CHAPTER III

PATH PROBLEMS ON HYPERGRAPHS

The following path problems on hypergraphs are presented in this chapter

(i) enumeration of all independent node sets,

(ii) enumeration of all matchings (independent edge sets),

(iii) enumeration of all simple paths between two given nodes,

(iv) enumeration of all elementary paths between two given nodes.

3.1 Hypergraphs

A **hypergraph** $H = (X, E)$ consists of a finite set $X = \{1, 2, \ldots, n\}$, where elements are called **nodes** and a set E of nonempty subsets of X, the elements E_i, $i = 1, 2, \ldots, m$, of which have a property

$$\bigcup_{i=1}^{m} E_i = X^{\dagger}$$

and are called **edges** [2].

The representation of a hypergraph $H = (X, E)$ by a diagram is given as follows.

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† This property is not required for our work.

The nodes of H are represented by small squares. Let $|E_i|$ denote the number of elements of an edge $E_i \in E$. An edge E_i with $|E_i| > 2$ is drawn as a curve encircling all nodes of E_i. An edge E_i with $|E_i| = 2$, is drawn as a curve connecting its two nodes. An edge E_i with $|E_i| = 1$, is drawn as a loop as in a digraph.

As an illustration, Figure 3.1.1 represents the hypergraph $H = (X, E)$ where

$X = \{1, 2, \ldots, 8\}$ and

$E = \{\{1, 2, 3, 4\}, \{2\}, \{4, 5, 6\}, \{3, 5\}, \{1, 7, 8\}, \{6, 8\}, \{8\}, \{6, 7\}\}$.

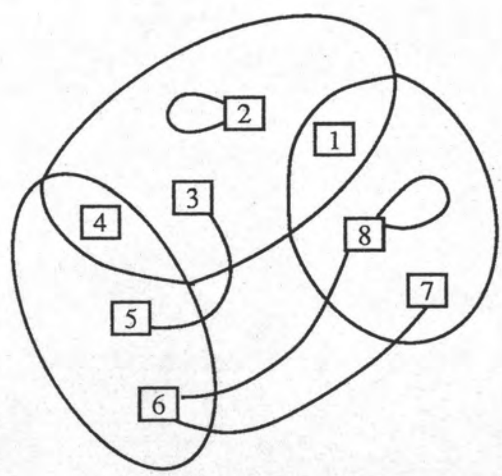

Figure 3.1.1

The **representative digraph of a hypergraph** $H = (X, E)$ is a digraph $G = (X, U)$ whose node set is the set X and arc $(i, j) \in U$ for all $i, j \in X$ if and only if either $\{i\} \in E$ when $i = j$ or $i, j \in B$ for some $B \in E$ when $i \neq j$.

Figure 3.1.2 shows the representative digraph of the hypergraph H of Figure 3.1.1.

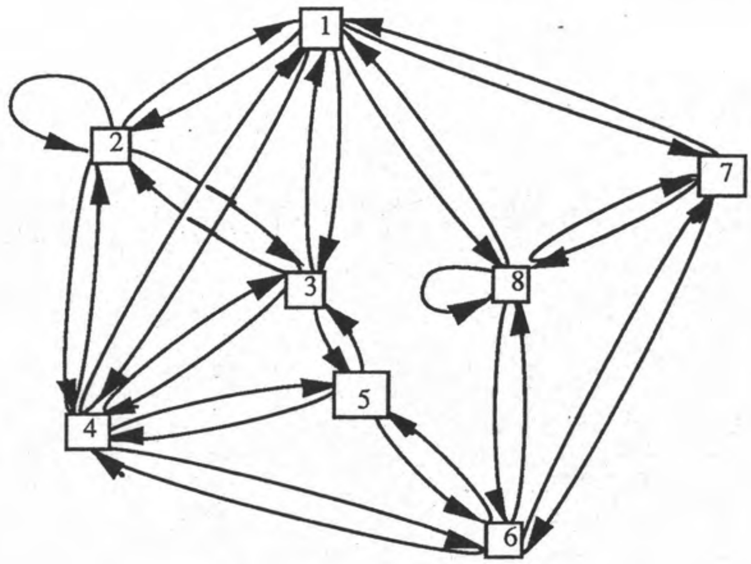

Figure 3.1.2

In a hypergraph H, a **path of order q from node i_1 to node i_{q+1}** is an alternating sequence of nodes and edges of the form

$$i_1 E_{j_1} i_2 E_{j_2} i_3 \ldots i_q E_{j_q} i_{q+1}$$

such that

(i) $i_1, i_2, \ldots, i_{q+1}$ are nodes of the hypergraph H,

(ii) $E_{j_1}, E_{j_2}, \ldots, E_{j_q}$ are edges of the hypergraph H,

(iii) $i_k, i_{k+1} \in E_{j_k}$ for $k = 1, 2, \ldots, q$.

The nodes i_1 and i_{q+1} are called the **initial** and **terminal nodes** respectively.

A **simple path** is a path whose edges are all distinct.

An **elementary path** is a simple path whose nodes are all distinct.

A **null path** is a path of order zero which the initial and terminal nodes are the same.

A hypergraph H = (X, *E*) is **connected** if every pair of distinct nodes is joined by a path in H, otherwise it is **disconnected.**

An **independent node set** of a hypergraph H = (X, *E*) is a set of nodes S $\subseteq$ X such that no two distinct nodes in S are elements of some edges in *E*.

A **matching (independent edge set)** of a hypergraph H = (X, *E*) is a set of edges $M \subseteq E$ such that for every two distinct edges E and E' in M, $E \cap E' = \emptyset$.

Example 3.1.1 Consider a connected hypergraph H = (X, *E*) in Figure 3.1.3 where 1, 2, . . . , 12 are nodes of H and **A**, **B**, . . . , **M** are edges of H.

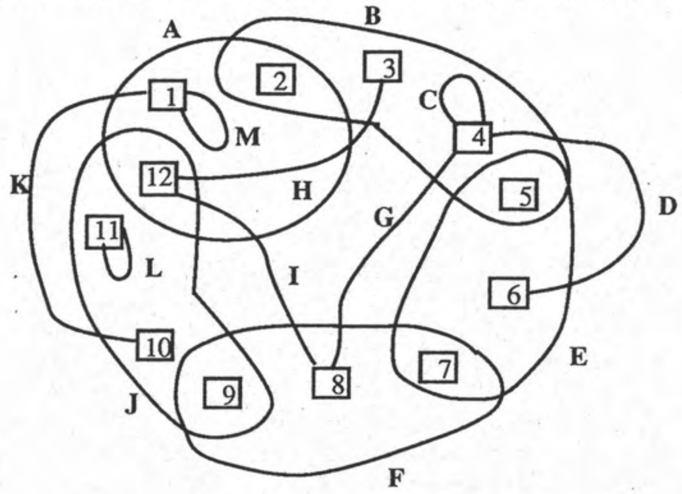

Figure 3.1.3

Some simple paths from node 10 to node 6 are

10J12A2B3H12I8F7E6,

10J12A2B4C4D6,

10J12A2B3H12I8G4D6,

10J9F8G4C4D6,

10J12A2B3H12I8F7E6.

Some elementary paths from node 1 to node 8 are

1A2B3H12I8,

1A12H3B5E7F8,

1K10J12H3B4D6E7F8,

1K10J9F7E6D4G8,

1A2B5E6D4G8.

Since for any two distinct nodes i and j in S = { 1, 3, 6, 8, 11}, {i, j} is not a subset of any edges of H, therefore S is an independent node set of H. Also for any two distinct edges α and β of *M* = {**A**, **C**, **E**, **L**}, α ∩ β = Ø, thus *M* is a matching of H.

3.2 Enumeration of All Independent Node Sets

Let $\mathscr{P}(\Sigma)$ be the power set of an alphabet Σ and let B be any subset of $\mathscr{P}(\Sigma)$. Let S be any member of $\mathscr{P}(\Sigma)$. The member S is **independent** in B if any two distinct elements in S are not the elements in any members of B . Let $I_B = \{S$ / $S \in \mathscr{P}(\Sigma)$ and S is independent in $B\}$.

Proposition 3.2.1 Let P_{11} be the power set of the set I_B. The addition is defined by

$A \oplus B = A \cup B$ for all $A, B \in P_{11}$,

and the multiplication is defined by

$A \otimes \varnothing = \varnothing = \varnothing \otimes A$ for all $A \in P_{11}$,

$A \otimes \{\varnothing\} = A = \{\varnothing\} \otimes A$ for all $A \in P_{11}$,

and $A \otimes B = A \cup B \cup \{a \cup b \in I_B \ / \ a \in A, b \in B\}$ for all $A, B \in P_{11} - \{\varnothing, \{\varnothing\}\}$.

Then $(P_{11}, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$ and unit $\{\varnothing\}$.

Proof: To show that $(P_{11}, \oplus, \otimes)$ is a semiring, we must show that the addition and multiplication are associative and the multiplication is distributive over the addition.

It is clear that the addition is associative.

To show that the multiplication is associative, let A, B and C be any elements in P_{11}.

Case 1: $A \neq \varnothing$ and $B \neq \varnothing$ and $C \neq \varnothing$.

