CHAPTER IV

CONVOLUTION OPERATORS

ON

HOMOGENEOUS SPACES

1. Convolution Operators on $L^2(\mathbb{T})$

By way of introduction, suppose that $\{\lambda_n\}$ is a bounded sequence of complex numbers and suppose that there exists a g in $L^2(\mathbb{T})$ such that for all integers n, $\lambda_n = c_n(g)$.

Note that a necessary and sufficient condition for such a g to exist is that $\sum_n |\lambda_n|^2 < \infty$ according to the Riesz - Fisher Theorem. By Theorem 3.2 there is a stationary continuous linear operator P on $L^2(\mathbb{T})$ such that

(1) $Pf \sim \sum_n c_n(g)\, c_n(f)\, E_n$ for all f in $L^2(\mathbb{T})$.

For each $\dot{x} \in \mathbb{T}$, the assignment $\dot{t} \longmapsto h(\dot{t}) = \overline{g(\dot{x}-\dot{t})}$ defines a function on $\mathbb{T}$ with

$$
\begin{aligned}
c_n(h) &= \langle h, E_n \rangle \\
&= \int_{\mathbb{T}} \overline{g(\dot{x}-\dot{t})}\, E_{-n}(\dot{t})\, d\dot{t} \\
&= \int_{\mathbb{T}} \overline{g(\dot{x}+\dot{t})}\, E_n(\dot{t})\, dt \\
&= E_n(-\dot{x}) \int_{\mathbb{T}} \overline{g(\dot{t})}\, E_n(\dot{t})\, dt
\end{aligned}
$$

$$= E_n\,(-\dot{x})\;\overline{\int_{\mathbb{T}} g(\dot{t})\;E_{-n}(\dot{t})d\dot{t}}$$

$$= \overline{E_n(\dot{x})\;\;c_n(g)}$$

Now, let μ be any complex number, by Parseval's formula, we get

$$(2)\quad \|f+\mu h\|^2 = \sum_n |c_n(f)+\mu c_n(h)|^2$$

$$= \sum_n (c_n(f)+\mu c_n(h)(\overline{c_n(f)} + \overline{\mu c_n(h)})$$

By identifying with the scalar product, we get

$$\|f+\mu h\|^2 = \langle f+\mu h,\ f+\mu h\rangle$$

$$= \|f\|^2 + |\mu|^2\ \|h\|^2 + 2\ \mathrm{Re}\,\bar{\mu}\ \langle f,h\rangle$$

Thus $2\ \mathrm{Re}\,\bar{\mu}\ \langle f,h\rangle \;=\; 2\ \mathrm{Re}\,\bar{\mu}\ \sum c_n(f)\ \overline{c_n(h)}$

Take $\mu = 1$; we have

$$\mathrm{Re}\ \langle f,h\rangle = \mathrm{Re}\ \sum c_n(f)\ \overline{c_n(h)}$$

Take $\mu = i$; we have

$$\mathrm{Re}\text{-}i\langle f,\ h\rangle = \mathrm{Re} - i\ \sum c_n(f)\ \overline{c_n(h)},$$ which

implies $\mathrm{Im}\,\langle f,\ h\rangle \;=\; \mathrm{Im}\ \sum c_n(f)\ \overline{c_n(h)}.$

Hence

$$(3)\qquad \langle f,\ h\rangle = \sum c_n(f)\ \ \overline{c_n(h)}$$

$$= \sum c_n(f)\ c_n(g)\ E_n(\dot{x})$$

Thus it follows from (1) and (3) that

$$Pf(\dot{x}) = \langle f,h\rangle = \int_{\mathbb{T}} g(\dot{x}-\dot{t})\ f(\dot{t})d\dot{t}.$$

We are then led to the definition of convolution.

1.1 Definition. Let f and g be two functions in $L^2(\mathbb{T})$. The convolution $f*g$ of f and g is a function on $\mathbb{T}$ defined by

$$\dot{x} \longmapsto f*g(\dot{x}) = \int_{\mathbb{T}} f(\dot{t})\, g(\dot{x}-\dot{t})\, d\dot{t}$$

Immediately, we must show that $f*g$ is well defined.

1.2 Proposition. Let $f, g \in L^2(\mathbb{T})$. Then

(a) $c_n(f*g) = c_n(f)\, c_n(g)$.

(b) $f*g$ is actually a continuous function on $\mathbb{T}$.

Proof. (a) This follows immediately from Eq(3) in the introduction.

(b) First we will show that $\sum_{n\in\mathbb{Z}} c_n(f)\, c_n(g)\, E_n(x)$ converges uniformly in x.

Consider $\sum_{n\in\mathbb{Z}} |c_n(f)\, c_n(g)\, E_n(\dot{x})| = \sum_{n\in\mathbb{Z}} |c_n(f)|\,|c_n(g)|$.

