CHAPTER V

APPLICATIONS TO GEOMETRY

In this chapter, we apply the results in previous chapters to solve some problems in geometry. The main results in this chapter are due to L.Carlitz (see [2]). However, for the proof of the main results we follow the method considered in F.R.Jung [7] for the problem of similar nature. Throughout this chapter, F denotes a finite field of odd order q and of odd characteristic p.

The results of Corollary 4.8 lead to the following theorems.

5.1 Theorem. Let S_n denote an n-dimensional affine space with F as base field. If $n \geqslant 4$, there are no hyperplanes of S_n contained in the complement of the quadric $Q_n(a)$ defined by

$$a_1x_1^2+\dots+a_nx_n^2 = a \qquad (a_1\dots a_n \neq 0).$$

Proof. By the first statement of Corollary 4.8, a hyperplane $L_{n-1}(b)$ of S_n defined by

$$b_1x_1+\dots+b_nx_n = b$$

always has a common point with $Q_n(a)$ if $n \geqslant 4$. Therefore if $n \geqslant 4$, there are no hyperplanes of S_n contained in the complement of $Q_n(a)$.

5.2 Theorem. Let T_n denote an n-dimensional projective space with base field F. If $n \geqslant 3$, a quadric Q_n of T_n defined by

$$(5\text{-}1) \qquad a_0x_0^2+a_1x_1^2+\dots+a_nx_n^2 = 0 \qquad (a_0a_1\dots a_n \neq 0)$$

has at least one point in common with a given hyperplane,

(5-2) $$L_n : b_0x_0 + b_1x_1 + \dots + b_nx_n = 0.$$

Proof. Since Q_n has a point in common with L_n if and only if $N_{s,n+1}(0,0) > 1$, the theorem follows from the second assertion of Corollary 4.8.

Let Q_n denote the quadric of T_n defined by (5-1). By a quadric, we shall mean a diagonal quadric; there is no loss in generality in making such an assumption (for example, see L.E.Dickson [5, §§168]). If $\Psi(a)$ denotes the Legendre symbol in F, that is, $\Psi(a) = -1, -1$, or 0 according as a is a square, a non-square or zero in F, then we define the exterior of Q_n as the set of points $(x_0, x_1, \dots, x_n)$ of T_n such that

$$\Psi(Q_n(x_0, x_1, \dots, x_n)) = +1 ,$$

where $Q_n(x_0, x_1, \dots, x_n) = a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2$. Similarly, the interior of Q_n is the set of points of T_n such that

$$\Psi(Q_n(x_0, x_1, \dots, x_n)) = -1.$$

For a given hyperplane L_n defined by

$$b_0x_0 + b_1x_1 + \dots + b_nx_n = 0,$$

where at least one of the b_j is non-zero, we let $N_E(L_n)$ denote the number of points of L_n in the exterior of Q_n and $N_I(L_n)$ the number of points of L_n in the interior of Q_n. The numbers $N_E(L_n)$ and $N_I(L_n)$ are determined explicitly in Theorem 5.3. Moreover, we find as a direct consequence of Theorem 5.3 that $N_E(L_n) = N_I(L_n)$ or $N_E(L_n) + N_I(L_n) = q^{n-1}$. Finally, we determine the number of points in the interior and in the exterior of Q_n (see Theorem 5.8).

We consider the sum

$$(5\text{-}3) \qquad S = \sum \Psi(a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2),$$

where the summation is over all $x_i \in F$ such that

$$(5\text{-}4) \qquad b_0x_0 + b_1x_1 + \dots + b_nx_n = 0.$$

Clearly,

$$(5\text{-}5) \qquad S = \sum_{a \in F} \Psi(a)N(a),$$

where $N(a)$ denotes the number of solutions of the system of equations

$$a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2 = a$$

$$b_0x_0 + b_1x_1 + \dots + b_nx_n = 0.$$

If $\xi(a) = -1$ or $q-1$ according as $a \neq 0$ or $a = 0$, and if

$$(5\text{-}6) \qquad A = a_0a_1\dots a_n, \quad B = \sum_{i=0}^{n} \frac{b_i^2}{a_i},$$

then by Theorem 4.7, we obtain $N(a)$ as follows.

<u>Case 1.</u> $(B \neq 0 = D)$.

$$N_{s,t}(a,0) = \begin{cases} q^{t-2} + q^{k-1}(q-1)\Psi((-1)^kAB) & \text{if } t = 2k+1, \\ q^{t-2} & \text{if } t = 2k. \end{cases}$$

Using $t = n+1$ and put $k = m$, we get

$$N(a) = \begin{cases} q^{2m-1} + q^{m-1}(q-1)\Psi((-1)^mAB) & \text{if } n = 2m, \\ q^{2m'} & \text{if } n = 2m'+1, \end{cases}$$

where $m' = m-1$.

