CHAPTER II

DIRECT LIMITS OF ABELIAN GROUPS

The materials of this chapter are drawn from references [1], [11]

<u>Convention</u>. Throughout the chapter, <u>group means abelian group</u>.

The main purpose of this chapter is to characterize the locally cyclic groups in terms of direct limits. Thus we study some basic properties of direct limit groups. Since we would like the chapter and the whole thesis to be relatively self-contained, we begin these discussion by studying the notions of direct product and direct sum of groups.

If G is a group and $S \subseteq G$, the symbol $[S]$ will denote the subgroup of G generated by S ; i.e., the set of elements of G which are finite linear combinations of elements of S with integer coefficients. In particular, if S consists of only a finite number of elements $s_1, s_2, \ldots, s_n$, of G, we will denote $[S]$ by $[s_1, s_2, \ldots, s_n]$.

<u>Definition 2.1</u>. Let $\{B_i\}_{i \in I}$ be a family of subgroups of a group A. A is said to be the <u>internal direct sum of the subgroups B_i</u> if the following conditions hold :

(i) $A = \left[\bigcup_{i \in I} B_i\right]$

and (ii) for each $i \in I$,

$$B_i \cap \left[\bigcup_{j \in I \setminus \{i\}} B_j\right] = 0$$

(where, as usual, 0 denotes both the 0 of A and the trivial subgroup of A) .

Definition 2.2. Let $\{B_i\}_{i \in I}$ be a family of groups. The symbol $\underset{i \in I}{X} B_i$ is used to denote the cartesian product of the B_i . Recall that the elements of $\underset{i \in I}{X} B_i$ can be considered as functions $f : I \longrightarrow \cup \{ B_i / i \in I \}$ with $f(i) \in B_i$ for each $i \in I$ or as ordered tuples $\langle b_i \rangle_{i \in I}$ where the i^{th} entry or component b_i comes from B_i . We will use both notations interchangeably.

The direct product of the B_i, denoted by $\prod_{i \in I} B_i$, consists of the set $\underset{i \in I}{X} B_i$ and the binary operation "+" on $\underset{i \in I}{X} B_i$ defined by

$$(f + g)(i) = f(i) + g(i)$$

where $f, g \in \underset{i \in I}{X} B_i$ and $i \in I$. Note that the addition operation on the right-hand side of the equation is the given group operation of B_i . Moreover, it can easily be checked that $\prod_{i \in I} B_i$ is a group.

In the direct product $\prod_{i \in I} B_i$ of the B_i , we pick out a particular set consists of all those elements of $\prod_{i \in I} B_i$ all of whose components are 0, except possibly for a finite number of components. This set, denoted by $\sum_{i \in I} B_i$, can easily be shown to be a subgroup of $\prod_{i \in I} B_i$, called the external direct sum of the B_i .

For each $i \in I$, the assignment ρ_i sending an element $b \in B_i$ into the element $\rho_i(b) \in \prod_{i \in I} B_i$ all of whose components are 0 except the i^{th} one which is b can easily seen to be an isomorphism of B_i onto a subgroup B_i' of $\prod_{j \in I} B_j$ consisting of elements all of whose components are 0 except possibly the i^{th} component. Moreover, it can easily be checked that the subgroup $\left[\bigcup_{i \in I} B_i'\right]$ generated by $\bigcup_{i \in I} B_i'$ is indeed the internal direct sum of the B_i'. Thus we shall always identify the B_i with the B_i' and the $\sum_{i \in I} B_i$ with the $\left[\bigcup_{i \in I} B_i'\right]$.

We now come to one of the most important applications of direct sum, but first, we need some preliminary definitions

<u>Definition 2.3</u>. Let G be a group. Let

$$tG = \{ x \in G / x \text{ has finite order} \}.$$

Then it can easily check that tG is a subgroup of G, called the <u>torsion subgroup of G</u>. Moreover, G is said to be <u>torsion</u> if $G = tG$ and <u>torsion - free</u> if $tG = 0$.

<u>Theorem 2.4</u>. If G is torsion, then

$$G = \sum_p G_p$$

where the sum over all prime numbers and where

$$G_p = \{ x \in G \ / \text{ the order of } x \text{ is a power of } p \}.$$

G_p is called the p-primary component or subgroup of G.

