CHAPTER V

AN OPERATOR IN HARMONIC ANALYSIS

1. Harmonic Analysis on $L^2(\mathbb{T})$

For $f \in L^2(\mathbb{T})$, the Fourier series of f is given by

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi ikx}$$

where

$$c_k = \hat{f}(k) = (f, E_k) = \int_0^1 f(t)\, e^{-2\pi ikt}\, dt,$$

$$k = 0, \pm 1, \pm 2, \ldots .$$

1.1 Lemma. (Bessel's inequality)

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \leq \|f\|_2^2 = \int_0^1 |f(x)|^2\, dx$$

holds for function f in $L^2(\mathbb{T})$.

Proof. Since, for any g in $L^2(\mathbb{T})$, $(g,g) \geq 0$, we have

$$0 \leq \left(\sum_{k=-n}^{n} c_k E_k - f,\ \sum_{k=-n}^{n} c_k E_k - f\right)$$

$$= \left(\sum_{k=-n}^{n} c_k E_k, \sum_{k=-n}^{n} c_k E_k\right) - \left(\sum_{k=-n}^{n} c_k E_k, f\right)$$

$$- \left(f, \sum_{k=-n}^{n} c_k E_k\right) + (f, f)$$

$$= \sum_{k=-n}^{n} |c_k|^2 - \sum_{k=-n}^{n} |c_k|^2 - \sum_{k=-n}^{n} |c_k|^2 + \|f\|_2^2$$

That is, $\sum_{k=-n}^{n} |c_k|^2 \leq \|f\|_2^2$.

Hence $\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \leq \|f\|_2^2$ follows by letting $n \longrightarrow \infty$.

The proof is complete.

By an application of Theorem 4.3.2, we can reverse the inequality in Bessel's inequality as follow :

1.2 Lemma. If $f \in L^2(\mathbb{T})$ then

$$\|f\|_2^2 \leq \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 .$$

Proof. Since $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ and $c_k = \hat{f}(k)$, $k = 0, \pm 1, \ldots$, then the (C, 1) means of f are given by

$$\sigma_n(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) c_k e^{2\pi ikx} .$$

By Theorem 3.2.2, $\{E_n\}$ is an orthogonal set so that

$$\int_0^1 |\sigma_n(x)|^2 dx = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right)^2 |c_k|^2$$

$$\leq \sum_{k=-\infty}^{\infty} |c_k|^2 .$$

Since $\sigma_n(x) \longrightarrow f(x)$ almost everywhere, by Theorem 4.3.2, Fatou's lemma implies

$$\int_0^1 |f(x)|^2 dx \leq \varliminf_{n \to \infty} \int_0^1 |\sigma_n(x)|^2 dx \leq \sum_{k=-\infty}^{\infty} |c_k|^2 .$$

Hence

$$\|f\|_2^2 \leq \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 .$$

This completes the proof.

Together with Bessel's inequality, Lemma 1.2 gives us the following relation, known as Parseval's formula :

$$\| f \|_2^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 .$$

1.3 Theorem. (Uniqueness Theorem) If two functions f and g of $L^2(\mathbb{T})$ have the same Fourier coefficients, then they are equal almost everywhere.

Proof. Since the Fourier coefficients of function f - g are all 0 and $\{E_n\}$ is total in $L^2(\mathbb{T})$, by Theorem 3.2.4 . Then we have f(x) - g(x) = 0 almost everywhere. This completes the proof.

1.4 Theorem. (Riesz - Fisher) Suppose f belongs to $L^2(\mathbb{T})$. Then its Fourier series converges to f in the L^2-norm; that is,

$$\| f - S_n \|_2 = \left(\int_0^1 |f(x) - S_n(x)|^2 \, dx \right)^{1/2}$$

$$= \left(\int_0^1 \left| f(x) - \sum_{k=-n}^{n} \hat{f}(k) \, e^{2\pi ikx} \right|^2 dx \right)^{1/2}$$

tends to 0 as n tends to ∞ . Furthermore,

$$\| f \|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2} = \left(\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \right)^{1/2} = \| \hat{f} \|_2 .$$

If a sequence $\{c_k\}$ satisfies $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$, then there exists a function f in $L^2(\mathbb{T})$ such that $c_k = \hat{f}(k)$ for all integer k.

<u>Proof.</u> Consider

$$\lim_{n\to\infty}\left\|\sum_{k=-n}^{n} c_kE_k - f\right\|_2^2 = \lim_{n\to\infty}\left(\sum_{k=-n}^{n} c_kE_k - f, \sum_{k=-n}^{n} c_kE_k - f\right)$$

$$= \lim_{n\to\infty}\left(\|f\|_2^2 - \sum_{k=-n}^{n}|c_k|^2\right) = 0$$

where $c_k = (f, E_k)$ is the k-th Fourier coefficient of f, $k = 0, \pm1, \pm2, \ldots$

Thus the partial sums $S_n = \sum_{k=-n}^{n} c_kE_k = \sum_{k=-n}^{n} \hat{f}(k)\,E_k$ converges in the L^2-norm to f in $L^2(\mathbb{T})$ with $\|f\|_2 = \|\hat{f}\|_2$.

