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นางสาวยมลพร ธรรมานุสาร

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ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

FORMULA FOR A CORRELATION COEFFICIENT BETWEEN UNDERLYING COMMODITY PRICE AND ITS CONVENIENCE YIELD UNDER SCHWARTZ MODEL

Miss Yamonporn Thummanusarn

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Applied Mathematics and Computational Science
Department of Mathematics and Computer Science
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สินค้าโภคภัณฑ์เป็นกลุ่มสินค้าประเภทหนึ่งที่มีลักษณะเป็นวัตถุดิบที่นำมาใช้ในการอุปโภคบริโภค เช่น น้ำมันดิบ สินค้าทางการเกษตร ทองคำ เป็นต้น ดังนั้นในแต่ละประเทศ ราคาของสินค้าโภคภัณฑ์จึงขึ้นอยู่กับอุปสงค์และอุปทานของประชากรส่วนใหญ่ในประเทศ อย่างไรก็ตามยังมีตัวแปรหนึ่งที่มีผลต่อราคาสินค้าโภคภัณฑ์ซึ่งในทางเศรษฐศาสตร์เรียกตัวแปรนี้ว่า ผลตอบแทนความสะดวกที่ใช้อธิบายผลประโยชน์ที่ได้จากการถือครองสินค้าโภคภัณฑ์แทนที่จะถือครองตราสารอนุพันธ์ จากทฤษฎีการเก็บรักษา กล่าวว่า อัตราผลตอบแทนความสะดวกแปรผกผันกับปริมาณสินค้าคงคลังของสินค้าโภคภัณฑ์ เนื่องจากราคาสินค้าโภคภัณฑ์ขึ้นกับปริมาณสินค้าคงคลังของสินค้าโภคภัณฑ์ ดังนั้นการทราบค่าสหสัมพันธ์ระหว่างราคาสินค้าโภคภัณฑ์กับผลตอบแทนความสะดวกจะนำไปสู่การอนุมานเกี่ยวกับปริมาณสินค้าคงคลังของสินค้าโภคภัณฑ์ซึ่งเป็นตัวแปรสำคัญในการกำหนดนโยบายด้านราคาของสินค้าโภคภัณฑ์ของรัฐบาล จากเหตุผลดังกล่าวนี้ วิทยานิพนธ์นี้นำเสนอสูตรสำหรับการคำนวณหาสัมประสิทธิ์สหสัมพันธ์ระหว่างราคาสินค้าโภคภัณฑ์กับผลตอบแทนความสะดวก เพื่อนำไปสู่การอนุมานปริมาณสินค้าคงคลังของสินค้าโภคภัณฑ์ในแต่ละช่วงเวลาซึ่งโดยปกติแล้วจะไม่สามารถสังเกตปริมาณดังกล่าวได้จากตลาดสินค้าโภคภัณฑ์

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YAMONPORN THUMMANUSARN: FORMULA FOR A CORRELATION COEFFICIENT BETWEEN UNDERLYING COMMODITY PRICE AND ITS CONVENIENCE YIELD UNDER SCHWARTZ MODEL. ADVISOR: ASST. PROF. KHAMRON MEKCHAY, Ph.D., CO-ADVISOR: ASST. PROF. SANAE RUJIVAN, Ph.D., 38 pp.

Commodity is a marketable item usually use as inputs in productions of other goods such as crude oil, agricultural good, gold, etc. The price of commodity for each country usually depends on demand and supply of people in that country. However, there is a factor that effects the commodity price in economic called convenience yield, which is used to describe the benefit of holding a physical good, rather than the derivative product. The theory of storage tells us that the convenience yield varies inversely with the inventory of commodity. Since commodity price depend on level of the inventory, finding the covariance between commodity price and convenience yield can imply the inventory of commodity, which is an important factor for government to make policies and decisions in commodity price. This thesis offers a formula for a correlation coefficient between underlying commodity price and its convenience yield, which can be used to determine the inventory that is not observable from commodity market.

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Department:	Mathematics and Computer Science	Student's Signature	....
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CHAPTER I
INTRODUCTION

Commodities are the products that play a central role in economy which are necessary for our everyday life such as crude oil, agricultural goods, and metals, etc. Since commodity is a product that is considered to be the same whether it comes from different producers or manufacturers, commodity price for each country is usually depending on the demand and supply of people in that country; when the demand is high, producers can charge high prices for goods. The promise of earning large profits from high prices inspires producers to produce more goods. This will cause excess supply, and by the law of demand, if prices are too high, only a few consumers will purchase the goods, and demand will go unmet. Also in some seasons, the production is oversupply and the producers will get lose. On the other hand, if in some seasons the production is low, then the price will increase and some customers will not be able to effort. Since commodity is essential for our life, if commodity spot price is too high the government will put a limit on any increases in price, and when the price is too low the government will set a lower limit for prices. However, there are some producers that hoard products to sell when the price is increasing, which makes the government policy useless.

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We use wheat and rice as our food, oil in transportation for dairy life and business, metals in industrial, and much more usage of commodities that are essential for human. Therefore, government have to control the commodities price using policies to have standard prices for the market, so the prices of commodities are not too low for producers to sell or not too high for consumers to buy. To make the commodities price policies, the government needs to know the inventories of the commodities, because if there are high inventories but the government set the standard commodities price too high, absolutely, the market need to sell commodities at the price lower than the standard price, otherwise no one will buy their commodities at the standard price. On the other hand, if there are low inventories of the commodities but the standard price is low, the sellers will be unsatisfied to sell the items, and they will keep the commodities until the prices increase, or will protest the government. However, inventories of commodities are hard to observe from the markets. Thus, this thesis will offer an alternative way to imply the inventories of commodities in the markets.

Theory of storage [17] provides that the convenience yield varies inversely with the level of inventories. Since convenience yield is a hidden factor of the commodities price, therefore, an exact data of the convenience yield is not known. However, the only available data in the market is the commodities price, which will be used to predict the level of inventories.

Commodities differ from other financial assets because they have the mean reversion property; it does not matter whether the price is increasing dramatically or decreasing gradually, in the end, the price will return to the mean price. Gibson and Schwartz [3] introduced a two-factor model (Schwartz two-factor model) that gives the relation between the commodity spot price and the convenience yield. For the first factor, the commodity spot price follows a geometric Brownian motion (GBM), where the logarithm of the commodity spot price follows the mean reverting process of the Ornstein-Uhlenbeck (OU) type. For the second factor, the convenience yield follows a mean reverting process. Moreover, these two factors have a constant volatility in which the commodity price and the convenience yield follow a joint stochastic process with constant correlation. Schwartz two-factor model [13] is probably the most famous term structure model of commodity prices. It was extended by many researchers to have sophisticated model ([5],[6],[8],[9],[11],[14]-[16],[18]).

In this work, we find the correlation between the underlying commodity price and its convenience yield based on the Schwartz's two-factor model, which involves the finding of expectations of these two factors, including the expectation of their product. The key idea to get the closed form formula for the expectations, and therefore obtain the correlations, is to employ the Feynman-Kac formula [6], where the expectations are obtained via the solutions of partial differential equations. According to the theory of storage [17], the convenience yield varies inversely with the inventory of commodity, therefore, finding the correlation will imply the relation between the commodity spot price and its inventory.

The thesis is organized as follows. In Chapter 2, we provide the definitions of financial words and some background knowledges for the correlation coefficient, the Schwartz two-factor model, the Ito lemma, the Pearson's correlation formula, and the Feynman-Kac formula. In Chapter 3, we obtain the main result by calculating the correlation between commodity spot price and its convenience yield. In Chapter 4, we confirm our formula with some numerical examples using parameters that estimated by Schwartz [13] for 3 commodities, oil, copper, and gold. Moreover, we discuss the behaviors of the correlations as time approach the final time. Furthermore, we analyze the graphs that we obtain to predict the inventory level. And finally, in Chapter 5, we conclude the result of the thesis with comments and suggestions.

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CHAPTER II

BACKGROUND KNOWLEDGE

In this Chapter, we provided the definition of financial terms and some background knowledges for the correlation coefficients, the Schwartz two-factor model, the Ito lemma, the Pearson's correlation formula and the Feynman-Kac formula.

2.1 Theory of storage

This section aim to provide the brief review of theory of storage [17]. Which the important term convenience yield arising from inventories of storable commodities.

2.1.1 Inventories

Inventory is a quantity of goods or raw materials owned and stored by a business that is intended either for resale or put to manufacturing process. In this thesis, we focus on inventory of commodity. Inventory is an important factor in commodities markets. For manufacturing industries or producers, they holds inventories to reduce cost of adjusting production over times, and also to keep the process run continuously, running out of only one item can prevent a manufacturer from completing the production of its finished goods. Since commodities is important for production/manufacturing process, producers/manufacturers need to hold inventories but they need to pay for the cost of storage. Thus, some of them decided to hold but some of them not. Therefore, the inventories are hard to observe from the market. Considering for the price of commodities, when there are high inventories, the commodities price must be low and conversely when there are low inventories, the commodities price must be high. Thus, the level of inventories of commodities in the market plays an important role in determining price.