<u>Proof</u>. Let G be a torsion group and for each prime p, let G_p consist of all g in G whose order is a power of the prime p. Since 0 is in G_p, G_p is not empty. If $x, y \in G_p$, then

there exist non-zero integers m,n such that

$$p^n x = 0 = p^m y .$$

Let $r = \max.(m,n)$. Then

$$p^r(x \pm y) = 0 ;$$

i.e., the order of $x \pm y$ is a power of p, and hence G_p is a subgroup of G . Moreover, G_p is obvious p-primary .

Now we will show that G is isomorphic to $\sum_{p \in \mathbb{P}} G_p$, where $\mathbb{P}$ is the set of prime numbers .

To show that $G = \left[\bigcup_{p \in \mathbb{P}} G_p \right]$, let a non-zero g be in G . Since G is torsion, the order n of g is finite. Let

$$n = p_1^{r_1} p_2^{r_2} \cdots\cdots p_k^{r_k} ,$$

for some $p_1, p_2, \ldots, p_k \in \mathbb{P}$, positive integers $r_1, r_2, \ldots, r_k$ and let

$$n_i = n/p_i^{r_i}$$

for $i = 1,2,\ldots, k$. Then the greatest common divisor of the $n_1, n_2, \ldots, n_k$ is 1 and, therefore, there exist integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$1 = \sum_{i=1}^{k} \alpha_i n_i ,$$

and

$$g = \sum_{i=1}^{k} \alpha_i n_i g .$$

Now

$$p_i^{r_i} \alpha_i n_i g = \alpha_i n g = 0 ,$$

and so the order of $\alpha_i n_i g$ divides $p_i^{r_i}$; i.e ., the order of $\alpha_i n_i g$ is a power of p_i , and hence $\alpha_i n_i g$

is in G_{p_i} . Thus g is in $\left[\bigcup_{p\in\mathbb{P}} G_p\right]$; i.e, $G \subseteq \left[\bigcup_{p\in\mathbb{P}} G_p\right]$. Since the reverse inclusion is obvious, $G = \left[\bigcup_{p\in\mathbb{P}} G_p\right]$.

Next we will show that $G_q \cap \left[\bigcup_{p\neq q} G_p\right] = 0$ for each $q \in \mathbb{P}$. Let $x \in G_q \cap \left[\bigcup_{p\neq q} G_p\right]$. Then

$$x = \sum_{i=1}^{n} \alpha_i g_i$$

for some $g_i \in G_{p_i}$ and $\alpha_i \in \mathbb{Z}$ $(i = 1, 2, \ldots, n)$. Since g_i and, therefore, $\alpha_i g_i$ are in G_{p_i} , the order of $\alpha_i g_i$ is $p_i^{r_i}$ for some positive integers r_i . Hence

$$\left(\prod_{i=1}^{n} p_i^{r_i}\right) x = 0 .$$

But $x \in G_q$ so that the order of x is a power of q , say q^r . Since $q, p_1, p_2, \ldots, p_n$ are all distinct, we must have $r = r_1 = r_2 = \ldots = r_n = 0$ and $x = 0$.

Hence G is the direct sum of the G_p .

Example and Definition. The additive group $\mathbb{Q}/\mathbb{Z}$ of rationals modulo the additive group of integers $\mathbb{Z}$, is clearly torsion so that

$$\mathbb{Q}/\mathbb{Z} = \sum_p (\mathbb{Q}/\mathbb{Z})_p ,$$

by Theorem 2.4. The p-primary component $(\mathbb{Q}/\mathbb{Z})_p$ and any group isomorphic to $(\mathbb{Q}/\mathbb{Z})_p$ are all denoted by $\mathbb{Z}(p^\infty)$ and such groups are said to be of type p^∞.

Note that $\mathbb{Z}(p^\infty) = (\mathbb{Q}/\mathbb{Z})_p$ is isomorphic to the additive group generated by the elements

c_1 , c_2 ....

such that

$$pc_1 = 0 \ , \ pc_2 = c_1 \ , \ \ldots\ldots, \ pc_n = c_{n-1} \ , \ \ldots\ldots$$

It is clear that the order of c_n is p^n and that every element of this group is an integral multiple of c_n for some n. (Cf. [1]) .

