CHAPTER V

A METHOD FOR FINDING A MAGIC SQUARE OF ANY EVEN ORDER

5.1 Most rules for the construction of magic squares of any even order fall into two categories, according as the even order n is singly even, that is of the form 2(2m+1); or doubly even, that is of the form 4m.

We shall use some properties of the magic squares discussed in chapter IV for finding the magic squares of even order.

From 4.1 we know that from every given pair of orthogonal Latin squares (this is often referred to in the literature as a Graeco-Latin square or Euler square) a quasi-magic square can be obtained, and by a suitable arrangement on the two main diagonals of the orthogonal Latin squares we can get a magic square.

Our first step for finding a magic square of any even order is therefore to construct a pair of orthogonal Latin squares of even order. This can be done for every even order[1] except order 6. But it is not easy to construct a pair of orthogonal Latin squares of singly even order.

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[1]Ryser, Herbert John. Combinatorial Mathematics (New Jersey: The Mathematical Association of America, 1963), pp. 84-85.

Hence we shall not use this method for the singly even orders. However, we can easily construct a pair of orthogonal Latin squares for orders that are integral powers of 2 by using properties of finite fields which are known as Galois fields and designated by the notation $GF(p^u)$, where p is a prime and u is a positive integer. In a Galois field, if N denotes the number of elements in the field, it is well known that we must have $N = p^u$. Now since there is only one prime even number, that is 2, we therefore can find the Galois field of 2^u elements from which the pair of orthogonal Latin squares of order 2^u can be obtained[2]. And by selecting a pair of orthogonal Latin squares such that the two main diagonals are also magic, as in 2.14, we can get a magic square.

5.2 We shall now proceed to construct the Galois field of 4 elements. This can be done by adjoining to GF(2) (a field with two elements) a root of an irreducible quadratic polynomial[3]. In GF(2) x^2+x+1 is the irreducible quadratic polynomial. Suppose there were a number t in GF(2) which

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[2]Mann, H.B. Analysis and Design of Experiments. (New York: Dover Publications, Inc., 1949), p.95.

[3]Johnson, Richard E. University Algebra. (New Jersey: Prentice-Hall, Inc., 1966), pp. 107-108.

satisfied $t^2+t+1 = 0$ in GF(2). This t is called a Galois imaginary. Consider the set of expressions

$$a_0+a_1t+a_2t^2+\ldots\ldots+a_{n-1}t^{n-1}+a_nt^n, \qquad (a_i \in F)$$

Note that $t^2 = -t-1 = t+1$

$$t^3 = t.t^2 = t(t+1) = t^2+t = t+1+t = 1$$

etc.

Hence any polynomial in t can be written in the form a+bt. Since there are two numbers in GF(2) which are 0 and 1, the set of all elements of the form a+bt has 4 elements 0, 1, t, 1+t. We can easily verify that this set of elements forms a field.

When the elements of a Galois field are known then the construction of a pair of orthogonal Latin squares can be simplified considerably by constructing 2 addition tables with suitable elements in the diagonals as follows:

In table 1 put 0, 1, t, 1+t at the head of the columns and t times these elements, viz, 0, t, 1+t, 1 at the beginning of the rows.

In table 2 put 0, 1, t, 1+t at the head of the columns and (1+t) times these elements, viz, 0, 1+t, 1, t at the beginning of the rows.

These two addition tables are:

Table 1.

+	0	1	t	1+t
0	0	1	t	1+t
t	t	1+t	0	1
1+t	1+t	t	1	0
1	1	0	1+t	t

Table 2.

+	0	1	t	1+t
0	0	1	t	1+t
1+t	1+t	t	1	0
1	1	0	1+t	t
t	t	1+t	0	1

Figure 5.1 Two additional tables of GF(2^2).

Each table is a Latin square in which both main diagonals contain the complete set of elements. Putting these two tables together and replacing 0 by 0, 1 by 1, t by 2 and 1+t by 3 we get the following orthogonal Latin square.

00	11	22	33
23	32	01	10
31	20	13	02
12	03	30	21

Figure 5.2 A pair of orthogonal Latin squares of order 4.