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2.1.2 Instantaneous convenience yields

Brennan and Schwartz's article provided the definition of instantaneous convenience yield as follows:

"*The convenience yield is the flow of services that accrues to an owner of the physical commodity but not to the owner of a contract for future delivery of the commodity. Recognizing the time lost and the costs incurred in transporting a commodity from one location to another, the convenience yield may be thought of as the value of being able to profit from temporary local shortages of the commodity through ownership of the physical commodity. The profit may arise either from local price variations or from the ability to maintain a production process as a result of ownership of an inventory of raw material*".

For example, assume that we are producers. Thus, we need physical goods to produce our products. We have two choices, in order to reduce the risk of price fluctuations, one is to own the contract or derivative instrument and another one is to hold an underlying product or physical goods. If we choose to hold physical goods, we have to pay for the cost of storage and we may get lose from perishable goods. In fact, when there are lack of goods, the option of holding physical goods has an advantage because the producers will has an ability to keep a production process running. This advantage that we obtained from holding an underlying product or physical good, rather than the contract or derivative product, is call the convenience yield.

The Kaldor–Working hypothesis [17] postulates that the convenience yield depends inversely upon the level of inventories. In other words, if inventories to be stored are exceptionally large, the cost of storage may exceed the return for carrying physical goods which lead to the low value of convenience yield. On the other hand, if inventories are quite low, the return for carrying physical goods may greater than the cost of storage which lead to the high value of convenience yield.

2.1.3 Mean-reversion property

The mean reversion is the property that prices will eventually return toward the mean or average value, the long-run mean. Commodity prices neither grow nor decline on average over time, but they fluctuate around their long-run mean. The mean-reversion property is one of the main properties that researchers take into account when researching on commodities ([2], [10]).

2.2 Stochastic model for commodity prices

One way to study about commodities price behavior is by considering stochastic model for commodity prices. The first model for commodity prices was introduced by Brennan and Schwartz [1] where the spot commodity price follows a GBM and the convenience yield is treated as a dividend yield. However, this model is inappropriate because it does not take into account the mean-reversion property of the spot commodity prices and neglects the inventory-dependence property of the convenience yield.

Gibson and Schwartz [3] introduced a two-factor with constant volatility model where the spot price and the convenience yield follow a joint stochastic process with constant correlation. The spot price of commodity follows a GBM and the convenience yield follows a mean reverting stochastic process of the OU type. In the spot price process, they added the convenience yield to consider as a dividend yield. And in the convenience yield process, the OU process relies on the hypothesis that there is a regeneration property of inventories, that is to say, there is an appropriate level of stocks which satisfies the needs of industry under normal conditions. This appropriate level can be guaranteed the existence by the behavior of the operators in the physical market. When the convenience yield is low, there are plenty of stocks and the operators sustain a high storage cost compared with the benefits related to holding the commodity. Therefore, if the holders are logical, they will try to cut these surplus stocks. Conversely, when the stocks are rare the operators tend to reconstitute them. Schwartz [13] introduced difference of this model where the convenience yield is mean reverting and interferes in the commodity price dynamics. He also tests the model with several commodities. Schwartz [13] three-factor model, Miltersen and Schwartz [7] and Hilliard and Reis [5] add a third stochastic factor to the model to account for stochastic interest rates. Nevertheless, the inclusion of stochastic interest rates in the commodity price models does not have a significant impact in the pricing of commodity options and futures in practice. Accordingly, interest rate can be assumed deterministic.

In this thesis, we assume that the commodity spot prices follow the Schwartz's two-factor model, Gibson and Schwartz [3], the relations between the commodity spot price and the instantaneous convenience yield, where logarithm of the spot price and the convenience yield satisfy the mean reverting property. The model can be written as follow:

$$dS_t = (r - \delta_t)S_t dt + \sigma_1 S_t dW_t^{(1)}, \tag{2.1}$$

$$d\delta_t = (\kappa(\alpha - \delta_t) - \lambda)dt + \sigma_2 dW_t^{(2)}, \tag{2.2}$$

where S_t is the commodity spot prices at time $t > 0,$ δ_t is the instantaneous convenience yield at time $t,$ $\left(W_t^{(1)}, W_t^{(2)}\right)_{t\geq 0}$ is the 2-dimensional Wiener process with $dW_t^{(1)} dW_t^{(2)} = \rho dt,$ where ρ denotes the correlation coefficient between the two Brownian motions, r is the risk-free interest rate, κ is the speed of mean- reversion, λ is a risk premium of the market, σ_1 is a parameter showing a volume of a fluctuation of a commodity spot prices, and σ_2 is a parameter showing a volume of fluctuation instantaneous convenience yield. The stochastic models (2.1) and (2.2) include 2 stochastic processes, $\left(S_t(\omega)\right)_{t\geq 0}$ and $\left(\delta_t(\omega)\right)_{t\geq 0}$, for $\omega \in \Omega,$ where S_t is expressed by the Black-Scholes model whose volatility depended on S_t, and δ_t is expressed by the OU model that converges to a long term mean $\alpha_0 = \alpha - \lambda/\kappa$ of instantaneous convenience yield.

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2.3 Ito's lemma

Suppose that the value of a variable x follows the Ito process

$$dx = a(x,t)dt + b(x,t)dz, \tag{2.3}$$

where z is the Wiener process and $a(x,t)$ and $b(x,t)$ are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Ito's lemma shows that if G is a twice continuously and differentiable in x, and once in t, then

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz,$$

where z is the same Wiener process as in (2.3).

(Proof see [4]).

2.4 Pearson's correlation

We use Pearson's correlation to compute the correlation coefficient between commodity spot price and its convenience yield which formula is given below.

The correlation of two random variables X and Y, denoted by $Corr(X,Y)$, is defined by, given that $\mathrm{Var}(X)\mathrm{Var}(Y)$ is positive,

$$Corr(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}},$$

where

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y],\ \mathrm{Var}(X) = E\left[(X - E[X])^2\right]$$

(see [12]).

2.5 Feynman-Kac formula

It is a method for calculation of expectations of a function or a functional of diffusion process on the boundary. This expectation can be computed by using a solution to the corresponding partial differential equation with a given boundary condition

Theorem (Feynman-Kac)

Let $T \in (0,\infty)$ be a fixed time horizon and D be a domain in $\mathbb{R}^n$. Let $X(t)$ be an *n*-dimension Ito's process satisfying the stochastic differential equation for $t > s \geq 0,$

$$dX(t) = \mu(t, X(t))dt + \sum_{j=1}^{m} \sigma_j(t, X(t))dW_j^{\mathbb{Q}}(t), \qquad X(s) = x \in D,$$

where $\mu(t, X(t)):[0,T]\times D \to \mathbb{R}^n$ is continuous, $W_j^{\mathbb{Q}}(t)$ is a $\mathbb{Q}$-Brownian motion, and $\sigma_j(t, X(t)):[0,T]\times D \to \mathbb{R}^n, \;\; j=1,\ldots,m$. In other words,

$$d\begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix} = \begin{pmatrix} \mu_1(t, X(t)) \\ \vdots \\ \mu_n(t, X(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(t, X(t)) & \cdots & \sigma_{1m}(t, X(t)) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(t, X(t)) & \cdots & \sigma_{nm}(t, X(t)) \end{pmatrix} \begin{pmatrix} dW_1^{\mathbb{Q}}(t) \\ \vdots \\ dW_m^{\mathbb{Q}}(t) \end{pmatrix}.$$

Let

$$v = \begin{pmatrix} \sigma_{11}(t, X(t)) & \cdots & \sigma_{1m}(t, X(t)) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(t, X(t)) & \cdots & \sigma_{nm}(t, X(t)) \end{pmatrix}.$$

Then, the generator of the process is

$$\mathcal{A} = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial X_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(vv^T\right)_{ij} \frac{\partial^2}{\partial X_i \partial X_j}, \tag{2.4}$$

where, for notational convenience, $\mu_i = \mu_i(t, X(t))$, and $\left(vv^T\right)_{ij}$ is an element (i, j) of the matrix vv^T of the size $(n \times n)$. Let $g: D \to [0,\infty)$ be a given measurable bounded function and $u(t,x)$ be a differentiable function of t and twice differentiable function of x defined by

$$u(t,x) := E\left[g\left(X(T)\right)\middle| X(t)=x\right].$$

Then, $u(t,x)$ is the solution of the partial differential equation,

$$\frac{\partial u}{\partial t} + \mathcal{A}u = 0, \quad \text{on}\,(0,T)\times D \tag{2.5}$$

with the boundary condition

$$u(T,x) = g(x), \qquad \text{for}\; x \in D$$

(See [6] for the complete proof).