Subcase 1.1: $A \neq \{\varnothing\}$ and $B \neq \{\varnothing\}$ and $C \neq \{\varnothing\}$. Then

$A \otimes (B \otimes C) = A \otimes (B \cup C \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\})$

$= A \cup B \cup C \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in B \cup C \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\}\}$

$= A \cup B \cup C \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in C\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in \{b \cup c \in I_B \ / \ b \in B, c \in C\}\}$

$= A \cup B \cup C \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in C\} \cup \{a \cup b \cup c \in I_B \ / \ a \in A, b \in B, c \in C\}$

$= A \cup B \cup C \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in C\} \cup \{x \cup c \in I_B \ / \ x \in \{a \cup b \in I_B \ / \ a \in A, b \in B\}, c \in C\}$

$= A \cup B \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup C \cup \{a \cup x \in I_B \ / \ a \in A, x \in C\} \cup \{b \cup c \in I_B \ / \ b \in B, c \in C\} \cup \{x \cup c \in I_B \ / \ x \in \{a \cup b \in I_B \ / \ a \in A, b \in B\}, c \in C\}$

$= A \cup B \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup C \cup \{x \cup c \in I_B \ / \ x \in A \cup B \cup \{a \cup b \in I_B \ / \ a \in A, b \in B\}, c \in C\}$

$= (A \otimes B) \cup C \cup \{x \cup c \in I_B \ / \ x \in A \otimes B, c \in C\}$

$= (A \otimes B) \otimes C.$

Subcase 1.2: $A = \{\varnothing\}$ and $B \neq \{\varnothing\}$ and $C \neq \{\varnothing\}$. Then $A \otimes (B \otimes C) = B \otimes C = (A \otimes B) \otimes C$.

Subcase 1.3: $A \neq \{\varnothing\}$ and $B = \{\varnothing\}$ and $C \neq \{\varnothing\}$. Then $A \otimes (B \otimes C) = A \otimes C = (A \otimes B) \otimes C$.

Subcase 1.4: $A \neq \{\varnothing\}$ and $B \neq \{\varnothing\}$ and $C = \{\varnothing\}$. Then $A \otimes (B \otimes C) = A \otimes B = (A \otimes B) \otimes C$.

Subcase 1.5: $A = \{\varnothing\}$ and $B = \{\varnothing\}$ and $C \neq \{\varnothing\}$. Then $A \otimes (B \otimes C) = C = (A \otimes B) \otimes C$.

Subcase 1.6: $A = \{\varnothing\}$ and $B \neq \{\varnothing\}$ and $C = \{\varnothing\}$. Then $A \otimes (B \otimes C) = B = (A \otimes B) \otimes C$.

Subcase 1.7: $A \neq \{\varnothing\}$ and $B = \{\varnothing\}$ and $C = \{\varnothing\}$. Then $A \otimes (B \otimes C) = A = (A \otimes B) \otimes C$.

Subcase 1.8: $A = \{\varnothing\}$ and $B = \{\varnothing\}$ and $C = \{\varnothing\}$. Then $A \otimes (B \otimes C) = \{\varnothing\} = (A \otimes B) \otimes C$.

Case 2: $A = \varnothing$ and $B \neq \varnothing$ and $C \neq \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Case 3: $A \neq \varnothing$ and $B = \varnothing$ and $C \neq \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Case 4: $A \neq \varnothing$ and $B \neq \varnothing$ and $C = \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Case 5: $A = \varnothing$ and $B = \varnothing$ and $C \neq \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Case 6: $A = \varnothing$ and $B \neq \varnothing$ and $C = \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Case 7: $A \neq \varnothing$ and $B = \varnothing$ and $C = \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Case 8: $A = \varnothing$ and $B = \varnothing$ and $C = \varnothing$. Then $A \otimes (B \otimes C) = \varnothing = (A \otimes B) \otimes C$.

Therefore, the multiplication is associative.

To show that the multiplication is distributive over the addition, let A, B and C be any elements in P_{11}.

Case 1: $A = \varnothing$. Then $A \otimes (B \oplus C) = \varnothing = (A \otimes B) \oplus (A \otimes C)$.

Case 2: $A = \{\varnothing\}$. Then $A \otimes (B \oplus C) = B \oplus C = (A \otimes B) \oplus (A \otimes C)$.

Case 3: $A \neq \varnothing$ and $A \neq \{\varnothing\}$. Then

$$
\begin{aligned}
A \otimes (B \oplus C) &= A \otimes (B \cup C) \\
&= A \cup B \cup C \cup \{a \cup x \in I_B \ / \ a \in A, x \in B \cup C\} \\
&= A \cup B \cup C \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup \{a \cup x \in I_B \ / \ a \in A, x \in C\} \\
&= A \cup B \cup \{a \cup x \in I_B \ / \ a \in A, x \in B\} \cup A \cup C \cup \{a \cup x \in I_B \ / \ a \in A, x \in C\} \\
&= (A \otimes B) \cup (A \otimes C) \\
&= (A \otimes B) \oplus (A \otimes C).
\end{aligned}
$$

Similarly, we can show that $(B \oplus C) \otimes A = (B \otimes A) \oplus (C \otimes A)$. Therefore, the multiplication is distributive over the addition.

Therefore, $(P_{11}, \oplus, \otimes)$ is a semiring.

It is clear that the addition is idempotent and commutative.

The set P_{11} contains the elements $\varnothing$ and $\{\varnothing\}$ such that for each $A \in P_{11}$,

$$A \oplus \varnothing = A = \varnothing \oplus A \ ,$$

$$A \otimes \{\varnothing\} = A = \{\varnothing\} \otimes A \text{ and}$$

$$A \otimes \varnothing = \varnothing = \varnothing \otimes A.$$

Therefore, $(P_{11}, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$ and unit $\{\varnothing\}$. #

Let H = (X, E) be a connected hypergraph and let $\mathcal{B}$ be the set of all independent node sets of H. The algebraic structure which turns out to be appropriate for the determination of the set $\mathcal{B}$, is a path algebra (P_{11}, ⊕, ⊗) in Proposition 3.2.1. The reasoning is based on the fact that the set $\mathcal{B}$ is an element of P_{11}, when Σ = X and $B = E$ (because $\mathcal{B}$ is a subset of the power set of the node set X and for any member F of $\mathcal{B}$, any two distinct nodes in F are not elements of any edges in E). The set $\mathcal{B}$ will be determined by an algebraic formulation in terms of the simple dipaths on the representative digraph G = (X, U) of H. First, we define a function $v:U \rightarrow P_{11}$ by for each arc (i, j) ∈ U,

$$v(i, j) = \begin{cases} \{\{i\},\{j\}\} & \text{if } i \neq j, \\ \{\{i\}\} & \text{if } i = j. \end{cases}$$

Therefore, for each arc (i, j) ∈ U, v(i, j) is a set of independent node sets of H. Let μ be any simple dipath from node i_0 to node i_r in G and we define

$$v(\mu) = \begin{cases} \{\emptyset\} & \text{if } \mu \text{ is a null dipath}, \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r) \\ & \text{is a non-null dipath}, \end{cases}$$

where

$$v(i_j, i_{j+1}) \otimes v(i_{j+1}, i_{j+2}) = v(i_j, i_{j+1}) \cup v(i_{j+1}, i_{j+2}) \cup \{\alpha \cup \beta \in I_B \;/\; \alpha \in v(i_j, i_{j+1}) \text{ and } \beta \in v(i_{j+1}, i_{j+2})\}$$

for j = 0, 1, 2, . . . , r - 2.

For a connected hypergraph H = (X, E) which has only one node i, we have that $\mathcal{B}$ = {∅, {i}}. The following results are concerned with a connected hypergraph containing at least two nodes.

Proposition 3.2.2 Let i and j be any nodes of the connected hypergraph H = (X, *E*). The set $\mathcal{B}$ of all independent node sets of H is equal to the formal sum

$$\oplus\sum_{\mu\in T_{ij}^q} v(\mu) = \begin{cases} \bigcup_{\mu\in T_{ij}^q} v(\mu) & \text{if } T_{ij}^q \neq \varnothing, \\ \varnothing & \text{if } T_{ij}^q = \varnothing, \end{cases}$$

where q is the maximum order of simple dipaths from node i to node j in G and T_{ij}^q = $\{\mu \;/\; \mu$ is a dipath of order r, $0 \le r \le q$, from node i to node j in G$\}$.

Before Proposition 3.2.2 is proved, we need the following two lemmas.

Lemma 3.2.3 Let $\mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r)$ be any non-null simple dipath from node i_0 to node i_r in G. Then $v(\mu) = \{D \;/\; D \subseteq \{i_0, i_1, i_2 \ldots, i_r\}$, $D \neq \varnothing$ and D is an independent node set of H$\}$.

Proof: The proof is obvious, because of the definitions of the function v and the multiplication $\otimes$.#

Lemma 3.2.4 In the representative digraph G = (X, U) of a connected hypergraph H = (X, *E*),

(i) there exists a simple cycle C containing all arcs of G and

(ii) the cycle C consists of disjoint simple cycles such that each simple cycle contains nodes in only one edge in *E*.

