CHAPTER I

EULER'S ANGLES AND THE QUATERNIONS

1.1 Euler's Angles

Euler's angles are used to specify the orientation of a rigid body in space. We suppose that one point O in the body is fixed in space, and the body is free to rotate in any arbitrary manner about O.

Let OX, OY, OZ be axes fixed in space and let OX', OY', OZ' be axes with the same origin fixed in the body as in figure 1. Let θ be the angle ZOZ' measured from OZ toward OX. Let ϕ be the angle between the planes XOZ and $Z'OZ$, measured from OX toward OY. In the case where OZ' coincides with OZ, ϕ is the angle between OX and OX', measured in the same direction as a rotation from OX toward OY. Let ψ be the angle between the planes $Z'OZ$ and $X'OZ'$ measured in the same direction as a rotation from OX' toward OY'. Then θ, ϕ and ψ are the Euler angles for the body.

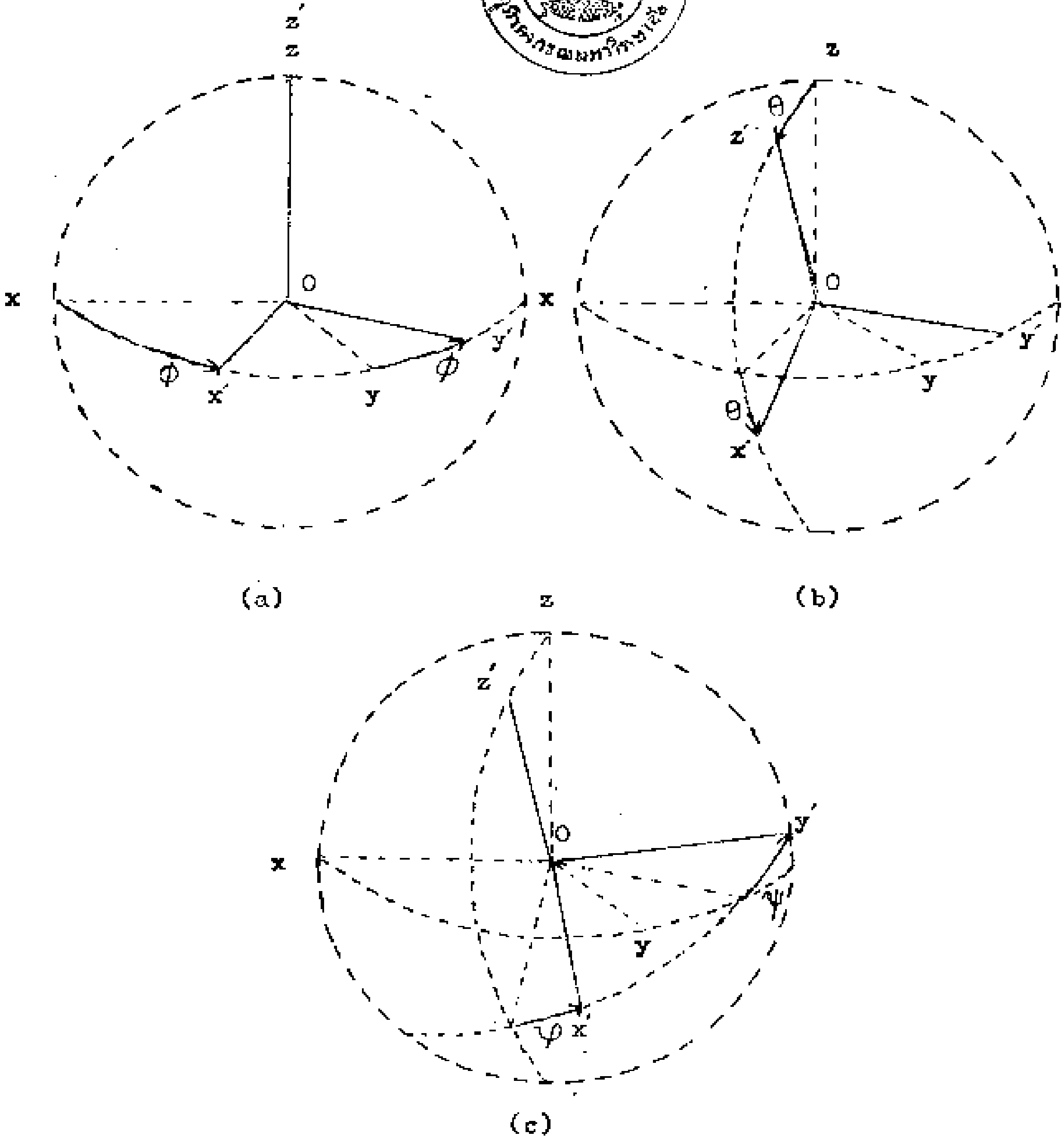

Figure 1

1.2 Quaternions

A quaternion is a hypercomplex number in the form

$$q = d.1 + a.i + b.j + c.k$$

where d, a, b, c are real parameters called the coefficients of the quaternion. The elements i, j, k are the basis elements which have the properties :

1) $1^2 = 1, \quad i.1 = 1.i = i,$

$j.1 = 1.j = j, \quad k.1 = 1.k = k$

2) $i^2 = j^2 = k^2 = -1$

3) $ij = k, \quad jk = i, \quad ki = j$

4) $ji = -k, \quad kj = -i, \quad ik = -j.$

It is obvious that the commutative law for multiplication is not obeyed. The coefficient of 1 is called the "scalar part" of the quaternion. The sum ai+bj+ck is called the "vector part" of the quaternion.

Let $p = d + ai + bj + ck$

and $q = w + xi + yj + zk$

be any two quaternions. Then the product of the two quaternions is:

$$q' = p.q = w' + x'i + y'j + z'k$$

$$= (d + ai + bj + ck)(w + xi + yj + zk)$$

$$= (dw - ax - by - cz) + (aw + dx + bz - cy)\, i$$

$$+ (bw + dy + cx - az)\, j + (cw + dz + ay - bx)\, k$$

On the other hand

$$q.p = (w + xi + yj + zk)(d + ai + bj + ck)$$

$$= (wd - ax - by - cz) + (aw + dx - bz + cy)\, i$$

$$+ (bw + dy - cx + az)\, j + (cw + dz - ay + bx)\, k$$

Although the commutative law fails for mutliplication the distributive and associative laws hold :

$$p(q + q_1) = pq + pq_1$$