This is equivalent to a pair of orthogonal Latin squares of q and s as in 4.1 and both values of q and s run through the integers from 0 to 3 in every row, every column and both main diagonals. We can therefore replace the ordered pair (q,s) by the number m given by m = nq+s+1. This can be simply done by replacing the ordered pair (0,0) by the integer 1, (0,1) by 2, (0,2) by 3 and so on. We then get the following magic square.

1	6	11	16
12	15	2	5
14	9	8	3
7	4	13	10

Figure 5.3 A magic square of order 4.

Now when the magic square of 4^{th} order is known we can construct magic squares of order 4g, where g is any odd number by using the properties of magic squares discussing in 4.3. That is we can now construct magic squares of orders 4, 12, 16, 20, 28, 36, 44, and so on.

5.3 We shall now consider the construction of a magic square of order 8. We can do this in a way similar to that used for constructing the magic square of order 4. That is we first find the Galois field of 8 elements by adjoining to GF(2) a root of an irreducible polynomial of degree 3. Since x^3+x+1 is an irreducible polynomial in GF(2), we can use it to construct a Galois field of 8 elements as before, and we obtain 0, 1, t, t^2, 1+t, $t+t^2$, $1+t+t^2$, $1+t^2$. It can be easily verified that all of these elements form a field.

Now we construct 2 addition tables such that the main diagonals of both tables contain a complete set of elements of this Galois field. These two addition tables can be constructed using:

(1). $0,1,t,t^2,1+t,t+t^2,1+t+t^2,1+t^2$ at the head of the columns and t^2 times these elements, viz, $0,t^2,1+t,t+t^2,1+t+t^2,1+t^2,1,t$ at the beginning of the rows.

(2). $0,1,t,t^2,1+t,t+t^2,1+t+t^2,1+t^2$ at the head of the columns and 1+t times these elements, viz, $0,1+t,t+t^2,1+t+t^2,1+t^2,1,t,t^2$ at the beginning of the rows.

These two additional tables are:

+	0	1	t	t^2	$1+t$	$t+t^2$	$1+t+t^2$	$1+t^2$
0	0	1	t	t^2	$1+t$	$t+t^2$	$1+t+t^2$	$1+t^2$
t^2	t^2	$1+t^2$	$t+t^2$	0	$1+t+t^2$	t	$1+t$	1
$1+t$	$1+t$	t	1	$1+t+t^2$	0	$1+t^2$	t^2	$t+t^2$
$t+t^2$	$t+t^2$	$1+t+t^2$	t^2	t	$1+t^2$	0	1	$1+t$
$1+t+t^2$	$1+t+t^2$	$t+t^2$	$1+t^2$	$1+t$	t^2	1	0	t
$1+t^2$	$1+t^2$	t^2	$1+t+t^2$	1	$t+t^2$	$1+t$	t	0
1	1	0	$1+t$	$1+t^2$	t	$1+t+t^2$	$t+t^2$	t^2
t	t	$1+t$	0	$t+t^2$	1	t^2	$1+t^2$	$1+t+t^2$

+	0	1	t	t^2	$1+t$	$t+t^2$	$1+t+t^2$	$1+t^2$
0	0	1	t	t^2	$1+t$	$t+t^2$	$1+t+t^2$	$1+t^2$
$1+t$	$1+t$	t	1	$1+t+t^2$	0	$1+t^2$	t^2	$t+t^2$
$t+t^2$	$t+t^2$	$1+t+t^2$	t^2	t	$1+t^2$	0	1	$1+t$
$1+t+t^2$	$1+t+t^2$	$t+t^2$	$1+t^2$	$1+t$	t^2	1	0	t
$1+t^2$	$1+t^2$	t^2	$1+t+t^2$	1	$t+t^2$	$1+t$	t	0
1	1	0	$1+t$	$1+t^2$	t	$1+t+t^2$	$t+t^2$	t^2
t	t	$1+t$	0	$t+t^2$	1	t^2	$1+t^2$	$1+t+t^2$
t^2	t^2	$1+t^2$	$t+t^2$	0	$1+t+t^2$	t	$1+t$	1

Figure 5.4 Two additional tables of $GF(2^3)$.

Putting these two additional tables together and replacing 0 by 0, 1 by 1, t by 2, t^2 by 3, 1+t by 4, $t+t^2$ by 5, $1+t+t^2$ by 6 and $1+t^2$ by 7, we get the following orthogonal Latin square.