Proof: Note that G is connected. It follows by the construction of the representative digraph G of H that for any nodes i and j in edge E' of *E*, if (i, j) ∈ U then there exists node k ∈ E' such that (j, k) ∈ U. Since there exists edge D ∈ *E* such that D contains at least two nodes, thus G has a simple cycle Z whose nodes are contained in D. Let $G_1 = (X, U_1)$ be the digraph obtained by the removal of the arcs of Z. If G_1 has no arcs, then we have the claim. Otherwise, for any nodes i and j in edge E' of *E*, if (i, j) ∈ U_1 then there exists node k ∈ E' such that (j, k) ∈ U_1. Since G is connected , then G_1 has a simple cycle Z_1 with a node , say node s , in common with Z , whose nodes in Z_1 are contained in some edge D_1 of *E*. The dipath beginning at node s and consisting of the cycles Z and Z_1 in succession is a simple cycle containing the arcs of these two cycles. Let G_2 = (X, U_2) be the digraph obtained by the removal of the arcs of Z and Z_1. If G_2 has no arcs, then we have the claim; otherwise the same argument is applied to G_2 and so on. Since U is finite, eventually a digraph G_n which has no arcs, is obtained and we have the claim.#

Proof: (of Proposition 3.2.2) Let i and j be any nodes in X. It is clear that $\oplus\sum_{\mu\in T^q_{ij}} v(\mu) \subseteq \mathcal{B}$. We will show that $\mathcal{B} \subseteq \oplus\sum_{\mu\in T^q_{ij}} v(\mu)$. Let $F \in \mathcal{B}$. Then F is an independent node set of H.

Case 1: $F = \varnothing$. Clearly, $F \in \oplus\sum_{\mu\in T^q_{ij}} v(\mu)$.

Case 2: $F \neq \varnothing$. By Lemma 3.2.4, there exists a simple cycle C containing all arcs of G and this cycle consists of disjoint simple cycles such that each simple cycle contains nodes in only one edge in *E*. Since distinct nodes in F are contained in distinct edges in *E*, thus distinct nodes in F are contained in disjoint simple cycles. Clearly, there exists simple cycles C_i and C_j on C such that C_i and C_j contain node i and node j respectively. Therefore, there exists a simple dipath μ' = $(i_0, i_1)(i_1, i_2) \ldots (i_{s-1}, i_s)$ such that $i_0 = i$, $i_s = j$ and μ' contains all nodes in F and $F \subseteq \{i_0, i_1, i_2, \ldots, i_s\}$. By Lemma 3.2.3, we get $v(\mu') = \{D \,/\, D \subseteq \{i_0, i_1, i_2, \ldots, i_s\}$, $D \neq \varnothing$ and D is an independent node set of H$\}$.

Thus, $F \in v(\mu') \subseteq \oplus\sum_{\mu\in T^q_{ij}} v(\mu)$.

Therefore, $\mathcal{B} \subseteq \oplus\sum_{\mu\in T^q_{ij}} v(\mu)$. Hence, $\mathcal{B} = \oplus\sum_{\mu\in T^q_{ij}} v(\mu)$.#

Proposition 3.2.5 The arc-value matrix A of the representative digraph G = (X, U, v) of H over P_{11} is stable.

Proof: Let i and j be any nodes in X. Then the set of all independent node sets of H is the formal sum

$$\oplus\sum_{\mu\in T_{ij}^{m_{ij}}} v(\mu)$$

where m_{ij} is the maximum order of simple dipaths from node i to node j in G and $T_{ij}^{m_{ij}} = \{\mu \,/\, \mu$ is a dipath of order r, $0 \le r \le m_{ij}$, from node i to node j in G$\}$. Let m be $\max\{m_{ij} \,/\, i, j \in X\}$. Then $\oplus\sum_{\mu\in T_{ij}^{m_{ij}}} v(\mu) = \oplus\sum_{\mu\in T_{ij}^{m}} v(\mu) = \oplus\sum_{\mu\in T_{ij}^{r}} v(\mu)$ for all integer $r \ge m$.

By (2.1.3), $\oplus\sum_{k=0}^{m} A^k = \oplus\sum_{k=0}^{r} A^k$ for all integer $r \ge m$. Therefore, A is stable. #

Example 3.2.1 Enumeration of all independent node sets.

Let us consider a connected hypergraph H = (X, *E*) of Figure 3.2.1(a), where **A**, **B**, . . . , **K** are the edges of H. In order to enumerate all independent node sets of H, we first get a representative digraph G = (X, U, v) of H over P_{11} and then we obtain the arc-value matrix A of G as shown in Figure 3.2.1(b).

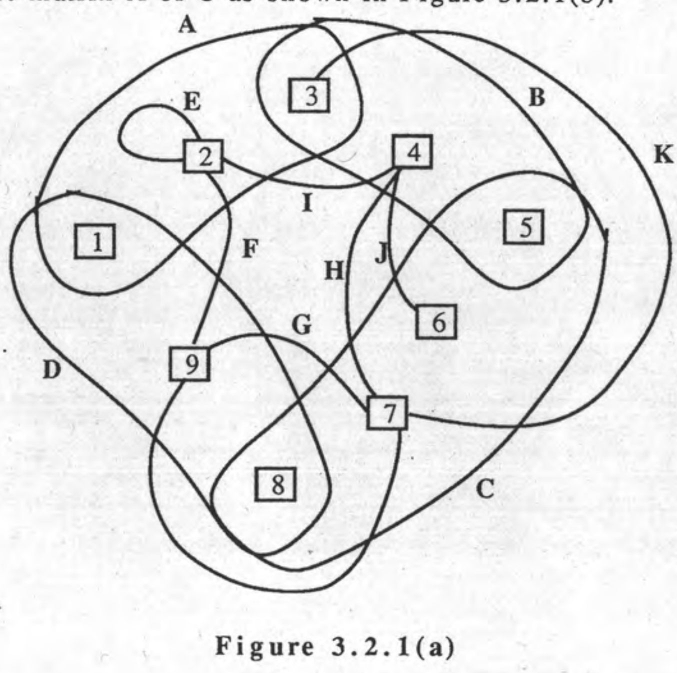

Figure 3.2.1(a)

$$\begin{bmatrix}
\varnothing & \{\{1\},\{2\}\} & \{\{1\},\{3\}\} & \varnothing & \varnothing & \varnothing & \varnothing & \{\{1\},\{8\}\} & \{\{1\},\{9\}\} \\
\{\{1\},\{2\}\} & \{\{2\}\} & \{\{2\},\{3\}\} & \{\{2\},\{4\}\} & \varnothing & \varnothing & \varnothing & \varnothing & \{\{2\},\{9\}\} \\
\{\{1\},\{3\}\} & \{\{2\},\{3\}\} & \varnothing & \{\{3\},\{4\}\} & \{\{3\},\{5\}\} & \varnothing & \{\{3\},\{7\}\} & \varnothing & \varnothing \\
\varnothing & \{\{2\},\{4\}\} & \{\{3\},\{4\}\} & \varnothing & \{\{4\},\{5\}\} & \{\{4\},\{6\}\} & \{\{4\},\{7\}\} & \varnothing & \varnothing \\
\varnothing & \varnothing & \{\{3\},\{5\}\} & \{\{4\},\{5\}\} & \varnothing & \{\{5\},\{6\}\} & \{\{5\},\{7\}\} & \{\{5\},\{8\}\} & \varnothing \\
\varnothing & \varnothing & \varnothing & \{\{4\},\{6\}\} & \{\{5\},\{6\}\} & \varnothing & \{\{7\},\{6\}\} & \{\{6\},\{8\}\} & \varnothing \\
\varnothing & \varnothing & \{\{3\},\{7\}\} & \{\{4\},\{7\}\} & \{\{5\},\{7\}\} & \{\{7\},\{6\}\} & \varnothing & \{\{7\},\{8\}\} & \{\{7\},\{9\}\} \\
\{\{1\},\{8\}\} & \varnothing & \varnothing & \varnothing & \{\{5\},\{8\}\} & \{\{6\},\{8\}\} & \{\{7\},\{8\}\} & \varnothing & \{\{8\},\{9\}\} \\
\{\{1\},\{9\}\} & \{\{2\},\{9\}\} & \varnothing & \varnothing & \varnothing & \varnothing & \{\{7\},\{9\}\} & \{\{8\},\{9\}\} & \varnothing
\end{bmatrix}$$

Figure 3.2.1(b)

Proposition 3.2.2 shows that for any nodes i and j in X, the set of all independent node sets of H is the formal sum $\oplus\sum_{\mu\in T_{ij}^{q}} v(\mu)$ where q is the maximum order of simple dipaths from node i to node j in G and T_{ij}^{q} = {μ / μ is a dipath of order r, $0 \le r \le q$, from node i to node j in G}. Let the set of all independent node sets of H be denoted by y_{ij} for all i, j ∈ X. Therefore, for fix j = 5, $y_{i5} = \oplus\sum_{\mu\in T_{i5}^{q}} v(\mu)$. From Proposition 3.2.5, the arc-value matrix A of G is stable. By (2.2.2), y_{i5} is the entry of the solution $\mathbf{y} = A^{*}\,\mathbf{b}$ of the matrix equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$ where

$$\mathbf{y} = [y_{15}\ y_{25}\ y_{35}\ y_{45}\ y_{55}\ y_{65}\ y_{75}\ y_{85}\ y_{95}]'$$

and $$\mathbf{b} = [\varnothing\ \ \varnothing\ \ \varnothing\ \ \varnothing\ \ \{\varnothing\}\ \ \varnothing\ \ \varnothing\ \ \varnothing\ \ \varnothing]'$$

is the 5<u>th</u> column vector of the unit matrix E of a path algebra $(M_9(P_{11}), \oplus, \otimes)$. Thus, $\mathbf{y} = \mathbf{b}^{(9)}$ by the Jordan elimination method, and the set of all independent node sets of H is y_{45} (fix i = 4).

