CHAPTER IV

GENERALIZED SKEWNESS AND CHROMATIC NUMBERS OF GRAPHS

In this chapter we will generalize the concept of skewness of graphs and prove theorems dealing with chromatic numbers and our generalized skewness.

<u>Definition</u> 4.1 The <u>t-skewness</u> of a graph G, denoted by $\mu_t(G)$, is the minimum number of edges whose removal makes G embeddable into a surface S_t.

<u>Remark</u> 4.2 If $G' = G - v$, where v is a vertex of G and G' is a subgraph of G, then $\mu_t(G') \leq \mu_t(G)$.

<u>Theorem</u> 4.3 Let G be a connected graph. For any non-negative integer k and any positive integer t, if $\mu_t(G) < \binom{k+t+1}{2} - 6t$, then $\chi(G) \leq k + t + 3$.

<u>Proof</u>. Let $k \geq 0$ and $t > 0$ be given. Let G be a connected graph with n vertices and m edges satisfying $\mu_t(G) < \binom{k+t+1}{2} - 6t$. We want to show that $\chi(G) \leq k + t + 3$.

$$\text{Define} \quad \alpha(x) = \frac{7 + \sqrt{1 + 8x + 48t}}{2},$$

$$H(x) = [\alpha(x)],$$

where [y] denotes the greatest integer less than or equal to y.

Observe that α is strictly increasing. So

$$\alpha(\mu_t(G)) < \alpha\left(\binom{k+t+1}{2} - 6t\right)$$

$$= \alpha\left(\frac{(k+t+1)(k+t)}{2} - 6t\right)$$

$$= \frac{7 + \sqrt{1 + 8\left(\frac{(k+t+1)(k+t)}{2} - 6t\right) + 48t}}{2}$$

$$= \frac{7 + \sqrt{1 + 4(k+t+1)(k+t) - 48t + 48t}}{2}$$

$$= \frac{7 + \sqrt{1 + 4(k+t)^2 + 4(k+t)}}{2}$$

$$= \frac{7 + (1 + 2(k+t))}{2}$$

$$= \frac{2(k+t) + 8}{2}$$

$$= k + t + 4 .$$

By definition of H, we see that $H(\mu_t(G)) \leq \alpha(\mu_t(G))$. Hence we have

$$H(\mu_t(G)) < k + t + 4 .$$

Hence

(1) $$H(\mu_t(G)) \leq k + t + 3 .$$

We shall complete our proof by showing that

$$\chi(G) \leq H(\mu_t(G)).$$

Observe that $\chi(G) \leq n$, hence if $n \leq H(\mu_t(G))$, then

$\chi(G) \leq H(\mu_t(G))$. So it remains to consider the case $n > H(\mu_t(G))$.

Let d be the average degree of vertices of G. By Theorem 3.1.1 and Corollary 3.3.3 , we have

$$\begin{aligned} dn &= 2m \\ &= 2(m - \mu_t(G)) + 2\mu_t(G) \\ &\leq 2(3(n - (2 - 2t)) + 2\mu_t(G) \\ &= 6n - 6(2 - 2t) + 2\mu_t(G) \\ &= 6n + 2\mu_t(G) + 12(t - 1). \end{aligned}$$

Hence

$$(2) \qquad d \leq 6 + \frac{2}{n}(\mu_t(G) + 6(t - 1)).$$

Since $n > H(\mu_t(G))$, then

$$(3) \qquad n > \alpha(\mu_t(G)).$$

From (2) and (3) , we have

$$\begin{aligned} (4) \qquad d &\leq 6 + \frac{2}{\alpha(\mu_t(G))}(\mu_t(G) + 6(t - 1)) \\ &= 6 + \frac{2}{\frac{7 + \sqrt{1 + 8\mu_t(G) + 48t}}{2}}(\mu_t(G) + 6(t - 1)) \\ &= 6 + \frac{4(7 - \sqrt{1 + 8\mu_t(G) + 48t})}{49 - (1 + 8\mu_t(G) + 48t)}(\mu_t(G) + 6(t - 1)) \\ &= 6 + \frac{4(7 - \sqrt{1 + 8\mu_t(G) + 48t})}{48 - 8\mu_t(G) - 48t}(\mu_t(G) + 6(t - 1)) \\ &= 6 - \frac{4(7 - \sqrt{1 + 8\mu_t(G) + 48t})}{8(\mu_t(G) + 6(t - 1))}(\mu_t(G) + 6(t - 1)) \end{aligned}$$

