อนันต์เมื่อปราศจากการเลือก

นายสุภกุญช์ พนาสวัสดิ์วงศ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2556

INFINITY WITHOUT CHOICE

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Mathematics

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2013

Thesis Title	INFINITY WITHOUT CHOICE
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Field of Study	Mathematics
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สุภกุญช์ พนาสวัสดิ์วงศ์: อนันต์เมื่อปราศจากการเลือก. (INFINITY WITHOUT CHOICE)
อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ. ดร.พิมพ์เพ็ญ เวชชาชีวะ, 33 หน้า.

เมื่อมีสัจพจน์การเลือก ถ้า A เป็นเซตอนันต์แล้วจะมีฟังก์ชันแบบหนึ่งต่อหนึ่งจาก ω ไปยัง A เขียนแทนด้วย $\omega \preceq A$ ซึ่งทำให้ได้ว่ามีฟังก์ชันจาก A ไปทั่วถึง ω เขียนแทนด้วย $\omega \preceq^* A$ เมื่อปราศจากสัจพจน์การเลือกสิ่งเหล่านี้ไม่อาจพิสูจน์ได้ เซตอนันต์จึงถูกจำแนกเป็นหลายประเภทเมื่อไม่มีสัจพจน์ดังกล่าว เรากล่าวว่าเซต A เป็นเซตอนันต์แบบเดเดคินต์ ถ้า $\omega \preceq A$ และเป็นเซตอนันต์แบบเดเดคินต์อย่างอ่อน ถ้า $\omega \preceq^* A$ มิฉะนั้นแล้วเรากล่าวว่า A เป็นเซตจำกัดแบบเดเดคินต์ และเซตจำกัดแบบเดเดคินต์อย่างอ่อน ตามลำดับ และเราเรียก A ว่าเซตเดเดคินต์ ถ้า A เป็นเซตจำกัดแบบเดเดคินต์ที่เป็นอนันต์ และเรียก A ว่าเซตเดเดคินต์อย่างอ่อน ถ้า A เป็นเซตจำกัดแบบเดเดคินต์อย่างอ่อนที่เป็นอนันต์ เราศึกษาจำนวนเชิงการนับของเซตเหล่านี้ว่าสมบัติใดบ้างที่สามารถพิสูจน์ได้จาก ZF และถ้า ZF ต้องกันแล้วสมบัติของจำนวนเชิงการนับใดที่ต้องกันกับ ZF ซึ่งทำให้เราจะได้ว่านิเสธของสมบัติเหล่านั้นไม่สามารถพิสูจน์ได้เมื่อปราศจากสัจพจน์การเลือก

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต . . . .

สาขาวิชา . . . . คณิตศาสตร์ . . . . ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก . . . . .

ปีการศึกษา . . . . 2556 . . . .

5572152423: MAJOR MATHEMATICS

KEYWORDS: CARDINAL NUMBERS / AXIOM OF CHOICE / SET THEORY WITH ATOMS

SUPAKUN PANASAWATWONG: INFINITY WITHOUT CHOICE.

ADVISOR: ASST. PROF. PIMPEN VEJJAJIVA, Ph.D., 33 pp.

With the Axiom of Choice (AC), if a set A is infinite, then there is a one-to-one function from ω into A, written $\omega \preceq A$, and so there is a function from A onto ω, written $\omega \preceq^* A$. Without AC, these no longer hold. Therefore, there are many kinds of infinite sets in the absence of AC. We call A Dedekind-infinite if $\omega \preceq A$ and weakly Dedekind-infinite if $\omega \preceq^* A$, otherwise A is called Dedekind-finite and weakly Dedekind-finite, respectively. A is a Dedekind set if A is infinite Dedekind-finite and A is a weakly Dedekind set if A is infinite weakly Dedekind-finite. We investigate the cardinals of such sets and show which properties can be proved from ZF and, provided that ZF is consistent, which properties are consistent with ZF and therefore their negations cannot be proved without AC.

Department	Mathematics and Computer Science	Student's Signature	....
Field of Study	... Mathematics....................	Advisor's Signature	....
Academic Year	... 2013...........................		

Acknowledgements

I would like to express my deepest gratitude to my advisor, Assistant Professor Pimpen Vejjajiva, for her insightful suggestions and useful critiques during the conceptual planning and development of this thesis. I also would like to thank the members of the thesis committee, Associate Professor Ajchara Harnchoowong, Professor Chawewan Ratanaprasert, and Athipat Thamrongthanyalak, for their useful comments. Moreover, I would like to express my appreciation to my parents for supporting me. The Scholarship from the Graduate School, Chulalongkorn University to commemorate the 72nd Anniversary of his Majesty King Bhumibala Aduladeja is gratefully acknowledged.

Contents

CHAPTER I

INTRODUCTION

Any two sets have the same size or the same *cardinality* if there is a bijection between them. We write $|A|$ for the cardinality of a set A. For any sets A and B, we write $A \approx B$ if there is a bijection from A onto B, write $A \preceq B$ if there is a one-to-one function from A into B, and write $A \preceq^* B$ if there is a function from B onto A. We say $|A|$ is *less than or equal* to $|B|$, written $|A| \leq |B|$, if $A \preceq B$ and say $|A|$ is *less than* $|B|$, written $|A| < |B|$, if $|A| \leq |B|$ but $|A| \neq |B|$.

The Axiom of Choice (AC) implies that every set can be well-ordered. As a result, every two cardinals are comparable. It follows that if A is an infinite set, then $\aleph_0 \leq |A|$ and so $\aleph_0 \leq^* |A|$. However, AC causes some paradoxical situations. For example, the existence of a non-measurable subset of $\mathbb{R}$ and the Banach-Tarski paradox (see [9]). To avoid these paradoxes, we have to reject AC.

Without AC, we cannot guarantee that $\aleph_0 \leq |A|$ for every infinite set A. Moreover, we cannot even assert that $\aleph_0 \leq^* |A|$ for every infinite set A. Therefore, without AC, there are many kinds of infinite sets. We call a set A *Dedekind-infinite* if $\aleph_0 \leq |A|$ and *weakly Dedekind-infinite* if $\aleph_0 \leq^* |A|$, otherwise A is called *Dedekind-finite* and *weakly Dedekind-finite*, respectively. A is a *Dedekind* set if A is infinite Dedekind-finite and A is a *weakly Dedekind* set if A is infinite weakly Dedekind-finite. The cardinality of a Dedekind-finite set is called a *Dedekind-finite* cardinal. Similarly for the cardinality of other sets defined above.

It is interesting to know that which properties of these different kinds of sets and cardinals can be proved from ZF and which properties are consistent with ZF, provided that ZF is consistent, and therefore their negations cannot be proved from ZF.

We investigate such sets and their cardinals and show which properties can be proved from ZF. Furthermore, we show that some relations between cardinals are consistent with ZF, provided that ZF is consistent, by showing that such relations hold in some specific permutation models.