The column vector $\mathbf{b}^{(9)}$ obtained by the equation (2.2.3) is given below.

$b^{(9)}=$

{{1},ϕ,{3},{2},{5},{1,4},{4},{1,5},{2,5},{1,6},{6},{6,2},{6,3},{1,7},{7},{7,2},{8},{8,2},{8,3},{8,4},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

{{2},ϕ,{3},{1},{5},{4},{4,1},{2,5},{1,5},{2,6},{6},{6,1},{6,3},{2,7},{7},{7,1},{8},{8,2},{8,3},{8,4},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

{ϕ,{1},{3},{2},{5},{4},{4,1},{5,2},{5,1},{3,6},{6},{6,2},{6,1},{7},{7,2},{7,1},{8},{8,2},{8,3},{8,4},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

{ϕ,{2},{4},{4,1},{1},{3},{5},{5,2},{5,1},{6},{6,2},{6,1},{6,3},{7},{7,2},{7,1},{2,8},{8},{8,3},{8,4},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

{ϕ,{3},{5},{5,1},{5,2},{1},{1,4},{2},{4},{6},{6,3},{6,1},{6,2},{7},{7,1},{7,2},{8},{8,3},{8,2},{8,4},{3,9},{9},{9,4},{9,5},{9,6},{9,6,3}}

{ϕ,{4},{4,1},{6},{6,2},{6,3},{6,1},{2},{1},{3},{5},{5,1},{5,2},{7},{7,2},{7,1},{8},{8,4},{8,2},{8,3},{4,9},{9},{9,3},{9,5},{9,6},{9,6,3}}

{ϕ,{3},{3,6},{7},{7,1},{7,2},{1},{2},{4},{4,1},{5},{5,1},{5,2},{6},{6,1},{6,2},{8},{8,3},{8,2},{8,4},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

{ϕ,{1},{1,4},{1,5},{8},{8,2},{2},{8,3},{3},{8,4},{4},{5},{5,2},{6},{6,1},{6,2},{6,3},{7},{7,1},{7,2},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

{ϕ,{1},{1,4},{1,5},{1,6},{1,7},{9},{2},{9,3},{3},{9,4},{4},{5},{5,2},{2,6},{6},{6,3},{7},{7,2},{9,5},{9,6},{9,6,3},{8},{8,2},{8,3},{8,4}}

Therefore, the set of all independent node sets of H is y_{45}.

$y_{45}=$ {ϕ,{2},{4},{4,1},{1},{3},{5},{5,2},{5,1},{6},{6,2},{6,1},{6,3},{7},{7,2},{7,1},{2,8},{8},{8,3},{8,4},{9},{9,3},{9,4},{9,5},{9,6},{9,6,3}}

3.3 Enumeration of All Matchings (Independent edge sets)

Let Σ be any alphabet and let $\mathcal{P}(\Sigma)$ be the power set of Σ. Let I_Σ be the set of all pairwise disjoint subsets of $\mathcal{P}(\Sigma)$.

Proposition 3.3.1 Let P_{12} be the power set of the set I_Σ. The addition is defined by

$$A \oplus B = A \cup B \qquad \text{for all } A, B \in P_{12},$$

and the multiplication is defined by

$A \otimes \varnothing = \varnothing = \varnothing \otimes A$ for all $A \in P_{12}$,

$A \otimes \{\varnothing\} = A = \{\varnothing\} \otimes A$ for all $A \in P_{12}$,

and $A \otimes B = A \cup B \cup \{a \cup b \in I_\Sigma \ / \ a \in A,\ b \in B\}$ for all $A, B \in P_{12} - \{\varnothing, \{\varnothing\}\}$.

Then $(P_{12}, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$ and unit $\{\varnothing\}$.

Proof: The proof is similar to Proposition 3.2.1. #

Let $H = (X, E)$ be a connected hypergraph and let $\mathcal{M}$ be the set of all matchings of H. The algebraic structure which turn out to be appropriate for the determination of the set $\mathcal{M}$, is a path algebra $(P_{12}, \oplus, \otimes)$ in Proposition 3.3.1. The reasoning is based on the fact that the set $\mathcal{M}$ is an element of P_{12}, when $\Sigma = X$ (because $\mathcal{M}$ is a subset of the power set of E and for any member M of $\mathcal{M}$, M is a pairwise disjoint subset of E). The set $\mathcal{M}$ will be determined by an algebraic formulation in terms of the simple dipaths on the representative digraph $G = (X, U)$ of H. First, we define a function $v: U \to P_{12}$ by for each arc $(i, j) \in U$,

$$v(i, j) = \begin{cases} \{\{\alpha\} \ / \ \alpha \in E \text{ and } i, j \in \alpha\} & \text{if } i \neq j, \\ \{\{i\}\} & \text{if } i = j. \end{cases}$$

Therefore, for each arc $(i, j) \in U$, $v(i, j)$ is a set of matchings of H. Let μ be any simple dipath from node i_0 to node i_r in G and we define

$$v(\mu) = \begin{cases} \{\varnothing\} & \text{if } \mu \text{ is a null dipath,} \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \dots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \dots (i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

where

$$v(i_j, i_{j+1}) \otimes v(i_{j+1}, i_{j+2}) = v(i_j, i_{j+1}) \cup v(i_{j+1}, i_{j+2}) \cup \{\alpha \cup \beta \in I_\Sigma \ / \ \alpha \in v(i_j, i_{j+1}),\ \beta \in v(i_{j+1}, i_{j+2})\}$$

for $j = 0, 1, 2, \dots, r - 2$.

Because of the definitions of the function v and the multiplication $\otimes$, for each simple dipath $\mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r)$, $v(\mu) = \{D \,/\, D \subseteq \{ \alpha \,/\, \{\alpha\} \in \bigcup_{j=0}^{r-1} v(i_{0+j}, i_{1+j}) \}, D \neq \varnothing$ and D is a matching of H$\}$.

Let i and j be any nodes in X. We will show $\mathcal{M} = \oplus\sum_{\mu \in T_{ij}^q} v(\mu)$, where

$$\oplus\sum_{\mu \in T_{ij}^q} v(\mu) = \begin{cases} \bigcup_{\mu \in T_{ij}^q} v(\mu) & \text{if } T_{ij}^q \neq \varnothing, \\ \varnothing & \text{if } T_{ij}^q = \varnothing, \end{cases}$$

q is the maximum order of simple dipaths from node i to node j in G and $T_{ij}^q = \{\mu \,/\, \mu$ is a dipath of order r, $0 \le r \le q$, from node i to node j in G$\}$.

It is clear that $\oplus\sum_{\mu \in T_{ij}^q} v(\mu) \subseteq \mathcal{M}$.

To show that $\mathcal{M} \subseteq \oplus\sum_{\mu \in T_{ij}^q} v(\mu)$, let $M \in \mathcal{M}$. Then M is a matching of H.

Case 1: $M = \varnothing$. Clearly, $M \in \oplus\sum_{\mu \in T_{ij}^q} v(\mu)$.

Case 2: $M \neq \varnothing$. By Lemma 3.2.4, there exists a simple cycle C containing all arcs of G and this cycle consists of disjoint simple cycles such that each simple cycle contains nodes in only one edge in *E*. Therefore, for each edge $\alpha \in M$ there exists a simple cycle $\mu = (i_0, i_1)(i_1, i_2) \ldots (i_{s-1}, i_s)$ such that $i_s = i_0$ and $\{\alpha\} \in v(i_k, i_{k+1})$ for $k = 0, 1, 2, \ldots, s-1$. Clearly, there exists simple cycles C_i and C_j on C such that C_i and C_j contain node i and node j respectively. Since M contains disjoint edges of H, thus there exists a simple dipath $\mu' = (i_0, i_1)(i_1, i_2) \ldots (i_{t-1}, i_t)$

such that $i_0 = i$, $i_t = j$ and for each edge $\alpha \in M$, $\{\alpha\} \in v(i_k, i_{k+1})$ for some $k \in \{0, 1, 2, \ldots, t-1\}$. Therefore, $M \subseteq \{\alpha \,/\, \{\alpha\} \in \bigcup_{j=0}^{t-1} v(i_{0+j}, i_{1+j})\}$. Since $v(\mu') = \{D \,/\, D \subseteq \{\alpha \,/\, \{\alpha\} \in \bigcup_{j=0}^{t-1} v(i_{0+j}, i_{1+j})\}$, $D \neq \varnothing$ and D is a matching of H$\}$,

thus $M \in v(\mu') \subseteq \oplus\sum_{\mu \in T_{ij}^{q}} v(\mu)$.

Therefore, $\mathcal{M} \subseteq \oplus\sum_{\mu \in T_{ij}^{q}} v(\mu)$.