<u>Definition 2.5</u>. By a <u>directed</u> <u>set</u> we mean a partially ordered set I , say ordered by $\leqslant$, with the additional property : to each pair i , j , $\in$ I, there is a $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

A <u>direct</u> <u>system</u> or family <u>of</u> <u>groups</u> is a triple $(A_i \ , \ \pi_i^j \ ; \ I)$ where I is a directed set, $\{A_i\}_{i \in I}$ a family of groups indexed by I and where for each pair i , $j \in I$ with $i \leqslant j$, we have a homomorphism

$$\pi_i^j : A_i \longrightarrow A_j$$

moreover, the following conditions are satisfied :

(i) for each $i \in I$,

$$\pi_i^i : A_i \longrightarrow A_i$$

is the identity map ;

(ii) if i , j , $k \in I$ with $i \leqslant j$ and $j \leqslant k$, then

$$\pi_j^k \ \pi_i^j = \pi_i^k \ .$$

Suppose that $(A_i \ , \pi_i^j \ ; \ I)$ is a direct system of groups . Let

$$A = \sum_{i \in I} A_i$$

and let B be the subgroup of A generated by elements of the form

$$a_i - \pi_i^j(a_i)$$

where $a_i \in A_i$ and $i \leq j$. We define the <u>direct limit</u> of $(A_i , \pi_i^j ; I)$ to be the quotient group A/B of A modulo B ; it will be denoted by $\varinjlim_I A_i$ or $\varinjlim A_i$ if the indexing set I is clear from context. We now enumerate some elementary properties of the direct limit which will be needed in our ensuing discussions .

<u>Property (a)</u>. An element

$$a = a_{i_1} + a_{i_2} + \ldots\ldots + a_{i_n}$$

of A is in B if and only if there is a $j \in I$ with $j \geqslant i_1$, i_2 , ..., i_n and such that

$$\pi_{i_1}^j(a_{i_1}) + \pi_{i_2}^j(a_{i_2}) + \ldots\ldots + \pi_{i_n}^j(a_{i_n}) = 0 .$$

<u>Proof</u>. Let $a \in B$. Then

$$a = \sum_{k=1}^{m} n_k(a_{i_k} - \pi_{i_k}^{j_k}(a_{i_k}))$$

for some i_1 , i_2 , ..., i_m , j_1 , j_2 , ..., $j_m \in I$ with $i_k \leq j_k$, $k = 1 , 2 , \ldots, m$ and for some n_1 , n_2 , ..., $n_m \in \mathbb{Z}$ and for some $a_{i_k} \in A_{i_k}$ $(k = 1 , 2 , \ldots, m)$. Since I is directed , we can find a $p \in I$ with $p \geqslant j_1$, j_2 , ..., j_m . Hence

$$\pi_{i_k}^p(a_{i_k}) - \pi_{j_k}^p \pi_{i_k}^{j_k}(a_{i_k}) = \pi_{i_k}^p(a_{i_k}) - \pi_{i_k}^p(a_{i_k}) = 0$$

for $k = 1 , 2 , \ldots, m$ so that

$$\sum_{k=1}^{m} \left\{ \pi^{p}_{i_k}(n_k a_{i_k}) - \pi^{p}_{j_k} \pi^{j_k}_{i_k}(n_k a_{i_k}) \right\} = 0.$$

Thus the condition is satisfied.

Conversely, if

$$a = a_{i_1} + a_{i_2} + \ldots\ldots + a_{i_n}$$

is an element in A with

$$\sum_{k=1}^{n} \pi^{j}_{i_k}(a_{i_k}) = 0$$

for some $j \in I$ such that $j \geqslant i_1, i_2, \ldots, i_n$, then

$$\begin{aligned} a &= a - 0 \\ &= \sum_{k=1}^{n} a_{i_k} - \sum_{k=1}^{n} \pi^{j}_{i_k}(a_{i_k}) \\ &= \sum_{k=1}^{n} (a_{i_k} - \pi^{j}_{i_k}(a_{i_k})) \end{aligned}$$

is in B , as to be proved.

<u>Property (b)</u>. There exist homomorphisms

$$\pi_j : A_j \longrightarrow \varinjlim A_i$$

$(j \in I)$ such that the following diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\ \pi^{j}_{i}\ } & A_j \\ & {\scriptstyle \pi_i} \searrow \quad \swarrow {\scriptstyle \pi_j} & \\ & \varinjlim A_k & \end{array}$$

commute for all $i, j \in I$ with $i \leqslant j$.