and $$pqq_1 = p(qq_1) = (pq)q_1$$

We define the <u>conjugate value</u> $\bar{p}$ of p by the equation

$$\bar{p} = d - ai - bj - ck.$$

<u>1.3 The Relation between Euler's Angles and Querternions</u>

It is clear from figure that the axes OX', OY', OZ' can be moved from an orientation coincident with the fixed axes OX , OY , OZ to the orientation specified by the Euler's angles ϕ, θ, ψ by the following process :

1. Rotate the axes OX', OY' about OZ through an angle ϕ
2. Rotate the axes OZ', OX' about OY' through an angle θ
3. Rotate the axes OX', OY' about OZ' through an angle ψ

After the three rotations ϕ, θ, ψ the point (x,y,z) becomes (x', y', z') where

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Thus by matrix multiplication :-

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi & \cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi & -\cos\psi\sin\theta \\ -\sin\psi\cos\psi\cos\theta - \cos\psi\sin\phi & \sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or

$$\left.\begin{aligned} x' &= (\cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi)\,x \\ &\quad + (\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi)\,y - \cos\psi\sin\theta\,.z \\ y' &= (-\sin\psi\cos\phi\cos\theta - \cos\psi\sin\phi)\,x \\ &\quad + (-\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi)\,y + \sin\psi\sin\theta\,.\,z \\ z' &= \sin\theta\cos\phi\,.x + \sin\theta\sin\phi\,.y + \cos\theta\,.z \end{aligned}\right\} \quad \ldots\ldots(1)$$

We shall first show that a rotation in two dimensions can be represented by a complex number. Let $Z = x + iy$ denote the point (x, y) and let $Z' = x' + iy'$ denote the point (x', y'). If θ is the angle through which the axes OX', OY' have been rotated relative to OX, OY we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2)$$

This can be written in the form

$$Z' = UZ\,,$$

where $u = \cos\theta - i\sin\theta$. For, the equation

$$Z' = U.Z \quad \text{becomes}$$

$$x' + y'i = (\cos\theta - i\sin\theta)(x + iy)$$

$$= x\cos\theta + y\sin\theta - x\sin\theta\,.i + y\sin\theta\; i$$

Then by equating real and imaginary parts on both sides of the equation we obtain the same transformation as (2)