Hence, $\mathcal{M} = \oplus\sum_{\mu \in T_{ij}^{q}} v(\mu)$. #

Proposition 3.3.2 The arc-value matrix A of the representative digraph G = (X, U, v) of H over P_{12} is stable.

Proof: The proof is similar to Proposition 3.2.5. #

Example 3.3.1 Enumeration of all matchings (independent edge sets).

Let us consider a connected hypergraph H = (X, *E*) of Figure 3.3.1(a). In order to enumerate all matchings of H, we first get a representative digraph G = (X, U, v) of H over P_{12} and then we obtain the arc-value matrix A of G as shown in Figure 3.3.1(b).

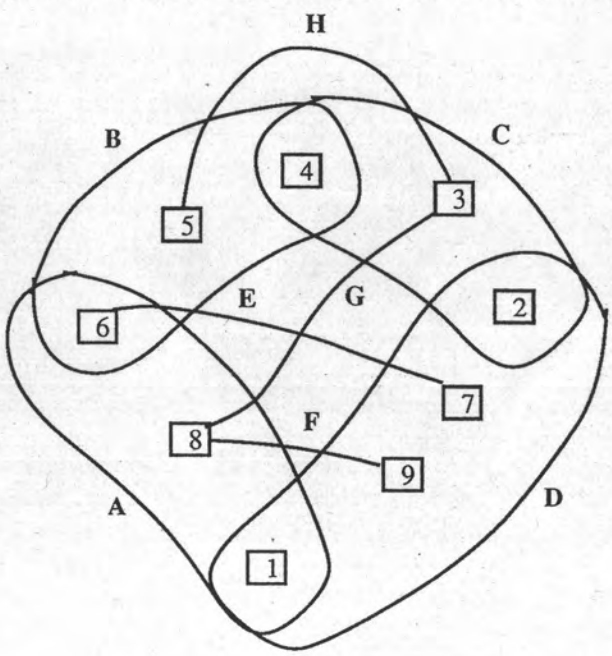

Figure 3.3.1(a)

$$
\begin{bmatrix}
\emptyset & \{\{D\}\} & \emptyset & \emptyset & \emptyset & \{\{A\}\} & \{\{D\}\} & \{\{A\}\} & \{\{D\}\} \\
\{\{D\}\} & \emptyset & \{\{C\}\} & \{\{C\}\} & \emptyset & \emptyset & \{\{D\}\} & \emptyset & \{\{D\}\} \\
\emptyset & \{\{C\}\} & \emptyset & \{\{C\}\} & \{\{H\}\} & \emptyset & \emptyset & \{\{G\}\} & \emptyset \\
\emptyset & \{\{C\}\} & \{\{C\}\} & \emptyset & \{\{B\}\} & \{\{B\}\} & \emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \{\{H\}\} & \{\{B\}\} & \emptyset & \{\{B\}\} & \emptyset & \emptyset & \emptyset \\
\{\{A\}\} & \emptyset & \emptyset & \{\{B\}\} & \{\{B\}\} & \emptyset & \{\{E\}\} & \{\{A\}\} & \emptyset \\
\{\{D\}\} & \{\{D\}\} & \emptyset & \emptyset & \emptyset & \{\{E\}\} & \emptyset & \emptyset & \{\{D\}\} \\
\{\{A\}\} & \emptyset & \{\{G\}\} & \emptyset & \emptyset & \{\{A\}\} & \emptyset & \emptyset & \{\{F\}\} \\
\{\{D\}\} & \{\{D\}\} & \emptyset & \emptyset & \emptyset & \emptyset & \{\{D\}\} & \{\{F\}\} & \emptyset
\end{bmatrix}
$$

Figure 3.3.1(b)

We have shown that for any nodes i and j in X, the set of all matchings of H is the formal sum $\underset{\mu \in T_{ij}^{q}}{\oplus\sum} v(\mu)$ where q is the maximum order of simple dipaths from node i to node j in G and $T_{ij}^{q} = \{\mu \,/\, \mu$ is a dipath of order r, $0 \le r \le q$, from node i to node j in G$\}$. Let the set of all matchings of H be denoted by y_{ij} for all i, j ∈ X. Therefore, for fix j = 1, $y_{i1} = \underset{\mu \in T_{i1}^{q}}{\oplus\sum} v(\mu)$. From Proposition 3.3.2, the arc-value matrix A of G is stable. By (2.2.2), y_{i1} is the entry of the solution $\mathbf{y} = A^{*}\mathbf{b}$ of the matrix equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$ where

$$\mathbf{y} = [y_{11}\ y_{21}\ y_{31}\ y_{41}\ y_{51}\ y_{61}\ y_{71}\ y_{81}\ y_{91}]'$$

$$\mathbf{b} = [\{\emptyset\}\ \emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset]'$$

is the first column vector of the unit matrix E of a path algebra $(M_9(P_{12}), \oplus, \otimes)$. Thus, $\mathbf{y} = \mathbf{b}^{(9)}$ by the Jordan elimination method, and the set of all matchings of H is y_{21} (fix i = 2).

The column vector $\mathbf{b}^{(9)}$ obtained by the equation (2.2.3) is given below.

$b^{(9)} =$

{{D},ϕ,{C},{D,H},{D,B},{H},{B},{A},{A,C},{A,H},{E},{C,E},{E,H},{D,G},{G},{D,G,B},{G,B},{F},{F,C},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{D},{C},{D,H},{D,B},{H},{B},{A},{A,C},{A,H},{C,E},{E},{E,H},{D,G},{G},{D,G,B},{G,B},{F},{F,C},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{C},{D},{H},{H,D},{D,B},{B},{C,A},{A},{A,H},{C,E},{E},{E,H},{G},{G,D},{G,D,B},{G,B},{G,E},{C,F},{F},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{C},{D},{B},{B,D},{D,H},{H},{C,A},{A},{A,H},{C,E},{E},{E,H},{G},{D,G},{D,G,B},{G,B},{C,F},{F},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{H},{H,D},{C},{D},{D,B},{B},{H,A},{A},{A,C},{H,E},{E},{C,E},{G},{D,G},{D,G,B},{G,B},{H,F},{F},{F,C},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{A},{A,C},{A,H},{D},{D,B},{D,H},{C},{B},{H},{E},{E,H},{E,C},{G},{D,G},{D,G,B},{G,B},{F},{F,C},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{D},{D,H},{D,B},{C},{C,E},{C,A},{H},{B},{E},{E,H},{A},{A,H},{D,G},{G},{D,G,B},{G,B},{F},{F,C},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}

{ϕ,{A},{A,C},{A,H},{D},{D,G},{D,G,B},{D,H},{D,B},{G},{C},{G,B},{H},{B},{E},{E,C},{E,H},{F},{F,C},{F,H},{F,B};{F,E},{F,C,E},{F,E,H}}

{ϕ,{D},{D,H},{C},{D,B},{H},{B},{A},{A,C},{A,H},{E},{C,E},{E,H},{F},{F,C},{F,H},{F,B},{F,E},{F,C,E},{F,E,H},{D,G},{G},{G,B},{G,D,B}}

.

Therefore, the set of all matchings of H is y_{21}.

$y_{21} =$ {ϕ,{D},{C},{D,H},{D,B},{H},{B},{A},{A,C},{A,H},{C,E},{E},{E,H},{D,G},{G},{D,G,B},{G,B},{F},{F,C},{F,H},{F,B},{F,E},{F,C,E},{F,E,H}}.

The following definitions are needed for the last two problems.

Let Σ be any alphabet. A **chain** over Σ is a finite sequence of the form

$$c_1 C_1 c_2 C_2 c_3 \ . \ . \ . \ c_{n-1} C_{n-1} c_n \qquad (3.4.1)$$

where (i) $c_1, c_2, \ldots, c_n$ are letters in Σ,

(ii) $C_1, C_2, \ldots, C_{n-1}$ are nonempty subsets of Σ,

(iii) $c_k, c_{k+1} \in C_k$ for $k = 1, 2, \ldots, n-1$.

The chain (3.4.1) is said to be **empty** if $n = 0$, and it is denoted by Ω.

The chain (3.4.1) is said to be **simple** if $C_1, C_2, \ldots, C_{n-1}$ are all distinct.

The chain (3.4.1) is said to be **elementary** if it is a simple chain which has properties

(1) $c_1, c_2, \ldots, c_{n-1}$ are all distinct when $c_1 = c_n$,

(2) $c_1, c_2, \ldots, c_n$ are all distinct when $c_1 \neq c_n$.

Let α and β be any chains over Σ such that

$\alpha = c_1C_1c_2C_2c_3 \ldots c_mC_mc_{m+1}$ and

$\beta = d_1D_1d_2D_2d_3 \ldots d_nD_nd_{n+1}$.

The concatenation Θ of α and β is defined by

$$\alpha \,\Theta\, \beta = \begin{cases} c_1C_1c_2C_2c_3 \ldots c_mC_md_1D_1d_2D_2d_3 \ldots d_nD_nd_{n+1} & \text{if } c_{m+1} = d_1, \\ \Omega & \text{if } c_{m+1} \neq d_1, \end{cases}$$

and $\alpha \,\Theta\, \Omega = \alpha = \Omega \,\Theta\, \alpha$. Then the concatenation Θ is an associative binary operation on the set of all chains over Σ.