This thesis is arranged as follows. Chapter II provides some basic concepts needed for later chapters. Chapter III gives definitions of different kinds of infinite sets in the absence of AC as well as summarizes some known results. Chapter IV gives new results that can be proved from ZF. Chapter V contains consistency results concerning relations between some cardinals.

CHAPTER II

PRELIMINARIES

This chapter gives some background needed for this thesis.

Some background in Set Theory

All the work in the first three chapters is done in Zermelo-Fraenkel Set Theory (ZF) (for more details on ZF see [3]). All basic notions in set theory are defined in the usual way. All proofs in this chapter will be omitted. They can be found in any elementary set theory textbook (for example, [3, 15]).

2.1 Cardinal numbers

Definition 2.1.1. For any sets A and B, we say A is *equinumerous* to B, written $A \approx B$, if there is a bijection from A onto B.

The *cardinality* of a set A, denoted by $|A|$, is the number of all elements of A whose exact definition will be given later. Two sets have the same cardinality if they are equinumerous, i.e., for any sets A and B,

$$|A| = |B| \text{ iff } A \approx B.$$

Definition 2.1.2. A set $\mathfrak{m}$ is a *cardinal (number)* if $\mathfrak{m} = |A|$ for some set A and we say that A is of *cardinality* $\mathfrak{m}$.

Finite and infinite sets

Each natural number is constructed so that it is the set of all smaller natural numbers, namely, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}, \ldots$ and so on. Let ω denote the set of all natural numbers. The construction of natural numbers as well as their basic properties will be omitted and will be used in the ordinary way. Full details can be found in [3].

Definition 2.1.3. A set is *finite* if it is equinumerous to some natural number. A set is *infinite* if it is not finite.

Theorem 2.1.4. *Every finite set is equinumerous to a unique natural number.*

From the above theorem, we obtain the definition for cardinalities of finite sets.

Definition 2.1.5. For a finite set A, $|A|$ is the unique natural number which is equinumerous to A.

Remark. $\mathfrak{n}$ is a finite cardinal iff $\mathfrak{n} \in \omega$.

Cardinal arithmetic

Definition 2.1.6. Let $\mathfrak{m}$ and $\mathfrak{n}$ be any cardinals, say $\mathfrak{m} = |A|$ and $\mathfrak{n} = |B|$. We define

1. $\mathfrak{m} + \mathfrak{n} = |A \cup B|$ where $A \cap B = \emptyset$,
2. $\mathfrak{m} \cdot \mathfrak{n} = |A \times B|$,
3. $\mathfrak{m}^{\mathfrak{n}} = |{}^B A|$ where ${}^B A = \{f \mid f$ is a function from B into $A\}$.

Theorem 2.1.7. *For any set A, $|\mathcal{P}(A)| = |{}^A 2| = 2^{|A|}$.*

Theorem 2.1.8. *For any cardinals $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{p}$,*

1. $\mathfrak{m} + \mathfrak{n} = \mathfrak{n} + \mathfrak{m}$ *and* $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n} \cdot \mathfrak{m}$,

2. $\mathfrak{m} + (\mathfrak{n} + \mathfrak{p}) = (\mathfrak{m} + \mathfrak{n}) + \mathfrak{p}$ *and* $\mathfrak{m} \cdot (\mathfrak{n} \cdot \mathfrak{p}) = (\mathfrak{m} \cdot \mathfrak{n}) \cdot \mathfrak{p}$,

3. $\mathfrak{m} \cdot (\mathfrak{n} + \mathfrak{p}) = (\mathfrak{m} \cdot \mathfrak{n}) + (\mathfrak{m} \cdot \mathfrak{p})$,

4. $\mathfrak{m}^{\mathfrak{n}+\mathfrak{p}} = \mathfrak{m}^{\mathfrak{n}} \cdot \mathfrak{m}^{\mathfrak{p}}$,

5. $(\mathfrak{m} \cdot \mathfrak{n})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{p}} \cdot \mathfrak{n}^{\mathfrak{p}}$,

6. $(\mathfrak{m}^{\mathfrak{n}})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{n} \cdot \mathfrak{p}}$.

Ordering cardinal numbers

Notation. For any sets A and B, we write $A \preceq B$ if there is a one-to-one function from A into B, and we write $A \preceq^* B$ if there is a function from B onto A.

Theorem 2.1.9. *For any sets A and B, if $A \preceq B$, then $A \preceq^* B$.*

Remark. The converse of the above theorem is not necessarily true.

Definition 2.1.10. Let $\mathfrak{m}$ and $\mathfrak{n}$ be any cardinals, say $\mathfrak{m} = |A|$ and $\mathfrak{n} = |B|$. We define $\mathfrak{m}$ to be *less than or equal to* $\mathfrak{n}$, written $\mathfrak{m} \leq \mathfrak{n}$, if $A \preceq B$,

$\mathfrak{m}$ to be *less than* $\mathfrak{n}$, written $\mathfrak{m} < \mathfrak{n}$, if $\mathfrak{m} \leq \mathfrak{n}$ and $\mathfrak{m} \neq \mathfrak{n}$, and

$\mathfrak{m} \leq^* \mathfrak{n}$ if $A \preceq^* B$.

Notation. We may write $\mathfrak{m} \geq \mathfrak{n}$ if $\mathfrak{n} \leq \mathfrak{m}$. Similarly for $>$ and $\geq^*$.

Theorem 2.1.11. *For any cardinals $\mathfrak{m}$ and $\mathfrak{n}$, if $\mathfrak{m}$ is finite and $\mathfrak{n}$ is infinite, then $\mathfrak{m} < \mathfrak{n}$.*

Theorem 2.1.12 (Cantor's Theorem)**.** *For all cardinals $\mathfrak{m}$, $\mathfrak{m} < 2^{\mathfrak{m}}$.*

Theorem 2.1.13. *Let $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{p}$ be cardinals. Then*

1. $\mathfrak{m} \leq \mathfrak{m}$. (*Reflexivity*)

2. *if* $\mathfrak{m} \leq \mathfrak{n}$ *and* $\mathfrak{n} \leq \mathfrak{p}$, *then* $\mathfrak{m} \leq \mathfrak{p}$. (*Transitivity*)

3. *if* $\mathfrak{m} \leq \mathfrak{n}$ *and* $\mathfrak{n} \leq \mathfrak{m}$, *then* $\mathfrak{m} = \mathfrak{n}$. (*Antisymmetry*)

Theorem 2.1.14. *Let* $\mathfrak{m}$, $\mathfrak{n}$, *and* $\mathfrak{p}$ *be any cardinals. If* $\mathfrak{m} \leq \mathfrak{n}$, *then*

1. $\mathfrak{m} + \mathfrak{p} \leq \mathfrak{n} + \mathfrak{p}$.

