CHAPTER II

GEODESIC DIFFERENTIAL EQUATION

From differential geometry we know that geodesic curves satisfy the following type of differential equation

$$(1) \qquad \frac{d^2\psi^i}{dt^2} = \sum_{j=1}^{n} \sum_{k=1}^{n} G^i_{jk}(\vec{\psi})\frac{d\psi^j}{dt}\frac{d\psi^k}{dt}$$ where G^i_{jk} is c^1 function

on an open subset D of R^n for all $i = 1,2,\ldots,n$; as a result, we call this differential equation the geodesic differential equation.

Notation

2-1.1 $\vec{\psi}$ is an n-vector valued function of real-valued functions

$\psi^1(t),\ldots,\psi^n(t)$ denoted by $\vec{\psi}(t) = (\psi^1(t),\ldots,\psi^n(t))$.

2-1.2 $\frac{d\psi^i}{dt} = \dot{\psi}^i$, $\frac{d^2\psi^i}{dt^2} = \ddot{\psi}^i$ for $1 \leq i \leq n$

2-1.3 $G^i_{j_1\ldots j_k}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_k} = \sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n} G^i_{j_1\ldots j_k}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_k}$

where for each $i = 1,2,\ldots,n$; for each $k = 1,2,3,\ldots$

$G^i_{j_1\ldots j_k}$ is defined on some subset of R^n.

Many of the geometric properties of geodesics come from the fact that solutions to differential equations of the form (1) satisfy a certain functional equation given below. For each $1 \leq i \leq n$,

$\psi^i(\vec{P},\alpha\vec{V},t)$ exists iff $\psi^i(\vec{P},\vec{V},\alpha t)$ exists and

$$\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t). \tag{2}$$

We will prove (2) in chapter III page 58

Let $H^i(\vec{\psi},\mathring{\vec{\psi}},t) = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k$ where G^i_{jk} is analytic on D for $1 \leq i,j, k \leq n$.

Then H^i is defined on $D \times R^n \times R$ and H^i is an analytic function on $D \times R^n \times R$.

2-2 Properties of the solution curves to the geodesic differential equation

[2-2.1] Given any initial point $\vec{P}_0$ in D, any t_0 in R, then there exists a neighbourhood U of the zero vector at $\vec{P}_0$ such that $\vec{\psi}(\vec{P}_0,\vec{V},t_0)$ is defined for all $\vec{V} \in U$.

Proof Let t_0 be any element in R. Let $\vec{P}_0$ be any initial point in D. Since $(\vec{P}_0,\vec{0},0)$ is a point on the domain of definition of $\vec{H}$, hence the fundamental theorem for second order ordinary differential equations implies that there exists a neighbourhood W of the zero vector at $\vec{P}_0$, and there exists an interval $I = (-r,r)$ such that $\vec{\psi}(\vec{P}_0,\vec{V},t)$ exists for all $\vec{V} \in W$ and for all $t \in I$.

For any $t_0 \varepsilon R$, we then choose a real number $\alpha \neq 0$ such that $|\alpha t_0| < r$. Hence αt_0 is in I. By the above statement, $\vec{\psi}(\vec{P}_0, \vec{V}, \alpha t_0)$ exists for all $\vec{V} \varepsilon W$. Since for $1 \leq i \leq n$, the solution curve ψ^i satisfies the functional equation (2). Therefore, $\vec{\psi}(\vec{P}_0, \alpha\vec{V}, t_0)$ exists for all $\vec{V} \varepsilon W$.

That is, $\vec{\psi}(\vec{P}_0, \vec{W}, t_0)$ exists for all $\vec{W} = \alpha\vec{V} \varepsilon \alpha W$.

But we know that W is a neighbourhood of zero vector at $\vec{P}_0$, hence αW is a neighbourhood of zero vector at $\vec{P}_0$. Let us denote αW by U and $\vec{W}$ by $\vec{V}$. Then we are done.

[2-2.2] Given any initial point $\vec{P}_0$ in D, and initial vector $\vec{V}_0$ at $\vec{P}_0$, then the paths $\vec{\psi}(\vec{P}_0, \vec{V}_0, t)$ and $\vec{\psi}(\vec{P}_0, \alpha\vec{V}_0, t)$ agree as point sets for all $\alpha \varepsilon R - \{0\}$. That is the images of the two functions are the same sets.

<u>Proof</u> Fix $\alpha_0 \varepsilon R - \{0\}$.