3.4 Enumeration of All Simple Paths between Two Given Nodes

Proposition 3.4.1 Let Σ be any alphabet and let the set of all simple chains over Σ be denoted by S^*. Let P_{13} be the power set of the set S^*. For any A and B in P_{13}, the addition is defined by

$$A \oplus B = A \cup B$$

and the multiplication is defined by

$$A \otimes B = \{ \alpha \,\Theta\, \beta \in S^* \,/\, \alpha \in A \text{ and } \beta \in B\},$$

where $\alpha \,\Theta\, \beta$ is the concatenation of the simple chains α and β. Then $(P_{13}, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$ and unit $\{\Omega\}$.

Proof: To show that $(P_{13}, \oplus, \otimes)$ is a semiring, we must show that the addition and multiplication are associative, and the multiplication is distributive over the addition.

It is clear that the addition is associative. Since the concatenation Θ is associative, so the multiplication is associative.

To show that the multiplication is distributive over the addition, let A, B and C be any elements in P_{13}.

$A \otimes (B \oplus C) = A \otimes (B \cup C)$

$= \{a \Theta b \,/\, a \in A, b \in B \cup C, a \Theta b \in S^*\}$

$= \{ a \Theta b \,/\, (a \in A, b \in B, a \Theta b \in S^*)$ or $(a \in A, b \in C, a \Theta b \in S^*) \}$

$= \{ a \Theta b \,/\, a \in A, b \in B, a \Theta b \in S^* \} \cup \{ a \Theta b \,/\, a \in A, b \in C, a \Theta b \in S^* \}$

$= (A \otimes B) \cup (A \otimes C)$

$= (A \otimes B) \oplus (A \otimes C)$.

Similarly, we can show that $(B \oplus C) \otimes A = (B \otimes A) \oplus (C \otimes A)$. Therefore, the multiplication is distributive over the addition.

Therefore, $(P_{13}, \oplus, \otimes)$ is a semiring.

It is clear that the addition is idempotent and commutative.

The set P_{13} contains the elements $\varnothing$ and $\{\Omega\}$ such that for each $A \in P_{13}$,

$A \oplus \varnothing = A \cup \varnothing = A$,

$A \otimes \{\Omega\} = A = \{\Omega\} \otimes A$ and

$A \otimes \varnothing = \varnothing = \varnothing \otimes A$.

Therefore, $(P_{13}, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$ and unit $\{\Omega\}$. #

Let $H = (X, \mathcal{E})$ be a hypergraph and let $\mathcal{S}$ be the set of all non-null simple paths from node i to node j in H. The algebraic structure which turns out to be appropriate for the determination of the set $\mathcal{S}$, is a path algebra $(P_{13}, \oplus, \otimes)$ in Proposition 3.4.1. The reasoning is based on the fact that for each simple path α in $\mathcal{S}$, α is a simple chain over an alphabet $\Sigma = X$. Thus, the set $\mathcal{S}$ is a subset of the set S^* of all simple chains over Σ and so the set $\mathcal{S}$ is an element of P_{13} which is the power set of S^*. The set $\mathcal{S}$ will be determined by an algebraic formulation in terms of the simple dipaths on the representative digraph $G = (X, U)$ of H. Note that for each simple chain α over $\Sigma = X$ if $\alpha = c_1 C_1 c_2 C_2 c_3 \ldots c_{n-1} C_{n-1} c_n$ and $C_i \in \mathcal{E}$ for $i = 1, 2, \ldots, n-1$ then α is a simple path from node c_1 to node c_n in H.

Then we define a function $v: U \to P_{13}$ by for each arc $(i', j') \in U$,

$$v(i', j') = \begin{cases} \{i'Cj' \ / \ C \in \mathcal{E} \text{ and } i', j' \in C\} & \text{if } i' \neq j', \\ \\ \{i'\{i'\}i'\} & \text{if } i' = j'. \end{cases}$$

Therefore, for each arc $(i', j') \in U$, $v(i', j')$ is a set of simple paths from node i' to node j' in H. Let μ be any simple dipath from node i_0 to node i_r in G such that $i_0 = i$ and $i_r = j$ and we define

$$v(\mu) = \begin{cases} \{\Omega\} & \text{if } \mu \text{ is a null dipath,} \\ \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

where

$$v(i_m, i_{m+1}) \otimes v(i_{m+1}, i_{m+2}) = \{ \alpha \ \Theta \ \beta \in S^* \ / \ \alpha \in v(i_m, i_{m+1}) \text{ and } \beta \in v(i_{m+1}, i_{m+2})\}$$

for $m = 0, 1, 2, \ldots, r - 2$. Let m be any element in $\{0, 1, 2, \ldots, r - 2\}$. By the definition of the multiplication $\otimes$, we have that $v(i_m, i_{m+1}) \otimes v(i_{m+1}, i_{m+2})$ is a set of simple paths from node i_m to node i_{m+2} in H. Therefore, $v(\mu)$ is a set of simple paths from node i to node j in H.

We will show that $\mathscr{S} = \oplus\sum_{\mu \in W_{ij}^q} v(\mu)$ where

$$\oplus\sum_{\mu \in W_{ij}^q} v(\mu) = \begin{cases} \bigcup_{\mu \in W_{ij}^q} v(\mu) & \text{if } W_{ij}^q \neq \varnothing, \\ \\ \varnothing & \text{if } W_{ij}^q = \varnothing, \end{cases}$$

q is the maximum order of simple dipaths from node i to node j in G and $W_{ij}^q = \{\mu \ /$ μ is a dipath of order r, $1 \leq r \leq q$, from node i to node j in G$\}$.

It is clear that $\underset{\mu \in W_{ij}^{q}}{\oplus \sum} v(\mu) \subseteq \mathcal{S}$.

To show that $\mathcal{S} \subseteq \underset{\mu \in W_{ij}^{q}}{\oplus \sum} v(\mu)$, let $\alpha = c_1 C_1 c_2 C_2 c_3 \ldots c_{n-1} C_{n-1} c_n \in \mathcal{S}$ be such that $c_1 = i$ and $c_n = j$. Since $c_i C_i c_{i+1} \in v(c_i, c_{i+1})$ for $i = 1, 2, \ldots, n-1$, thus there exists a simple dipath $\mu' = (c_1, c_2)(c_2, c_3) \ldots (c_{n-1}, c_n)$ such that $\alpha \in v(c_1, c_2) \otimes v(c_2, c_3) \otimes \ldots \otimes v(c_{n-1}, c_n) = v(\mu')$. Since $v(\mu') \subseteq \underset{\mu \in W_{ij}^{q}}{\oplus \sum} v(\mu)$, thus $\alpha \in \underset{\mu \in W_{ij}^{q}}{\oplus \sum} v(\mu)$. Therefore, $\mathcal{S} \subseteq \underset{\mu \in W_{ij}^{q}}{\oplus \sum} v(\mu)$.

Hence, $\mathcal{S} = \underset{\mu \in W_{ij}^{q}}{\oplus \sum} v(\mu)$.

Proposition 3.4.2 The arc-value matrix A of the representative digraph G = (X, U, v) of H over P_{13} is stable.

Proof: Let i and j be any nodes in X. Then the set of all simple paths from node i to node j in H is the formal sum $\underset{\mu \in T_{ij}^{m_{ij}}}{\oplus \sum} v(\mu)$ where m_{ij} is the maximum order of simple dipaths from node i to node j in G and $T_{ij}^{m_{ij}} = \{\mu \,/\, \mu$ is a dipath of order r, $0 \le r \le m_{ij}$, from node i to node j in G$\}$. Let m be $\max\{m_{ij} \,/\, i, j \in X\}$. Then $\underset{\mu \in T_{ij}^{m_{ij}}}{\oplus \sum} v(\mu) = \underset{\mu \in T_{ij}^{m}}{\oplus \sum} v(\mu) = \underset{\mu \in T_{ij}^{r}}{\oplus \sum} v(\mu)$ for all integer $r \ge m$. By (2.1.3), $\oplus \sum_{k=0}^{m} A^k = \oplus \sum_{k=0}^{r} A^k$ for all integer $r \ge m$. Therefore, A is stable. #

Example 3.4.1 Enumeration of all simple paths between two given nodes.

Let us consider a hypergraph H = (X, *E*) of Figure 3.4.1(a). In order to enumerate all non-null simple paths from node 2 to node 8 in H, we first get a representative digraph G = (X, U, v) over P_{13} and then we obtain the arc-value matrix A of G as shown in Figure 3.4.1(b).