We shall now try to represent the rotation shown in figure 1 (a), in a similar quaternion form. Let $q = xi + yj + zk$ denote the point (x, y, z), and let $q' = x'i + y'j + z'k$ denote the point (x', y', z'). The relation is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ (3)$$

Suppose that $p = d + ai + bj + ck$ is the quaternion that represents this rotation, and suppose that the transformation may be written

$$q' = p.q \qquad \ldots\ldots\ldots\ldots\ (4)$$

We have

$$x'i + y'j + z'k = (-ax - by - cz) + (dx + bz - cy)\,i + (dy - az + cx)j + (dz + ay - bx)\,k$$

We require

$$\begin{aligned} -ax - by - cz &= 0 \\ dx + bz - cy &= \cos\phi\, x + \sin\phi\, .y \\ dy - az + cx &= -\sin\phi\, x + \cos\phi\, .y \\ dz + ay - bx &= z \end{aligned}$$

From the fourth equation $d = 1$, $a = 0$, $b = 0$, and then from the first equation $c = 0$. Now the second and third equations are satisfied only when $\phi = 0$. Hence the quaternion equation $q' = pq$ cannot represent the rotation shown in figure 1(a).

It has been shown by Cayley (1, pp. 69-70) that a rotation in three dimensions can be expressed by the quaternion equation

$$q' = p \cdot q \cdot \bar{p} , \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

where p, q, q' are quaternions, and $\bar{p}$ is the quaternion conjugate of p.

$$q = xi + yj + zk ,$$

$$p = d + ai + bj + ck ,$$

$$\bar{p} = d - ai - bj - ck ,$$

$$q' = x'i + y'j + z'k .$$

The rotation in figure 1(a) is represented by equation (3). Which may also be written

$$\left. \begin{aligned} x' &= x \cos\phi + y \sin\phi \\ y' &= -x \sin\phi + y \cos\phi \\ z' &= z \end{aligned} \right\} \qquad \ldots\ldots\ldots\ldots (6)$$

Suppose this tranformation can be written in quaternion form

$$q' = p_1 \, q \, \bar{p}_1 ,$$

$$x'i + y'j + z'k = (d+ai+bj+ck)(xi+yj+zk)(d-ai-bj-ck) \ldots\ldots\ldots (7)$$

By quaternion multiplication, and comparing coefficients of the quaternion basis elements 1, i, j, k we get

$$\left. \begin{aligned} x' &= (d^2+a^2-b^2-c^2)x+2(ab-cd)y+2(ac+bd)z \\ y' &= 2(ab+cd)x+(d^2+b^2-c^2-a^2)y+2(bc-ad)z \\ z' &= 2(ac-bd)x+2(bc+ad)y+(d^2+c^2-a^2-b^2)z \end{aligned} \right\} \qquad \ldots\ldots\ldots (8)$$

Since x, y, z are arbitrary , the coefficients from the systems (6) and (8) are equal. We get nine equations :-

$$d^2 + a^2 - b^2 - c^2 = \cos\phi \quad \ldots\ldots\ldots\ldots (9)$$

$$2(ab - cd) = \sin\phi \quad \ldots\ldots\ldots\ldots (10)$$

$$2(ac + bd) = 0 \quad \ldots\ldots\ldots\ldots (11)$$

$$2(ab + cd) = -\sin\phi \quad \ldots\ldots\ldots\ldots (12)$$

$$d^2 + b^2 - a^2 - c^2 = \cos\phi \quad \ldots\ldots\ldots\ldots (13)$$

$$2(bc - ad) = 0 \quad \ldots\ldots\ldots\ldots (14)$$

$$2(ac - bd) = 0 \quad \ldots\ldots\ldots\ldots (15)$$

$$2(bc + ad) = 0 \quad \ldots\ldots\ldots\ldots (16)$$

$$d^2 + c^2 - a^2 - b^2 = 1 \quad \ldots\ldots\ldots\ldots (17)$$

From (11) and (15) $\quad ac = 0 , \quad bd = 0 \quad \ldots\ldots\ldots\ldots (18)$

From (14) and (16) $\quad bc = 0, \quad ad = 0 \quad \ldots\ldots\ldots\ldots (19)$

From (10) and (12) $\quad ab = 0 , \quad cd = -\frac{1}{2}\sin\phi \quad \ldots\ldots (20)$

