So all 0-(a-b), 0-(a-c), 0-(a-d), 0-(a-e), (a-c)-(a-b),(a-d)-(a-b),(a-e)-(a-b), (a-d)-(a-c),(a-e)-(a-c), and (a-e)-(a-d) are not in H. Therefore, the 5-subset $S = \{0, (a-b),(a-c),(a-d),(a-e)\}$ has all its lines colored blue. For convenience, let A = (a-b), B = (a-c), C = (a-d), D = (a-e). Hence $S = \{0,A,B,C,D\}$. So all A,B,C,D, A-B, A-C, A-D, B-C, B-D, C-D are not in H.

Suppose that $2 \in S$. Then 3,7 and 10 are not in S. Thus only 4,6,9 and 11 can be in S.

If $4 \in S$, then $9 \notin S$. It follows that the only possible elements of S are 0,2,4,6,11. Hence $S = \{0,2,4,6,11\}$. Observe that 6,11 are in S. Thus $11-6 = 5 \notin H$, which is a contradiction.

Similary, we can show that each of the followings $6 \in S$, $9 \in S$, $11 \in S$ leads to a contradiction. Hence the supposition that $2 \in S$ leads to a contradiction. Therefore, $2 \notin S$.

Suppose that $3 \in S$. Thus 4,11 are not in S. Therefore, only 6,7,9,10 can be in S.

If $6 \in S$, then $7 \notin S$. It follows that the only possible elements of S are 0,3,6,9,10. Hence $S = \{0,3,6,9,10\}$. Observe that 9,10 are in S. Thus $10-9 = 1 \notin H$, which is a contradiction.

Similary, we can show that each of the followings $7 \in S$, $9 \in S$, $10 \in S$ leads to a contradiction. Hence the supposition that $3 \in S$ leads to a contradiction. Therefore, $3 \notin S$.

Suppose that $6 \in S$. Thus 7,11 are not in S. Therefore, only 4,9,10 can be in S. Hence $S = \{0,4,6,9,10\}$. Observe that 9,10 are in S. Thus $10-9 = 1 \notin H$, which is a contradiction. Therefore, $6 \notin S$.

Suppose that $9 \in S$. Thus $10 \notin S$. Hence only 4,7,11 can be in S. Therefore, $S = \{0,4,7,9,11\}$. Observe that 4,9 are in S. Thus $9-4 = 5 \notin H$, which is a contradiction. Therefore, $9 \notin S$.

Hence S must be $\{0,4,7,10,11\}$. Thus $11-10 = 1 \notin H$, which is a contradiction. Therefore, there exists no 5-subset $\{0,A,B,C,D\}$ with all its lines colored blue. Hence there exists no 5-subset of F with all its lines colored blue. Thus no 3-subset of F has all its lines colored red and no 5-subset of F has all its lines colored blue. Therefore, $N(3,5;2) > 13$. Hence we have $N(3,5;2) = 14$.

It follows from Corollary 3.3.2 that $N(3,5;2) \leq N(3,4;2) + N(2,5;2)$. Thus $N(3,4;2) \geq 9$. Therefore, $N(3,4;2) = 9$.

Q.E.D.

3.4.3 Theorem $N(4,4;2) = 18$.

Proof : It follows from Theorems 2.4.1 and 3.4.2 that $N(3,4;2) = N(4,3;2) = 9$. Hence, by Corollary 3.3.2, we have $N(4,4;2) \leq 18$. We have to show that $N(4,4;2) > 17$. Let $(F,+,\cdot)$ be the field of residue classes modulo 17, i.e. $F = GF(17)$. Observe that $H = \{1,2,4,8,9,13,15,16\}$ is a multiplicative subgroup of F such that

$h \in H$ implies that $-h \in H$. Let us label the points of K_{17} by the elements of F. Let the lines of K_{17} be colored by the 2-coloring induced by H.