<u>Proof</u>. Define

$$\pi_i : A_i \longrightarrow \varinjlim A_k$$

by

$$a_i \longmapsto a_i + B .$$

Then Π_i is clearly a homomorphism . Let i , $j \in I$ with $i \leqslant j$ and let $a_i \in A_i$. Since I is directed , we can find a $k \in I$ such that i , $j \leqslant k$ so that

$$\Pi_i^k(a_i) - \Pi_j^k \Pi_i^j(a_i) = \Pi_i^k(a_i) - \Pi_i^k(a_i) = 0$$

and , by Property (a) ,

$$a_i - \Pi_i^j(a_i) \in B ;$$

i.e. ,
$$\Pi_i(a_i) = \Pi_j \Pi_i^j(a_i) ,$$
as to be proved .

<u>Property (c)</u>. If $a_* \in \varinjlim A_k$, then $a_* = a_i + B = \Pi_i(a_i)$ for some $i \in I$ and $a_i \in A_i$.

<u>Proof.</u> Let $a_* \in \varinjlim A_k = A/B$. Then

$$a_* = (a_{i_1} + a_{i_2} + \ldots\ldots + a_{i_n}) + B$$

for some $a_{i_k} \in A_k$ $(k = 1 , 2 , \ldots , n)$. Pick $i \in I$ so that $i \geqslant i_1 , i_2 , \ldots , i_n$. Then

$$\sum_{k=1}^{n} (a_{i_k} - \Pi_{i_k}^{i}(a_{i_k}))$$

is in B so that

$$\left(\sum_{k=1}^{n} a_{i_k} \right) + B = \left(\sum_{k=1}^{n} \Pi_{i_k}^{i}(a_{i_k}) \right) + B$$

$$= a_i + B$$

where

$$a_i = \sum_{k=1}^{n} \Pi_{i_k}^{i}(a_{i_k})$$

is in A_i , as to be proved .

The following is a restatement of Property(c) .

<u>Property (d)</u>. $\varinjlim A_k = \bigcup_{k \in I} \Pi_k \left[A_k \right]$.

<u>Property (e)</u>. If $\Pi_i(a_i) = 0$ for $a_i \in A_i$, then there is an $j \in I$, $j \geqslant i$ with $\Pi_i^j(a_i) = 0$.

<u>Proof.</u> If $\Pi_i(a_i) = a_i + B = 0$ in A/B , then $a_i \in B$ and Property (e) then follows from Property (a) .

<u>Property (f)</u>. If each of the Π_i^j is one-to-one , then so is each of the Π_i .

<u>Proof</u>. Suppose $\Pi_i(a_i) = 0$ for some $a_i \in A_i$. Then $\Pi_i^j(a_i) = 0$ for some $j \in I$ with $j \geqslant i$, by Property (e) . Since Π_i^j is one-to-one , $a_i = 0$.

<u>Property (g)</u>. If each of the Π_i^j is onto , then so is each of the Π_i .

<u>Proof</u>. Let $a_* \in \varinjlim A_k$. Then $a_* = \Pi_j(a_j) = a_j + B$ for some $j \in I$ and $a_j \in A_j$, by Property (d) . We will show that Π_i is onto , for $i \in I$. Since I is directed , we can find a $k \in I$ with $k \geqslant i , j$, so that

$$\Pi_j(a_j) = \Pi_k \Pi_j^k(a_j)$$

by Property (b) . By the surjectivity of Π_i^k , we can find an $a_i \in A_i$ with

$$\Pi_i^k(a_i) = \Pi_j^k(a_j) .$$

Thus

Thus

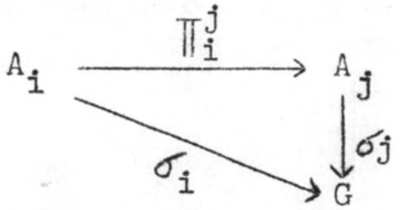
$$a_* = \pi_j(a_j) = \pi_k \pi_j^k(a_j) = \pi_k \pi_i^k(a_i)$$
$$= \pi_i(a_i)$$

by Property (b) . Hence π_i is onto .

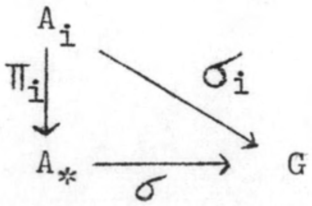
We are now in a position to prove a characteristic property —— the so-called universal property —— of the direct limits .

