CHAPTER VII

LOCALLY CYCLIC DECOMPOSABLE ABELIAN GROUPS

The materials of this chapter are drawn from reference [2].

The problem as to which abelian groups are locally cyclic decomposable is completely settled in this chapter with the help of the main theorem of the preceeding chapter.

7.1 <u>Lemma</u>. Let G be an abelian group and let a be a non-zero element of G which is of infinite order. Then the set

$$\langle a \rangle = \{ x \in G \,/\, mx \in [a] \quad \text{for some integer} \quad m \neq 0 \}$$

is a torsion-free maximal locally cyclic subgroup of G.

<u>Proof</u> : As in the proof of the theorem 5.1, the set $\langle a \rangle$ is easily seen to be a torsion-free abelian group, and is isomorphic to a subgroup of the additive group of the rationals. Hence $\langle a \rangle$ is locally cyclic by Theorem 3.8.

If there is an x in G such that x and a belong to the same cyclic subgroup, then x is in $\langle a \rangle$ so that $\langle a \rangle$ can not be contained in any other proper locally cyclic subgroup of G. Hence $\langle a \rangle$ is a maximal locally cyclic subgroup of G.

7.2 <u>Theorem</u>. An abelian group G is locally cyclic decomposable if and only if its torsion subgroup tG is.

<u>Proof</u> : Since subgroups of a locally cyclic decomposable group is locally cyclic decomposable by Theorem 4.4 we only need to prove one implication.

Suppose tG is locally cyclic decomposable and let $\{G_k / k \in K\}$ be the locally cyclic decomposition of tG. It then follows easily from Lemma 7.1 that the set

$$\{G_k / k \in K\} \cup \{\langle a \rangle / a \in G \text{ is of infinite order}\},$$

where $\langle a \rangle$ is defined as in Lemma 7.1, is the required locally cyclic decomposition of G.

7.3 <u>Theorem</u>. An abelian group G is locally cyclic decomposable if and only if either

(a) its torsion subgroup tG is locally cyclic,

or (b) there exists a prime p such that tG is the set-theoretical union of a disjoint family of p-cocyclic subgroups of G.

<u>Proof</u> : This follows immediately from Theorem 7.2 with the aid of Theorem 6.4 .