<u>Case 2</u>. $(B \neq 0 \neq D)$.

$$N_{s,t}(a,0) = \begin{cases} q^{t-2} - q^{k-1}\,\Psi((-1)^k AB) & \text{if } t = 2k+1, \\ q^{t-2} + q^{k-1}\,\Psi((-1)^k AD) & \text{if } t = 2k. \end{cases}$$

Since $D = -aB$, we get

$$N(a) = \begin{cases} q^{2m-1} - q^{m-1}\,\Psi((-1)^m AB) & \text{if } n = 2m, \\ q^{2m'} + q^{m'}\,\Psi((-1)^{m'} aAB) & \text{if } n = 2m'+1. \end{cases}$$

<u>Case 3</u>. $(B = 0 = a)$.

$$N_{s,t}(a,0) = \begin{cases} q^{t-2} + q^{k-1}(q-1)\,\Psi((-1)^k A) & \text{if } t = 2k, \\ q^{t-2} & \text{if } t = 2k+1. \end{cases}$$

Consequently,

$$N(a) = \begin{cases} q^{2m'} + q^{m'}(q-1)\,\Psi((-1)^{m'+1} A) & \text{if } n = 2m'+1, \\ q^{2m-1} & \text{if } n = 2m. \end{cases}$$

<u>Case 4</u>. $(B = 0 \neq a)$.

$$N_{s,t}(a,0) = \begin{cases} q^{t-2} - q^{k-1}\,\Psi((-1)^k A) & \text{if } t = 2k, \\ q^{t-2} + q^{k}\,\Psi((-1)^k aA) & \text{if } t = 2k+1. \end{cases}$$

Consequently,

$$N(a) = \begin{cases} q^{2m'} - q^{m'}\,\Psi((-1)^{m'+1} A) & \text{if } n = 2m'+1, \\ q^{2m-1} + q^{m}\,\Psi((-1)^m aA) & \text{if } n = 2m. \end{cases}$$

Thus when $B \neq 0$, it follows from Case 1 and 2 that

$$(5\text{-}7)\qquad N(a) = \begin{cases} q^{2m-1} + q^{m-1}\,\Psi((-1)^m AB)\,\xi(a) & \text{if } n = 2m, \\ q^{2m'} + q^{m'}\,\Psi((-1)^{m'} aAB) & \text{if } n = 2m'+1. \end{cases}$$

Substituting from (5-7) in (5-5) we get, when $B \neq 0$,

$$(5\text{-}8)\qquad S = \begin{cases} 0 & \text{if } n = 2m, \\ q^{m'}(q-1)\,\Psi((-1)^{m'} AB) & \text{if } n = 2m'+1. \end{cases}$$

When $B = 0$, it follows from Case 3 and 4 that

$$(5\text{-}9)\qquad N(a) = \begin{cases} q^{2m-1} + q^{m}\,\Psi((-1)^m aA) & \text{if } n = 2m, \\ q^{2m'} + q^{m'}\,\Psi((-1)^{m'+1} A)\,\xi(a) & \text{if } n = 2m'+1. \end{cases}$$

It follows that, when $B = 0$,

$$(5\text{-}10)\qquad S = \begin{cases} q^{m}(q-1)\,\Psi((-1)^m A) & \text{if } n = 2m, \\ 0 & \text{if } n = 2m'+1. \end{cases}$$

Let $N_E'(L_n)$ denote the number of solutions $x_0, x_1, \ldots, x_n$ of

$$(5\text{-}11)\qquad b_0x_0 + b_1x_1 + \ldots + b_nx_n = 0$$

such that $\Psi(a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2) = +1$ and $N_I'(L_n)$ denote the number of solutions of (5-11) such that $\Psi(a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2) = -1$.