Note. The idea of Feynman-Kac for relating stochastic differential equation to partial differential equation is as follow [4], based on the Black-Scholes partial differential equation.

Consider the stock (or asset) where its price S follows the Ito process

$$dS = \mu S dt + \sigma S dz, \tag{2.6}$$

where μ is the stock's expected rate of return and σ is the volatility of the stock price, both are constant and z is the Wiener process. Let f be the price of derivative contingent on S, thus the variable f must be a function of S and t. Hence, from Ito lemma

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dz. \tag{2.7}$$

The discrete versions of equations (2.6) and (2.7) are

$$\Delta S = \mu S \Delta t + \sigma S \Delta z, \tag{2.8}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z, \tag{2.9}$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. To eliminate the uncertainty Δz for both ΔS and Δf, one can set up a "risk-less" portfolio to balance this term. For example, considering the portfolio that the holder short one derivative and long an amount $\frac{\partial f}{\partial S}$ of share at time t. The value $\prod$ of the portfolio at this time is the sum of the value of holding stock $\frac{\partial f}{\partial S}S$ and selling value $-f$ of derivative

$$\prod = -f + \frac{\partial f}{\partial S}S. \tag{2.10}$$

Consider the change in the value of the portfolio $\Delta\prod$ in a small change of time Δt (as still holding $\frac{\partial f}{\partial S}$ of stock),

$$\Delta\prod = -\Delta f + \frac{\partial f}{\partial S}\Delta S. \tag{2.11}$$

Substituting (2.8) and (2.9) into (2.11) yields the change $\Delta\prod$ that is risk-less (no uncertainty Δz)

$$\Delta\prod = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t. \tag{2.12}$$

By no-arbitrage argument, the change in value $\Delta\prod$ must come from the risk-free interest rate only, namely

$$\Delta\prod = r\prod\Delta t, \tag{2.13}$$

where r is the risk-free interest rate. Finally, by substituting from equations (2.10) and (2.12) into (2.13), we obtain

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf, \tag{2.14}$$

which is the partial differential equation for the (fair) price f. For example, for futures derivative, the price (value) of the future $f(t,S)$ is the solution of partial differential equation (2.13) with the terminal condition $f(T,S)=S(T)=S_T$. Alternatively, one can obtain the (fair) price f from risk-neutral valuation. For this example, the price f will be the expected value of $S(T)$ given the information at time t, namely

$$f(t,S)=E^{\mathbb{Q}}\left[S_T|S\right].$$

Therefore, with this example and motivation, one can relate the conditional expectation as the solution of the partial differential equation, which is generalized into theorem known as Feynman-Kac theorem.

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CHAPTER III
CORRELATION COEFFICIENT BETWEEN COMMODITY PRICE AND ITS CONVENIENCE YIELD

In this chapter, we provide the correlation formula according to the Pearson's correlation between the commodity price and its convenience yield. Then, according to the formula, we find the expectation of the first and second powers of S_t and δ_t and the expectation of $S_t\delta_t$, as presented in the lemmas below. To find the expectations we use the method for finding the expectations of random processes by solving partial differential equations which is the well-known result called the Feynman-Kac formula.

3.1 Correlation coefficient

In this work, we are interested in finding the correlation between the commodity price and its convenience yield, $C_t^r(T)$. In this case,

$$C_t^r(T) := \frac{E_t^{\mathbb{Q}}[S_T\delta_T] - E_t^{\mathbb{Q}}[S_T]\,E_t^{\mathbb{Q}}[\delta_T]}{\sqrt{\left(E_t^{\mathbb{Q}}[(S_T - E_t^{\mathbb{Q}}[S_T])^2]\right)\left(E_t^{\mathbb{Q}}[(\delta_T - E_t^{\mathbb{Q}}[\delta_T])^2]\right)}},$$

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which we rewrite it as

$$C_t^r(T) := \frac{E_t^{\mathbb{Q}}[S_T\delta_T] - E_t^{\mathbb{Q}}[S_T]\,E_t^{Q}[\delta_T]}{\sqrt{\left(E_t^{\mathbb{Q}}[S_T{}^2] - \left(E_t^{\mathbb{Q}}[S_T]\right)^2\right)\left(E_t^{\mathbb{Q}}[\delta_T{}^2] - \left(E_t^{\mathbb{Q}}[\delta_T]\right)^2\right)}}, \tag{3.1}$$

where $t \in [0,T)$ and $E_t^{\mathbb{Q}}\left[\cdot\right] = E^{\mathbb{Q}}\left[\cdot \mid F_t\right] = E^{\mathbb{Q}}\left[\cdot \mid S_t\delta_t\right]$ denotes the conditional expectation with respect to a filtration up to time t, F_t.

In this work, the calculation of the expectations are obtained via the well-known result called the Feynman-Kac formula [6], i.e., finding the expectations of random processes by solving partial differential equations.

Lemma 3.1 Expectations of the convenience yield

For stochastic process for the convenience yield (2.2), the expectations of the first and the second powers of the convenience yield at the terminal time T are

$$E^{\mathbb{Q}}\left[\delta_T \mid \delta_t = \delta\right] = \delta A_{1,1}(\tau) + A_{1,0}(\tau),$$
$$E^{\mathbb{Q}}\left[\delta_T{}^2 \mid \delta_t = \delta\right] = \delta^2 A_{2,2}(\tau) + \delta A_{2,1}(\tau) + A_{2,0}(\tau),$$

where t is the initial time and $\tau = T - t.$

Proof

Define

$$u(t,\delta) = E\left[\delta_T{}^n \mid \delta_t = \delta\right], \quad t \geq 0,\ T > 0,\ \delta \in \mathbb{R}.$$

By applying the Feynman-Kac formula for (2.2) gives us that $u(t,\delta)$ is the solution of the partial differential equation

$$\frac{\partial u}{\partial t} + \left(\kappa(\alpha - \delta) - \lambda\right)\frac{\partial u}{\partial \delta} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial \delta^2} = 0 \tag{3.2}$$

with the condition

$$u(T,\delta) = \delta^n. \tag{3.3}$$

<u>Case 1</u>, $n = 1$

Following the idea of Schwartz's solution producer [13] for solving (3.2) corresponding to stochastic process (2.1), the solution of the above partial differential equation can be obtained from

$$u(\tau,\delta) = \delta A_{1,1}(\tau) + A_{1,0}(\tau), \tag{3.4}$$

where $\tau = T - t.$

Note Since in this thesis we are interested in the solution as the expectation of the convenience yield as shown in [13] to have the form of (3.4).

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By substituting (3.4) into (3.2), we obtain that $A_{1,0}(\tau)$ and $A_{1,1}(\tau)$ should satisfy the differential equation

$$(\alpha\kappa-\delta\kappa-\lambda)A_{1,1}(\tau)-\delta A'_{1,1}(\tau)-A'_{1,0}(\tau)=0, \tag{3.5}$$

where from the condition (3.3) gives

$$A_{1,0}(0)=0,$$

$$A_{1,1}(0)=1.$$

Then, we separate (3.5) into a system of first order ordinary differential equations by considering the coefficient of δ and the constant term,

$$-\kappa A_{1,1}(\tau)-A'_{1,1}(\tau)=0, \tag{3.6}$$

$$\alpha\kappa A_{1,1}(\tau)-\lambda A_{1,1}(\tau)-A'_{1,0}(\tau)=0. \tag{3.7}$$

Solve (3.6) with $A_{1,1}(0)=1$, then we obtain

$$A_{1,1}(\tau)=\mathrm{e}^{-\kappa\tau}$$

and substituting into (3.7) with $A_{1,0}(0)=0,$ we get

$$A_{1,0}(\tau)=\frac{\mathrm{e}^{-\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)(\alpha\kappa-\lambda)}{\kappa}.$$

Finally, we have (3.4)

<u>Case 2</u>, $n=2$

To solve this partial differential equation, based on $n=1$ and [13] for the terminal condition $u(T,\delta)=\delta^2,$ we assume the solution in the form

$$u(\tau,\delta)=\delta^2 A_{2,2}(\tau)+\delta A_{2,1}(\tau)+A_{2,0}(\tau) \tag{3.8}$$

where $\tau=T-t.$

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Note Similarly to case $n=1$, since in this thesis we are interested in the solution as the expectation of the second power of the convenience yield therefore, after a modification the solution is expect to have the form of (3.8).