00	11	22	33	44	55	66	77
34	72	51	06	60	27	43	15
45	26	13	62	07	70	31	54
56	65	37	24	73	01	10	42
67	53	76	41	35	14	02	20
71	30	64	17	52	46	25	03
12	04	40	75	21	63	57	36
23	47	05	50	16	32	74	61

Figure 5.5 A pair of orthogonal Latin squares of order 8.

From this we get the following magic square.

1	10	19	28	37	46	55	64
29	59	42	7	49	24	36	14
38	23	12	51	8	57	26	45
47	54	32	21	60	2	9	35
56	44	53	34	30	13	3	17
58	25	53	16	43	39	22	4
11	5	33	62	18	52	48	31
20	40	6	41	15	27	61	50

Figure 5.6 A magic square of order 8.

Now when a magic square of 8^{th} order is known we can construct magic squares of order 8h, where h is any number greater than 2. For h = 2 we can construct magic squares from magic squares of order 4 as discussed earlier in 4.3. Hence by the method discussed in 5.2, 5.3 and using some properties of magic squares in 4.3 we can construct magic squares of every doubly even order.

5.4 Now consider the construction of magic squares of singly even order. Since there are no orthogonal Latin square of order 6, and it is rather difficult to construct orthogonal Latin squares of singly even order, we shall not use the previous method directly to construct magic squares of singly even order. But we shall make use of the previous method for constructing doubly even order magic squares first and then by the help of the bordered squares method due to Frénicle[4], we can obtain magic squares of singly even order. The method is as follows:

In constructing a magic square of singly even order n, we first construct a magic square of doubly even order n-2 by the method discussed in 5.2 or 5.3. Then add 2n-2 to every number in it. By 4.2 we get a magic square of order n-2 with the $(n-2)^2$ consecutive integers from the number 2n+1 up to n^2-2n+2. The common sum is $\frac{(n-2)(n^2+1)}{2}$.

Now to construct a magic square of order n, we border this magic square on every side by one row. We

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[4]Ball, W.W. Rouse. Mathematical Recreations and Essays. (London: Macmillan and Co. Ltd. 1931), pp.145-146.

then get another 4n-4 cells. These 4n-4 cells must be filled with the numbers 1, 2, ..., 2n-2 and n^2-2n+3, n^2-2n+4, ..., n^2-1, n^2 in such a way that the common sum of this new square must be equal to $\frac{n(n^2+1)}{2}$.

Consider the numbers 1, 2, ..., 2n-2 and n^2-2n+3, n^2-2n+4, ..., n^2-1, n^2. We can see that the sum of each pair 1 and n^2, 2 and n^2-1, and so on is equal to n^2+1, which when added to the common sum of the $(n-2)^{th}$ order magic square gives the common sum of the n^{th} order magic square to be constructed. We therefore ought to put the numbers in each pair at opposite ends of each row, column and diagonal. Since there are 2n-2 pairs of these numbers, and there are n-2 rows, n-2 columns and 2 diagonals, which makes altogether 2n-2 lines, we have just enough pairs for constructing the required magic square.

It is only remains to make the sum of the numbers in each of the border rows and columns equal to $\frac{n(n^2+1)}{2}$ too. If this has been done we then get a magic square of singly even order.

<u>Example</u>. We may construct a magic square of order 10 from the magic square of order 8 on page 44 of 5.3 by adding 2n-2 or 18 to every number in it. Then border this magic square on every side by one row. Put the numbers in each of the pairs 1 and 100, 2 and 99, ..., 18 and 83 at opposite ends of each row, column and diagonal in such a way that the sum of the numbers in each of the border lines

is the same common sum. This can be easily made by trial and error. We then get the following 10^{th} order magic square.

3	2	4	13	91	92	93	94	95	18
5	19	28	37	46	55	64	73	82	96
15	47	77	60	25	67	42	54	32	86
16	56	41	30	69	26	75	44	63	85
17	65	72	50	39	78	20	27	53	84
90	74	62	81	52	48	31	21	35	11
89	76	43	71	34	61	57	40	22	12
87	29	23	51	80	36	70	66	49	14
100	38	58	24	59	33	45	79	68	1
83	99	97	88	10	9	8	7	6	98

Figure 5.7 A magic square of order 10.