From (9) and (13) $\quad d^2 - c^2 = \cos\phi , \quad a^2 - b^2 = 0$ or $a^2 = b^2 \quad \ldots(21)$

From (20) and (21) $\quad a = 0 , \quad b = 0$

Equations (18) and (19) are now useless :

There remain only

$$cd = -\frac{1}{2}\sin\phi \quad \ldots\ldots\ldots\ldots (22)$$

$$d^2 - c^2 = \cos\phi \quad \ldots\ldots\ldots\ldots (23)$$

From (17) $\quad d^2 + c^2 = 1 \quad \ldots\ldots\ldots\ldots (24)$

From (22) and (24) $\quad d^2 + 2cd + c^2 = 1 - \sin\phi$

$$d + c = \pm\sqrt{1 - \sin\phi} \quad \ldots\ldots\ldots\ldots (25)$$

and $d^2 - 2cd + c^2 = 1 + \sin\phi$

$$d - c = \pm\sqrt{1 + \sin\phi} \quad \ldots\ldots\ldots\ldots (26)$$

From (25) and (26) $\quad d = \pm\frac{1}{2}\left[\sqrt{1-\sin\phi} + \sqrt{1+\sin\phi}\right] \quad \ldots\ldots\ldots (27)$

$$c = \pm\frac{1}{2}\left[\sqrt{1-\sin\phi} - \sqrt{1+\sin\phi}\right] \quad \ldots\ldots\ldots (28)$$

Where the signs outside the brackets in (27) and (28) must be the same. It is easy to verify that (27) and (28) satisfy (23).

So $p_1 = \pm \frac{1}{2}\left[\left\{ \sqrt{1-\sin\phi} + \sqrt{1+\sin\phi} \right\} + \left\{ \sqrt{1-\sin\phi} - \sqrt{1+\sin\phi} \right\} k \right].$

For convenience we choose :-

$$p_1 = \frac{1}{2}\left[\left\{ \sqrt{1-\sin\phi} + \sqrt{1+\sin\phi} \right\} + \left\{ \sqrt{1-\sin\phi} - \sqrt{1+\sin\phi} \right\} k \right] \quad \ldots(29)$$

Similarly, rotates about OY' through an angle θ, we get :-

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We obtain nine equations which give the solution :-

$$a = c = 0,$$

$$d = \pm \frac{1}{2}\left[\sqrt{1-\sin\theta} + \sqrt{1+\sin\theta} \right],$$

$$b = \pm \frac{1}{2}\left[\sqrt{1-\sin\theta} - \sqrt{1+\sin\theta} \right].$$

So $p_2 = \frac{1}{2}\left[\left(\sqrt{1-\sin\theta} + \sqrt{1+\sin\theta} \right) + \left(\sqrt{1-\sin\theta} - \sqrt{1+\sin\theta} \right) j \right] \quad \ldots(30)$

And also when rotates about OZ' through an angle ψ, we get

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We obtain nine equations which give the solution :-

$$a = b = 0,$$

$$d = \pm \frac{1}{2}\left[\sqrt{1-\sin\psi} + \sqrt{1+\sin\psi} \right]$$

$$c = \pm \frac{1}{2}\left[\sqrt{1-\sin\psi} - \sqrt{1+\sin\psi} \right]$$

$$p_3 = \frac{1}{2}\left[\left(\sqrt{1 - \sin\psi} + \sqrt{1 + \sin\psi} \right) + \left(\sqrt{1 - \sin\psi} - \sqrt{1 + \sin\psi} \right) k \right] . \qquad \ldots\ldots\ldots\ldots(31)$$

__Example__ Consider the successive rotations in figure (2), namely a rotation (1) of $\frac{\pi}{2}$ about the OX' axis and a rotation (2) of $-\frac{\pi}{2}$ about the OZ' axis.