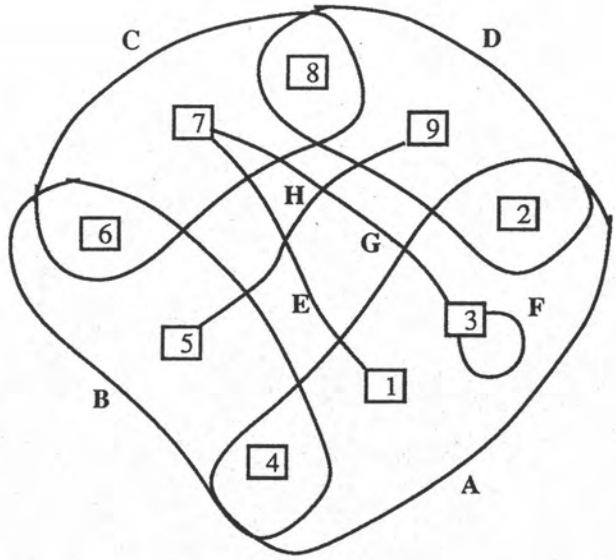

Figure 3.4.1(a)

$$\begin{bmatrix}
\emptyset & \{1A2\} & \{1A3\} & \{1A4\} & \emptyset & \emptyset & \{1E7\} & \emptyset & \emptyset \\
\{2A1\} & \emptyset & \{2A3\} & \{2A4\} & \emptyset & \emptyset & \emptyset & \{2D8\} & \{2D9\} \\
\{3A1\} & \{3A2\} & \{3F3\} & \{3A4\} & \emptyset & \emptyset & \{3G7\} & \emptyset & \emptyset \\
\{4A1\} & \{4A2\} & \{4A3\} & \emptyset & \{4B5\} & \{4B6\} & \emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset & \{5B4\} & \emptyset & \{5B6\} & \emptyset & \emptyset & \{5H9\} \\
\emptyset & \emptyset & \emptyset & \{6B4\} & \{6B5\} & \emptyset & \{6C7\} & \{6C8\} & \emptyset \\
\{7E1\} & \emptyset & \{7G3\} & \emptyset & \emptyset & \{7C6\} & \emptyset & \{7C8\} & \emptyset \\
\emptyset & \{8D2\} & \emptyset & \emptyset & \emptyset & \{8C6\} & \{8C7\} & \emptyset & \{8D9\} \\
\emptyset & \{9D2\} & \emptyset & \emptyset & \{9H5\} & \emptyset & \emptyset & \{9D8\} & \emptyset
\end{bmatrix}$$

Figure 3.4.1(b)

We have shown that the set of all non-null simple paths from node i to node j in H is the formal sum $\oplus \sum_{\mu \in W_{ij}^q} v(\mu)$ where q is the maximum order of simple dipaths from node i to node j in G and $W_{ij}^q = \{\mu \ / \ \mu$ is a dipath of order r, $1 \le r \le q$, from node i to node j in G$\}$. Let the set of all non-null simple paths from node i to node 8 in H be denoted by y_{i8}. Therefore, $y_{i8} = \oplus \sum_{\mu \in W_{i8}^q} v(\mu)$. From Proposition 3.4.2, the arc-value matrix A is stable. By (2.2.2), y_{i8} is the entry of the solution $\mathbf{y} = A^{*}\mathbf{b}$ of the matrix equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$ where

$$\mathbf{y} = [y_{18}\ y_{28}\ y_{38}\ y_{48}\ y_{58}\ y_{68}\ y_{78}\ y_{88}\ y_{98}]'$$

and $$\mathbf{b} = [\varnothing\quad \{2D8\}\quad \varnothing\quad \varnothing\quad \varnothing\quad \{6C8\}\quad \{7C8\}\quad \varnothing\quad \{9D8\}]'$$

is the 8th column vector of A. Thus, $\mathbf{y} = \mathbf{b}^{(9)}$ by the Jordan elimination method, and the set of all non-null simple paths from node 2 to node 8 in H is y_{28}.

The column vector $\mathbf{b}^{(9)}$ obtained by the equation (2.2.3) is given below.

$\mathbf{b}^{(9)} =$ [

{1A2D8,1A4B6C8,1E7C8,1E7G3A2D8,1E7C6B4A2D8,1E7G3A4B6C8,1E7G3F3A4B6C8,1A3G7C8,1A3F3G7C8,1A2D9H5B6C8,1A4B5H9D8,1E7G3A2D9H5B6C8,1E7G3F3A2D9H5B6C8,1E7G3A4B5H9D8,1E7G3F3A4B5H9D8}

{2D8,2A4B6C8,2A1E7C8,2A3G7C8,2A3F3G7C8,2D9H5B6C8,2D9H5B4A1E7C8,2D9H5B4A3G7C8,2A4B5H9D8,2A1E7C6B5H9D8,2A3G7C6B5H9D8,2A3F3G7C6B5H9D8}

{3A2D8,3F3A2D8,3A4B6C8,3G7C8,3G7E1A2D8,3G7C6B4A2D8,3G7E1A4B6C8,3A1E7C8,3A2D9H5B6C8,3A4B5H9D8,3G7E1A2D9H5B6C8,3G7E1A4B5H9D8,3G7C6B5H9D8}

{4A2D8,4B6C8,4A1E7C8,4A3G7C8,4B6C7E1A2D8,4B6C7G3A2D8,4A2D9H5B6C8,4B5H9D8,4B5H9D2A1E7C8,4B5H9D2A3G7C8,4B5H9D2A3F3G7C8,4A1E7C6B5H9D8,4A3G7C6B5H9D8}

{5B4A2D8,5B6C8,5B4A1E7C8,5B4A3G7C8,5B6C7E1A2D8,5B6C7G3A2D8,5H9D8,5H9D2A4B6C8,5H9D2A1E7C8,5H9D2A3G7C8,5H9D2A3F3G7C8}

{6C8,6B4A2D8,6C7E1A2D8,6C7G3A2D8,6B4A1E7C8,6B4A3G7C8,6B5H9D8,6B5H9D2A1E7C8,6B5H9D2A3G7C8,6B5H9D2A3F3G7C8,6C7E1A4B5H9D8}

{7C8,7E1A2D8,7G3A2D8,7C6B4A2D8,7E1A4B6C8,7G3A4B6C8,7G3F3A4B6C8,7G3A1E7C8,7G3F3A1E7C8,7E1A3G7C8,7E1A3F3G7C8,7E1A2D9H5B6C8,7G3A2D9H5B6C8,7G3F3A2D9H5B6C8,7E1A4B5H9D8,7G3A4B5H9D8,7G3F3A4B5H9D8}

{8C6B4A2D8,8D2A4B6C8,8C7G3A2D8,8D2A1E7C8,8D2A3G7C8,8D2A3F3G7C8,8D9H5B6C8,8D9H5B4A1E7C8,8D9H5B4A3G7C8,8C6B5H9D8,8C7E1A4B5H9D8}

{9D8,9H5B4A2D8,9D2A4B6C8,9H5B6C8,9D2A1E7C8,9D2A3G7C8,9D2A3F3G7C8,9H5B4A1E7C8,9H5B4A3G7C8}

] .

Therefore, the set of all non-null simple paths from node 2 to node 8 is y_{28}.

$y_{28} =$ {2D8,2A4B6C8,2A1E7C8,2A3G7C8,2A3F3G7C8,2D9H5B6C8,2D9H5B4A1E7C8,2D9H5B4A3G7C8,2A4B5H9D8,2A1E7C6B5H9D8,2A3G7C6B5H9D8,2A3F3G7C6B5H9D8}.

3.5 Enumeration of All Elementary Paths between Two Given Nodes.

Proposition 3.5.1 Let Σ be any alphabet and let the set of all elementary chains over Σ be denoted by E^*. Let P_{14} be the power set of E^*. For any A and B in P_{14}, the addition is defined by

$$A \oplus B = A \cup B$$

and the multiplication is defined by

$$A \otimes B = \{ \alpha \Theta \beta \in E^* / \alpha \in A \text{ and } \beta \in B\},$$

where $\alpha \Theta \beta$ is the concatenation of the elementary chains α and β. Then $(P_{14}, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$ and unit $\{\Omega\}$.

Proof: The proof is similar to Proposition 3.4.1. #

Let H = (X, E) be a hypergraph and let $\mathscr{E}$ be the set of all non-null elementary paths from node i to node j in H. The algebraic structure which turns out to be appropriate for the determination of the set $\mathscr{E}$, is a path algebra $(P_{14}, \oplus, \otimes)$ in Proposition 3.5.1. The reasoning is based on the fact that for each elementary path α in $\mathscr{E}$, α is an elementary chain over an alphabet $\Sigma = X$. Thus, the set $\mathscr{E}$ is a subset of the set E^* of all elementary chains over Σ and so the set $\mathscr{E}$ is an element of P_{14} which is the power set of E^*. The set $\mathscr{E}$ will be determined by an algebraic formulation in terms of the simple dipaths on the representative digraph G = (X, U) of H. Note that for each elementary chain α over $\Sigma = X$ if $\alpha = c_1C_1c_2C_2c_3 \ldots c_{n-1}C_{n-1}c_n$ and $C_i \in E$ for i = 1, 2, . . . , n-1 then α is an elementary path from node c_1 to node c_n in H. Then we define a function $v: U \to P_{14}$ by for each arc $(i', j') \in U$,

$$v(i', j') = \begin{cases} \{i'Cj' / C \in E \text{ and } i', j' \in C\} & \text{if } i' \neq j', \\ \\ \{i'\{i'\}i'\} & \text{if } i' = j'. \end{cases}$$

Therefore, for each arc $(i', j') \in U$, $v(i', j')$ is a set of elementary paths from node i' to node j' in H. Let μ be any simple dipath from node i_0 to node i_r in G such that $i_0 = i$ and $i_r = j$ and we define

$$v(\mu) = \begin{cases} \{\Omega\} & \text{if } \mu \text{ is a null dipath,} \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

where

$$v(i_m, i_{m+1}) \otimes v(i_{m+1}, i_{m+2}) = \{ \alpha \Theta \beta \in E^* \,/\, \alpha \in v(i_m, i_{m+1}) \text{ and } \beta \in v(i_{m+1}, i_{m+2})\}$$

for $m = 0, 1, 2, \ldots, r - 2$. Let m be any element in $\{0, 1, 2, \ldots, r - 2\}$. By the definition of the multiplication $\otimes$, we have that $v(i_m, i_{m+1}) \otimes v(i_{m+1}, i_{m+2})$ is a set of elementary paths from node i_m to node i_{m+2} in H. Therefore, $v(\mu)$ is a set of elementary paths from node i to node j in H.