Due to the Schwarz inequality and the fact that $\sum_{n\in\mathbb{Z}} |c_n(f)|^2$ and $\sum_{n\in\mathbb{Z}} |c_n(g)|^2$ converge, it follows that for any integer n

$$\sum_{k=-n}^{n} |c_k(f)\, c_k(g)| = \sum_{k=-n}^{n} |c_k(f)|\,|c_k(g)|$$

$$\leq \sqrt{\sum_{k=-n}^{n} |c_k(f)|^2} \cdot \sqrt{\sum_{k=-n}^{n} |c_k(g)|^2}$$

$$\leq \sqrt{\sum_{k\in\mathbb{Z}} |c_k(f)|^2} \cdot \sqrt{\sum_{k\in\mathbb{Z}} |c_k(g)|^2} < +\infty .$$

So that $\sum_{n\in\mathbb{Z}} c_n(f)\, c_n(g)\, E_n(\dot{x})$ converges uniformly.
Part (a) shows that $f * g$ is the uniform limit of $\sum c_n(f)\, c_n(g)\, E_n$. Therefore $f * g$ is continuous.

1.3 Proposition. Let g be a given function on $L^2(\mathbb{T})$. Then the operator $P : f \to f * g$ is a stationary continuous linear operator on $L^2(\mathbb{T})$.

Proof. We will show first that P is stationary. We have for all $\dot{x} \in \mathbb{T}$,

$$\begin{aligned} U_{\dot{h}}(Pf)(\dot{x}) &= U_h(f*g)(x) \\ &= f*g(\dot{x}+\dot{h}) \\ &= \int_{\mathbb{T}} f(\dot{t})\, g(\dot{x}+\dot{h}-\dot{t})\, d\dot{t} \\ &= \int_{\mathbb{T}} f(\dot{t}+\dot{h})\, g(\dot{x}-\dot{t})\, d\dot{t} \\ &= \int_{\mathbb{T}} U_{\dot{h}}f(\dot{t})\, g(\dot{x}-\dot{t})\, d\dot{t} \\ &= (U_{\dot{h}}f*g)(\dot{x}) \\ &= P(U_{\dot{h}}f)(\dot{x}). \end{aligned}$$

So that $U_{\dot{h}}(f*g) = U_{\dot{h}}f*g$.

For any $\alpha, \beta \in \mathbb{C}$ and $f, h \in L^2(\mathbb{T})$, we have

$$\begin{aligned} P(\alpha f+\beta h)(\dot{x}) &= ((\alpha f+\beta h)*g)(\dot{x}) \\ &= \int_{\mathbb{T}} (\alpha f+\beta h)(\dot{t})\, g(\dot{x}-\dot{t})\, d\dot{t} \end{aligned}$$

$$= \int_{\mathbb{T}} (\alpha f(\dot{t})\, g(\dot{x}-\dot{t}) + \beta\, h(\dot{t})\, g(\dot{x}-\dot{t}))\, d\dot{t}$$
$$= \alpha \int_{\mathbb{T}} f(\dot{t})\, g(\dot{x}-\dot{t}) d\dot{t} + \beta \int_{\mathbb{T}} h(\dot{t})\, g(\dot{x}-\dot{t}) d\dot{t}$$
$$= \alpha (f * g)(\dot{x}) + \beta\, (h * g)(\dot{x})$$
$$= \alpha\, P(f)(\dot{x}) + \beta\, P(h)(\dot{x}).$$

Since P is linear, we have

$$\sum_{n \in \mathbb{Z}} |c_n(f)\, c_n(g)|^2 \leq \sum_{n \in \mathbb{Z}} (\operatorname{Sup}_n |c_n(g)|^2)\, |c_n(f)|^2$$
$$= \operatorname{Sup}_n |c_n(g)|^2 \sum_n |c_n(f)|^2$$

and so

$$(\sum_n |c_n(f)\, c_n(g)|^2)^{\frac{1}{2}} \leq \operatorname{Sup}_n |c_n(g)| \, (\sum |c_n(f)|^2)^{\frac{1}{2}}$$
$$= \operatorname{Sup}_n |c_n(g)| \, \|f\|_2 .$$

It follows from Proposition 1.2 (a) that

$$\|Pf\|_2 \leq \operatorname{Sup}_n |c_n(g)| \, \|f\|_2$$

for all $f \in L^2(\mathbb{T})$. Thus P is continuous.

Hence the proposition is now proved.

Note. we see that stationary continuous linear operators from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ can be obtained via convolution by two functions of $L^2(\mathbb{T})$. The question is whether or not all stationary operators from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ come from convolution operators. The answer is negative. However we shall not construct any example to support the answer since the usual construction relies on the Riemann - Lebesgue Theorem concerning the behavior of Fourier coefficients.

1.4 Proposition. Let f, g and h be three functions in $L^2(\mathbb{T})$. The following properties are true.

(a) $f * g = g * f$.

(b) $f * (g * h) = (f * g) * h$.

Proof. Property (a) follows from Proposition 1.2 (a) and the uniqueness theorem for Fourier series represintation of functions in $L^2(\mathbb{T})$.

Since $g * h$ and $f * g$ are continuous, they also belongs to $L^2(\mathbb{T})$. So that

$$\begin{aligned} c_n(f * (g * h)) &= c_n(f)\,(c_n(g * h)) \\ &= c_n(f)\,(c_n(g)\,c_n(h)) \\ &= (c_n(f)\,c_n(g))\,c_n(h) \\ &= c_n(f * g) * h \end{aligned}$$

and property (b) holds again by the Uniqueness theorem •

1.5 Theorem. Let g be a given function in $L^2(\mathbb{T})$ and P be the convolution operator : $f \longmapsto f * g$

The kernel of P is the closure of the linear space generated by all those E_n for which $c_n(g) = 0$.