By substituting (3.8) into (3.2), we obtain that $A_{1,0}(\tau)$ and $A_{1,1}(\tau)$ satisfy the differential equation

$$\begin{aligned}\sigma_2{}^2 A_{2,2}(\tau)+(\alpha\kappa-\delta\kappa-\lambda)\left(A_{2,1}(\tau)+2\delta A_{2,2}(\tau)\right)-A'_{2,0}(\tau)- \\ \delta A'_{2,1}(\tau)-\delta^2 A'_{2,2}(\tau) \;=\; 0 \qquad (3.9)\end{aligned}$$

where from the condition (3.3) gives

$$A_{2,0}(0)=0,$$

$$A_{2,1}(0)=0,$$

$$A_{2,2}(0)=1.$$

Then, we separate (3.9) into a system of first order ordinary differential equations by considering the coefficient of δ^2, δ, and the constant term,

$$-2\kappa A_{2,2}(\tau)-A'_{2,2}(\tau)=0, \qquad (3.10)$$

$$-\kappa A_{2,1}(\tau)+2\alpha\kappa A_{2,2}(\tau)-2\lambda A_{2,2}(\tau)-A'_{2,1}(\tau)=0, \qquad (3.11)$$

$$\alpha\kappa A_{2,1}(\tau)-\lambda A_{2,1}(\tau)+\sigma_2{}^2 A_{2,2}(\tau)-A'_{2,0}(\tau)=0. \qquad (3.12)$$

Solve (3.10) with $A_{2,2}(0)=1,$ then we obtain,

$$A_{2,2}(\tau)=\mathrm{e}^{-2\kappa\tau}$$

and substituting into (3.11) with $A_{2,1}(0)=0,$ we get

$$A_{2,1}(\tau)=\frac{2\mathrm{e}^{-2\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)(\alpha\kappa-\lambda)}{\kappa}$$

and then, substituting $A_{2,2}(\tau)$and $A_{2,1}(\tau)$ into (3.12) with $A_{2,0}(0)=0,$ we get

$$A_{2,0}(\tau)=\frac{e^{-2\kappa\tau}\left(-1+e^{\kappa\tau}\right)}{2\kappa^2}\times\Big[e^{\kappa\tau}\left(\sigma_2^2\kappa+2\alpha^2\kappa^2-4\alpha\kappa\lambda+2\lambda^2\right)$$
$$+\sigma_2^2\kappa-2\alpha^2\kappa^2+4\alpha\kappa\lambda-2\lambda^2\Big].$$

Finally, we have (3.8). □

Lemma 3.2 Expectations of the commodity spot price

For system of stochastic processes (2.1) and (2.2) of the Schwartz two-factor model, the expectations of the first and the second powers of the commodity spot price at the terminal time T, are

$$E^{\mathbb{Q}}\left[S_T \middle| S_t=s,\delta_t=\delta\right]=se^{B_{1,0}(\tau)+\delta B_{1,1}(\tau)},$$
$$E^{\mathbb{Q}}\left[S_T{}^2 \middle| S_t=s,\delta_t=\delta\right]=s^2e^{B_{2,0}(\tau)+\delta B_{2,1}(\tau)},$$

where t is the initial time and $\tau=T-t$.

Proof.

Define

$$u(t,s,\delta)=E\left[S_T{}^n \middle| S_t=s,\delta_t=\delta\right],\ \ t\geq 0,\ T>0,\ \ s\in\mathbb{R},\ \ \delta\in\mathbb{R}.$$

By applying the Feynman-Kac formula in 2-dimension for the system of equations (2.1) and (2.2), we get $u(t,s,\delta)$ as the solution of the partial differential equation

$$\frac{\partial u}{\partial t}+(r-\delta)s\frac{\partial u}{\partial s}+\left(\kappa(\alpha-\delta)-\lambda\right)\frac{\partial u}{\partial\delta}+\frac{1}{2}\sigma_1{}^2\frac{\partial^2u}{\partial s^2}+\frac{1}{2}\sigma_2{}^2\frac{\partial^2u}{\partial\delta^2}$$
$$+\sigma_1 s\sigma_2\rho\frac{\partial^2u}{\partial s\partial\delta}=0, \qquad (3.13)$$

satisfying the condition

$$u(T,s,\delta)=s^n. \qquad (3.14).$$

The derivation of the partial differential equation is shown in appendix.

<u>Case 1</u>, $n=1$

Following the idea of Schwartz's solution producer [13], the solution of the above partial differential equation (3.13) can be assumed of the form

$$u(\tau,s,\delta)=se^{B_{1,0}(\tau)+\delta B_{1,1}(\tau)}, \tag{3.15}$$

where $\tau=T-t.$

Note Since in this thesis we are interested in the solution as the expectation of the commodity price as shown in [13] to have the form of (3.15).

By substituting (3.15) into (3.13), we obtain that $B_{1,0}(\tau)$ and $B_{1,1}(\tau)$ satisfy the differential equation

$$-\frac{1}{2}\mathrm{e}^{B_{1,0}(\tau)+\delta B_{1,1}(\tau)}s\Big[-2(\alpha\kappa-\delta\kappa-\lambda+\sigma_1\sigma_2\rho)B_{1,1}(\tau)-\sigma_2{}^2B_{1,1}(\tau)^2+ 2\big(-r+\delta+B_{1,0}'(\tau)+\delta B_{1,1}'(\tau)\big)\Big] = 0. \tag{3.16}$$

where from the condition (3.14)

$$B_{1,0}(0) = 0,$$

$$B_{1,1}(0) = 0.$$

Since $\mathrm{e}^{B_{1,0}(\tau)+\delta B_{1,1}(\tau)}\neq 0$, we divided (3.16) by $\mathrm{e}^{B_{1,0}(\tau)+\delta B_{1,1}(\tau)}$ to obtain

$$-\frac{1}{2}s\Big[-2(\alpha\kappa-\delta\kappa-\lambda+\sigma_1\sigma_2\rho)B_{1,1}(\tau)-\sigma_2{}^2B_{1,1}(\tau)^2+ 2\big(-r+\delta+B_{1,0}'(\tau)+\delta B_{1,1}'(\tau)\big)\Big] = 0 \tag{3.17}$$

Then, we separate (3.17) into a system of first order ordinary differential equations by considering the coefficient of δ, and the constant term,

$$-s-s\kappa B_{1,1}(\tau)-sB_{1,1}'(\tau)=0, \tag{3.18}$$

$$rs+s\alpha\kappa B_{1,1}(\tau)-s\lambda B_{1,1}(\tau)+s\sigma_1\sigma_2\rho B_{1,1}(\tau)+\frac{1}{2}s\sigma_2{}^2B_{1,1}^2(\tau)-sB_{1,0}'(\tau)=0. \tag{3.19}$$

Solve (3.18) with $B_{1,1}(0)=0,$ then we obtain,

$$B_{1,1}(\tau)=-\frac{\mathrm{e}^{-\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)}{\kappa}$$

and substituting into (3.19), we get

$$B_{1,0}(\tau)=-\frac{\mathrm{e}^{-2\kappa\tau}}{4\kappa^3}\Big[\Big(\left(\alpha\kappa^2-\kappa\lambda\right)4\mathrm{e}^{\kappa\tau}+\Big(-\alpha\kappa^2+\kappa\lambda+\left(\alpha\kappa^3-r\kappa^3-\kappa^2\lambda\right)\tau\Big)4\mathrm{e}^{2\kappa\tau}\Big){\sigma_2}^2+$$
$$\Big(\left(\kappa\rho\right)4\mathrm{e}^{\kappa\tau}+\left(-\kappa\rho+\kappa^2\rho\tau\right)4\mathrm{e}^{2\kappa\tau}\Big)\sigma_1\sigma_2+\left(1-4\mathrm{e}^{\kappa\tau}+3\mathrm{e}^{2\kappa\tau}-2\mathrm{e}^{2\kappa\tau}\kappa\tau\right)\Big].$$

Finally, we have (3.15)

<u>Case 2</u>, $n=2$

To solve this partial differential equation (3.13) for $n=2,$ we modify the idea [13] for terminal condition $u(T,s,\delta)=s^2,$ by assuming that the solution has the form

$$u(\tau,s,\delta)=s^2e^{B_{2,0}(\tau)+\delta B_{2,1}(\tau)}, \tag{3.20}$$

where $\tau=T-t.$

Note Similarly to case $n=1$, since in this thesis we are interested in the solution as the expectation of the second power of the commodity price therefore, after a modification the solution is expect to have the form of (3.20).

