CHAPTER IV

LOCALLY CYCLIC DECOMPOSABLE GROUPS

The materials of this chapter are drawn from references [4] , [8] .

This chapter contains some preliminaries about locally cyclic decomposable group. But the main theorem of this chapter is the fact that any group has at most one non-trivial locally cyclic decomposition, and that if such a decomposition exists, it coincides with the set of all maximal locally cyclic subgroups of the given group.

4.1 Definition. A group is locally cyclic decomposable if it has a locally cyclic decomposition, that is, a family $\{G_k / k \in K\}$ of subgroups of G such that

i. $G = \bigcup_{k \in K} G_k$;

ii. if $G_j \neq G_k$ implies $G_j \cap G_k = \{1\}$,

where 1 denotes the identity element of G and

iii. each G_k is locally cyclic.

4.2 Definition. A subgroup of a given group is a maximal locally cyclic subgroup if it is locally cyclic and is not properly contained in any other locally cyclic subgroup of the given group.

4.3 Theorem. A group G is locally cyclic decomposable if and only if every two elements a_1, a_2 of G such that $[a_1]\cap[a_2] \neq \{1\}$, there exists a third element a_3 of G such that $a_1, a_2 \in [a_3]$.

Proof: Sufficiency : Define a relation '$\equiv$' in $G \setminus \{1\}$ as follows : For each $a_1, a_2 \in G \setminus \{1\}$, we put $a_1 \equiv a_2$ if and only if there exists $a_3 \in G$ such that $[a_1], [a_2] \subset [a_3]$. Clearly $\equiv$ is both reflexive and symmetric. To show that it is transitive, suppose that $a_1 \equiv a_2$, $a_2 \equiv a_3$. Then there exist b_1, b_2 of G such that $[a_1], [a_2] \subset [b_1]$, $[a_2], [a_3] \subset [b_2]$. Since $[b_1], [b_2] \supset [a_2]$ we see that $[b_1] \cap [b_2] \neq \{1\}$ hence by the assumption of the theorem there exists $b_3 \in G$ such that $[b_1], [b_2] \subset [b_3]$. Therefore $[a_1], [a_3] \subset [b_3]$. This means that $a_1 \equiv a_3$. Hence $\equiv$ is an equivalence relation and decomposes $G \setminus \{1\}$ into a collection $\{G_\delta\}_{\delta \in \Delta}$ of equivalence classes. For each $\delta \in \Delta$, we put $X_\delta = G_\delta \cup \{1\}$. We want to show that X_δ is locally cyclic for all $\delta \in \Delta$.

Let $a_1, a_2 \in X_\delta \setminus \{1\}$, we have $[a_1], [a_2] \subset [a_3]$ for some $a_3 \in G$. Let $a \in [a_3]$, where $a \neq 1$. Then $[a_1], [a] \subset [a_3]$ implies that $a \equiv a_\delta$. Consequently $a \in G_\delta$. Therefore $[a_3] \subset X_\delta$. Hence X_δ is locally cyclic and G is locally cyclic decomposable.

Necessity : Suppose G has a locally cyclic decomposition $\{G_k\}_{k \in K}$.

Let $a_1, a_2 \in G$ such that $[a_1] \cap [a_2] \neq \{1\}$. Suppose that $a_1 \in G_{k_1}$, $a_2 \in G_{k_2}$ such that $k_1 \neq k_2$. Then $G_{k_1} \cap G_{k_2} \neq \{1\}$, which is a contradiction. Hence $a_1, a_2 \in G_{k_0}$ for some $k_0 \in K$. Since G_{k_0} is locally cyclic, there exists $a_3 \in G_{k_0} \subset G$ such that $a_1, a_2 \in [a_3]$. This proves the necessity.

4.4 <u>Theorem</u>. Every subgroup of a locally cyclic decomposable group is locally cyclic decomposable.

<u>Proof</u> : Let G' be any subgroup of a locally cyclic decomposable group G. For each $a_1, a_2 \in G'$ such that $[a_1] \cap [a_2] \neq \{1\}$, there exists $a_3 \in G$ such that $a_1, a_2 \in [a_3]$, by Theorem 4.3 . Hence $a_1, a_2 \in [a_3] \cap G'$. Since $[a_3] \cap G'$ is a subgroup of the cyclic group $[a_3]$, it follows that $[a_3] \cap G'$ is also cyclic. Again by Theorem 4.3 , G' is locally cyclic decomposable.

To prove the main theorem, we need two more lemmas.

4.5 <u>Lemma</u>. No locally cyclic group can have a non - trivial locally cyclic decomposition whose members are proper subgroups, where a locally cyclic decomposition $\{G_k\}$ is said to be non - trivial if each $G_k \neq \{1\}$.

Proof: Let G be a locally cyclic group. We may assume that $G \neq \{1\}$, if not, G will not have any proper non - trivial subgroup and we will have nothing to prove. Suppose that $\{G_k / k \in K\}$ is a locally cyclic decomposition of G where $G_k \subsetneq G$.