By the similar proof in Section 3.4, we have that the set of all non-null elementary paths from node i to node j in H is the formal sum $\oplus\sum_{\mu\in W_{ij}^q} v(\mu)$ where

$$\oplus\sum_{\mu\in W_{ij}^q} v(\mu) = \begin{cases} \bigcup_{\mu\in W_{ij}^q} v(\mu) & \text{if } W_{ij}^q \neq \varnothing, \\ \varnothing & \text{if } W_{ij}^q = \varnothing, \end{cases}$$

q is the maximum order of simple dipaths from node i to node j in G and $W_{ij}^q = \{\mu \,/\, \mu$ is a dipath of order r, $1 \le r \le q$, from node i to node j in G$\}$.

Proposition 3.5.2 The arc-value matrix A of the representative digraph G = (X, U, v) of H over P_{14} is stable.

Proof: The proof is similar to Proposition 3.4.2. #

Example 3.5.1 Enumeration of all elementary paths between two given nodes.

Let us consider a hypergraph H = (X, *E*) of Figure 3.5.1(a). In order to enumerate all non-null elementary paths from node 7 to node 3 in H, we first get a representative digraph G = (X, U, v) over P_{14} and then we obtain the arc-value matrix A of G as shown in Figure 3.5.1(b).

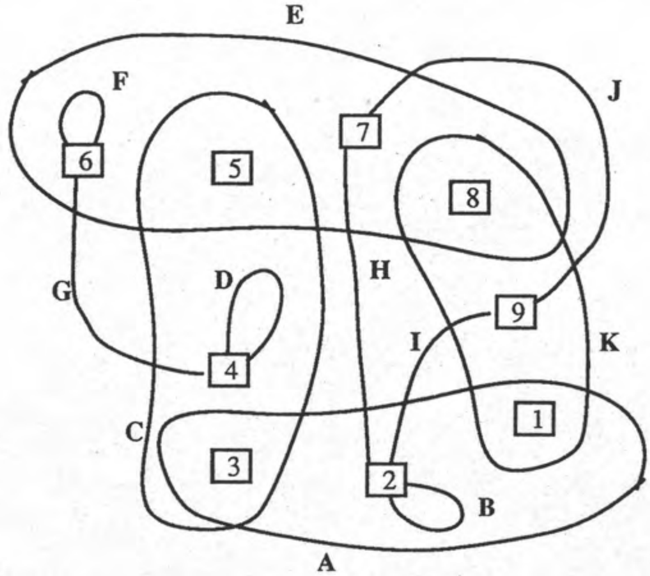

Figure 3.5.1(a)

Ø	{1A2}	{1A3}	Ø	Ø	Ø	Ø	{1K8}	{1K9}
{2A1}	{2B2}	{2A3}	Ø	Ø	Ø	{2H7}	Ø	{2I9}
{3A1}	{3A2}	Ø	{3C4}	{3C5}	Ø	Ø	Ø	Ø
Ø	Ø	{4C3}	{4D4}	{4C5}	{4G6}	Ø	Ø	Ø
Ø	Ø	{5C3}	{5C4}	Ø	{5E6}	{5E7}	{5E8}	Ø
Ø	Ø	Ø	{6G4}	{6E5}	{6F6}	{6E7}	{6E8}	Ø
Ø	{7H2}	Ø	Ø	{7E5}	{7E6}	Ø	{7E8}	{7J9}
{8K1}	Ø	Ø	Ø	{8E5}	{8E6}	{8E7}	Ø	{8K9}
{9K1}	{9I2}	Ø	Ø	Ø	Ø	{9J7}	{9K8}	Ø

Figure 3.5.1(b)

We have shown that the set of all non-null elementary paths from node i to node j in H is the formal sum $\oplus\sum_{\mu\in W_{ij}^{q}} v(\mu)$ where q is the maximum order of simple dipaths from node i to node j in G and $W_{ij}^{q} = \{\mu \ / \ \mu$ is a dipath of order r, $1 \le r \le q$, from node i to node j in G$\}$. Let the set of all non-null elementary paths from node i to node 3 in H be denoted by y_{i3}. Therefore, $y_{i3} = \oplus\sum_{\mu\in W_{i3}^{q}} v(\mu)$. From Proposition 3.5.2, the arc-value matrix A is stable. By (2.2.2), y_{i3} is the entry of the solution $\mathbf{y} = A^{*}\mathbf{b}$ of the matrix equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$ where

$$y = [y_{13}\ y_{23}\ y_{33}\ y_{43}\ y_{53}\ y_{63}\ y_{73}\ y_{83}\ y_{93}]'$$

and $$\mathbf{b} = [\{1A3\}\ \{2A3\}\ \varnothing\ \{4C3\}\ \{5C3\}\ \varnothing\ \varnothing\ \varnothing\ \varnothing]'$$

is the third column vector of A. Thus, $y = \mathbf{b}^{(9)}$ by the Jordan elimination method, and the set of all non-null elementary paths from node 7 to node 3 in H is y_{73}.

The column vector $\mathbf{b}^{(9)}$ obtained by the equation (2.2.3) is given below.

$\mathbf{b}^{(9)}$ =

{1A3,1A2H7E5C3,1A2H7E6G4C3,1K8E5C3,1K8E6G4C3,1K8E7H2A3,1K9I2A3,1K9J7H2A3,1K9J7E5C3,1K9J7E6G4C3,1K9I2H7E5C3,1K9I2H7E6G4C3,1A2I9J7E5C3,1A2I9J7E6G4C3,1A2I9K8E6G4C3,1A2H7J9K8E6G4C3}

{2A3,2H7E5C3,2H7E6G4C3,2A1K8E5C3,2A1K8E6G4C3,2H7E8K1A3,2I9K1A3,2I9J7E5C3,2I9J7E6G4C3,2I9K8E6G4C3,2I9J7E8K1A3,2A1K9J7E5C3,2A1K9J7E6G4C3,2H7J9K1A3,2H7J9K8E6G4C3}

{3A2H7E5C3,3A2H7E6G4C3,3C5E7H2A3,3C4G6E7H2A3,3A1K8E5C3,3A1K8E6G4C3,3C5E8K1A3,3C4G6E8K1A3,3A1K9J7E5C3,3A1K9J7E6G4C3,3A1K9I2H7E5C3,3A1K9I2H7E6G4C3,3A2I9J7E5C3,3A2I9J7E6G4C3,3A2I9K8E6G4C3,3A2H7J9K8E6G4C3,3C5E7J9K1A3,3C5E7J9I2A3}

{4C3,4G6E5C3,4C5E7H2A3,4G6E7H2A3,4C5E8K1A3,4G6E8K1A3,4C5E7J9K1A3,4C5E7J9I2A3,4C5E7H2I9K1A3}

{5C3,5E6G4C3,5E7H2A3,5C4G6E7H2A3,5E8K1A3,5C4G6E8K1A3,5E7J9K1A3,5E7J9I2A3,5E7H2I9K1A3}

{6G4C3,6E5C3,6E7H2A3,6G4C5E7H2A3,6E8K1A3,6G4C5E8K1A3,6E7J9K1A3,6E7J9I2A3,6E7H2I9K1A3}

{7H2A3,7E5C3,7E6G4C3,7E8K1A3,7H2A1K8E5C3,7H2A1K8E6G4C3,7J9K1A3,7J9I2A3,7J9K8E6G4C3,7J9I2A1K8E6G4C3,7H2I9K1A3,7H2I9K8E6G4C3}

{8K1A3,8E5C3,8E6G4C3,8E7H2A3,8K1A2H7E6G4C3,8K9I2A3,8K9J7H2A3,8K9J7E5C3,8K9J7E6G4C3,8K9I2H7E5C3,8K9I2H7E6G4C3,8K1A2I9J7E5C3,8K1A2I9J7E6G4C3}

{9K1A3,9I2A3,9J7H2A3,9J7E5C3,9J7E6G4C3,9I2H7E5C3,9I2H7E6G4C3,9K1A2H7E5C3,9K1A2H7E6G4C3,9K8E6G4C3,9K8E7H2A3,9I2A1K8E6G4C3,9J7E8K1A3,9J7H2A1K8E6G4C3,9I2H7E8K1A3}

Therefore, the set of all non-null elementary paths from node 7 to 3 is y_{73}.

y_{73} = {7H2A3,7E5C3,7E6G4C3,7E8K1A3,7H2A1K8E5C3,7H2A1K8E6G4C3,7J9K1A3,7J9I2A3,7J9K8E6G4C3,7J9I2A1K8E6G4C3,7H2I9K1A3,7H2I9K8E6G4C3}.