By substituting (3.20) into (3.13), we obtain that $B_{2,0}(\tau)$ and $B_{2,1}(\tau)$ satisfy

$$-\frac{\mathrm{e}^{B_{2,0}(\tau)+\delta B_{2,1}(\tau)}s^2}{2}\Big[2\left(-\alpha\kappa+\delta\kappa+\lambda-2\sigma_1\sigma_2\rho\right)B_{2,1}(\tau)-{\sigma_2}^2B_{2,1}(\tau)^2+$$
$$2\left(-2r-{\sigma_1}^2+2\delta+B'_{2,0}(\tau)+\delta B'_{2,1}(\tau)\right)\Big]=0, \tag{3.21}$$

where from the condition (3.14)

$$B_{2,0}(0)=0,$$
$$B_{2,1}(0)=0.$$

Since $e^{B_{2,0}(\tau)+\delta B_{2,1}(\tau)} \neq 0$, we divided (3.21) by $e^{B_{2,0}(\tau)+\delta B_{2,1}(\tau)}$ to obtained

$$-\frac{s^2}{2}\Big[2\left(-\alpha\kappa+\delta\kappa+\lambda-2\sigma_1\sigma_2\rho\right)B_{2,1}(\tau)-\sigma_2{}^2B_{2,1}(\tau)^2+ \\ 2\left(-2r-\sigma_1{}^2+2\delta+B'_{2,0}(\tau)+\delta B'_{2,1}(\tau)\right)\Big] = 0. \tag{3.22}$$

Then, we separate (3.22) into a system of the first order ordinary differential equations by considering the coefficient of $\delta,$ and constant term,

$$-2s^2-s^2\kappa B_{2,1}(\tau)-s^2B'_{2,1}(\tau)=0, \tag{3.23}$$

$$s^2\Big(2r+\sigma_1{}^2+\alpha\kappa B_{2,1}(\tau)-\lambda B_{2,1}(\tau)+2\sigma_1\sigma_2\rho B_{2,1}(\tau) \\ +\frac{1}{2}\sigma_2{}^2B_{2,1}(\tau)^2-B'_{2,0}(\tau)\Big)=0. \tag{3.24}$$

Solve (3.23) with $B_{2,1}(0)=0,$ then we obtain,

$$B_{2,1}(\tau)=-\frac{2e^{-\kappa\tau}\left(-1+e^{\kappa\tau}\right)}{\kappa}$$

and substituting into (3.24) with $B_{2,2}(0)=0,$ we get

$$B_{2,0}(\tau)=-\frac{e^{-2\kappa\tau}}{\kappa^3}\Big[\left(1-4e^{\kappa\tau}+3e^{2\kappa\tau}-2e^{2\kappa\tau}\kappa\tau\right)+\left(4e^{\kappa\tau}\kappa\rho+4e^{2\kappa\tau}\left(-\kappa\rho+\kappa^2\rho\tau\right)\right)\sigma_1\sigma_2+ \\ \left(2e^{\kappa\tau}\kappa\left(\alpha\kappa-\lambda\right)+e^{2\kappa\tau}\kappa\left(-2\alpha\kappa+2\lambda-2r\kappa^2\tau-\sigma_1{}^2\kappa^2\tau+2\alpha\kappa^2\tau-2\kappa\lambda\tau\right)\right)\sigma_2{}^2\Big].$$

Finally, we have (3.20) □

Lemma 3.3 Expectations of product of the convenience yield and the commodity price

For the system of stochastic processes (2.1) and (2.2) of the Schwartz two-factor model, the expectations of the product of convenience yield and commodity spot price at the terminal time T, are

$$E^{\mathbb{Q}}\left[S_T\delta_T \left| S_t = s, \delta_t = \delta\right.\right] = s\left(C_0(\tau) + \delta C_1(\tau)\right)e^{\delta C_2(\tau)},$$

where t is the initial time and $\tau = T - t.$

Proof

In this lemma, in order to find the required expectation, we used the Ito's lemma [4] to transform the process of the commodity spot price (2.1) via the transformation $X = \ln(S_t)$ to obtain the system

$$dX_t = \left(r - \delta_t - \frac{\sigma_1^{\ 2}}{2}\right)dt + \sigma_1 dW_t^{(1)}. \tag{3.25}$$

Define

$$u(t,x,\delta) = E\left[e^{X_T}\delta_T \left| X_t = x, \delta_t = \delta\right.\right],\ t \ge 0,\ T > 0,\ x \in \mathbb{R},\ \ \delta \in \mathbb{R}.$$

By applying the Feynman-Kac formula in 2-dimension, we get $u(t,x,\delta)$ as the solution of PDE corresponding to the system (3.25) and (2.2)

$$\begin{aligned}\frac{\partial u}{\partial t} + \left(r - \delta - \frac{\sigma_1^{\ 2}}{2}\right)\frac{\partial u}{\partial x} + \left(\kappa(\alpha - \delta) - \lambda\right)\frac{\partial u}{\partial \delta} + \frac{1}{2}\sigma_1^{\ 2}\frac{\partial^2 u}{\partial x^2} + \\ \frac{1}{2}\sigma_2^{\ 2}\frac{\partial^2 u}{\partial \delta^2} + \sigma_1\sigma_2\rho\frac{\partial^2 u}{\partial x \partial \delta} \quad = \quad 0,\end{aligned} \tag{3.26}$$

with condition

$$u(T,x,\delta) = e^x\delta. \tag{3.27}$$

Similarly, to solve partial differential equation (3.26) with condition $u(T,x,\delta)=e^{x}\delta,$ we modify the idea of [13] by assuming the solution has the form

$$u(\tau,x,\delta)=\mathrm{e}^{(x+C_2(\tau)\delta)}\left(C_0(\tau)+C_1(\tau)\right) \tag{3.28}$$

where $\tau=T-t.$

Note Similarly to lemmas 3.1-3.2 since in this thesis we are interested in the solution as the expectation of the product of the convenience yield and the commodity price therefore, after a modification the solution is expect to have the form of (3.28).

By substituting (3.28) into (3.26), we obtain that $C_0(\tau), C_1(\tau)$ and $C_2(\tau)$ satisfy,

$$\begin{aligned}&\frac{1}{2}\mathrm{e}^{x+\delta C_2(\tau)}\Big[-2\left(C_0'(\tau)+\delta C_1'(\tau)\right)+C_0(\tau)\Big(C_2(\tau)^2\sigma_2^2+2C_2(\tau)(\alpha\kappa-\delta\kappa-\lambda+\rho\sigma_1\sigma_2)\\&+2\left(r-\delta-\delta C_2'(\tau)\right)\Big)+C_1(\tau)\Big(\delta C_2(\tau)^2\sigma_2^2+2C_2(\tau)\big(-\delta(-\alpha\kappa+\delta\kappa+\lambda)+\delta\rho\sigma_1\sigma_2\\&+\sigma_2^2\big)+2\left(r\delta-\delta^2+\alpha\kappa-\delta\kappa-\lambda+\rho\sigma_1\sigma_2-\delta^2C_2'(\tau)\right)\Big)\Big]=0,\end{aligned} \tag{3.29}$$

where from the condition (3.27) gives

$$C_0(0)=0,\ C_1(0)=1,\ C_2(0)=0.$$

Since $\mathrm{e}^{x+\delta C_2(\tau)}\neq 0$, we divided (3.29) by $\mathrm{e}^{x+\delta C_2(\tau)}$ and separate it into the system of first order ODE, by considering the coefficient of δ^2, δ and the constant term as follow,

$$-C_1(\tau)\left(1+\kappa C_2(\tau)+C_2'(\tau)\right)=0 \tag{3.30}$$

$$\begin{aligned}&\Big(C_1(\tau)\Big(r-\kappa+\frac{1}{2}C_2(\tau)\left(2\alpha\kappa-2\lambda+2\rho\sigma_1\sigma_2+C_2(\tau)\sigma_2^2\right)\Big)\\&\qquad-C_1'(\tau)-C_2(\tau)\left(1+\kappa C_2(\tau)+C_2'(\tau)\right)\Big)=0\end{aligned} \tag{3.31}$$

$$\begin{aligned}&C_1(\tau)\left(\alpha\kappa-\lambda+\rho\sigma_1\sigma_2+C_2(\tau)\sigma_2^2\right)-C_0'(\tau)\\&+C_0(\tau)\left(r+\frac{1}{2}C_2(\tau)^2\sigma_2^2+C_2(\tau)(\alpha\kappa-\lambda+\rho\sigma_1\sigma_2)\right)=0\end{aligned} \tag{3.32}$$

(i) Consider the coefficient of δ^2 in (3.30)

$$-C_1(\tau)\left(1+\kappa C_2(\tau)+C_2'(\tau)\right)=0,$$

To solve (3.30), we may assume

$$\left(1+\kappa C_2(\tau)+C_2'(\tau)\right)=0,$$

with the condition $C_2(0)=0,$ we obtain,

$$C_2(\tau)=-\frac{\mathrm{e}^{-\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)}{\kappa}.$$