<u>Theorem 2.6.</u> Let $(A_i\ ,\ \pi_i^j\ ;\ I)$ be a direct system of groups and let A_* be the direct limit of this system .

If G is a group and if $\sigma_i : A_i \longrightarrow G$ are homomorphisms such that for each pair $i\ ,\ j \in I$ with $i \leq j$, the diagram



commutes , then there is one and only one homomorphism $\sigma : A_* \longrightarrow G$ such that for each $i \in I$, the diagram

(*) 

commutes .

Moreover , the direct limit A_* , together with the π_i , is completely determined by this property , up to isomorphism .

<u>Proof</u>. Let $a_* \in A_*$. By Property (d) , there exists an

$i \in I$ and an $a_i \in A_i$ such that

$$a_* = \Pi_i(a_i) = a_i + B .$$

Define

$$\sigma(a_*) = \sigma_i(a_i) .$$

We will show that σ is a map from A_* into G .

If a_* in A_* is represented by the cosets $a_i + B$ and $a_j + B$ for some $a_i \in A_i$ and $a_j \in A_j$, then

$$a_i + B = a_j + B ,$$

so that

$$a_i - a_j$$

is in B . By property (a) , we may choose $k \in I$ with $k \geqslant i$, j and

$$\Pi_i^k(a_i) - \Pi_j^k(a_j) = 0 ,$$

and so

$$\sigma_k (\Pi_i^k(a_i) - \Pi_j^k(a_j)) = 0 .$$

From the commutativity of the given diagrams it follows that

$$\sigma_i(a_i) = \sigma_j(a_j) ,$$

and hence σ is a well-defined map from A_* into G .

Next we will show that σ is a homomorphism . Let a_* , b_* be in A_* and

$$a_* = a_i + B ,$$

$$b_* = a_j + B$$

for some $a_i \in A_i$ and $a_j \in A_j$ in accordance with Property (d) . Choose $k \geqslant i$, j and from the commutativity of the given diagrams ,

$$\sigma_i(a_i) = \sigma_k \pi_i^k(a_i) ,$$
$$\sigma_j(a_j) = \sigma_k \pi_j^k(a_j) .$$

We have

$$a_* = a_i + B = \pi_i(a_i) = \pi_k \pi_i^k(a_i) ,$$
$$b_* = a_j + B = \pi_j(a_j) = \pi_k \pi_j^k(a_j)$$

by Property (b) . Therefore ,

$$a_* + b_* = \pi_k(\pi_i^k(a_i) + \pi_j^k(a_j)) = (\pi_i^k(a_i) + \pi_j^k(a_j)) + B ,$$

and so

$$\sigma(a_* + b_*) = \sigma_k(\pi_i^k(a_i) + \pi_j^k(a_j)) = \sigma_i(a_i) + \sigma_j(a_j)$$
$$= \sigma(a_*) + \sigma(b_*) .$$

Hence σ is homomorphism .

For any $a_i \in A_i$,

$$\pi_i(a_i) = a_i + B = a_*$$

for some $a_* \in A_*$ by Property (d) , and so

$$\sigma \pi_i(a_i) = \sigma(a_*) = \sigma_i(a_i) .$$

Thus the required diagram is commutative .

If there exists a homomorphism $\sigma' : A_* \longrightarrow G$ also making all diagrams of the form (*) commutative , then

$$\sigma \pi_i = \sigma_i = \sigma' \pi_i$$

and so

$$(\sigma - \sigma') \pi_i = 0$$

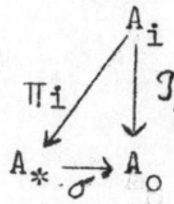
for every $i \in I$. By Property (d) ,

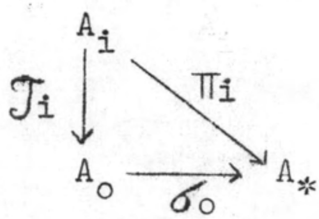
$$\sigma - \sigma' = 0 ,$$

and so σ is unique .