Alternatively it is the space of functions in $L^2(\mathbb{T})$ orthogonal to all those E_n for which $c_n(g) \neq 0$.

<u>Proof</u>. First we will prove the second half of the theorem. Assume $f \in \operatorname{Ker} P$, then $Pf = 0$. And so for any integer m,

$$
\begin{aligned}
0 = c_m(Pf) &= \Big\langle \sum_{n \in \mathbb{Z}} c_n(f)\, c_n(g)\, E_n,\ E_m \Big\rangle \\
&= \Big\langle \lim_{N \to \infty} \sum_{k=-N}^{N} c_k(f)\, c_k(g)\, E_k,\ E_m \Big\rangle \\
&= \lim_{N \to \infty} \Big\langle \sum_{k=-N}^{N} c_k(f)\, c_k(g)\, E_k,\ E_m \Big\rangle \\
&= \lim_{N \to \infty} c_m(f)\, c_m(g) \| E_m \| \\
&= c_m(f)\, c_m(g).
\end{aligned}
$$

Which implus that $c_m(f) = 0$ for all m such that $c_m(g) \neq 0$. So that f belongs to the space of functions in $L^2(\mathbb{T})$ orthogonal to all those E_m for which $c_m(g) \neq 0$.

Conversely, assume f belongs to the space of functions in $L^2(\mathbb{T})$ orthogonal to all those E_n for which $c_n(g) \neq 0$. And so for E_n such that $c_n(g) \neq 0$, $c_n(f) = \langle f, E_n \rangle = 0$. Thus $c_n(g)\, c_n(f) = 0$ for all n. And therefore $Pf = \sum_{n \in \mathbb{Z}} c_n(f)\, c_n(g)\, E_n = 0$. Which implies $f \in \operatorname{Ker} P$.

To prove the first half, assume f is in the closure of the linear space generated by all those E_n for which $c_n(g) = 0$. That is $f = \lim_{n \to \infty} \sum_{k=-n}^{n} \lambda_k E_{m_k}$, where E_{m_k}

is such that $c_{m_k}(g) \neq 0$. And so $c_n(f)\, c_n(g) = 0$ for all n in $\mathbb{Z}$, and $Pf = \sum_{n \in \mathbb{Z}} c_n(f)\, c_n(g)\, E_n = 0$. Therefore $f \in \mathrm{Ker}\ P$.

Conversely, assume $f \in \mathrm{Ker}\ P$; then $Pf = \sum_{n \in \mathbb{Z}} c_n(f)\, c_n(g)\, E_n = 0$. By the proof of the second half of the theorem, $0 = \langle Pf, E_m \rangle = c_m(f)\, c_m(g)$. So that $c_m(g) = 0$ whenever $c_m(f) \neq 0$. Let $I = \{ m_k \mid k \in \mathbb{Z} \}$ be the set of indicies such that $c_{m_k}(g) \neq 0$. Then $f \sim \sum_{k \in \mathbb{Z}} c_{m_k}(f)\, E_{m_k}$. And therefore f is in the closure of the linear space generated by all those E_n for which $c_n(g) \neq 0$.

Actually, the structure of $L^2(\mathbb{T})$ under convolution can be summarized by the introduction of a new terminology.

1.6 Definition. A normed space E of complex valued functions over $\mathbb{T}$ is called a commutative convolution algebra if:

(1) $f * g$ is a function in E whenever f and g are in E.

(2) If g is in E, then the mapping $f \longmapsto f * g$ is linear from E into E.

(3) $f * (g * h) = (f * g) * h$ for any f, g and h in E.

(4) $f * g = g * f$ for any f, g in E.

If in addition we get

(5) $\|f * g\|_E \leq \|f\|_E \|g\|_E$ for any f, g in E,

Then E is called a <u>commutative Banach algebra for convolution</u>.

1.7 <u>Theorem</u>. $L^2(\mathbb{T})$ is a commutative Banach algebra for convolution.

<u>Proof</u>. Properties (1), (2), (3) and (4) are the contents of previous propositions. Thus we need only to show that $\|f*g\|_2 \leq \|f\|_2 \|g\|_2$. But

$$\begin{aligned}\|f*g\|_2^2 &= \sum |c_n(f)|^2 \, |c_n(g)|^2 \\ &\leq \sum (\sup_n \; c_n(g)^{\,2}) \sum |c_n(f)|^2 \\ &= \sup_n \; |c_n(g)|^2 \; \sum_n c_n(f)^{\,2} \\ &\leq (\sum_{n\in\mathbb{Z}} |c_n(g)|^2)\,(\sum_{n\in\mathbb{Z}} |c_n(f)|^2)\end{aligned}$$

which yields

$$\|f*g\|_2 \leq \|f\|_2 \|g\|_2 ,$$

1.8 <u>Remark</u>. $L^2(\mathbb{T})$ is not an integral domain under convolution; that is, there are f and g in $L^2(\mathbb{T})$ not equal to zero almost everywhere but $f*g = 0$. Since $f*g = 0$ means $c_n(f)\, c_n(g) = 0$ only, we can find such f and g in $L^2(\mathbb{T})$. For example, let

$$f(\dot{t}) = 1 \quad , \quad g(\dot{t}) = e^{2\pi i t} \quad , \quad \dot{t} \in \mathbb{T}.$$

Then

$$c_n(f) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

$$c_n(g) = \int_0^1 e^{2\pi it} . e^{-2\pi int} dt$$

$$= \int_0^1 e^{2\pi i(1-n)t} dt$$

$$= \frac{1}{2\pi i(1-n)} \left[e^{2\pi i(1-n)t} \right]_0^1$$

$$= 0 \qquad n \neq 1$$

Thus f and g are in $L^2(\mathbb{T})$, not equal to zero almost everywhere and $f * g = 0$.