2. $\mathfrak{m} \cdot \mathfrak{p} \leq \mathfrak{n} \cdot \mathfrak{p}$.

3. $\mathfrak{m}^{\mathfrak{p}} \leq \mathfrak{n}^{\mathfrak{p}}$.

4. $\mathfrak{p}^{\mathfrak{m}} \leq \mathfrak{p}^{\mathfrak{n}}$, *if* $\mathfrak{m} \neq 0$ *or* $\mathfrak{p} \neq 0$.

Remark. It is not necessarily true that $\leq$ in the above theorem can be replaced by $<$. For example, $1 < 2$ but $1 + \aleph_0 = \aleph_0 = 2 + \aleph_0$ where $\aleph_0 = |\omega|$ (see Corollary 2.3.13).

Theorem 2.1.15. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be any cardinals. If* $\mathfrak{m} \leq \mathfrak{n}$, *then there exists a cardinal* $\mathfrak{p}$ *such that* $\mathfrak{m} + \mathfrak{p} = \mathfrak{n}$.

2.2 Ordinals

Definition 2.2.1. A set A is a *transitive set* if every member of A is a subset of A.

Definition 2.2.2. For any set A, let $\in_A = \{\langle x, y\rangle \in A \times A \mid x \in y\}$.

Definition 2.2.3. A set α is an *ordinal* if α is transitive and $\in_\alpha$ is a well-ordering on α.

Example. Every natural number and ω are ordinals.

Theorem 2.2.4. *Every well-ordered set is isomorphic to a unique ordinal.*

Definition 2.2.5. For any ordinal α, its *successor* $\alpha + 1$ is defined by

$$\alpha + 1 = \alpha \cup \{\alpha\}.$$

Definition 2.2.6. For ordinals α and β,

α is *less than* β, written $\alpha < \beta$, if $\alpha \in \beta$,

α is *less than or equal to* β, written $\alpha \leq \beta$, if $\alpha < \beta$ or $\alpha = \beta$.

Notation. We may write $\beta > \alpha$ for $\alpha < \beta$ and write $\beta \geq \alpha$ for $\alpha \leq \beta$.

Remark. For any ordinal α, $\alpha + 1$ is the least ordinal greater than α.

Definition 2.2.7. Let α be an ordinal.

α is a *successor ordinal* if $\alpha = \beta + 1$ for some ordinal β.

α is a *limit ordinal* if $\alpha \neq 0$ and α is not a successor ordinal.

A collection of sets with some property is called a *class*. Some class is too big to be a set since its existence causes some paradoxes. A class which is not a set is called a *proper class*. For example, the class of all sets, denoted by **V**, is a proper class.

Definition 2.2.8. Let **ON** denote the class of all ordinals.

2.3 Cardinal numbers and the Axiom of Choice

The Axiom of Choice (AC) was introduced by Zermelo in 1904. It states that every set can be well-ordered. The full theory ZFC is ZF with AC.

Cardinal numbers with AC

The following theorem follows straightforwardly from Theorem 2.2.4.

Theorem 2.3.1 (Numeration Theorem)**.** *The Axiom of Choice implies that every set is equinumerous to some ordinal.*

The converse of Theorem 2.1.9 also holds if we assume AC.

Theorem 2.3.2. (AC) *For any sets A and B, if $A \preceq^* B$, then $A \preceq B$.*

Definition 2.3.3. (AC) The *cardinality* of a set A is the least ordinal equinumerous to A.

Cardinal numbers without AC

Without the Axiom of Choice, we cannot guarantee that a set is equinumerous to some ordinal. One may define the cardinality of a set A by

$$|A| = \{B \mid B \approx A\}$$

but, for $A \neq \emptyset$, it is too big to be a set. It is a proper class.

A concept of constructing a set is that every set is constructed from sets that have already been defined. This idea can be obtained by iterating various set-theoretic operations starting from 0. The class of well-founded sets is an example of such classes that is built up by such construction.

Definition 2.3.4. For each ordinal α, we define $R(\alpha)$ recursively as follows.

$$\begin{aligned} R(0) &= 0, \\ R(\alpha+1) &= \mathcal{P}(R(\alpha)), \\ R(\alpha) &= \bigcup_{\beta<\alpha} R(\beta) \quad (\alpha \text{ is a limit}), \end{aligned}$$

and let

$$\mathbf{WF} = \bigcup_{\alpha \in \mathbf{ON}} R(\alpha).$$

We call each element of **WF** a *well-founded set*.

Note that **WF** is a proper class while each $R(\alpha)$ is a set.

The Axiom of Foundation is one of Zermelo-Fraenkel axioms. It states that every set is a well-founded set, i.e., $\mathbf{V} = \mathbf{WF}$. Therefore, in ZF, every set is in $R(\alpha)$ for some ordinal α.

Definition 2.3.5. For any infinite set A, we define

$$|A| = \mathfrak{C}(A) \cap R(\alpha),$$

where $\mathfrak{C}(A)$ is the class $\{B \mid A \approx B\}$ and α is the least ordinal such that $\mathfrak{C}(A) \cap R(\alpha) \neq \emptyset$.

Alephs

Definition 2.3.6. The cardinal number of an infinite well-ordered set is called an *aleph*.

Remark. $\aleph_0$ is the least aleph.

Theorem 2.3.7. *The class of all alephs is a proper class.*

Lemma 2.3.8. *For any infinite cardinal* $\mathfrak{m}$ *and aleph* $\aleph$*, if* $\mathfrak{m} \leq \aleph$*, then* $\mathfrak{m}$ *is also an aleph.*

Theorem 2.3.9. *If* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are alephs, then* $\mathfrak{m} \leq \mathfrak{n}$ *or* $\mathfrak{n} \leq \mathfrak{m}$.

Theorem 2.3.10. *Every nonempty class of alephs has a least element.*

Theorem 2.3.11 (Hartogs' Theorem)**.** *For every cardinal* $\mathfrak{m}$*, there exists a least aleph, denoted by* $\aleph(\mathfrak{m})$*, such that* $\aleph(\mathfrak{m}) \nleq \mathfrak{m}$.

Theorem 2.3.12. *For all alephs* $\aleph$*,* $\aleph \cdot \aleph = \aleph$.

Corollary 2.3.13. *If* $\mathfrak{m}$ *is an aleph and* $\mathfrak{n}$ *is a cardinal such that* $\mathfrak{n} \leq \mathfrak{m}$*, then*

1. $\mathfrak{m} + \mathfrak{n} = \mathfrak{m}$
2. $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m}$*, if* $\mathfrak{n} \neq 0$.

Theorem 2.3.14. *The* Axiom of Choice *holds iff every infinite cardinal is an aleph.*

More details about alephs can be found in [9].