$$= 6 - \frac{7 - \sqrt{1 + 8\mu_t(G) + 48t}}{2}$$

$$= \frac{5 + \sqrt{1 + 8\mu_t(G) + 48t}}{2}$$

$$= \frac{7 + \sqrt{1 + 8\mu_t(G) + 48t}}{2} - 1$$

$$= \alpha(\mu_t(G)) - 1 \ .$$

Note that

$$[\alpha(\mu_t(G))] \leq \alpha(\mu_t(G)) < [\alpha(\mu_t(G))] + 1 \ .$$

Hence

$$(5) \qquad \alpha(\mu_t(G)) - 1 < [\alpha(\mu_t(G))] = H(\mu_t(G)) \ .$$

From (4) and (5), we have

$$d < H(\mu_t(G)) \ .$$

Thus G must contain a vertex v of degree $\leq H(\mu_t(G)) - 1$.

Let $G' = G - v$, then G' has $n' = n - 1$ vertices.

If $n' = H(\mu_t(G))$, then G' can be colored by at most $H(\mu_t(G))$ colors. Since $\deg v \leq H(\mu_t(G)) - 1$, at least 1 color among these $H(\mu_t(G))$ colors is not used to color the vertices adjacent to v. By assigning one of the remaining colors to v, we obtain a coloring of G in which at most $H(\mu_t(G))$ colors are used. That is we have $\chi(G) \leq H(\mu_t(G))$.

If $n' > H(\mu_t(G))$, then $n' > \alpha(\mu_t(G))$. Let d' be the average degree of vertices of G', then

$$d' \leq 6 + \frac{2}{n'}(\mu_t(G') + 6(t-1)).$$

By Remark 4.2, we have $\mu_t(G') \leq \mu_t(G)$. Hence we have

$$d' \leq 6 + \frac{2}{n'}(\mu_t(G) + 6(t-1)).$$

By the same arguments as before, we have

$$d' < H(\mu_t(G)).$$

Thus G' must contain a vertex v' of degree $\leq H(\mu_t(G)) - 1$.

Let $G'' = G' - v'$, then G'' has $n'' = n' - 1$ vertices.

If $n'' = H(\mu_t(G))$, then G'' can be colored by $H(\mu_t(G))$ colors.

If $n'' > H(\mu_t(G))$, we may repeat the same arguments in a finite number of steps, and finally get a graph $G^{(r)}$ with $n^{(r)} = H(\mu_t(G))$ vertices. Hence $G^{(r)}$ can be colored by at most $H(\mu_t(G))$ colors. This coloring induces the coloring of $G^{(r-1)}$, $G^{(r-2)}$, ..., G by the same number of colors. Thus G can be colored by at most $H(\mu_t(G))$ colors. That is, we have $\chi(G) \leq H(\mu_t(G))$.

Hence by (1), we have

$$\chi(G) \leq k + t + 3 .$$

<u>Q.E.D.</u>

<u>Lemma</u> 4.4 If there exists a maximal S_t-embeddable graph H with $m(t)$ vertices, then for any $n > m(t)$, there exists a maximal S_t-embeddable graph with n vertices.

<u>Proof</u>. Let H be a maximal S_t-embeddable graph with $m(t)$ vertices. By introducing a new vertex in any triangle (region) of the embedding of H, and joining this new vertex to the vertices of this triangle.

We obtain a new graph H' with $m(t) + 1$ vertices. By continuing the process, we can obtain a graph $H^{(r)}$, where $r = n - m(t)$, with n vertices. Clearly, $H^{(r)}$ is maximal S_t-embeddable.

Q.E.D.