1.9 Remark. The convolution algebra $L^2(\mathbb{T})$ has no unity. That is; there exists no function g in $L^2(\mathbb{T})$ such that $f * g = f$ for all f in $L^2(\mathbb{T})$. If such g exists then $c_n(f)\, c_n(g) = c_n(f)$ for all n and since we can always find f such that $c_n(f) \neq 0$ for all n. For example,

let $f(\dot{t}) = e^{\frac{1}{2}\pi it}$ $(\dot{t} \in \mathbb{T})$. Then

$$c_n(f) = \langle f, E_n \rangle$$

$$= \int_0^1 f(t)\, \overline{E_n(t)}\, dt$$

$$= \int_0^1 e^{\frac{\pi i t}{2}} \cdot e^{-2\pi i n t}\, dt$$

$$= \int_0^1 e^{\pi i(\frac{1}{2} - 2n)t}\, dt$$

$$= \frac{1}{\frac{\pi i}{2}(1-4n)} \left[e^{\pi i \left(\frac{1-2n}{2}\right) t} \right]_0^1$$

$$= \frac{2}{\pi i\,(1-4n)} \left[e^{\frac{\pi i}{2}(1-2n)} - 1 \right]$$

which is not zero for all n. So that, we must have $c_n(g) = 1$ for all $n \in \mathbb{Z}$. This, however, is impossible since $\lim_{n \to \infty} c_n(g) = 0$. This latter fact is true just because $|c_n(g)|$ is n^{th} term of the convergent series $\sum |c_n(g)|$.

2. Convolution in Homogeneous space:

We have already seen that $L^2(\mathbb{T})$ is a Banach algebra for convolution and that $f * g$ is far more regular than f and g. In fact, if f, $g \in L^2(\mathbb{T})$, then $f * g$ is a continuous function having a uniform absolutely convergent Fourier series. These results can be extended in $L^p(\mathbb{T})$.

2.1 Definition. Let f and g be two functions in $L^1(\mathbb{T})$. The convolution of f and g is defined by

$$f * g(\dot{x}) = \int_{\mathbb{T}} f(\dot{x} - \dot{y})\, g(\dot{y})\, d\dot{y}.$$

As in the case of $L^2(\mathbb{T})$, we must show that $f * g$ is an element in $L^1(\mathbb{T})$.

2.2 <u>Theorem</u>. Let $f, g \in L^1(\mathbb{T})$. Then

(a) $$\int_{\mathbb{T}} \left| f(\dot{x} - \dot{t})\, g(\dot{t}) \right| d\dot{t} < \infty$$

for almost all $\dot{x}$ in $\mathbb{T}$. For these $\dot{x}$, define

(b) $$h(\dot{x}) = \int_{\mathbb{T}} f(\dot{x} - \dot{t})\, g(\dot{t})\, d\dot{t}.$$

Then $h \in L^1(\mathbb{T})$ and

(c) $$\| h \|_1 \leq \| f \|_1 \| g \|_1 .$$

Of course, h is just $f * g$.

<u>Proof</u>. If f is a measurable characteristic function, then by the fact that the σ - algebra of all Lebesgue measurable sets is the completion of the σ - algebra of all Borel sets, there exists a Borel function f_o such that $f_o = f$ a.e on $\mathbb{T}$. Since a simple function is a finite linear combination of measurable characteristic functions, there exists a Borel function $f_o = f$ a.e for any simple function f. If $f \geq o$ is measurable, and if $\{ s_n \}$ is a sequence of Lebesgue measurable simple functions which converges pointwise to f, there are Borel simple functions t_n such that $t_n = s_n$ a.e and such that $t_n(\dot{x}) = 0$ at those x at which $t_n(\dot{x}) \neq s_n(\dot{x})$. Then $f_o(\dot{x}) = \lim_{n \to \infty} t_n(\dot{x})$ exists for every $\dot{x}$, f_o is a Borel function, and $f_o = f$ a.e. Hence we obtain

by the usual trick that for any f and g in $L^1(\mathbb{T})$ there exists Borel function f_0 and g_0 such that $f = f_0$ a.e and $g = g_0$ a.e.

The integrals in (a) and (b) are unchanged, for every $\dot{x}$, if we replace f by f_0 and g by g_0. Thus we may assume, to begin with, that f and g are Borel functions.

To apply Fubini's Theorem, we shall first prove that the function F defined by

$$F(\dot{x}, \dot{y}) = f(\dot{x} - \dot{y})\, g(\dot{y})$$

is a Borel function on $\mathbb{T} \times \mathbb{T}$.