Then it is clear that

$$(5\text{-}12)\qquad N_E'(L_n) = \frac{1}{2}\sum\{1 + \Psi(a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2)\} - \frac{1}{2}M,$$

$$(5\text{-}13)\qquad N_I'(L_n) = \frac{1}{2}\sum\{1 - \Psi(a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2)\} - \frac{1}{2}M,$$

where in each case the summation is over all $x_0, x_1, \ldots, x_n$ that satisfy (5-11) and M is the number of solutions of the system of equations

$$(5\text{-}14) \qquad \begin{cases} a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2 = 0 \\ b_0x_0 + b_1x_1 + \ldots + b_nx_n = 0. \end{cases}$$

In view of (5-3), (5-12) and (5-13) may be replaced by

$$(5\text{-}15) \qquad N_E'(L_n) = \frac{1}{2}(q^n + S - M)$$

$$(5\text{-}16) \qquad N_I'(L_n) = \frac{1}{2}(q^n - S - M).$$

The number M is determined by using Theorem 4.7 or can be easily obtained from (5-7) and (5-9). Therefore when $B \neq 0$,

$$(5\text{-}17) \qquad M = \begin{cases} q^{2m-1} + q^{m-1}(q-1)\,\Psi((-1)^m AB) & \text{if } n = 2m, \\ q^{2m'} & \text{if } n = 2m'+1; \end{cases}$$

when $B = 0$ we have

$$(5\text{-}18) \qquad M = \begin{cases} q^{2m-1} & \text{if } n = 2m, \\ q^{2m'} + q^{m'}(q-1)\,\Psi((-1)^{m'+1} A) & \text{if } n = 2m'+1. \end{cases}$$

If follows from (5-8) and (5-17) that when $B \neq 0$

$$(5\text{-}19) \qquad S+M = \begin{cases} q^{2m-1} + q^{m-1}(q-1)\,\Psi((-1)^m AB) & \text{if } n = 2m, \\ q^{2m'} + q^{m'}(q-1)\,\Psi((-1)^{m'} AB) & \text{if } n = 2m'+1, \end{cases}$$

$$(5\text{-}20) \qquad S-M = \begin{cases} -q^{2m-1} - q^{m-1}(q-1)\,\Psi((-1)^m AB) & \text{if } n = 2m, \\ -q^{2m'} + q^{m'}(q-1)\,\Psi((-1)^{m'} AB) & \text{if } n = 2m'+1; \end{cases}$$

when B = 0 we get using (5-10) and (5-18)

$$(5\text{-}21)\qquad S+M = \begin{cases} q^{2m-1} + q^{m}(q-1)\,\Psi((-1)^{m}A) & \text{if } n = 2m, \\ q^{2m'} + q^{m'}(q-1)\,\Psi((-1)^{m'+1}A) & \text{if } n = 2m'+1, \end{cases}$$

$$(5\text{-}22)\qquad S-M = \begin{cases} -q^{2m-1} + q^{m}(q-1)\,\Psi((-1)^{m}A) & \text{if } n = 2m, \\ -q^{2m'} - q^{m'}(q-1)\,\Psi((-1)^{m'+1}A) & \text{if } n = 2m'+1. \end{cases}$$

From the definition of $N_E(L_n)$, $N_I(L_n)$, $N_E'(L_n)$ and $N_I'(L_n)$ it is clear that

$$(5\text{-}23)\qquad N_E'(L_n) = (q-1)N_E(L_n) \quad , \quad N_I'(L_n) = (q-1)N_I(L_n).$$

Thus substituting from (5-19),(5-20),(5-21) and (5-22) in (5-15) and (5-16) we obtain the explicit values of $N_E(L_n)$ and $N_I(L_n)$ as follows.

Case $B \neq 0$.

For $n = 2m$,
$$(q-1)N_E(L_n) = \frac{1}{2}\left\{q^{2m} - q^{2m-1} - q^{m-1}(q-1)\,\Psi((-1)^{m}AB)\right\}$$
$$= \frac{1}{2}\left\{q^{2m-1}(q-1) - q^{m-1}(q-1)\,\Psi((-1)^{m}AB)\right\}.$$

Then
$$N_E(L_n) = \frac{1}{2}\left\{q^{2m-1} - q^{m-1}\,\Psi((-1)^{m}AB)\right\}.$$

For $n = 2m'+1$,
$$(q-1)N_E(L_n) = \frac{1}{2}\left\{q^{2m'+1} - q^{2m'} + q^{m'}(q-1)\,\Psi((-1)^{m'}AB)\right\}$$
$$= \frac{1}{2}\left\{q^{2m'}(q-1) + q^{m'}(q-1)\,\Psi((-1)^{m'}AB)\right\}.$$

Then
$$N_E(L_n) = \frac{1}{2}\left\{q^{2m'} + q^{m'}\,\Psi((-1)^{m'}AB)\right\}.$$

For $n = 2m$, $(q-1)N_I(L_n) = \frac{1}{2}\{q^{2m} - q^{2m-1} - q^{m-1}(q-1)\ \Psi((-1)^m AB)\}$