In order to establish the final statement of the theorem , assume that there is a group A_o and homomorphisms

$\mathcal{T}_i : A_i \longrightarrow A_o$ have the property stated for A_* and π_i. From the first part of the theorem, we have a homomorphism $\sigma : A_* \longrightarrow A_o$, a homomorphism $\sigma_o : A_o \longrightarrow A_*$, making the diagrams





commute ; i.e.,

$$\mathcal{T}_i = \sigma \pi_i$$

and

$$\pi_i = \sigma_o \mathcal{T}_i .$$

We obtained

$$\pi_i = \sigma_o \sigma \pi_i .$$

Similarly,

$$\mathcal{T}_i = \sigma \sigma_o \mathcal{T}_i .$$

It now follows from Property (d) that $\sigma\sigma_o$ is the identity homomorphism of A_o and $\sigma_o\sigma$ the identity homomorphism of A_*. Hence A_* and A_o are isomorphic, as to be proved.

To illuminate the concept of direct limit, we will give a few simple examples.

Example 1. Any group A is the direct limit of its finitely generated subgroups. Let $\{A_i\}_{i \in I}$ be the system of all finitely generated subgroups of A where the index set I is partially ordered by $\leq$ as follows :

$i \leq j$ if and only if $A_i \subseteq A_j$.

Let $\pi_i^j : A_i \longrightarrow A_j$ $(i \leq j)$ be the inclusion map, Then

$(A_i \ , \ \pi_i^j \ ; \ I)$ can easily seen to be a directed system . We claim that the direct limit $A_* = \varinjlim A_j$ is isomorphic to A .

Proof. Define a map $\rho : A \longrightarrow A_*$ as follows . For $a \in A$, $[a]$, the subgroup of A generated by a , is finitely generated , let π_a be the cannonical projection from $[a]$ to A_* given by Property (b) .
Define

$$\rho(a) \ = \ \pi_a(a) \ .$$

To show that ρ is a homomorphism , let a , $b \in A$.
Then

$$[a] \ , \ [b] \ \subseteq \ [a,b] \ \equiv \ A_k$$

so if we let π_a^k , and π_b^k be the respective maps from $[a]$ into A_k and from $[b]$ into A_k we have

$$0 = \pi_k(a + b) - \pi_k \pi_a^k(a) - \pi_k \pi_b^k(b) = \pi_k(a + b) - \pi_a(a) - \pi_b(b)$$

in A_* by Property (b) ; hence

$$\rho(a + b) \ = \ \rho(a) + \rho(b) \ .$$

Since the inclusion maps π_i^j are one-to-one , the projections π_i are one-to-one by Property (f) . It then follows from the definition of ρ that ρ is one-to-one as well .

To show that ρ is onto , let $a_* \in A_*$. Then there is an $a \in A_j$ for some $j \in I$, such that

$$\pi_j(a) \ = \ a_*$$

by Property (d). Hence we have

$$\pi_a^j \ : \ [a] \longrightarrow A_j$$

and by Property (b) ,

$$\rho(a) = \pi_a(a) = \pi_j \pi_a^j(a) = \pi_j(a) ;$$

i.e. , ρ is onto .

Hence ρ is an isomorphism of A onto A_* .

Example 2. Let $A = \sum_{i \in I} C_i$, where the C_i are groups and the index set I is linearly ordered by $\leqslant$. Let

$$A_i = \sum_{\substack{k \in I \\ k < i}} C_k$$

and if $i \leqslant j$, let

$$\pi_i^j : A_i \longrightarrow A_j$$

be the natural injection . Then it can easily seen that $(A_i , \pi_i^j ; I)$ is a direct system . We claim that the direct limit $A_* = \varinjlim A_j$ is again isomorphic to A .

Proof. Let $a_* \in A_*$. By Property (d) , we can find a $j \in I$ and an $a_j \in A_j$ with $a_* = \pi_j(a_j)$. Define

$$\rho(a_*) = a_j .$$

Since we consider A_j as a subgroup of A , a_j is an element in A .

Since the maps π_i^j are one-to-one , the maps π_i are one-to-one by Property (f) . Suppose that $a_* \in A_*$ and

$$a_* = \pi_i(a_i) = \pi_j(a_j) ,$$

for some i , $j \in I$ and a_i , $a_j \in A$. Without loss of generality , we may assume that $i \leqslant j$, and so $\pi_i = \pi_j \big| A_i$.

Then

$$\pi_j(a_i) = \pi_j(a_j)$$

and , therefore , $a_i = a_j$. Hence ρ is a well-defined map from A_* to A .