Conversely, if a sequence $\{c_k\}$ satisfies $\sum_{k=1}^{\infty}|c_k|^2 < \infty$, then we shall prove that $S_n = \sum_{k=-n}^{n} c_kE_k$ converges in the L^2- norm. The only thing is to see the $\{S_n\}$ is a Cauchy sequence for the L^2- norm. If $m \geq n$, we get

$$\|S_m - S_n\|_2^2 = \sum_{n<|k|\leq m}|c_k|^2$$

and the expression on the right is the Cauchy remainder of a convergent series. Let f be a limit of $\{S_n\}$, in the L^2-norm. We shall show that $(f, E_m) = c_m$, for all integers m. For any integer m, we have

$$(f, E_m) = (\lim_{n\to\infty} S_n, E_m)$$

$$= \lim_{n\to\infty}(S_n, E_m)$$

$$= \lim_{n\to\infty}\left(\sum_{k=-n}^{n} c_kE_k, E_m\right)$$

$$= \lim_{n\to\infty}\left(\sum_{k=-n}^{n} c_k(E_k, E_m)\right) = c_m.$$

The uniqueness of f, as a function in $L^2(\Phi)$, is a consequence of Theorem 1.3.

The proof is complete.

2. Fatou's Theorem

2.1 Theorem. (Fatou's Theorem) If F is a bounded analytic function in the interior of the unit circle then the radial limits

$$\lim_{r\to 1} F(re^{2\pi i\theta})$$

exist for almost all θ in $[0, 1)$.

Proof. Suppose $F(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$ is an analytic function in the interior of the unit circle. Suppose, further, that F is bounded in this domain; say, $|F(z)| \leq B < \infty$ for $|z| < 1$. Let us write $z = re^{2\pi i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 1$. Using the orthogonality relations of $\{E_k\}$,

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} = \int_0^1 \Big(\sum_{k=0}^{\infty} a_k r^k e^{2\pi ik\theta}\Big)\Big(\sum_{k=0}^{\infty} \bar{a}_k r^k e^{-2\pi ik\theta}\Big)\, d\theta$$

$$= \int_0^1 |F(re^{2\pi i\theta})|^2\, d\theta \leq B^2 \quad \text{for } 0 \leq r < 1.$$

Letting $r \to 1$ we therefore obtain $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$. By Theorem 1.4 we can thus conclude that there exists an f belonging to $L^2(\Phi)$ such that $\hat{f}(k) = a_k$ for $k = 0,1,2,\dots$ and $\hat{f}(k) = 0$ for all negative integers k. This shows that

$$F(re^{2\pi i\theta}) = \sum_{k=0}^{\infty} \hat{f}(k)\, r^k e^{2\pi ik\theta} = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, r^k e^{2\pi ik\theta}, \quad 0 \leq r < 1,$$

are the Abel means of the Fourier series of f. By Theorem 4.3.2, therefore, $\lim_{r \to 1} F(re^{2\pi i\theta}) = f(\theta)$ for almost every θ. This completes the proof.

We shall use Theorem 2.1 (Fatou's Theorem) to define an important operator, the conjugate function operator, defined on integrable and 1-periodic functions. Suppose f is such a function. It follows from our discussion concerning the Poisson kernel and the conjugate Poisson kernel that the function G defined by

$$G(z) = \int_0^1 \frac{1 + re^{2\pi i(\theta - t)}}{1 - re^{2\pi i(\theta - t)}} f(t)\, dt$$

$$= \int_0^1 P(r, \theta - t) f(t)\, dt + i \int_0^1 Q(r, \theta - t) f(t)\, dt,$$

$z = re^{2\pi i\theta}$, is an analytic in the interior of the unit circle. We already know that the first expression in the last sum has radial limits, as $r \to 1$, for almost all θ. The following theorem asserts that this is also true for the second term.

2.2 <u>Theorem</u>. Suppose $f \in L^1(0, 1)$; then the limits, $\tilde{f}(\theta)$, as $r \to 1$, of

$$\tilde{A}(r, \theta) = \int_0^1 Q(r, \theta - t) f(t)\, dt = \int_0^1 \frac{2r \sin 2\pi(\theta - t)}{1 - 2r\cos 2\pi(\theta - t) + r^2} f(t)\, dt$$

exist for almost all θ. The function $\tilde{f}$ is called the <u>conjugate function of f</u>.

<u>Proof</u>. By decomposing f into its real and imaginary parts and considering separately the positive and negative parts of each

of these, we see that it suffices to prove the theorem for $f \geq 0$. Thus, letting $A(r,\theta)$ be the Poisson integral and $\tilde{A}(r, \theta)$ the conjugate Poisson integral of f, we obtain an analytic function for $|z| < 1$, $z = re^{2\pi i\theta}$, whose values lie in the right half-plane (by property (B') of the Poisson kernel in Chapter IV. Sec.4). Thus,

$$F(z) = e^{-A(r,\theta) - i\tilde{A}(r,\theta)}$$

is a bounded ($|F(z)| \leq 1$) analytic function in the interior of the unit circle. By Theorem 2.1, the radial limits of F exist almost everywhere. Since the radial limits of $A(r, \theta)$ also exist almost everywhere and are finite (they equal to $f(\theta)$), the limits of F must be nonzero almost everywhere. But this implies the existence of $\lim_{r \to 1} \tilde{A}(r, \theta)$ for almost all θ, and the theorem is proved.