$= \frac{1}{2}\{q^{2m-1}(q-1) - q^{m-1}(q-1)\ \Psi((-1)^m AB)\}.$

Then $N_I(L_n) = \frac{1}{2}\{q^{2m-1} - q^{m-1}\ \Psi((-1)^m AB)\}.$

For $n = 2m'+1$, $(q-1)N_I(L_n) = \frac{1}{2}\{q^{2m'+1} - q^{2m'} - q^{m'}(q-1)\ \Psi((-1)^{m'} AB)\}$

$= \frac{1}{2}\{q^{2m'}(q-1) - q^{m'}(q-1)\ \Psi((-1)^{m'} AB)\}.$

Then $N_I(L_n) = \frac{1}{2}\{q^{2m'} - q^{m'}\ \Psi((-1)^{m'} AB)\}.$

Case $B = 0$.

For $n = 2m$, $(q-1)N_E(L_n) = \frac{1}{2}\{q^{2m} - q^{2m-1} + q^m(q-1)\ \Psi((-1)^m A)\}$

$= \frac{1}{2}\{q^{2m-1}(q-1) + q^m(q-1)\ \Psi((-1)^m A)\}.$

Then $N_E(L_n) = \frac{1}{2}\{q^{2m-1} + q^m\ \Psi((-1)^m A)\}.$

For $n = 2m'+1$, $(q-1)N_E(L_n) = \frac{1}{2}\{q^{2m'+1} - q^{2m'} - q^{m'}(q-1)\ \Psi((-1)^{m'+1} A)\}$

$= \frac{1}{2}\{q^{2m'}(q-1) - q^{m'}(q-1)\ \Psi((-1)^{m'+1} A)\}.$

Then $N_E(L_n) = \frac{1}{2}\{q^{2m'} - q^{m'}\ \Psi((-1)^{m'+1} A)\}.$

For $n = 2m$, $(q-1)N_I(L_n) = \frac{1}{2}\{q^{2m} - q^{2m-1} - q^m(q-1)\ \Psi((-1)^m A)\}$

$= \frac{1}{2}\{q^{2m-1}(q-1) - q^m(q-1)\ \Psi((-1)^m A)\}.$

Then $N_I(L_n) = \frac{1}{2}\{q^{2m-1} - q^m\ \Psi((-1)^m A)\}.$

For $n = 2m'+1$, $\quad (q-1)N_I(L_n) = \frac{1}{2}\{q^{2m'+1} - q^{2m'} - q^{m'}(q-1)\,\Psi((-1)^{m'+1}A)\}$

$$= \frac{1}{2}\{q^{2m'}(q-1) - q^{m'}(q-1)\,\Psi((-1)^{m'+1}A)\}.$$

Then $\quad N_I(L_n) = \frac{1}{2}\{q^{2m'} - q^{m'}\Psi((-1)^{m'+1}A)\}.$

Hence we have the following theorem.

5.3 Theorem. (L.Carlitz [2, Theorem 1]). Let Q_n denote the nonsingular quadric defined by

$$a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2 = 0$$

and let L_n denote the hyperplane

$$b_0x_0 + b_1x_1 + \dots + b_nx_n = 0.$$

Furthermore, let A and B be defined as in (5-6). If $N_E(L_n)$ denotes the number of points of L_n in the exterior of Q_n and $N_I(L_n)$ denotes the number of points of L_n in the interior of Q_n then we have, when $B \neq 0$,

$$N_E(L_n) = \begin{cases} \frac{1}{2}\{q^{2m-1} - q^{m-1}\,\Psi((-1)^m AB)\} & \text{if } n = 2m, \\ \frac{1}{2}\{q^{2m'} + q^{m'}\,\Psi((-1)^{m'} AB)\} & \text{if } n = 2m'+1\ ; \end{cases}$$

$$N_I(L_n) = \begin{cases} \frac{1}{2}\{q^{2m-1} - q^{m-1}\,\Psi((-1)^m AB)\} & \text{if } n = 2m, \\ \frac{1}{2}\{q^{2m'} - q^{m'}\,\Psi((-1)^{m'} AB)\} & \text{if } n = 2m'+1. \end{cases}$$

When $B = 0$, we have

$$N_E(L_n) = \begin{cases} \frac{1}{2}\{q^{2m-1} + q^m \Psi((-1)^m A)\} & \text{if } n = 2m, \\ \frac{1}{2}\{q^{2m'} - q^{m'} \Psi((-1)^{m'+1} A)\} & \text{if } n = 2m'+1 \ ; \end{cases}$$

$$N_I(L_n) = \begin{cases} \frac{1}{2}\{q^{2m-1} - q^m \Psi((-1)^m A)\} & \text{if } n = 2m, \\ \frac{1}{2}\{q^{2m'} - q^{m'} \Psi((-1)^{m'+1} A)\} & \text{if } n = 2m'+1. \end{cases}$$

As immediate consequences of Theorem 5.3, we obtain the following theorems.