Since we assume $\left(1+\kappa C_2(\tau)+C_2'(\tau)\right)=0,$ the system of the first ODE above reduce to

$$\left(C_1(\tau)\left(r-\kappa+\frac{1}{2}C_2(\tau)\left(2\alpha\kappa-2\lambda+2\rho\sigma_1\sigma_2+C_2(\tau)\sigma_2^2\right)\right)-C_1'(\tau)\right)=0 \qquad (3.31)$$

$$\begin{aligned} &C_1(\tau)\left(\alpha\kappa-\lambda+\rho\sigma_1\sigma_2+C_2(\tau)\sigma_2^2\right)-C_0'(\tau) \\ &+C_0(\tau)\left(r+\frac{1}{2}C_2(\tau)^2\sigma_2^2+C_2(\tau)\left(\alpha\kappa-\lambda+\rho\sigma_1\sigma_2\right)\right)=0 \end{aligned} \qquad (3.32)$$

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(ii) substituting $C_2(\tau)$ into (3.31), we obtain

$$\begin{aligned} &C_1(\tau)\left(r-\kappa+\frac{\left(\mathrm{e}^{-\kappa\tau}-1\right)}{2\kappa}\left(2\alpha\kappa-2\lambda+2\rho\sigma_1\sigma_2+\frac{\left(\mathrm{e}^{-\kappa\tau}-1\right)}{\kappa}\sigma_2^2\right)\right) \\ &\qquad -C_1'(\tau)=0 \end{aligned} \qquad (3.33)$$

Solve (3.33) with $C_1(0)=0,$ we get,

$$C_1(\tau)=\mathrm{e}^{\frac{\alpha\kappa^2-\lambda\kappa+\rho\sigma_1\sigma_2\kappa-\frac{3}{4}\sigma_2^2-\mathrm{e}^{-\kappa\tau}\left(\alpha\kappa^2+\lambda\kappa-\rho\sigma_1\sigma_2\kappa+\sigma_2^2\right)-\mathrm{e}^{-2\kappa\tau}\frac{\sigma_2^2}{4}}{\kappa^3}+\frac{\tau\left(2\kappa\left(r\kappa-\alpha\kappa-\kappa^2+\lambda\right)-2\kappa\rho\sigma_1\sigma_2+\sigma_2^2\right)}{2\kappa^2}}.$$

(iii) substituting $C_1(\tau)$ and $C_2(\tau)$ into (3.32) we obtain,

$$\mathrm{e}^{\frac{\alpha\kappa^2-\lambda\kappa+\rho\sigma_1\sigma_2\kappa-\frac{3}{4}\sigma_2^2-\mathrm{e}^{-\kappa\tau}\left(\alpha\kappa^2+\lambda\kappa-\rho\sigma_1\sigma_2\kappa+\sigma_2^2\right)-\mathrm{e}^{-2\kappa\tau}\frac{\sigma_2^2}{4}}{\kappa^3}+\frac{\tau\left(2\kappa\left(r\kappa-\alpha\kappa-\kappa^2+\lambda\right)-2\kappa\rho\sigma_1\sigma_2+\sigma_2^2\right)}{2\kappa^2}}$$

$$\left(\alpha\kappa-\lambda+\rho\sigma_1\sigma_2+\frac{\left(\mathrm{e}^{-\kappa\tau}-1\right)}{\kappa}\sigma_2^2\right)-C_0'(\tau)+C_0(\tau)\left(r+\frac{1}{2}\frac{\left(\mathrm{e}^{-\kappa\tau}-1\right)^2}{\kappa^2}\sigma_2^2\right.$$

$$\left.+\frac{\left(\mathrm{e}^{-\kappa\tau}-1\right)}{\kappa}\left(\alpha\kappa-\lambda+\rho\sigma_1\sigma_2\right)\right)=0 \quad (3.34)$$

Solve (3.34) with $C_0(0)=0,$ we get,

$$C_0(\tau)=\frac{1}{2\kappa^2}\mathrm{e}^{\frac{\alpha\kappa^2-\lambda\kappa+\rho\sigma_1\sigma_2\kappa-\frac{3}{4}\sigma_2^2-\mathrm{e}^{-\kappa\tau}\left(\alpha\kappa^2+\lambda\kappa-\rho\sigma_1\sigma_2\kappa+\sigma_2^2\right)-\mathrm{e}^{-2\kappa\tau}\frac{\sigma_2^2}{4}}{\kappa^3}-2\kappa\tau+\frac{\tau\left(2\kappa\left(r\kappa-\alpha\kappa+\lambda\right)-2\kappa\rho\sigma_1\sigma_2+\sigma_2^2\right)}{2\kappa^2}}$$

$$\left(-1+\mathrm{e}^{\kappa\tau}\right)\left(2\mathrm{e}^{\kappa\tau}\alpha\kappa^2-2\mathrm{e}^{\kappa\tau}\kappa\lambda+2\mathrm{e}^{\kappa\tau}\kappa\rho\sigma_1\sigma_2+\sigma_2^2-\mathrm{e}^{\kappa\tau}\sigma_2^2\right).$$

Finally, we have (3.28). Then substituting the result with $x=\ln(s),$ we obtain

$$u(\tau,s,\delta)=s\left(C_0(\tau)+\delta C_1(\tau)\right)e^{\delta C_2(\tau)},$$

where

$$C_0(\tau)=\frac{1}{2\kappa^2}\mathrm{e}^{\frac{\alpha\kappa^2-\lambda\kappa+\rho\sigma_1\sigma_2\kappa-\frac{3}{4}\sigma_2^2-\mathrm{e}^{-\kappa\tau}\left(\alpha\kappa^2+\lambda\kappa-\rho\sigma_1\sigma_2\kappa+\sigma_2^2\right)-\mathrm{e}^{-2\kappa\tau}\frac{\sigma_2^2}{4}}{\kappa^3}-2\kappa\tau+\frac{\tau\left(2\kappa\left(r\kappa-\alpha\kappa+\lambda\right)-2\kappa\rho\sigma_1\sigma_2+\sigma_2^2\right)}{2\kappa^2}}$$

$$\left(-1+\mathrm{e}^{\kappa\tau}\right)\left(2\mathrm{e}^{\kappa\tau}\alpha\kappa^2-2\mathrm{e}^{\kappa\tau}\kappa\lambda+2\mathrm{e}^{\kappa\tau}\kappa\rho\sigma_1\sigma_2+\sigma_2^2-\mathrm{e}^{\kappa\tau}\sigma_2^2\right),$$

$$C_1(\tau)=\mathrm{e}^{\frac{\alpha\kappa^2-\lambda\kappa+\rho\sigma_1\sigma_2\kappa-\frac{3}{4}\sigma_2^2-\mathrm{e}^{-\kappa\tau}\left(\alpha\kappa^2+\lambda\kappa-\rho\sigma_1\sigma_2\kappa+\sigma_2^2\right)-\mathrm{e}^{-2\kappa\tau}\frac{\sigma_2^2}{4}}{\kappa^3}+\frac{\tau\left(2\kappa\left(r\kappa-\alpha\kappa-\kappa^2+\lambda\right)-2\kappa\rho\sigma_1\sigma_2+\sigma_2^2\right)}{2\kappa^2}},$$

$$C_2(\tau)=-\frac{\mathrm{e}^{-\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)}{\kappa}. \qquad \square$$

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Lemmas 3.1-3.3 and (2.3) lead us to the main theorem,

Theorem 3.4 Formula for a correlation coefficient between underlying commodity price and its convenience yield

A closed-form formula for a correlation coefficient between underlying commodity price and its convenience yield is

$$C_t^r[\tau]=\frac{\left(C_0(\tau)+\delta C_1(\tau)\right)e^{\delta C_2(\tau)}-e^{B_{1,0}(\tau)+\delta B_{1,1}(\tau)}\left[\delta A_{1,1}(\tau)+A_{1,0}(\tau)\right]}{\left(D(\tau)E(\tau)\right)^{\frac{1}{2}}},$$

where

$$D(\tau)=e^{B_{2,0}(\tau)+\delta B_{2,1}(\tau)}-e^{2\left(B_{1,0}(\tau)+\delta B_{1,1}(\tau)\right)},$$

$$E(\tau)=\left[\delta^2 A_{2,2}(\tau)+\delta A_{2,1}(\tau)+A_{2,0}(\tau)\right]-\left[\delta^2 A_{1,1}^2(\tau)+2\delta A_{1,1}(\tau)A_{1,0}(\tau)+A_{1,0}^2(\tau)\right],$$