Let $c_1 = \{\vec{\psi}(\vec{P}_0, \alpha_0\vec{V}_0, t) \mid \vec{\psi}(\vec{P}_0, \alpha_0\vec{V}_0, t) \text{ is defined}\}$.

Let $c_2 = \{\vec{\psi}(\vec{P}_0, \vec{V}_0, t) \mid \vec{\psi}(\vec{P}_0, \vec{V}_0, t) \text{ is defined}\}$.

In order to show $c_1 = c_2$, let Q_1 be any point on c_1.
Hence there exists $t_1 \varepsilon R$ such that $\vec{\psi}(\vec{P}_0, \alpha_0\vec{V}_0, t_1)$ is defined and

$$\vec{\psi}(\vec{P}_0, \alpha_0\vec{V}_0, t_1) = Q_1$$

Since the solution $\vec{\psi}$ of (1) satisfies the functional equation (2), hence $\vec{\psi}(\vec{P}_0, \vec{V}_0, \alpha_0 t_1)$ is defined and $\vec{\psi}(\vec{P}_0, \vec{V}_0, \alpha_0 t_1) = \vec{\psi}(\vec{P}_0, \alpha_0\vec{V}_0, t_1)$.
Thus there exists $t_2 = \alpha_0 t_1$ such that $\vec{\psi}(\vec{P}_0, \vec{V}_0, t_2)$ is defined and

$$\vec{\psi}(\vec{P}_0, \vec{V}_0, t_2) = Q_1 \text{ i.e. } Q_1 \varepsilon c_2$$

Let Q_2 be any point on c_2. Then there exists $t_1 \varepsilon R$ such that

$$\vec{\psi}(\vec{P}_0,\vec{V}_0,t_1) = Q_2 .$$

Then $\vec{\psi}(\vec{P}_0,\vec{V}_0,\alpha_0\frac{t_1}{\alpha_0})$ is defined.

Hence $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0, \frac{t_1}{\alpha_0})$ is defined and $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0, \frac{t_1}{\alpha_0}) = \vec{\psi}(\vec{P}_0,\vec{V}_0,t_1)$

Therefore there exists $t_2 = \frac{t_1}{\alpha_0}$ in R such that $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t_2)$ is defined and $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t_2) = Q_2$ i.e. $Q_2 \varepsilon c_1$.

Thus $c_1 = c_2$. Then the proof is complete.

Definition 2-2.3 Let $\vec{f}$ be a vector-valued function on an open subset U of R^n into an open subset V of R^n. Then $\vec{f}$ is said to be a bidifferential map if $\vec{f}$ is one to one from U onto V, and also $\vec{f}$ and $\vec{f}^{-1}$ are differentiable

[2-2.4] For any $t_0 \varepsilon R - \{0\}$, for any $\vec{P}_0$ in D, there exists a neighbourhood U of the zero vector at $\vec{P}_0$ such that $\vec{V} \to \vec{\psi}(\vec{P}_0,\vec{V},t_0)$ is a bidifferential map of U onto an open set .

Proof Let t_0 be any element in $R - \{0\}$.
Let $\vec{P}_0$ be any point in D.
By property [2-2.1], there exists a neighbourhood U of the zero vector at $\vec{P}_0$ such that for all $\vec{V} \varepsilon U$, $\vec{\psi}(\vec{P}_0,\vec{V},t_0)$ exists.
Let $\vec{f}$ be the map defined by $\vec{f}(\vec{V}) = \vec{\psi}(\vec{P}_0,\vec{V},t_0)$ where $\vec{V} \varepsilon U$
We will prove that $\vec{f}$ is a bidifferential map.
By theorem 1-1.11, $\vec{f}$ is a c^1 function, thus it is enough to prove that the jacobian of $\vec{f}$ at the zero vector is not zero because of the inverse function theorem.

Since for each $i = 1,2,\ldots,n$, ψ^i satisfies the functional equation

(2) $$\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t).$$

Differentiate (2) with respect to α, we obtain

$$v^j \frac{\partial\psi^i}{\partial v^j}(\vec{P},\alpha\vec{V},t) = \mathring{\psi}^i(\vec{P},\vec{V},\alpha t)t \qquad \text{for } 1 \le j \le n$$

Substitute $\alpha = 0$ into the above equation, we get

$$v^j \frac{\partial\psi^i}{\partial v^j}(\vec{P},\vec{0},t) = \mathring{\psi}^i(\vec{P},\vec{V},0)t = v^i t$$

Let $$\delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Hence for each $i = 1,2,\ldots, n$ we get