To each $E \subset \mathbb{T}$, we associate a set $\widetilde{E} \subset \mathbb{T} \times \mathbb{T}$ defined by

$$\widetilde{E} = \{ (\dot{x}, \dot{y}) : \dot{x} - \dot{y} \in E \}.$$

If E is open, so is $\widetilde{E}$. Let $\mathfrak{A}$ be the collection of all $E \subset \mathbb{T}$ for which $\widetilde{E}$ is a Borel set. Then $\mathfrak{A}$ is a σ - algebra in $\mathbb{T}$, since

(i) $\mathbb{T} \times \mathbb{T}$ is a Borel set, and hence $\mathbb{T}$ is in $\mathfrak{A}$

(ii) Let E be in $\mathfrak{A}$. Then $\widetilde{E}$ is a Borel set. But $\widetilde{E^c} = \widetilde{E}^c$, So that E^c is in $\mathfrak{A}$.

(iii) Let $\{E_n\}$ be a sequence in $\mathfrak{A}$. We proceed to show that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{A}$. It is enough to show that $\widetilde{\bigcup_{n=1}^{\infty} E_n} = \bigcup_{n=1}^{\infty} \widetilde{E}_n$. Let $(x, y) \in \widetilde{\bigcup_{n=1}^{\infty} E_n}$. Then $(x - y) \in \bigcup_{n=1}^{\infty} E_n$;

that is there is a positive integer N such that $(x-y) \in E_N$. So that $(x,y) \in \bigcup_{n=1}^{\infty} \tilde{E}_n$. Hence $\widetilde{\bigcup_{n=1}^{\infty} E_n} = \bigcup_{n=1}^{\infty} \tilde{E}_n$.

Conversely, assume $(x, y) \in \bigcup_{n=1}^{\infty} \tilde{E}_n$. Then there exists a positive integer k such that $(x, y) \in \tilde{E}_k$. So that $x - y \in E_k$, and hence in $\bigcup_{n=1}^{\infty} E_n$. Which implus $(x,y) \in \widetilde{\bigcup_{n=1}^{\infty} E_n}$. Thus $\bigcup_{n=1}^{\infty} \tilde{E}_n \subset \widetilde{\bigcup_{n=1}^{\infty} E_n}$. Consequently $\bigcup_{n=1}^{\infty} \tilde{E}_n = \widetilde{\bigcup_{n=1}^{\infty} E_n}$. It follows that $\tilde{E}$ is a Borel set in $\mathbb{T} \times \mathbb{T}$ whenever E is a Borel set in $\mathbb{T}$.

Now let V be any open set, and let $E = \{ \dot{x} : f(\dot{x}) \in V \}$. Then E is a Borel set in $\mathbb{T}$, and so is

$\{ (\dot{x}, \dot{y}) : f(\dot{x} - \dot{y}) \in V \} = \{ (\dot{x}, \dot{y}) : \dot{x} - \dot{y} \in E \} = \tilde{E}$.

This shows that the assignment $(\dot{x}, \dot{y}) \mapsto f(\dot{x} - \dot{y})$ defines a Borel function. Since the composition of Borel functions is a Borel function, $(\dot{x}, \dot{y}) \longmapsto g(\dot{y})$ defines a Borel function. Since the product of two Borel functions is a Borel function, our assertion concerning F is proved.

Next we observe that

$$(1) \quad \int_{\mathbb{T}} d\dot{y} \int_{\mathbb{T}} |F(\dot{x}, \dot{y})| \, d\dot{x} = \int_{\mathbb{T}} |g(\dot{y})| \, d\dot{y} \int_{\mathbb{T}} |f(\dot{x}-\dot{y})| \, d\dot{x}$$

$$= \|f\|_1 \, \|g\|_1 ,$$

since $\int_{\mathbb{T}} |f(\dot{x} - \dot{y})| \, d\dot{x} = \int_{\mathbb{T}} |f(x)| \, dx = \|f\|_1$

for every y , by the translation invarience of Lebesgue integration.

Thus $F \in L^1(\mathbb{T} \times \mathbb{T})$, and the Fubini's theorem implies that the integral in (b) exists for almost all $\dot{x}$ in $\mathbb{T}$ and $\dot{h} \in L^1(\mathbb{T})$.

Finally, (c) follows from

$$\|f * g\|_1 = \int_{\mathbb{T}} |f * g(\dot{x})|\, d\dot{x} \leq \int_{\mathbb{T}} d\dot{x} \int_{\mathbb{T}} |F(\dot{x}, \dot{y})|\, d\dot{y} = \|f\|_1 \|g\|_1$$

by (1). The proof is complete .

2.5 <u>Theorem</u>. $L^1(\mathbb{T})$ is a commutative Banach algebra for convolution.

<u>Proof</u>. Let f and $g \in L^1(\mathbb{T})$. Then

$$(f * g)(\dot{x}) = \int_{\mathbb{T}} f(\dot{x} - \dot{t})\, g(\dot{t})\, d\dot{t}.$$

Putting $\dot{x} - \dot{t} = \dot{u}$, we obtain

$$\begin{aligned} f * g(\dot{x}) &= \int_{\mathbb{T}} f(\dot{u})\, g(\dot{x} - \dot{u})\, d\dot{u} \\ &= \int_{\mathbb{T}} g(\dot{x} - \dot{u})\, f(\dot{u})\, d\dot{u} \\ &= g * f(x). \end{aligned}$$