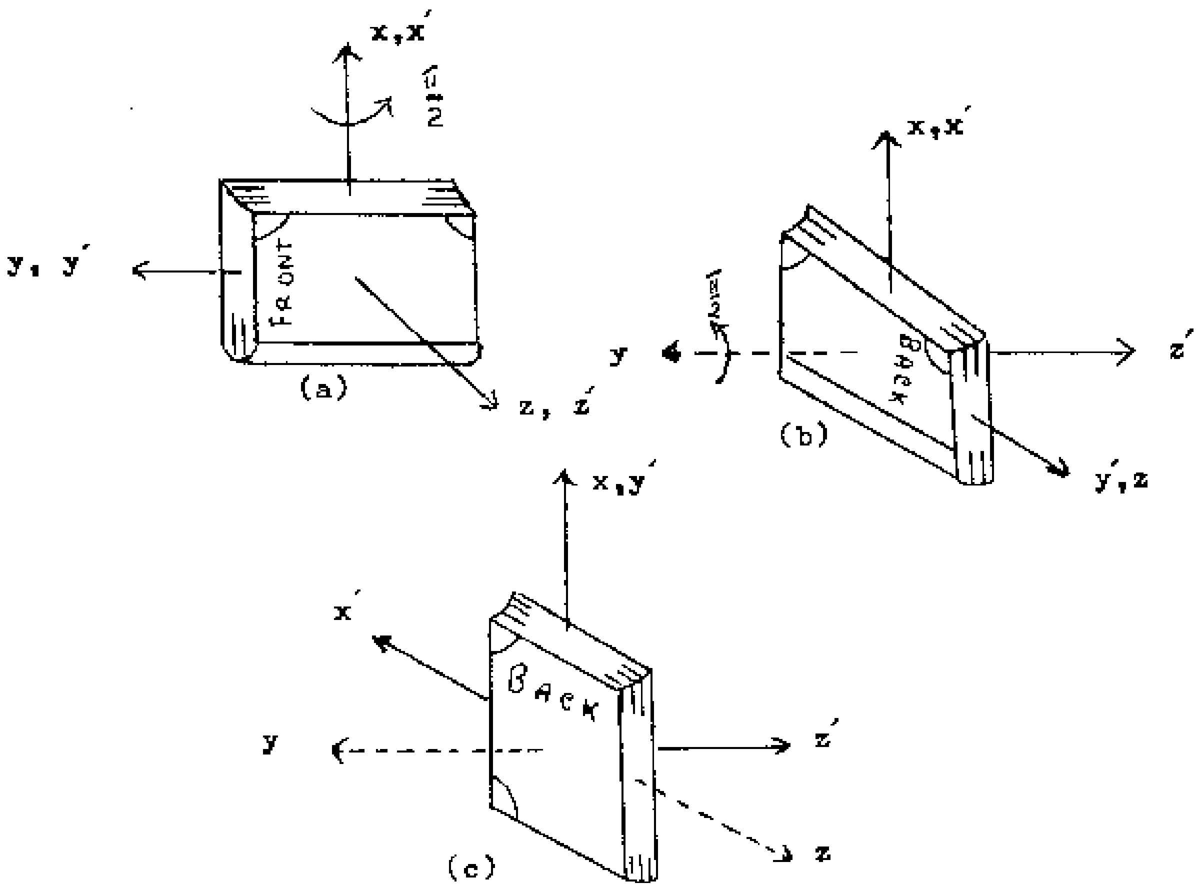

Figure 2

We shall show that the product of the quaternions for the succesive rotations (1) and (2) is the same as the quaternions for the resultant rotation .

For rotation (1), from figure 1(a) to figure 1(b) , the Euler angles are

$$\phi = \frac{3\pi}{2}, \quad \theta = \frac{\pi}{2}, \quad \psi = \frac{\pi}{2}$$

Then by (29)

$$P_1 = \frac{1}{2}\left[\left\{\sqrt{1 - \sin\frac{3\pi}{2}} + \sqrt{1 + \sin\frac{3\pi}{2}}\right\} + \left\{\sqrt{1 - \sin\frac{3\pi}{2}} - \sqrt{1 + \sin\frac{3\pi}{2}}\right\} k\right]$$

$$= \frac{1}{\sqrt{2}}(1 + k)$$

By (30) ,

$$P_2 = \frac{1}{2}\left[\left\{\sqrt{1 - \sin\frac{\pi}{2}} + \sqrt{1 + \sin\frac{\pi}{2}}\right\} + \left\{\sqrt{1 - \sin\frac{\pi}{2}} - \sqrt{1 + \sin\frac{\pi}{2}}\right\} j\right]$$

$$= \frac{1}{\sqrt{2}}(1 - j)$$

By (31) ,

$$P_3 = \frac{1}{2}\left[\left\{\sqrt{1 - \sin\frac{\pi}{2}} + \sqrt{1 + \sin\frac{\pi}{2}}\right\} + \left\{\sqrt{1 - \sin\frac{\pi}{2}} - \sqrt{1 + \sin\frac{\pi}{2}}\right\} k\right]$$