$A_{i,j}, B_{i,j}$ and C_i are defined as in the lemmas above and $\tau=T-t.$

Note. We will show that $D(\tau)>0$ and $E(\tau)>0.$

1. To show $D(\tau)>0$,

$$\mathrm{D}(\tau)=\mathrm{e}^{-\frac{2\mathrm{e}^{-\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)\delta}{\kappa}-\frac{\mathrm{e}^{-2\kappa\tau}((2\mathrm{e}^{\kappa\tau}\alpha\kappa^2-2\mathrm{e}^{2\kappa\tau}\alpha\kappa^2-2\mathrm{e}^{\kappa\tau}\kappa\lambda+2\mathrm{e}^{2\kappa\tau}\kappa\lambda-2\mathrm{e}^{2\kappa\tau}r\kappa^3\tau+2\mathrm{e}^{2\kappa\tau}\alpha\kappa^3\tau-2\mathrm{e}^{2\kappa\tau}\kappa^2\lambda\tau)+[2\mathrm{e}^{\kappa\tau}\kappa\rho\sigma_1\sigma_2-2\mathrm{e}^{2\kappa\tau}\kappa\rho\sigma_1\sigma_2+2\mathrm{e}^{2\kappa\tau}\kappa^2\rho\tau\sigma_1\sigma_2+\frac{1}{2}\sigma_2^2-2\mathrm{e}^{\kappa\tau}\sigma_2^2+\frac{3}{2}\mathrm{e}^{2\kappa\tau}\sigma_2^2-\mathrm{e}^{2\kappa\tau}\kappa\tau\sigma_2^2])}{\kappa^3}}$$

$$\left(\mathrm{e}^{\frac{-\mathrm{e}^{-2\kappa\tau}[2\mathrm{e}^{\kappa\tau}\kappa\rho\sigma_1\sigma_2-2\mathrm{e}^{2\kappa\tau}\kappa\rho\sigma_1\sigma_2+2\mathrm{e}^{2\kappa\tau}\kappa^2\rho\tau\sigma_1\sigma_2+\frac{1}{2}\sigma_2^2-2\mathrm{e}^{\kappa\tau}\sigma_2^2+\frac{3}{2}\mathrm{e}^{2\kappa\tau}\sigma_2^2-\mathrm{e}^{2\kappa\tau}\kappa\tau\sigma_2^2]+\mathrm{e}^{2\kappa\tau}\kappa^3\tau\sigma_1^2}{\kappa^3}}-1\right).$$

Since

$$\mathrm{e}^{-\frac{2\mathrm{e}^{-\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)\delta}{\kappa}-\frac{\mathrm{e}^{-2\kappa\tau}((2\mathrm{e}^{\kappa\tau}\alpha\kappa^2-2\mathrm{e}^{2\kappa\tau}\alpha\kappa^2-2\mathrm{e}^{\kappa\tau}\kappa\lambda+2\mathrm{e}^{2\kappa\tau}\kappa\lambda-2\mathrm{e}^{2\kappa\tau}r\kappa^3\tau+2\mathrm{e}^{2\kappa\tau}\alpha\kappa^3\tau-2\mathrm{e}^{2\kappa\tau}\kappa^2\lambda\tau)+[2\mathrm{e}^{\kappa\tau}\kappa\rho\sigma_1\sigma_2-2\mathrm{e}^{2\kappa\tau}\kappa\rho\sigma_1\sigma_2+2\mathrm{e}^{2\kappa\tau}\kappa^2\rho\tau\sigma_1\sigma_2+\frac{1}{2}\sigma_2^2-2\mathrm{e}^{\kappa\tau}\sigma_2^2+\frac{3}{2}\mathrm{e}^{2\kappa\tau}\sigma_2^2-\mathrm{e}^{2\kappa\tau}\kappa\tau\sigma_2^2])}{\kappa^3}}>0,$$

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It is enough to show that

$$I := \frac{-2e^{-\kappa\tau}\kappa\rho\sigma_1\sigma_2 + 2\kappa\rho\sigma_1\sigma_2 - 2\kappa^2\rho\tau\sigma_1\sigma_2 - \frac{1}{2}e^{-2\kappa\tau}\sigma_2^2 + 2e^{-\kappa\tau}\sigma_2^2 - \frac{3}{2}\sigma_2^2 + \kappa\tau\sigma_2^2}{\kappa^3}$$

$$+\frac{\kappa^3\tau\sigma_1^2}{\kappa^3} > 0.$$

Using the Taylor series approximation* for the exponential functions, we get

$$I \approx \frac{2\kappa\rho\sigma_1\sigma_2\left(1-\kappa\tau-(1-\kappa\tau+\frac{(\kappa\tau)^2}{2})\right)+\kappa^3\tau\sigma_1^2}{\kappa^3}$$

$$+\frac{\frac{\sigma_2^2}{2}\left(-3+2\kappa\tau+4\left(1-\kappa\tau+\frac{(\kappa\tau)^2}{2}-\frac{(\kappa\tau)^3}{6}\right)-\left(1-2\kappa\tau+\frac{(2\kappa\tau)^2}{2}-\frac{(2\kappa\tau)^3}{6}\right)\right)}{\kappa^3}$$

After simplify; we get

$$I \approx \tau\left(\sigma_1^2 - \rho\sigma_1\sigma_2\tau + \frac{\sigma_2^2}{2}\tau^2\right)$$

Since ρ can be positive or negative and $|\rho|<1,$ it is clear that $-\rho > -\sqrt{2}.$ Therefore,

$$0 \le \tau\left(\sigma_1 - \frac{\sigma_2\tau}{\sqrt{2}}\right)^2 = \tau\left(\sigma_1^2 - \sqrt{2}\sigma_1\sigma_2\tau + \frac{\sigma_2^2}{2}\tau^2\right) < \tau\left(\sigma_1^2 - \rho\sigma_1\sigma_2\tau + \frac{\sigma_2^2}{2}\tau^2\right)$$

which conclude that

$$I > 0.$$

Thus,

$$D(\tau) > 0.$$

*Note this approximation is good when $\kappa\tau < 1.$ For the case $\kappa\tau > 1$ is harder but can be check using similar idea.

2. To show $E(\tau) > 0$,

$$E(\tau) = \frac{\sigma_2^2 \left(1 - \mathrm{e}^{-2\kappa\tau}\right)}{2\kappa},$$

For all $\tau > 0$,

$$0 < \left(1 - \mathrm{e}^{-2\kappa\tau}\right) \le 1,$$

then

$$0 < E(\tau) \le \frac{\sigma_2^2}{2\kappa}.$$

Since the obtained formula for the correlation coefficient $C_t^r[T]$ involves many parameters in the formula, it is very complicated to show that the value $\left|C_t^r[T]\right| \le 1$ for the formula. However, we have illustrated in the next section that for some given examples of parameters, the calculated values $C_t^r[T]$ is in the range $[-1,1]$.

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CHAPTER IV
NUMERICAL EXAMPLES AND DISCUSSION

In this Chapter, we confirm our results with some numerical examples using parameters estimated by Schwartz [13] for 3 commodities; oil, copper, and gold, and we discuss the behaviors of the correlations as the final time T varies.

4.1 The correlations at various final time T

We plot the graph of the correlation coefficient, $C_t^r[T]$ (between underlying commodity price and its convenience yield) at different final time T, where the parameters were estimated by Schwartz [13] as shown in the table below.

Parameter	Oil	Copper	Gold
r	0.06	0.06	0.06
δ	0.1	0.1	0.25
κ	1.876	0.25	0.298
α	0.106	1.156	0.019
σ_1	0.393	0.248	0.107
σ_2	0.527	0.280	0.015
ρ	0.776	0.818	0.025
λ	0.198	0.256	0.008
t	0	0	0

Table 4.1: shows the set of parameters that were estimated by Schwartz [13] for 3 commodities; oil, copper, gold.

4.1.1 Oil

In this section, we plot the graph of the correlation coefficient $C_t^r[T]$ (between underlying commodity price and its convenience yield) at different final time T using the parameters of oil from the Table 4.1.

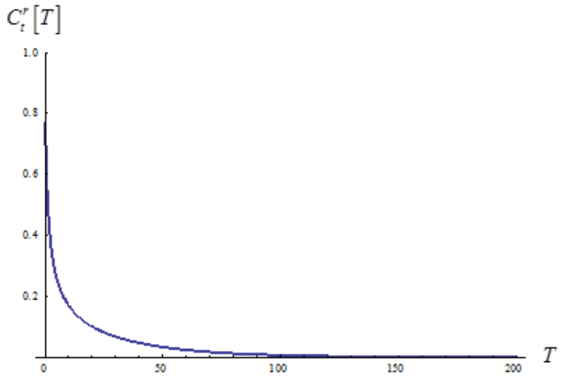


Figure 4.2: The correlation coefficient between commodity price and its convenience yield simulated at different final time T for oil using parameters from Table 4.1.

4.1.2 Copper

In this section, we plot the graph of the correlation coefficient $C_t^r[T]$ (between underlying commodity price and its convenience yield) at different final time T using the parameters of copper from the Table 4.1.