Let $h \in L^1(\mathbb{T})$. Then, by the definition of convolution on $L^1(\mathbb{T})$ and Fubini's Theorem, it follows that

$$\begin{aligned} (f * (g * h))(\dot{x}) &= \int_{\mathbb{T}} f(\dot{x} - \dot{t})\, (g * h)\, (\dot{t})\, d\dot{t} \\ &= \int_{\mathbb{T}} f(\dot{x} - \dot{t}) \left(\int g(\dot{t} - \dot{u})\, h(\dot{u})\, d\dot{u} \right) d\dot{t} . \end{aligned}$$

Putting $\dot{t} = \dot{s} + \dot{u}$, one gets

$$(f * (g * h))\, (\dot{x}) = \left(\int_{\mathbb{T}} f(\dot{x} - (\dot{s} + \dot{u}))\, d\dot{s} \right) \left(\int_{\mathbb{T}} g(\dot{s})\, h(\dot{u})\, d\dot{u} \right.$$

$$= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(\dot{x} - \dot{s} - \dot{u}) \quad g(\dot{s}) d\dot{s} \right) h(\dot{u}) \, d\dot{u}$$

$$= \int_{\mathbb{T}} f * g(\dot{x} - \dot{u}) \, h(\dot{u}) \, d\dot{u}$$

$$= \quad ((f * g) * h) \, (\dot{x})$$

Now we will show that for a given g in $L^1(\mathbb{T})$, the mapping $f \to f * g$ is linear from $L^1(\mathbb{T})$ into $L^1(\mathbb{T})$. By the definition of the convolution in $L^1(\mathbb{T})$ and the linearity of integration, it follows that for any $\alpha, \beta \in \mathbb{C}$,

$$((\alpha f + \beta h) * g) \, (\dot{x}) = \int_{\mathbb{T}} (\alpha f + \beta h) \, (\dot{x} - \dot{t}) \, g(\dot{t}) d\dot{t}$$

$$= \int_{\mathbb{T}} \alpha f(\dot{x} - \dot{t}) \, g(\dot{t}) d\dot{t} + \int_{\mathbb{T}} \beta h(\dot{x} - \dot{t}) \, g(\dot{t}) d\dot{t}$$

$$= \alpha \int_{\mathbb{T}} f(\dot{x} - \dot{t}) \, g(\dot{t}) d\dot{t} + \beta \int_{\mathbb{T}} h(\dot{x} - \dot{t}) \, g(\dot{t}) d\dot{t}$$

$$= \alpha \, (f * g) \, (\dot{x}) + \beta \, (h * g) \, (\dot{x}) \quad .$$

This and Theorem 2.4. show that $L^1(\mathbb{T})$ is a commutative Banach algebra for convolution.

2.6 <u>Theorem</u>. For $1 \leq p \leq +\infty$, $L^p(\mathbb{T})$ is a commutative Banach algebra for convolution.

<u>Proof</u>. Observe that for any $f \in L^p(\mathbb{T})$, we have $\|f\|_p^{\,p} = \int_{\mathbb{T}} |f(\dot{x})|^p d\dot{x} < +\infty$. This implies that $|f(\dot{x})| < +\infty$ a.e. and so,

$\|f\|_1 = \int_{\mathbb{T}} |f(\dot{x})| d\dot{x} < \infty$; i.e $f \in L^1(\mathbb{T})$.

Consequently, <u>$L^P(\mathbb{T}) \subset L^1(\mathbb{T})$</u>.

As we have already proved that $L^1(\mathbb{T})$ was a convolution algebra, the theorem will follow if we prove that when f is in $L^1(\mathbb{T})$ and g is in $L^P(\mathbb{T})$ then $f * g$ is in

$L^p(\mathbb{T})$. Let $f \in L^1(\mathbb{T})$, $g \in L^p(\mathbb{T})$ and h $L^q(\ )$, where q is the conjugate exponent of p. Consider the following integral $\int_{\mathbb{T}} \left(\int_{\mathbb{T}} g(\dot{x}-\dot{t})\, f(\dot{t})d\dot{t} \right) h(\dot{x})d\dot{x}$. We get that the double integral is absolutely convergent, because

$$\begin{aligned}
\int_{\mathbb{T}\times\mathbb{T}} |g(\dot{x}-\dot{t})\, f(\dot{t})\, h(\dot{x})|\, d\dot{x}d\dot{t} &= \int_{\mathbb{T}} |f(\dot{t})| \left(\int_{\mathbb{T}} |g(\dot{x}-\dot{t})\, h(\dot{x})| d\dot{x} \right) d\dot{t} \\
&\leq \int_{\mathbb{T}} |f(\dot{t})|\, \|g\|_p \|h\|_q d\dot{t} \\
&\leq \|f\|_1 \|g\|_p \|h\|_q
\end{aligned}$$

and so

$$\left| \int_{\mathbb{T}} f * g(\dot{x})\, h(\dot{x})d\dot{x} \right| \leq \|f\|_1 \|g\|_p \|h\|_q \ .$$

Using a mild converse of Hölder's inequality stating that if $f \in L^1(\mathbb{T})$ and if for all $h \in L^q(\mathbb{T})$ $(q \geq 1)$ we have $\left| \int_{\mathbb{T}} f(\dot{x})\, h(\dot{x})d\dot{x} \right| \leq A.\ \|h\|_q$ for some $A > 0$, then $f \in L^p(\mathbb{T})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\|f\|_p \leq A$, we can conclude that $f * g \in L^p(\mathbb{T})$ and

$$(*) \qquad \|f * g\|_p \leq \|f\|_1 \|g\|_p \ .$$

Applying Hölder's inequality to $g = 1$, we get $\|f\|_1 \leq \|f\|_p$.