5.4 <u>Theorem</u>. With the notation of Theorem 5.3 we have $N_E(L_n) = N_I(L_n)$ when $B \neq 0$ and $n = 2m$ or $B = 0$ and $n = 2m'+1$. In the remaining cases $N_E(L_n) + N_I(L_n) = q^{n-1}$.

5.5 <u>Theorem</u>. $N_E(L_n) = 0$ if and only if one of the following conditions holds.

(i) $B \neq 0$, $n = 1$ and $\Psi(AB) = -1$;

(ii) $B = 0$, $n = 1$ and $\Psi(-A) = +1$;

(iii) $B = 0$, $n = 2$ and $\Psi(-A) = -1$.

$N_I(L_n) = 0$ if and only if one of the following conditions is satisfied.

(i) $B \neq 0$, $n = 1$ and $\Psi(AB) = +1$;

(ii) $B = 0$, $n = 1$ and $\Psi(-A) = +1$;

(iii) $B = 0$, $n = 2$ and $\Psi(-A) = +1$.

5.6 <u>Theorem</u>. Let N_E denote the number of points in the exterior of Q_n. Let P be the number of solutions of the equation

(5-24) $$\Psi(a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2) = +1.$$

Then $P = (q-1)N_E$.

<u>Proof</u>. Let $x = (x_0, x_1, \dots, x_n)$ be a point in the exterior of Q_n. Then we get $\Psi(a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2) = +1$. Given β be any non-zero element of F and let $\beta x = (\beta x_0, \beta x_1, \dots, \beta x_n)$, then $\Psi(a_0(\beta x_0)^2 + a_1(\beta x_1)^2 + \dots + a_n(\beta x_n)^2) = \Psi(\beta^2(a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2))$ $= +1$. Hence βx is a solution of (5-24). Also, it is clear that any solution of (5-24) is of the form $\beta x = (\beta x_0, \beta x_1, \dots, \beta x_n)$ where $(x_0, x_1, \dots, x_n)$ is in the exterior of Q_n. Since the number of non-zero elements in F is $(q-1)$, we therefore have $(q-1)N_E = P$.

Similarly, we obtain

5.7 <u>Theorem</u>. The number of solutions of the equation

$$\Psi(a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2) = -1$$

is $(q-1)N_I$, where N_I denotes the number of points in the interior of Q_n.

Let N be the number of solutions of

(5-25) $$a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2 = 1.$$

It follows that $N_E = N/2$. For if $x = (x_0, x_1, \dots, x_n)$ is a solution of (5-25), then $\theta x = (\theta x_0, \theta x_1, \dots, \theta x_n)$ where $\theta \in F^*$ is also a solution of (5-24). Thus for every two solutions $\pm x = (\pm x_0, \pm x_1, \dots, \pm x_n)$ of (5-25), there are $(q-1)$ solutions of (5-24). Conversely, any solution of (5-24) is of the form θx, where x is a solution of (5-25) and $\theta \in F^*$. Thus $P = N(q-1)/2$. Consequently, by Theorem 5.6 we have $N_E = \frac{P}{q-1} = \frac{N}{2}$.

Similarly, N_I is half the number of solutions of

$$a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2 = \mu,$$

where μ is a fixed non-square of F.

By Theorem 4.6, we have therefore the following result.

5.8 Theorem. If Q_n denotes a non-singular quadric of discriminant A, that is, $A = a_0a_1 \dots a_n$, then

$$N_E = \begin{cases} \frac{1}{2}\{q^{2m} + q^m \Psi((-1)^m A)\} & \text{if } n = 2m, \\ \frac{1}{2}\{q^{2m'+1} - q^{m'} \Psi((-1)^{m'+1} A)\} & \text{if } n = 2m'+1 ; \end{cases}$$

$$N_I = \begin{cases} \frac{1}{2}\{q^{2m} - q^m \Psi((-1)^m A)\} & \text{if } n = 2m, \\ \frac{1}{2}\{q^{2m'+1} - q^{m'} \Psi((-1)^{m'+1} A)\} & \text{if } n = 2m'+1 ; \end{cases}$$

where $m' = m-1$.

5.9 Theorem. With the notation of Theorem 5.8 we have $N_E = N_I$ when n is odd and $N_E + N_I = q^n$ when n is even.