Some background in Logic

We assume some basic knowledge on first-order logic. For a fixed language $\mathscr{L}$, a structure $\mathbf{M}$ for $\mathscr{L}$, a sentence φ, and a set of sentences T of $\mathscr{L}$, we write $\mathbf{M} \vDash \varphi$ if φ is *true* in $\mathbf{M}$ or $\mathbf{M}$ is a *model* of φ, $\mathsf{T} \vdash \varphi$ if T *proves* φ, and Con(T) if T is *consistent*. All details about these terminologies can be founded in [4]. The following theorems will be needed in Chapter V (for proofs, see [10]).

Theorem 2.1. *Let* T *and* T' *be sets of sentences and* $\mathbf{M}$ *be a class. Suppose we can prove from* T *that* $\mathbf{M} \neq \emptyset$ *and* $\mathbf{M}$ *is a model for* T'*. Then if* T *is consistent, so is* T'*.*

Theorem 2.2. *For any sets of sentences* T *and any sentence* φ*,* $\mathsf{T} \cup \{\neg\varphi\}$ *is consistent if and only if* $\mathsf{T} \nvdash \varphi$*.*

CHAPTER III

INFINITY WITHOUT CHOICE

Without the Axiom of Choice, it is not necessarily true that for any infinite set A, $\omega \preceq A$ (see [1, 2, 13]). Therefore, without AC, there are many kinds of infinite sets and the following definitions are needed.

3.1 Definitions and some basic properties

Definition 3.1.1. A set A is *Dedekind-infinite* if $\omega \preceq A$. Otherwise A is *Dedekind-finite*. A cardinal number is *Dedekind-infinite (finite)* if it is the cardinality of a Dedekind-infinite (finite) set.

Remark. A Dedekind-infinite set is infinite but an infinite set is not necessarily Dedekind-infinite. Therefore a Dedekind-finite set could be infinite.

Definition 3.1.2. A *Dedekind set* is an infinite Dedekind-finite set. A cardinal number is a *Dedekind cardinal* if it is the cardinal number of a Dedekind set.

Theorem 3.1.3. [9] *A set A is Dedekind-finite iff there is no bijection from A onto a proper subset of A.*

The next corollary follows.

Corollary 3.1.4. *If a set A is Dedekind-finite and $B \subset A$, then $|B| < |A|$.*

Theorem 3.1.5. [9] *The union of a Dedekind-finite family of mutually disjoint Dedekind-finite sets is Dedekind-finite.*

Theorem 3.1.6. [14] *If $\mathfrak{m}$ and $\mathfrak{n}$ are Dedekind-finite cardinals, then so are $\mathfrak{m} + \mathfrak{n}$ and $\mathfrak{m} \cdot \mathfrak{n}$.*

The following theorem is needed for the proof of Theorem 5.3.1.7.

Theorem 3.1.7. [14] *For all Dedekind-finite cardinals $\mathfrak{p} \neq 0$, if $\mathfrak{m} < \mathfrak{n}$ where each of $\mathfrak{m}$ and $\mathfrak{n}$ is either a natural number or an aleph, then $\mathfrak{m} \cdot \mathfrak{p} < \mathfrak{n} \cdot \mathfrak{p}$.*

Remark. It is easy to see that for any cardinal $\mathfrak{m}$, if $2^{\mathfrak{m}}$ is Dedekind-finite, then $\mathfrak{m}$ is also Dedekind-finite since $\mathfrak{m} < 2^{\mathfrak{m}}$. But the converse is not necessarily true since it is consistent with ZF that there is a cardinal $\mathfrak{m}$ such that $\aleph_0 \not\leq \mathfrak{m}$ and $\aleph_0 \leq 2^{\mathfrak{m}}$ (see Subsection 5.3.2).

Without AC, it cannot be proved that every infinite set A can be mapped onto ω (see [8, 12]). This leads to the following definitions.

Definition 3.1.8. A set A is *weakly Dedekind-infinite* if $\omega \preceq^* A$. Otherwise A is *weakly Dedekind-finite*. A cardinal number is *weakly Dedekind-infinite (finite)* if it is the cardinality of a weakly Dedekind-infinite (finite) set.

Definition 3.1.9. A *weakly Dedekind set* is an infinite weakly Dedekind-finite set. A cardinal number is a *weakly Dedekind cardinal* if it is the cardinal number of a weakly Dedekind set.

Remarks.

1. Every Dedekind-infinite set is weakly Dedekind-infinite but the converse is not necessarily true (see [11, 13]).
2. Every finite set is weakly Dedekind-finite and so it is Dedekind-finite.

3.2 Cardinal relations in ZF

Notation. Let A be a set. We let

1. $\text{fin}(A)$ denote the set of all finite subsets of A.
2. $\inf(A)$ denote the set of all infinite subsets of A.
3. $\text{dfin}(A)$ denote the set of all Dedekind-finite subsets of A.
4. $\text{dinf}(A)$ denote the set of all Dedekind-infinite subsets of A.
5. $\text{dfin}^*(A)$ denote the set of all weakly Dedekind-finite subsets of A.
6. $\text{dinf}^*(A)$ denote the set of all weakly Dedekind-infinite subsets of A.
7. $\text{ded}(A)$ denote the set of all Dedekind subsets of A.
8. $\text{ded}^*(A)$ denote the set of all weakly Dedekind subsets of A.
9. $\text{part}(A)$ denote the set of all partitions of A.
10. $[A]^n$ denote the set of all n-element subsets of A, where $n \in \omega$.
11. A^n denote the set $\underbrace{A \times A \times \ldots \times A}_{n \text{ copies}}$, where $n \in \omega$.

The cardinals of those (1)–(10) above are denoted by the same notation as the corresponding sets with A replaced by $|A|$.

Theorem 3.2.1. *For any infinite cardinal* $\mathfrak{m}$*, we have the following:*

1. $\mathfrak{m} \leq \text{fin}(\mathfrak{m}) \leq \text{dfin}^*(\mathfrak{m}) \leq \text{dfin}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

2. $\text{ded}^*(\mathfrak{m}) \leq \text{ded}(\mathfrak{m}) \leq \text{inf}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

3. $\text{dinf}(\mathfrak{m}) \leq \text{dinf}^*(\mathfrak{m}) \leq \text{inf}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

4. $\text{ded}^*(\mathfrak{m}) \leq \text{dfin}^*(\mathfrak{m})$ *and* $\text{ded}(\mathfrak{m}) \leq \text{dfin}(\mathfrak{m})$.

5. $\text{fin}(\mathfrak{m}) \leq \text{inf}(\mathfrak{m})$.