Finally, we get $\|f * g\|_p \leq \|f\|_p \|g\|_p$.

2.7 <u>Theorem</u>. $A(\mathbb{T})$ is a Banach convolution algebra.

<u>Proof</u>. By the definition of $A(\mathbb{T})$, it follows from Proposition 1.2 (b) that $L^2(\mathbb{T}) * L^2(\mathbb{T}) \subset A(\mathbb{T})$. Since $A(\mathbb{T})$ is a subset of $L^2(\mathbb{T})$, $A(\mathbb{T}) * A(\mathbb{T}) \subset A(\mathbb{T})$. Thus the convolution has all the desired properties of a convolution algebra.

The inequality

$\|f*g\|_{A(\mathbb{T})} \leq \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}$ holds since

$$\sum_{n\in\mathbb{Z}} |c_n(f*g)| = \sum |c_n(f)||c_n(g)|$$
$$\leq (\sup_{n\in\mathbb{Z}} |c_n(f)|)\left(\sum_{n\in\mathbb{Z}} |c_n(g)|\right.$$
$$\leq (\sum_{n\in\mathbb{Z}} |c_n(f)|)(\sum_{n\in\mathbb{Z}} |c_n(g)|$$

and so, $\|f*g\|_{A(\mathbb{T})} \leq \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}$

Hence $A(\mathbb{T})$ is a Banach convolution algebra.

2.8 <u>Definition</u>. <u>A convolutor</u> is a homogeneous Banach space **E** of functions over $\mathbb{T}$ such that

(a) $\mathbf{E} \cap C(\mathbb{T})$ is dense in **E**.

(b) $L^1(\mathbb{T}) * \mathbf{E}$ is a subset of **E**.

(c) $\|f*g\|_{\mathbf{E}} \leq \|f\|_1 \|g\|_{\mathbf{E}}$ for any f in $L^1(\mathbb{T})$ any g in **E**.

2.9 <u>Theorem</u>. $L^p(\mathbb{T})$ for $1 \leq p < \infty$, $C(\mathbb{T})$ and $A(\mathbb{T})$ are convolutors.

<u>Proof</u>. We know that all these spaces are homogeneous Banach spaces. It then suffices to check the three other properties.

For $L^p(\mathbb{T})$, $\infty > p \geq 1$, 2.8 (a) follows immediately since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ and $L^p(\mathbb{T}) \cap C(\mathbb{T}) = C(\mathbb{T})$. The property 2.8 (b) follows since

$L^1(\mathbb{T}) * L^P(\mathbb{T}) \subset C(\mathbb{T})$ and so is a subset of $L^P(\mathbb{T})$. The property 2.8 (c) follows from the inequality ($*$) of Theorem 2.6.

For $C(\mathbb{T})$, property 2.8 (a) is trivial; property 2.8 (b) follows since $L^1(\mathbb{T}) * C(\mathbb{T}) \subset C(\mathbb{T})$; and so the only property we need to prove is 2.8 (c). We obtain

$$\|f * g\|_{C(\mathbb{T})} = \sup_{\dot{x} \in \mathbb{T}} |f * g(\dot{x})|$$

$$= \sup_{\dot{x} \in \mathbb{T}} \left| \int_{\mathbb{T}} f(\dot{x}-\dot{y})\, g(\dot{y})\, d\dot{y} \right|$$

Applying Holder's inequality and the fact that Lebesgue integration is invariant under translation, we then get.

$$\|f * g\|_{C(\mathbb{T})} \leq \|f\|_1 \, \|g\|_{C(\mathbb{T})}.$$

For $A(\mathbb{T})$, property 2.8 (a) follows immediately since $A(\mathbb{T}) \subset C(\mathbb{T})$, we get $A(\mathbb{T}) \cap C(\mathbb{T}) = A(\mathbb{T})$; property 2.8 (b) follows from proposition 1.2 (a), and so the only property we need to prove is 2.8 (e).

Consider

$$\sum_{n \in \mathbb{Z}} |c_n(f * g)| = \sum_{n \in \mathbb{Z}} |c_n(f)|\, |c_n(g)|.$$

But $|c_n(f)| \leq \|f\|_1$ for all n, so that

$$\|f * g\|_{A(\mathbb{T})} \leq \|f\|_1 \, \|g\|_{A(\mathbb{T})}.$$

Hence the theorem is now proved.

2.10 <u>Proposition</u>. Let f be a given function in $L^1(\mathbb{T})$. Then the operator $P : g \longrightarrow g * f$ is a continuous linear stationary operator on $L^1(\mathbb{T})$.

<u>Proof</u>. For any $g \in L^1(\mathbb{T})$, $\|g * f\|_1 \leq \|g\|_1 \| f \|_1$, and so $\|Pg\|_1 \leq \|f\|_1 \cdot \|g\|_1$. Hence P is a bounded operator from $L^1(\mathbb{T})$ into $L^1(\mathbb{T})$.