Let $a \in G_i$, $b \in G_j$, $i \neq j$ such that $a \neq 1$ and $b \neq 1$. Since G is locally cyclic so a, b $\in [c]$ for some $c \in G$. Thus $c \in G_k$ for some $k \in K$. Since $G_i \cap G_k \supset [a] \cap [c] = [a] \neq \{1\}$, then $G_i = G_k$. But then $G_i \cap G_j = G_k \cap G_j \supset [c] \cap [b] = [b] \neq \{1\}$ so that $G_i = G_j$, contradicting the choice of G_i and G_j.

Hence the Lemma is proved.

4.6 Lemma. If L is a locally cyclic subgroup of a group G, then there exists a maximal locally cyclic subgroup of G that contains L.

Proof : Let $\mathcal{M} = \{M \subseteq G / L \subseteq M$ and M is locally cyclic$\}$. We partially order $\mathcal{M}$ by inclusion ; i.e., $M_1, M_2 \in \mathcal{M}$, $M_1 \leq M_2$ if and only if $M_1 \subseteq M_2$; and let $\mathcal{C} \subset \mathcal{M}$ be a chain. In view of 2.2 , we can prove that $\mathcal{M}$ has a maximal element by showing that $\mathcal{C}$ has an upper bound.

Consider $\cup \mathcal{C}$, it is clear that $L \subseteq \cup \mathcal{C}$; moreover

$\bigcup\mathcal{C}$ is an ascending union, hence by Lemma 3.3, we know that $\bigcup\mathcal{C}$ is locally cyclic. Therefore $\bigcup\mathcal{C}$ is a locally cyclic subgroup of G that contains L, then $\bigcup\mathcal{C} \in \mathcal{M}$. Hence $\bigcup\mathcal{C}$ is an upper bound of $\mathcal{C}$. Then $\mathcal{M}$ has a maximal element, this proves the lemma.

Now we come to the uniqueness theorem for locally cyclic decomposable group.

4.7 Theorem. If a group G is locally cyclic decomposable, then it has exactly one non - trivial locally cyclic decomposition $\{G_k / k \in K\}$ which coincides with the collection of all its maximal locally cyclic subgroups.

Proof : Let G be a locally cyclic decomposable group. Since the case for which $G = \{1\}$ is trivial, we assume that $G \neq \{1\}$. Furthermore, we may assume that G is not locally cyclic, for otherwise it follows from Lemma 4.5 that G can not have a non - trivial locally cyclic decomposition whose members are proper subgroups of G, that is, $\{G\}$ is the only possible such locally cyclic decomposition.

Let $\{G_k / k \in K\}$ be a non - trivial locally cyclic decomposition of G. We first show that each G_k is a maximal locally cyclic subgroup of G. Suppose there is a G_{k_o} which is not a maximal locally cyclic subgroup of G. Let M be a maximal locally cyclic subgroup of G such that $\{1\} \subsetneq G_{k_o} \subsetneq M$, such M exists by Lemma 4.6. Let $a \in G_{k_o} \setminus \{1\}$ and $b \in M \setminus G_{k_o}$. Then $b \in G_m$ for some $m \in K$. Since both a and b belong to M, $a, b \in [c]$ for some $c \in G$ and there is an $n \in K$ such that $c \in G_n$. But $G_n \cap G_{k_o} \supset [c] \cap [a] = [a] \neq \{1\}$ so that $G_n = G_{k_o}$. Moreover, $G_m \cap G_{k_o} = G_m \cap G_n \supset [b] \cap [c] = [b] \neq \{1\}$ so that $G_m = G_{k_o}$. Hence $b \in G_{k_o}$ which is a contradiction. Thus G_{k_o} must be a maximal locally cyclic subgroup. Hence each G_k is a maximal locally cyclic subgroup of G.

We are left to show that each maximal locally cyclic subgroup of G is one of the G_k. Suppose to the contrary that there is a maximal locally cyclic subgroup M of G such that $M \neq G_k$ for all $k \in K$. Then $M \neq \{1\}$ and can not be contained in any G_j for any $j \in K$ so that we can find a pair $j, k \in K$ such that $M \cap G_j \setminus \{1\} \neq \emptyset$ and that $M \cap G_k \setminus G_j \neq \emptyset$. Let $a \in M \cap G_j \setminus \{1\}$ and $b \in M \cap G_k \setminus G_j$. Since $a, b \in M$, $a, b \in [c]$ for some $c \in G_m$ and for some $m \in K$. Then

$$G_j \cap G_m \supset [a] \cap [c] \supset [a] \neq \{1\}$$

so that $G_j = G_m$. Similarly $G_k = G_m$. Thus $G_j = G_m = G_k$ which is a contradiction since $b \in G_k \setminus G_j$. Hence every maximal locally cyclic subgroup of G is one of the G_k.

The theorem is now completely proved.