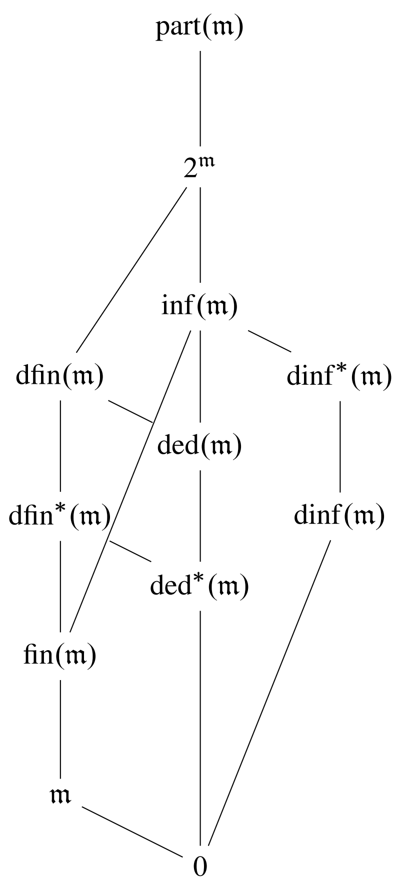
6. $2^{\mathfrak{m}} \leq \text{part}(\mathfrak{m})$.

7. *if* $\mathfrak{m} \geq 5$ *and* $\aleph_0 \not\leq 2^{\mathfrak{m}}$, *then* $2^{\mathfrak{m}} < \text{part}(\mathfrak{m})$.

8. $\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. (1)–(4) follow directly from the definitions. For the proof of (6)–(7), see Proposition 8.3 in [6] and for the proof of (8), see Theorem 3 in [5]. To prove (5), let A be an infinite set such that $\mathfrak{m} = |A|$. It is easy to see that the map $S \mapsto A \setminus S$ is an injection from $\text{fin}(A)$ into $\text{inf}(A)$, so $\text{fin}(\mathfrak{m}) \leq \text{inf}(\mathfrak{m})$. □

The relations (1)–(5) from Theorem 3.2.1 can be shown by the following diagram. The cardinal below a line is less than or equal to the one above it.



CHAPTER IV

SOME RESULTS IN ZF

This chapter gives new results concerning different kinds of finite and infinite sets that can be proved from ZF. We start with some basic properties.

It has been shown that if there is an injection from a set to some of its proper subsets, then such a set is Dedekind-infinite (see [9]). The following theorem shows the analogous property for weakly Dedekind-infinite sets.

Theorem 4.1. *For any set A, if there exists a surjection from a proper subset of A onto A, then A is weakly Dedekind-infinite.*

Proof. Let A be a set. Assume there are $S \subset A$ and a surjection $f: S \to A$. Define

$$f^{-(n+1)}[X] = f^{-1}[f^{-n}[X]] \quad \text{for all } n \in \omega \setminus \{0\} \text{ and } X \subseteq A.$$

Claim 1. For all $m, k \in \omega \setminus \{0\}$ and $X \subseteq A$,

$$f^{-(m+k)}[X] = f^{-k}[f^{-m}[X]].$$

Let $X \subseteq A$. We will prove by induction on k. The case $k = 1$ follows directly from the definition. Assume $k > 1$. Then $k = l + 1$ for some $l \in \omega \setminus \{0\}$. By the induction hypothesis, $f^{-(m+l)}[X] = f^{-l}[f^{-m}[X]]$. Thus

$$\begin{aligned} f^{-(m+k)}[X] &= f^{-(m+l+1)}[X] \\ &= f^{-1}[f^{-(m+l)}[X]] \\ &= f^{-1}[f^{-l}[f^{-m}[X]]] \\ &= f^{-(l+1)}[f^{-m}[X]] \\ &= f^{-k}[f^{-m}[X]]. \end{aligned}$$

Claim 2. For any $m, n \in \omega \setminus \{0\}$, if $m \neq n$, then $f^{-m}[A \setminus S] \cap f^{-n}[A \setminus S] = \emptyset$.

Let $m, n \in \omega \setminus \{0\}$. Assume $m < n$, then there exists $k \in \omega \setminus \{0\}$ such that $k + m = n$. By Claim 1, $f^{-n}[A \setminus S] = f^{-m}[f^{-k}[A \setminus S]]$. Since $f^{-k}[A \setminus S] \subseteq S$, $f^{-k}[A \setminus S] \cap (A \setminus S) = \emptyset$. Thus $f^{-m}[A \setminus S] \cap f^{-n}[A \setminus S] = f^{-m}[A \setminus S] \cap f^{-m}[f^{-k}[A \setminus S]] = \emptyset$.

Define $g: S \to \omega$ by

$$g(a) = \begin{cases} n & \text{if } a \in f^{-(n+1)}[A \setminus S] \text{ for some } n \in \omega; \\ 0 & \text{otherwise.} \end{cases}$$

By Claim 2, g is well-defined. Since $S \subset A$, $A \backslash S \neq \emptyset$. Since f is surjective, $f^{-1}[A \backslash S] \neq \emptyset$ and so, by induction, $f^{-(n+1)}[A \setminus S] \neq \emptyset$ for all $n \in \omega$. It follows that g is onto and so $\omega \preceq^* A$. □

In [9], it has been shown that the union of a Dedekind-finite family of mutually disjoint Dedekind-finite sets is Dedekind-finite. Such property for weakly Dedekind-finite sets is stronger since the union need not be disjoint.

Theorem 4.2. *A weakly Dedekind-finite union of weakly Dedekind-finite sets is weakly Dedekind-finite.*

Proof. Let I be a weakly Dedekind-finite set and, for each $i \in I$, let X_i be a weakly Dedekind-finite set. It is trivial if I is empty. Without loss of generality, we may assume all X_i's are nonempty.

Suppose $\bigcup_{i \in I} X_i$ is weakly Dedekind-infinite, i.e., there is a surjection $f: \bigcup_{i \in I} X_i \to \omega$. For each $i \in I$, since X_i is weakly Dedekind-finite, $f[X_i]$ must be a finite subset of ω (otherwise X_i can be mapped onto ω).

Suppose $\{|f[X_i]| \mid i \in I\}$ is infinite. Define $g: I \to \omega$ by $g(i) = |f[X_i]|$. Then $\operatorname{ran}(g)$ is an infinite subset of ω, so $\omega \approx \operatorname{ran}(g) \preceq^* I$, but I is weakly Dedekind-finite, a contradiction. Hence $\{|f[X_i]| \mid i \in I\}$ is finite.

Since $\bigcup_{i \in I} f[X_i] = \omega$ and each $f[X_i]$ is finite for all $i \in I$, I is infinite and there exists $n \in \omega$ such that $\Gamma := \{f[X_i] \mid |f[X_i]| = n \text{ and } i \in I\}$ is infinite.