Next, we shall show that there exists no 4-subset of F with all its lines colored by the same color. Suppose that there exists some 4-subset $\{a,b,c,d\}$ of F with all its lines of one color. By the same argument as in the proof of Theorem 3.2.3 we see that there exist A,B belonging to F such that the 4-subset $\{0,1,A,B\}$ has all its lines of one color. Then all of $1, A, B, 1-A, 1-B$ and $A-B$ are in H or all not in H. Since H contains 1, hence all of $1, A, B, 1-A, 1-B$ and $A-B$ are in H. In order that both A and $1-A$ are in H, A must be 2 or 9 or 16. By the same reason, B must be 2 or 9 or 16. Hence the only possibilities are $\{A,B\} = \{2,9\}$ or $\{2,16\}$ or $\{9,16\}$. In any case we see that $A-B \notin H$. Thus there exists no $A,B \in F$ such that the 4-subset $\{0,1,A,B\}$ has all its lines of one color. Hence there exists no 4-subset of F with all its lines colored by the same color. Thus $N(4,4;2) > 17$. Therefore, $N(4,4;2) = 18$.

Q.E.D.

3.5 On the Ramsey Numbers $N(3,3,\ldots,3;2)$

In this section we shall determine the values of the Ramsey numbers $N(3,3,3;2)$, $N(3,3,3,3;2)$.

For convenience, we let $t_\ell = N(q_1,q_2,\ldots,q_\ell;2)$, where $q_i = 3$ for all $i = 1,2,\ldots,\ell$.

3.5.1 Theorem $t_{\ell+1} \leq (\ell+1)(t_\ell - 1)+2.$

Proof : By Theorem 3.3.1, we have

$$t_{\ell+1} = N(3,3,\ldots,3;2) \leq q_1' + q_2' + \ldots + q_{\ell+1}' - ((\ell+1)-2) \text{ where}$$

$$q_1' = N(2,3,\ldots,3;2),$$

$$q_2' = N(3,2,3,\ldots,3;2),$$

$$q_{\ell+1}' = N(3,\ldots,3,2;2).$$

It follows from Theorem 2.4.1 and 2.4.2 that

$q_1' = q_2' = \ldots = q_{\ell+1}' = t_\ell$. Therefore,

$t_{\ell+1} \leq (\ell+1)t_\ell - ((\ell+1)-2)$. Hence $t_{\ell+1} \leq (\ell+1)(t_\ell-1)+2$. .

Q.E.D.

3.5.2. Theorem $N(3,3,3;2) = 17.$

Proof : By Theorem 3.5.1, we have $N(3,3,3;2) \leq 17$. Next, we shall show that $N(3,3,3;2) > 16$. Let $(F,+,\cdot)$ be the field of 16 elements. F can be taken to be the residue class ring $J[x] / (x^4 - x - 1)$, where $J = GF(2)$. The elements of F are $0,1,x,x^2,x^3,1+x,1+x^2,1+x^3,$ $1+x+x^2$, $1+x+x^3$, $x+x^2$, $x+x^3$, x^2+x^3, $1+x^2+x^3$, $x+x^2+x^3$, $1+x+x^2+x^3$. Observe that $H = \{1, x^3, x+x^3, x^2+x^3, 1+x+x^2+x^3\}$ is a multiplicative subgroup of F such that $h \in H$ implies that $-h \in H$. Note that H has

3 cosets 1H, xH, x^2H. Let us label the points of K_{16} by the elements of F. Let the lines of K_{16} be colored by a coloring induced by H. Since $h_1 - h_2 \neq 1$ for all $h_1, h_2 \in H$. It follows from Theorem 3.2.3 that there exists no 3-subset of F with all its lines colored by the same color. Thus $N(3,3,3;2) > 16$. Hence we obtain $N(3,3,3;2) = 17$.

Q.E.D.

3.5.3 Theorem $41 < N(3,3,3,3;2) \leqslant 66$.

Proof : By Theorem 3.5.1, we have $N(3,3,3,3;2) \leqslant 66$. Next, we shall show that $N(3,3,3,3;2) > 41$. Let $(F,+,\cdot)$ be the field of residue classes modulo 41, i.e. F = GF(41). Observe that $H = \{1,4,10,16,18,23,25,31,37,40\}$ is a multiplicative subgroup of F such that $h \in H$ implies that $-h \in H$. Note that H has 4 cosets 1H, 2H, 3H, 6H. Let us label the points of K_{41} by the elements of F. Let the lines of K_{41} be colored by a coloring induced by H. Since $h_1 - h_2 \neq 1$ for all $h_1, h_2 \in H$. It follows from Theorem 3.2.3 that there exists no 3-subset of F with all its lines colored by the same color. Therefore, $N(3,3,3,3;2) > 41$. Hence we obtain $41 < N(3,3,3,3;2) \leqslant 66$.

Q.E.D.