We will now show that P is linear. For any α, $\beta \in \mathbb{C}$ and g, $h \in L^1(\mathbb{T})$, we have

$$
\begin{aligned}
P(\alpha g+\beta\ h)(\dot{x}) &= ((\alpha g + \beta h) * f)(\dot{x}) \\
&= \int_{\mathbb{T}} (\alpha g + \beta h)(\dot{t})\ f(\dot{x}-\dot{t})d\dot{t} \\
&= \int_{\mathbb{T}} (\alpha g(\dot{t})\ f(\dot{x}-\dot{t}) + \beta h(\dot{t})\ f(\dot{x}-\dot{t}))d\dot{t} \\
&= \alpha \int_{\mathbb{T}} g(\dot{t})\ f(\dot{x}-\dot{t})\ d\dot{t} + \beta \int_{\mathbb{T}} h(\dot{t})\ f(\dot{x}-\dot{t})d\dot{t} \\
&= \alpha (g * f)(\dot{x}) + \beta (h * g)(\dot{x}) \\
&= \alpha P(g)(\dot{x}) + \beta\ P(h)(\dot{x})
\end{aligned}
$$

Next we will show that P is stationary. We have for all $\dot{x} \in \mathbb{T}$ and $\dot{h} \in \mathbb{T}$,

$$
\begin{aligned}
U_{\dot{h}}(P(g)(\dot{x}) &= U_{\dot{h}}(g * f)(\dot{x}) \\
&= g * f(\dot{x} + \dot{h}) \\
&= \int_{\mathbb{T}} g(\dot{t})\ f(\dot{x} + \dot{h} - \dot{t})d\dot{t} \\
&= \int_{\mathbb{T}} g(\dot{t} + \dot{h})\ f(\dot{x} - \dot{t})d\dot{t} \\
&= \int_{\mathbb{T}} U_{\dot{h}} g(\dot{t})\ f(\dot{x} - \dot{t})d\dot{t}
\end{aligned}
$$

$$= \quad (U_{\dot{h}} g * f)(\dot{x})$$

$$= \quad P(U_{\dot{h}} g)(\dot{x}) \; .$$

Hence P is stationary.

This completes the proof.

2.11 Proposition. Let f be a given function in $L^p(\top)$, $(1 \le p < \infty)$. Then the operator $P : g \longrightarrow g * f$ is a continuous linear stationary operator on $L^p(\top)$.

Proof. For any $g \in L^p(\top)$, we have $\|P(g)\|_p = \|g * f\|_p \le \|f\|_p \cdot \|g\|_p$. So that P is a bounded operator from $L^p(\top)$ into itself.

We will now show that P is linear.

This follows from 2.10 and the fact that $L^p(\top)$ $L^p(\top) \subset L^1(\top)$.

Next we proceed to show that P is stationary. We have for all $\dot{x} \in \top$ and $\dot{h} \in \top$,

$$U_{\dot{h}}(P(g))(\dot{x}) = U_{\dot{h}}(g * f)(\dot{x})$$

$$= g * f(\dot{x} + \dot{h})$$

$$= \int_{\top} g(\dot{t}) \, f(\dot{x} + \dot{h} - \dot{t}) d\dot{t}$$

$$= \int_{\top} g(\dot{t} + \dot{h}) \, f(\dot{x} - \dot{t}) d\dot{t}$$

$$= \int_{\top} U_{\dot{h}} g(\dot{t}) \, f(\dot{x} - \dot{t}) d\dot{t}$$

$$= \quad (U_{\dot h} g * f)(\dot x)$$

$$= \quad P(U_{\dot h} g)(\dot x) \,.$$

Hence P is stationary.

This completes the proof.

2.12 <u>Proposition</u>. Let f be a given function in $A(\mathbb{T})$. Then the operator $P : g \longrightarrow g * f$ is a continuous linear stationary operator on $A(\mathbb{T})$.

<u>Proof</u>. For any $g \in A(\mathbb{T})$, we have

$$\begin{aligned} \| P(g) \|_{A(\mathbb{T})} &= \| g * f \|_{A(\mathbb{T})} \\ &= \sum_n | c_n(g * f) | \\ &= \sum_n | c_n(g) | \, | c_n(f) | \\ &\leq \sum_n (\operatorname{Sup}_n | c_n(f) |) . | c_n(g) | \\ &= \operatorname{Sup}_n | c_n(f) | \, . \sum_n | c_n(g) | \\ &= \operatorname{Sup}_n | c_n(f) | \, . \| g \|_{A(\mathbb{T})} \end{aligned}$$

Since $A(\mathbb{T}) \subset L^1(\mathbb{T})$, in particular $A(\mathbb{T}) \subset C(\mathbb{T})$, the linearity of P follows from 2.10.

Next we will show that P is stationary. For any $\dot x \in \mathbb{T}$ we have

$$\begin{aligned} U_{\dot h}(P(g))(\dot x) &= U_{\dot h}(g * f)(\dot x) \\ &= g * f(\dot x + \dot h) \end{aligned}$$

$$= \int_{T} g(\dot{t})\, f(\dot{x} + \dot{h} - \dot{t})\,d\dot{t}$$

$$= \int_{T} g(\dot{t} + \dot{h})\, f(\dot{x} - \dot{t})\,d\dot{t}$$

$$= \int_{T} U_{\dot{h}}g(\dot{t})\, f(\dot{x} - \dot{t})\,d\dot{t}$$

$$= (U_{\dot{h}}g * f)(\dot{x})$$

$$= P(U_{\dot{h}}g)(\dot{x})$$

This completes the proof.