Since $\Gamma \subseteq [\omega]^n$ and $[\omega]^n \preceq \omega^n$ by the map $\{m_1, m_2, \ldots, m_n\} \mapsto \langle m_1, m_2, \ldots, m_n\rangle$ where $m_1 < m_2 < \ldots < m_n$ and ω^n is countable, Γ is countable. Thus $\omega \approx \Gamma$. Let $a \in \Gamma$. Define $g: I \to \Gamma$ by

$$g(i) = \begin{cases} f[X_i] & \text{if } f[X_i] \in \Gamma; \\ a & \text{otherwise.} \end{cases}$$

It is easy to see that g maps I onto Γ. Then $\omega \approx \Gamma \preceq^* I$ but I is weakly Dedekind-finite, a contradiction.

Hence $\bigcup_{i \in I} X_i$ is weakly Dedekind-finite. □

We have shown in [14] that if $\mathfrak{m}$ and $\mathfrak{n}$ are Dedekind-finite cardinals, then so are $\mathfrak{m} + \mathfrak{n}$ and $\mathfrak{m} \cdot \mathfrak{n}$. The following corollary shows that the property also holds for weakly Dedekind-finite cardinals.

Corollary 4.3. *If* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are weakly Dedekind-finite cardinals, then so are* $\mathfrak{m} + \mathfrak{n}$ *and* $\mathfrak{m} \cdot \mathfrak{n}$.

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be any weakly Dedekind-finite cardinals, say $\mathfrak{m} = |M|$ and $\mathfrak{n} = |N|$ where M and N are disjoint. By Theorem 4.2, $M \cup N$ is weakly Dedekind-finite and so is $\mathfrak{m} + \mathfrak{n}$.

For the case $\mathfrak{m} \cdot \mathfrak{n}$, since $M \times N = \bigcup\{M \times \{n\} \mid n \in N\}$ where each $M \times \{n\}$ is weakly Dedekind-finite (if not, M is weakly Dedekind-infinite) and N is weakly Dedekind-finite, by Theorem 4.2, $\mathfrak{m} \cdot \mathfrak{n}$ is weakly Dedekind-finite. □

Corollary 4.4. *Let $\mathfrak{m}$ be a cardinal and $n \in \omega$. If $\mathfrak{m}$ is (weakly) Dedekind-finite, then so is $\mathfrak{m}^n$. This statement also holds if we replace "(weakly) Dedekind-finite" by "(weakly) Dedekind-infinite" for the case $n \neq 0$.*

Proof. The case $\mathfrak{m}$ is (weakly) Dedekind-finite follows straightforwardly from Theorem 3.1.6 and Corollary 4.3 by induction.

The case $\mathfrak{m}$ is (weakly) Dedekind-infinite follows directly from the fact that $\mathfrak{m} \leq \mathfrak{m}^n$ for all $n \in \omega \setminus \{0\}$. □

Theorem 4.5. *For any cardinals $\mathfrak{m}$ and $\mathfrak{n}$, if $\mathfrak{m} \geq 2$ and $\mathfrak{n}$ is infinite, then $\mathfrak{m}^{\mathfrak{n}}$ is weakly Dedekind-infinite. In particular, $2^{\mathfrak{n}}$ is weakly Dedekind-infinite for all infinite cardinals $\mathfrak{n}$.*

Proof. Let $\mathfrak{m}$ be a cardinal such that $\mathfrak{m} \geq 2$ and $\mathfrak{n}$ be any infinite cardinal, say $\mathfrak{m} = |M|$ and $\mathfrak{n} = |N|$. Let $m, m' \in M$ be such that $m \neq m'$. Define $F \colon {}^N M \to \omega$ by

$$F(f) = \begin{cases} |f^{-1}[\{m\}]| & \text{if } f^{-1}[\{m\}] \text{ is finite;} \\ 0 & \text{otherwise.} \end{cases}$$

Since N is infinite, by Theorem 2.1.11, for each $k \in \omega$, there exists $N_k \subseteq N$ such that $|N_k| = k$ and then $F\left((N_k \times \{m\}) \cup ((N \setminus N_k) \times \{m'\})\right) = |N_k| = k$. Hence F is onto, so $\omega \preceq^* {}^N M$, i.e., $\mathfrak{m}^{\mathfrak{n}}$ is weakly Dedekind-infinite. □

Next we will give some results concerning relations between some cardinals.

Lemma 4.6. [5] $\operatorname{fin}(\alpha) \approx \alpha$ *for all infinite ordinals α.*

Corollary 4.7. *For any aleph $\aleph$,*

$$\begin{aligned} 0 = \operatorname{ded}^*(\aleph) = \operatorname{ded}(\aleph) < \aleph = \operatorname{fin}(\aleph) = \operatorname{dfin}^*(\aleph) = \operatorname{dfin}(\aleph) \\ < \operatorname{dinf}(\aleph) = \operatorname{dinf}^*(\aleph) = \operatorname{inf}(\aleph) = 2^{\aleph}. \end{aligned}$$

Proof. Let $\aleph$ be an aleph, say $\aleph = |\alpha|$ for some infinite ordinal α. Then $\operatorname{fin}(\aleph) = \aleph$ follows from Lemma 4.6.

Since $\aleph = \text{fin}(\aleph) \le \text{inf}(\aleph)$, by Theorem 2.1.15, there exists a cardinal $\mathfrak{p}$ such that $\aleph + \mathfrak{p} = \text{inf}(\aleph)$. Thus $2^{\aleph} = \text{fin}(\aleph) + \text{inf}(\aleph) = \aleph + \aleph + \mathfrak{p} = \aleph + \mathfrak{p} = \text{inf}(\aleph)$.

Since every infinite subset of α is Dedekind-infinite and so it is weakly Dedekind-infinite, $\text{dinf}(\aleph) = \text{dinf}^*(\aleph) = \text{inf}(\aleph)$. It follows that all weakly Dedekind-finite and Dedekind-finite subsets of α are finite. Hence $\text{dfin}^*(\aleph) = \text{dfin}(\aleph) = \text{fin}(\aleph)$ and $\text{ded}^*(\aleph) = \text{ded}(\aleph) = 0$. □

Theorem 3 in [5] shows that for all infinite cardinals $\mathfrak{m}$, $\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$. We show that this cannot be proved from ZF if we replace $\text{fin}(\mathfrak{m})$ by $\text{dfin}^*(\mathfrak{m})$. Moreover we show that $\text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$ provided that $\mathfrak{m}$ is weakly Dedekind-infinite. We first need the following lemma.