(3) $$\frac{\partial\psi^i}{\partial v^j}(\vec{P},\vec{0},t) = t\,\delta^i_j\,,\ i,\ j = 1,2,\ldots,n$$

Since

$$J_{\vec{f}}(\vec{0}) = \det \begin{pmatrix} \frac{\partial\psi^1}{\partial v^1}(\vec{P}_0,\vec{0},t_0) & \frac{\partial\psi^1}{\partial v^2}(\vec{P}_0,\vec{0},t_0) & \ldots & \frac{\partial\psi^1}{\partial v^n}(\vec{P}_0,\vec{0},t_0) \\ \frac{\partial\psi^2}{\partial v^1}(\vec{P}_0,\vec{0},t_0) & \frac{\partial\psi^2}{\partial v^2}(\vec{P}_0,\vec{0},t_0) & \ldots & \frac{\partial\psi^2}{\partial v^n}(\vec{P}_0,\vec{0},t_0) \\ \vdots & & & \\ \frac{\partial\psi^n}{\partial v^1}(\vec{P}_0,\vec{0},t_0) & \frac{\partial\psi^n}{\partial v^2}(\vec{P}_0,\vec{0},t_0) & \ldots & \frac{\partial\psi^n}{\partial v^n}(\vec{P}_0,\vec{0},t_0) \end{pmatrix}$$

Substitute (3) into the above equation, we get

$$J_{\vec{f}}(\vec{0}) = \det \begin{bmatrix} t_0 & & 0 \\ & \ddots & \\ 0 & & t_0 \end{bmatrix} = t_0^n \neq 0$$

Then the proof is complete.

In differential geometry property [2-2.4] is called the exponential property.

[2-2.5] For any compact neighbourhood U of the zero vector at $\vec{P}_0$ in D, there exists a neighbourhood V of zero in R such that $\vec{\psi}(\vec{P}_0,\vec{V},t)$ exists for all $\vec{V} \in U$ and for all $t \in V$.

<u>Proof</u> It is enough to show that for any open ball $B(\vec{0}_{\vec{P}_0}, r_1)$ there exists an interval V of zero in R such that $\vec{\psi}(\vec{P}_0,\vec{V},t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t \in V$ since U is a compact neighbourhood of $\vec{0}_{\vec{P}_0}$ in R^n, hence U is a closed and bounded subset of R^n (by Heine-Borel Theorem). Therefore, there exists an open ball $B(\vec{0}_{\vec{P}_0}, r)$ which contains U.

By the fundamental theorem for 2^{nd} order ordinary differential equations, there exists an open ball $B(\vec{0}_{\vec{P}_0}, r_2)$ and an interval I of zero in R such that $\vec{\psi}(\vec{P}_0,\vec{V},t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_2)$ and for all $t \in I$.

Let $B(\vec{0}_{\vec{P}_0}, r_1)$ be any open ball center at $\vec{0}_{\vec{P}_0}$, radius r_1. If $B(\vec{0}_{\vec{P}_0}, r_1) \subseteq B(\vec{0}_{\vec{P}_0}, r_2)$, then it is clear from the above statement that there exists an interval V of zero in R such that $\vec{\psi}(\vec{P}_0,\vec{V},t)$ is defined for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t \in V$, just take $V = I$.

If $B(\vec{0}_{\vec{P}_0}, r_2) \subset B(\vec{0}_{\vec{P}_0}, r_1)$, then choose $\alpha_0 \in R - \{0\}$ such that $|\alpha_0 r_1| < r_2$

For any $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$, we have $|\alpha_0\vec{V}| = |\alpha_0||\vec{V}| < |\alpha_0 r_1| < r_2$.

Therefore, $\alpha_0\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_2)$ for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$.

Hence $\vec{\psi}(\vec{P}_0, \alpha_0\vec{V}, t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t \in I$.

Since $\vec{\psi}$ satisfies the functional equation (2), hence $\vec{\psi}(\vec{P}_0, \vec{V}, \alpha_0 t)$ exists and $\vec{\psi}(\vec{P}_0, \alpha_0\vec{V}, t) = \vec{\psi}(\vec{P}_0, \vec{V}, \alpha_0 t)$.

Thus $\vec{\psi}(\vec{P}_0, \vec{V}, \alpha_0 t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t \in I$.

Let $V = \alpha_0 I$. Then V is an open interval of zero in R.

Therefore, we conclude that there exists a neighbourhood V of 0 in R such that $\vec{\psi}(\vec{P}_0, \vec{V}, t^*)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t^* \in V$. Then the proof is complete.