$$= \frac{1}{\sqrt{2}}(1 - k)$$

Then the quaternion for rotation (1) is

$$p' = p_3 \cdot p_2 \cdot p_1$$

$$= \frac{1}{\sqrt{2}}(1 - k) \cdot \frac{1}{\sqrt{2}}(1 - j) \cdot \frac{1}{\sqrt{2}}(1 + k)$$

$$= \frac{1}{\sqrt{2}}(1 - i)$$

For rotation (2) from figures 2(b) to 2(c) , the Euler angles are

$$\phi = -\frac{\pi}{2}, \quad \theta = 0, \quad \psi = 0,$$

and by the same method as that given above we find that the quaternion for rotation (2) is

$$p'' = \frac{1}{\sqrt{2}}(1 + k) .$$

The quaternion for the result of performing first rotation (1) and then rotation (2) is

$$p''p' = \frac{1}{\sqrt{2}}(1 + k) \cdot \frac{1}{\sqrt{2}}(1 - i)$$

$$= \frac{1}{2}(1 - i - j + k)$$

For rotation (3), from figures 2(a) to 2(c), the Euler angles are

$$\phi = \frac{3\pi}{2}, \quad \theta = \frac{\pi}{2}, \quad \psi = 0$$

Therefore the quaternion for the rotation is

$$p = \frac{1}{2}(1 - i - j + k)$$

Then, the quaternion which we get from the product of the quaternion for rotation (1) and quaternion for rotation (2) is equal to the quaternion for rotation (3) .

We shall now examine the result of performing rotation (1) and (2) in the reverse order by the succesive rotations as in figure 3.