Lemma 4.8. *For any set M and ordinal α, if $\omega \le \alpha \preceq \text{dfin}^*(M)$, then $\alpha \approx \Pi$ for some partition Π of M.*

Proof. Let M be any set and α be an ordinal such that $\omega \le \alpha \preceq \text{dfin}^*(M)$. Then there is a one-to-one α-sequence $\langle m_0, m_1, m_2, \ldots, m_\beta, \ldots\rangle_\alpha$ of $\text{dfin}^*(M)$. Define an equivalence relation $\sim$ on M by $x \sim y$ iff $\forall \beta < \alpha\, (x \in m_\beta \leftrightarrow y \in m_\beta)$. For any $x \in M$ and $0 < \mu < \alpha$, define $D_{x,\mu}$ by

$$D_{x,\mu} = \begin{cases} \bigcap_{\iota<\mu} \{m_\iota \mid x \in m_\iota\} & \text{if } x \in m_\beta \text{ for some } \beta < \mu; \\ M & \text{otherwise;} \end{cases}$$

and define $g\colon M \to \mathcal{P}(\alpha)$ by $g(x) = \{\mu < \alpha \mid x \in m_\mu \text{ and } D_{x,\mu} \nsubseteq m_\mu\}$.

Claim 1. $x \sim y$ iff $g(x) = g(y)$ for all $x, y \in M$.

It is easy to see that if $x \sim y$, then $g(x) = g(y)$. The converse also holds since if $x \not\sim y$ and β is the least ordinal such that $x \in m_\beta$ but $y \notin m_\beta$, then $y \in D_{y,\beta} = D_{x,\beta}$, so $D_{x,\beta} \nsubseteq m_\beta$ and thus $\beta \in g(x)$ but $\beta \notin g(y)$. Hence there is a one-to-one correspondence between $\{[x]_\sim \mid x \in M\}$ and $\{g(x) \mid x \in M\}$.

Claim 2. $g(x)$ is finite for all $x \in M$.

Suppose for a contradiction that $g(x)$ is infinite for some $x \in M$. Note that for all $\mu_1, \mu_2 \in g(x)$ such that $\mu_1 < \mu_2$, $D_{x,\mu_2} \subset D_{x,\mu_1}$ because $D_{x,\mu_2} \subseteq m_{\mu_1}$ and $D_{x,\mu_2} \subseteq D_{x,\mu_1} \nsubseteq m_{\mu_1}$. Since $g(x)$ is an infinite set of ordinals, there is a one-to-one ω-sequence $\langle \mu_0, \mu_1, \mu_2, \ldots\rangle_\omega$ of $g(x)$ where $\mu_0 < \mu_1 < \mu_2 < \ldots$. Then we have $m_{\mu_0} \supseteq D_{x,\mu_1} \supset D_{x,\mu_2} \supset \ldots$. Define $f\colon m_{\mu_0} \to \omega$ by

$$f(y) = \begin{cases} n & \text{if } y \in D_{x,\mu_{n+1}} \setminus D_{x,\mu_{n+2}} \text{ for some } n \in \omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a surjection, so $\aleph_0 \le^* |m_{\mu_0}|$ which contradicts $m_{\mu_0} \in \text{dfin}^*(M)$.

Hence $\{g(x) \mid x \in M\} \subseteq \text{fin}(\alpha)$. Since, by Lemma 4.6, $\text{fin}(\alpha) \approx \alpha$, $\{g(x) \mid x \in M\} \preceq \alpha$. Thus $\{g(x) \mid x \in M\} \approx \gamma$ for some $\gamma \leq \alpha$. Now we will show that $\alpha \preceq \gamma$. Since $\{[x]_\sim \mid x \in M\} \approx \{g(x) \mid x \in M\} \approx \gamma$ and there are $\alpha \geq \omega$ m_ι's which are union of $\sim$-classes, γ is infinite. Let $\eta: \{g(x) \mid x \in M\} \to \gamma$ be a bijection.

Suppose for a contradiction that $\{\eta(g(x)) \mid x \in m_\iota\}$ is infinite for some $\iota < \alpha$. Since $\{[x]_\sim \mid x \in m_\iota\} \approx \{g(x) \mid x \in m_\iota\} \approx \{\eta(g(x)) \mid x \in m_\iota\}$ which is an infinite set of ordinals, $\omega \preceq \{[x]_\sim \mid x \in m_\iota\}$, so there is a one-to-one ω-sequence $\langle [x_0]_\sim, [x_1]_\sim, [x_2]_\sim, \ldots \rangle_\omega$. Define a function $f': m_\iota \to \omega$ by

$$f'(x) = \begin{cases} n & \text{if } x \in [x_n]_\sim; \\ 0 & \text{otherwise.} \end{cases}$$

Again we get a contradiction from the fact that f' is a surjection but m_ι is weakly Dedekind-finite.

Hence $\{\eta(g(x)) \mid x \in m_\iota\}$ is a finite subset of γ for all $\iota < \alpha$. Define $h: \alpha \to \text{fin}(\gamma)$ by $h(\iota) = \{\eta(g(x)) \mid x \in m_\iota\}$. Since η is an injection and all m_ι's are distinct, h is an injection. Thus $\alpha \preceq \text{fin}(\gamma) \approx \gamma$. Hence $\alpha \approx \gamma$ and so $\{[x]_\sim \mid x \in M\}$ is the partition as desired. Finally, since the bijections $\alpha \approx \text{fin}(\alpha)$ and $\gamma \approx \text{fin}(\gamma)$ are canonical, so is the bijection $\{[x]_\sim \mid x \in M\} \to \alpha$. □

Theorem 4.9. *For any cardinal* $\mathfrak{m}$, $\aleph_0 \leq^* \mathfrak{m}$ *iff* $\text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Let M be a set such that $\mathfrak{m} = |M|$.

($\Leftarrow$) Assume $\text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$. Then $\text{dfin}^*(M) \subset \mathcal{P}(M)$. Thus there exists $X \subseteq M$ such that $\omega \preceq^* X$, so $\omega \preceq^* M$, i.e., $\aleph_0 \leq^* \mathfrak{m}$.

($\Rightarrow$) This follows closely to the proof of Theorem 3 in [5]. It is clear that $\text{dfin}^*(\mathfrak{m}) \leq 2^{\mathfrak{m}}$. Suppose $\aleph_0 \leq^* \mathfrak{m}$ and $\text{dfin}^*(\mathfrak{m}) = 2^{\mathfrak{m}}$. We will get a contradiction to Hartogs' Theorem by constructing a one-to-one α-sequence of $\text{dfin}^*(M)$ for any α.

Let $B: \text{dfin}^*(M) \to \mathcal{P}(M)$ be a bijection. Let $m_0 = B^{-1}(M)$ and for any $k \in \omega$, let $m_{k+1} = B^{-1}(m_k)$. Since $M \notin \text{dfin}^*(M)$ and B is an injection, the sequence $\langle m_0, m_1, m_2, \ldots \rangle_\omega$ is a one-to-one ω-sequence of $\text{dfin}^*(M)$.

Assume there exists a one-to-one α-sequence $\langle m_0, m_1, m_2, \ldots, m_\beta, \ldots \rangle_\alpha$ of $\text{dfin}^*(M)$. We will construct an m_α by Cantor's diagonal proof that $X \prec \mathcal{P}(X)$ for any X, as follows.