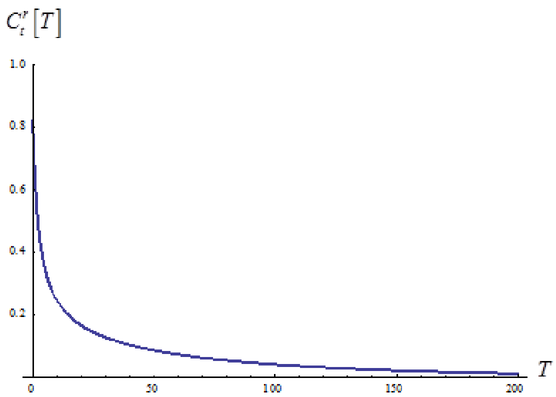


Figure 4.3: The correlation coefficient between commodity price and its convenience yield simulated at different final time T for copper using parameters from Table 4.1.

4.1.3 Gold

In this section, we plot the graph of the correlation coefficient $C_t^r[T]$ (between underlying commodity price and its convenience yield) at different final time T using the parameters of gold from the Table 4.1.

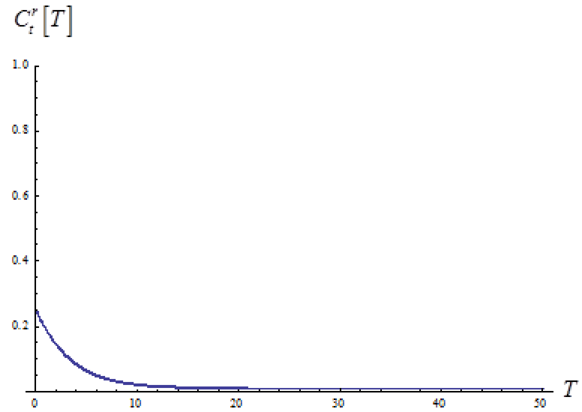
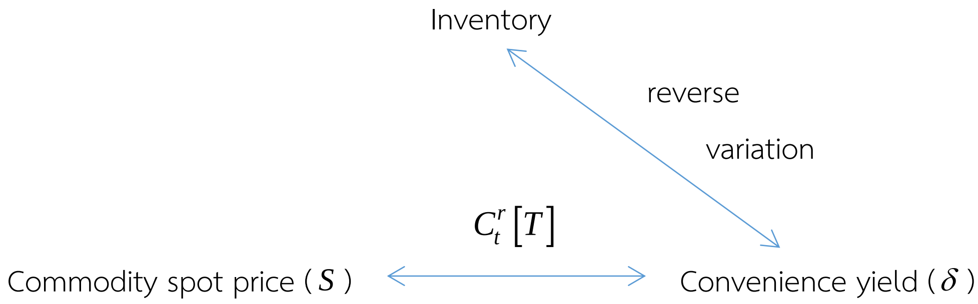


Figure 4.4: The correlation coefficient between commodity price and its convenience yield simulated at different final time T for gold using parameters from Table 4.1.

As shown in Figures 4.2 - 4.4, the correlation coefficients are close to ρ, the correlation of the increment of Brownian motion, as T is closed to zero, and the correlation coefficients tend to zero as T increases approaching infinity. Note also that the obtained values of $C_t^r[T]$ from the formula set of parameters are in the range $[-1,1]$ as expected for the correlation. These plotted results confirm the right behavior of the correlation because at the starting point the correlation of these two factors must equal to their given relation which is ρ. Moreover, since both the commodity spot price and the convenience yield have the mean reversion property, as the time approaching infinity these two factors will converge to their own means, and therefore the correlation between these two factors will go to zero.

4.2 Inventories



From the above diagram, we can predict the inventory level from the commodity spot price by finding the correlation between the commodity spot price and the convenience yield, $C_t^r[T]$. If the value of $C_t^r[T]$ is positive, then the inventory level will have the opposite trend to the commodity spot price but if the value of $C_t^r[T]$ is negative, then the inventory level will have the same trend as the commodity spot price.

As shown in Figure 4.2 - 4.4, we see that the correlation coefficients $C_t^r[T]$ for all three commodities stay positive for all T given that the parameter ρ is positive. Form the inventory diagram, we can say that the inventory level and the commodity spot price have the reverse relation if T is small which will be very small as T large.

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CHAPTER V

CONCLUSIONS

In this thesis, we derive the closed-form formula for the correlation coefficient, $C_t^r[T]$, between the commodity spot price, S_t, and its convenience yield, δ_t, based on the Schwartz two-factor model and the Pearson correlation. According to the Pearson correlation, we obtained the expectations of the first and the second powers of S_t, and δ_t, and the product $S_t\delta_t$ using the Feynman-Kac formula, which is the method of finding the expectations of random processes by solving partial differential equations. To solve the PDEs, we assumed the form of the solution and solved the unknown coefficients by solving a system of ODEs using the computer software "Mathematica 9" to obtain the solution. As the result, the obtained formula, $C_t^r[T]$, depends on several parameters such as interest rate, the convenience yield at initial time, the speed of mean reversion, the long-run mean of instantaneous convenience yield, a parameter showing a volume of a fluctuation of a commodity spot prices, a parameter showing a volume of fluctuation instantaneous convenience yield, the correlation of the increment of Brownian motion, a risk premium of the market, and the range of time $T-t$. To illustrate the results numerically, plots of the graphs between the correlation coefficient, $C_t^r[T]$, and T using parameters that were estimated by Schwartz [3] for oil, copper, and gold are shown in Figures 4.2-4.4.

As illustrated in the examples of Chapter 4 and according to the theory of storage (Working [17]) that the convenience yield varies inversely with the inventory of commodity, therefore, the correlation that we obtained can imply the relation between the commodity spot price and its inventory. In this case, the inventory level and the commodity spot price will have the reverse relation when T is small which will be very small as T large.

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APPENDIX

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The derivation of the partial differential equations (PDE)

We consider the processes (2.2) and (2.3) and the process for $Y_t = (X_t, \delta_t)$ can be written in terms of two independent Brownian motions Z_1 and Z_2 as

$$d\begin{pmatrix} X_t \\ \delta_t \end{pmatrix} = \begin{pmatrix} r-\delta_t-\dfrac{\sigma_1^{\,2}}{2} \\ \kappa(\alpha-\delta_t)-\lambda \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix} \qquad (6.1)$$

Here, we use the fact that the Brownian motions $W^{(1)}$ and $W^{(2)}$ have correlation ρ, which can be expressed in terms of two independent Brownian motions Z_1 and Z_2, namely, $dW^{(1)} = dZ_1$ and $dW^{(2)} = \rho dZ_1 + \sqrt{1-\rho^2}\, dZ_2$.

To obtain the generator in (2.4), we need the following matrix from (6.1)

$$\begin{aligned} vv^T &= \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \sigma_1 & \sigma_2\rho \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \\ &= \begin{pmatrix} \sigma_1^{\,2} & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^{\,2} \end{pmatrix}. \end{aligned}$$

The generator in (2.4) is therefore

$$\mathcal{A} = \left(r-\delta-\frac{\sigma_1^2}{2} \right)\frac{\partial}{\partial x} + \left(\kappa(\alpha-\delta)-\lambda\right)\frac{\partial}{\partial \delta} + \frac{1}{2}\left[\sigma_1^2 \frac{\partial^2}{\partial x^2} + \sigma_2^2 \frac{\partial^2}{\partial \delta^2} + 2\sigma_1\sigma_2\rho \frac{\partial^2}{\partial x \partial \delta} \right].$$

The PDE for $u = u(t, x, \delta)$ becomes (2.5)

$$\frac{\partial u}{\partial t} + \left(r-\delta-\frac{\sigma_1^2}{2} \right)\frac{\partial u}{\partial x} + \left(\kappa(\alpha-\delta)-\lambda\right)\frac{\partial u}{\partial \delta} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial \delta^2} + \sigma_1\sigma_2\rho \frac{\partial^2 u}{\partial x \partial \delta} = 0.$$

Similarly, for PDE (3.13) we consider Schwartz two-factor model (2.1) and (2.2). In this case, after applying the same process as above, the PDE for $u = u(t, s, \delta)$ of (2.1-2.2) becomes

$$\frac{\partial u}{\partial t}+(r-\delta)s\frac{\partial u}{\partial s}+\left(\kappa(\alpha-\delta)-\lambda\right)\frac{\partial u}{\partial \delta}+\frac{1}{2}\sigma_1^2\frac{\partial^2 u}{\partial s^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial \delta^2}+\sigma_1 s\sigma_2\rho\frac{\partial^2 u}{\partial s\partial \delta}=0,$$

as used in Lemma 3.2.

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