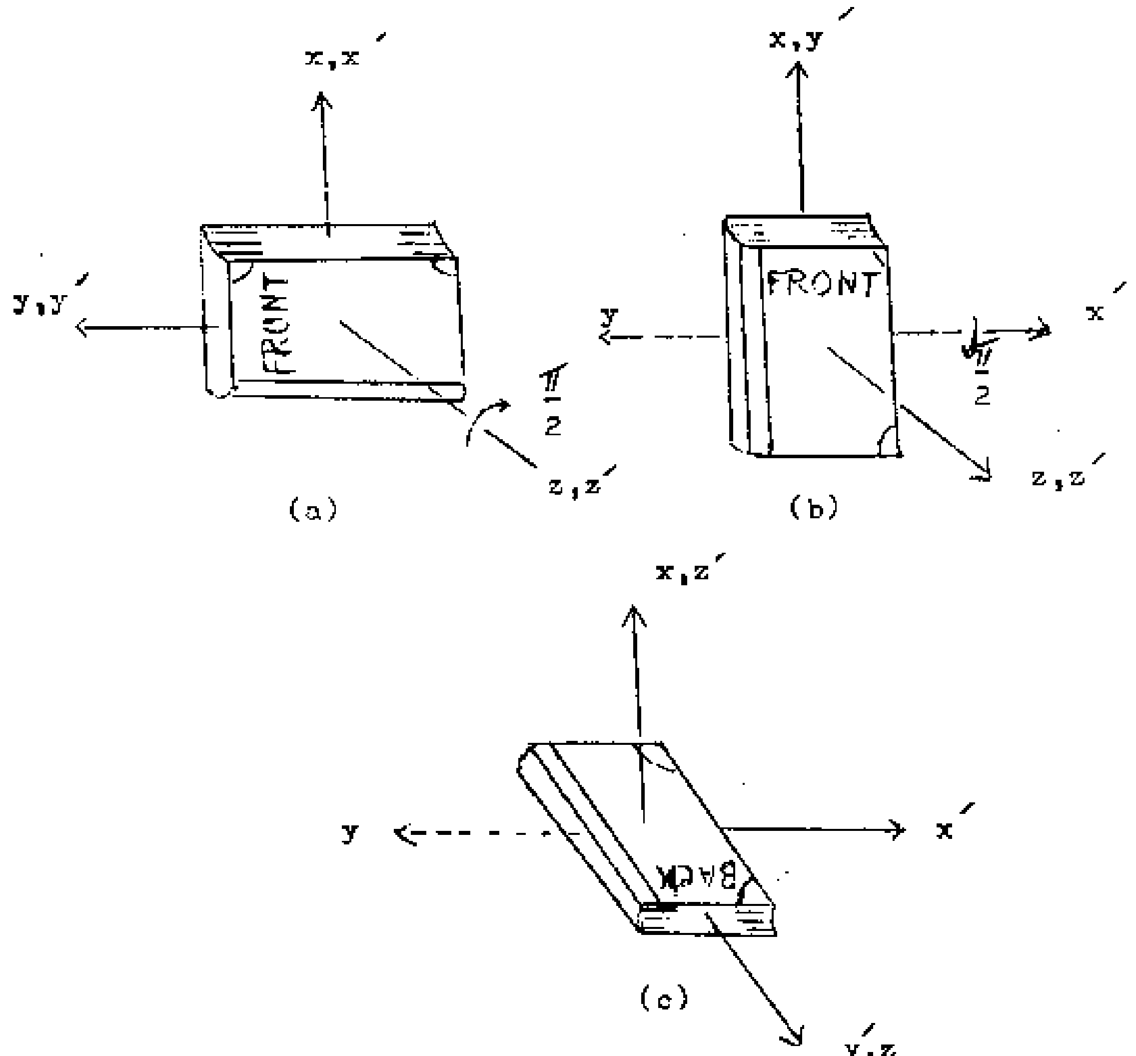

Figure 3

The quaternion for the result of performing first rotation (2) and then rotation (1) is

$$pp'' = \frac{1}{2}(1 - i) \cdot \frac{1}{2}(1 + k)$$

$$= \frac{1}{2}(1 - i + j + k)$$

For rotation (4), from figure 3(a) to 3(c), the Euler angles are

$$\phi = 0, \quad \theta = -\frac{\pi}{2}, \quad \psi = -\frac{\pi}{2}$$

By (29)

$$p_1 = \frac{1}{2}\left[\left\{\sqrt{1-\sin 0} + \sqrt{1+\sin 0}\right\} + \left\{\sqrt{1-\sin 0} - \sqrt{1+\sin 0}\right\}k\right]$$

$$= 1$$

By (30)

$$p_2 = \frac{1}{2}\left[\left\{\sqrt{1-\sin\frac{-\pi}{2}} + \sqrt{1+\sin\frac{-\pi}{2}}\right\} + \left\{\sqrt{1-\sin\frac{-\pi}{2}} - \sqrt{1+\sin\frac{-\pi}{2}}\right\} j\right]$$

$$= \frac{1}{\sqrt{2}}(1+j)$$

By (31)

$$p_3 = \frac{1}{2}\left[\left\{\sqrt{1-\sin\frac{-\pi}{2}} + \sqrt{1+\sin\frac{-\pi}{2}}\right\} + \left\{\sqrt{1-\sin\frac{-\pi}{2}} - \sqrt{1+\sin\frac{-\pi}{2}}\right\} k\right]$$

$$= \frac{1}{\sqrt{2}}(1 + k)$$

$$p = p_3 \cdot p_2 \cdot p_1$$

$$= \frac{1}{\sqrt{2}}(1 + k) \cdot \frac{1}{\sqrt{2}}(1 + j) \cdot 1$$

$$= \frac{1}{2}(1 - i + j + k)$$

Then, the quaternion which we get from rotation (4) is equal to the product of the quaternions which we get from rotations (2) and (1).

From this example we can see the noncommutative law of the multiplication of the quaternions and the product of the rotations.

Similarly , given the quaternion

$$p = d + ai + bj + ck ,$$

successive rotations through angles ϕ , θ , ψ , can be represented by the quaternion equation

$$q' = p \cdot q \cdot \bar{p}$$

By comparing coefficients of x, y, z from systems (1) and (8) we obtain

$$\cos\Theta = d^2 + c^2 - a^2 - b^2$$

$$\sin\Theta \cos\phi = 2(ac - bd) ,$$

$$-\sin\Theta \cos\Psi = 2(ac + bd) ,$$

Then from these three equations we get Euler's angles in terms of the coefficients of the quaternion as follows :

$$\Theta = \cos^{-1} (d^2 + c^2 - a^2 - b^2)$$

$$\phi = \cos^{-1} \frac{ac - bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

$$\Psi = \cos^{-1} \frac{-(ac + bd)}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$