To show that ρ is a homomorphism , let a_*, $b_* \in A_*$. Then

$$a_* = \pi_j(a_j)$$

and

$$b_* = \pi_k(a_k) ,$$

for some j , $k \in I$ and a_j , $a_k \in A$. Again we may assume that $j \leq k$, and so $\pi_j = \pi_k|A_j$. Hence

$$a_* + b_* = \pi_k(a_j) + \pi_k(a_k) = \pi_k(a_j + a_k) ,$$

so that

$$\rho(a_* + b_*) = a_j + a_k = \rho(a_*) + \rho(b_*) .$$

If $\rho(a_*) = 0$ for some $a_* \in A_*$, then $a_* = \pi_j(0)$, for some $j \in I$. Hence $a_* = 0$ and ρ is one-to-one .

To show that ρ is onto , let $a \in A$. Then $a \in A_j$ for some $j \in I$. Let

$$a_* = \pi_j(a) .$$

Then

$$\rho(a_*) = a ,$$

and hence ρ is onto .

Hence ρ is an isomorphism of A_* onto A .

We now come to our main applications of direct limits , but first we need another notion .

Definition 2.7. A group G is said to be locally cyclic if every finitely generated subgroup of it is cyclic ; or equivalently , every finite number of elements belong to a cyclic subgroup .

<u>Examples.</u> Subgroups of the additive groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are known to be locally cyclic (See [11]) , where $\mathbb{Q}$ and $\mathbb{Z}$ denote , respectively , the additive groups of the rationals and integers .

<u>Theorem 2.8.</u> A group G is locally cyclic if and only if it is a direct limit of cyclic groups .

<u>Proof.</u> Assume G is a direct limit of cyclic groups $[a_j]$, $j \in I$. Then $G = A_* = A/B$ where $A = \sum_{i \in I} [a_i]$ and B is the subgroup of A generated by elements of the form

$$b_i - \pi_i^j(b_i)$$

where $b_i \in [a_i]$ and $i \leqslant j$.

For any x_* , $y_* \in G$, by Property (d) ,

$$x_* = b_i + B$$

and

$$y_* = b_j + B ,$$

for some $b_i \in [a_i]$ and $b_j \in [a_j]$. Since I is directed , we can find a $k \in I$ with $k \geqslant i$, j , so that

$$\pi_i(b_i) = \pi_k \pi_i^k(b_i)$$

and

$$\pi_j(b_j) = \pi_k \pi_j^k(b_j)$$

by Property (b) . Then

$$x_* = b_i + B = \pi_i^k(b_i) + B = ma_k + B ,$$

for some integer m , so that

$$x_* = m(a_k + B) ;$$

and also

$$y_* = b_j + B = \Pi^k_j(b_j) + B = na_k + B ,$$

for some integer n , so that

$$y_* = n(a_k + B) .$$

Hence x_* and y_* belong to the cyclic subgroup $[a_k + B]$ of G , and so G is locally cyclic .

Conversely , suppose that G is locally cylic . Then each finitely generated subgroup of G is cyclic so that G is the direct limit of its cyclic subgroups by Example 1 , as to be proved .

Hence the theorem is completely proved .

Theorem 2.9. The direct limit of torsion (torsion-free) groups is again torsion (torsion-free) .

Proof. Assume A_* is the direct limit of torsion groups A_i , $i \in I$. Let $a_* \in A_*$. By Property (d) ,

$$a_* = a_i + B$$

with $a_i \in A_i$, for some $i \in I$ and B is the subgroup of A generated by elements of the form

$$a_j - \Pi^k_j(a_j)$$

where $a_j \in A_j$ and $j \leqslant k$. Since A_i is torsion ,

$$na_i = 0 ,$$

for some integer $n \neq 0$, and so

$$na_* = n(a_i + B) = na_i + B = 0$$

in A/B . Hence a_* is of finite order , and thus A_* is torsion .

Assume A_* is the direct limit of torsion-free groups A_i , $i \in I$. Let $a_* \in A_*$. Then we can write $a_* = a_i + B$, for some $a_i \in A_i$ and $i \in I$, by Property (d) . If a_* is of finite order , then

$$na_* = 0 ,$$

for some integer $n \neq 0$, and so

$$n(a_i + B) = na_i + B = 0 .$$

Hence na_i is in B and , therefore ,

$$n\pi_i^j(a_i) = 0 ,$$

for some $j \in I$ such that $j \geq i$, by Property (a) , contradicting the assumption that A_j is torsion-free . Thus a_* is of infinite order , and so A_* is torsion-free .

Hence the theorem is completely proved .