Lemma 4.5 For any positive integer t, let $m(t)$ be the smallest number of vertices of maximal S_t-embeddable graph. If $n \geq m(t)$, then $\mu_t(K_n) = \dfrac{n^2 - 7n + 12(1 - t)}{2}$

Proof. Since K_n has $\binom{n}{2}$ edges, then by Corollary 3.3.3

$$\binom{n}{2} - \mu_t(K_n) \leq 3(n - (2 - 2t)).$$

Therefore

$$(1) \qquad \mu_t(K_n) \geq \binom{n}{2} - 3(n - (2 - 2t))$$
$$= \frac{n^2 - n}{2} - 3(n - (2 - 2t))$$
$$= \frac{n^2 - n - 6n + 12(1 - t)}{2}$$
$$= \frac{n^2 - 7n + 12(1 - t)}{2} .$$

By Lemma 4.4, we can construct a graph H with n vertices which is a maximal S_t-embeddable. Then by Corollary 3.3.2 , H has exactly $3(n - (2 - 2t))$ edges. Hence H can be obtained from K_n by removal $\binom{n}{2} - 3(n - (2 -2t))$ edges. Thus

$$(2) \qquad \mu_t(K_n) \leq \binom{n}{2} - 3(n - (2 - 2t))$$
$$= \frac{n^2 - 7n + 12(1 - t)}{2} .$$

From (1) and (2), we have

$$\mu_t(K_n) = \frac{n^2 - 7n + 12(1 - t)}{2} .$$

<u>Q.E.D.</u>

<u>Theorem</u> 4.6 For any positive integer t, let m(t) be the smallest number of vertices of maximal S_t-embeddable graph. For any non-negative integer k such that $k + t + 3 \geq m(t)$ we have

$$\max. \{\chi(G) : G \text{ is a connected graph with } \mu_t(G) < \binom{k+t+1}{2} - 6t\}$$

$$= k + t + 3$$

$$= \max. \{n : \mu_t(K_n) < \binom{k+t+1}{2} - 6t \}.$$

<u>Proof</u>. Let t be any positive integer and k be any non-negative integer such that $k + t + 3 \geq m(t)$.

Then by Lemma 4.5, we have

$$(1) \quad \mu_t(K_{k+t+3}) = \frac{(k + t + 3)^2 - 7(k + t + 3) + 12(1 - t)}{2}$$

$$= \frac{(k + t + 3)(k + t + 3 - 7) + 12(1 - t)}{2}$$

$$= \frac{(k + t + 3)(k + t - 4) + 12(1 - t)}{2}$$

$$= \frac{(k + t)^2 - (k + t) - 12 + 12 - 12t}{2}$$

$$= \frac{(k + t)(k + t - 1)}{2} - 6t$$

$$= \binom{k+t}{2} - 6t$$

$$< \binom{k+t+1}{2} - 6t .$$

Therefore

$\chi(K_{k+t+3}) \in \{\chi(G)$: G is a connected graph with $\mu_t(G) < \binom{k+t+1}{2} - 6t\}$.

By using Theorem 4.3 and the fact that $\chi(K_{k+t+3}) = k + t + 3$, we have

(2) $\max.\{\chi(G)$: G is a connected graph with $\mu_t(G) < \binom{k+t+1}{2} - 6t\} = k + t + 3$.

We shall complete our proof by showing that $\max.\{n : \mu_t(K_n) < \binom{k+t+1}{2} - 6t\} = k + t + 3.$

Since

$$\mu_t(K_{k+t+3}) < \binom{k+t+1}{2} - 6t \text{ , and}$$

$$\begin{aligned}\mu_t(K_{k+t+4}) &= \frac{(k+t+4)^2 - 7(k+t+4) + 12(1-t)}{2} \\ &= \frac{(k+t+4)(k+t+4-7) + 12(1-t)}{2} \\ &= \frac{(k+t+4)(k+t-3) + 12(1-t)}{2} \\ &= \frac{(k+t)^2 + (k+t) - 12 + 12 - 12t}{2} \\ &= \frac{(k+t)(k+t+1)}{2} - 6t \\ &= \binom{k+t+1}{2} - 6t \text{ ,}\end{aligned}$$

then

(3) $k + t + 3 \in \{n : \mu_t(K_n) < \binom{k+t+1}{2} - 6t\}$

and

(4) $k + t + 4 \notin \{n : \mu_t(K_n) < \binom{k + t + 1}{2} - 6t\}$,

respectively.

By Remark 4.2, we see that

$$\mu_t(K_n) \geq \mu_t(K_{k + t + 4}) = \binom{k + t + 1}{2} - 6t$$

for all positive integers $n > k + t + 4$. Hence from this and (3) and (4), we have

(5) $\max.\{n : \mu_t(K_n) < \binom{k + t + 1}{2} - 6t\} = k + t + 3.$

Thus the theorem holds.

Q.E.D.