Since $\alpha \preceq \text{dfin}^*(M)$, by Lemma 4.8, there is a partition Π of M such that $\Pi \approx \alpha \approx \{m_\iota \mid \iota < \alpha\}$.

Let $H: \{m_\iota \mid \iota < \alpha\} \to \Pi$ be a bijection. Define $h: M \to \{m_\iota \mid \iota < \alpha\}$ by $h(x) = m_\iota$ if $x \in H(m_\iota)$. Let $F = B \circ h$, so $F: M \to \mathcal{P}(M)$ and, as in the usual proof of Cantor's Theorem, $M_\alpha := \{x \in M \mid x \notin F(x)\} \notin \text{ran}(F)$. Note that h is a surjection and so $B(m_\iota) \in \text{ran}(F)$

for all $\iota < \alpha$. Let $m_\alpha = B^{-1}(M_\alpha)$. Then $m_\alpha \notin \{m_\iota \mid \iota < \alpha\}$. Hence we have a one-to-one $(\alpha + 1)$-sequence of $\mathrm{dfin}^*(M)$. This completes the proof. □

Theorem 4.10. *For any cardinal* $\mathfrak{m}$ *and any aleph* $\aleph$*, if* $\aleph \leq \mathrm{dfin}^*(\mathfrak{m})$*, then* $2^{\aleph} \leq 2^{\mathfrak{m}}$.

Proof. Let $\mathfrak{m}$ be a cardinal number, say $\mathfrak{m} = |M|$, $\aleph$ be an aleph, and let α be the least ordinal such that $|\alpha| = \aleph$. Assume $\aleph \leq \mathrm{dfin}^*(\mathfrak{m})$. Then $\alpha \preceq \mathrm{dfin}^*(M)$. By Lemma 4.8, there exists a partition Π of M such that $\alpha \approx \Pi$. Let $f : \alpha \to \Pi$ be a bijection. Define a function $F : \mathcal{P}(\alpha) \to \mathcal{P}(M)$ by $F(x) = \bigcup f[x]$ for all $x \in \mathcal{P}(\alpha)$. Since f is an injection, so is F. Hence $\mathcal{P}(\alpha) \preceq \mathcal{P}(M)$, so $2^{\aleph} \leq 2^{\mathfrak{m}}$. □

Since $\aleph_0 \leq^* \mathfrak{m}$ for all infinite $\mathfrak{m}$ is a consequence of AC which is not provable in ZF (see [8, 12]), Theorem 4.9 tells us that $\mathrm{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$ cannot be proved from ZF for an arbitrary $\mathfrak{m}$. A condition that makes the statement provable from ZF is that $\mathrm{dfin}^*(\mathfrak{m})$ is Dedekind-infinite. This is Corollary 4.12 which immediately follows from Theorem 4.10, by letting $\aleph = \aleph_0$, and Corollary 4.11.

Corollary 4.11. *Let* $\mathfrak{m}$ *be a cardinal. The following are equivalent.*

1. $\aleph_0 \leq^* \mathfrak{m}$.
2. $\aleph_0 \leq 2^{\mathfrak{m}}$.
3. $2^{\aleph_0} \leq 2^{\mathfrak{m}}$.
4. $\mathrm{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. (1 ⇔ 2) follows from Lemma 4.11 in [7], (2 ⇔ 3) follows from Fact 8.1 in [6], and (1 ⇔ 4) follows from Theorem 4.9. □

Corollary 4.12. *For any cardinal* $\mathfrak{m}$*, if* $\aleph_0 \leq \mathrm{dfin}^*(\mathfrak{m})$*, then* $\mathrm{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Follows from Theorem 4.10 and Corollary 4.11. □

As mentioned earlier, it has been shown in [5] that $\mathrm{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$ for any infinite cardinal $\mathfrak{m}$. This fact also follows from Theorem 4.9 and Corollary 4.11.

Corollary 4.13. *For any infinite cardinal* $\mathfrak{m}$*,* $\mathrm{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Let $\mathfrak{m}$ be an infinite cardinal, say $\mathfrak{m} = |M|$. If $\aleph_0 \leq^* \mathfrak{m}$, then, by Theorem 4.9, $\mathrm{fin}(\mathfrak{m}) \leq \mathrm{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Assume $\aleph_0 \not\leq^* \mathfrak{m}$. By Corollary 4.11, $\aleph_0 \not\leq 2^{\mathfrak{m}}$, i.e., $\mathcal{P}(M)$ is Dedekind-finite. Since M is infinite, $\text{fin}(M) \subset \mathcal{P}(M)$. By Corollary 3.1.4, $\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$. □

Corollary 4.14. *For any cardinal* $\mathfrak{m}$, $\text{ded}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Let $\mathfrak{m}$ be a cardinal, say $\mathfrak{m} = |M|$. If $\mathfrak{m}$ is finite, then $\text{ded}^*(\mathfrak{m}) = 0 < 2^{\mathfrak{m}}$. Since $\text{ded}^*(\mathfrak{m}) \leq \text{dfin}^*(\mathfrak{m})$ and $\text{ded}^*(M) \subset \mathcal{P}(M)$, the proof for the case $\mathfrak{m}$ is infinite is similar to proof of the previous corollary. □

Corollary 4.15. *For any cardinal* $\mathfrak{m} \geq 5$, $\text{dfin}^*(\mathfrak{m}) < \text{part}(\mathfrak{m})$.

Proof. Let $\mathfrak{m}$ be a cardinal number such that $\mathfrak{m} \geq 5$. By Theorem 3.2.1(6), $2^{\mathfrak{m}} \leq \text{part}(\mathfrak{m})$ and so if $\text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$, we're done. Assume $\text{dfin}^*(\mathfrak{m}) \not< 2^{\mathfrak{m}}$. By Corollary 4.11, $\aleph_0 \not\leq 2^{\mathfrak{m}}$. By Theorem 3.2.1(7), $2^{\mathfrak{m}} < \text{part}(\mathfrak{m})$, so $\text{dfin}^*(\mathfrak{m}) < \text{part}(\mathfrak{m})$. □

Cantor's Theorem, stating that $\mathfrak{m} < 2^{\mathfrak{m}}$ for all cardinals $\mathfrak{m}$, can be improved for infinite cardinals as follows.

Theorem 4.16. *For any infinite cardinal* $\mathfrak{m}$, $\mathfrak{m} < \inf(\mathfrak{m})$.

Proof. Let $\mathfrak{m}$ be an infinite cardinal, say $\mathfrak{m} = |M|$. Suppose $\mathfrak{m} = \inf(\mathfrak{m})$. Then there exists a bijection $f\colon \inf(M) \to M$. We will show that there is an injection from **ON** into M which contradicts Hartogs' Theorem. Therefore we can conclude that $\mathfrak{m} < \inf(\mathfrak{m})$ since $\mathfrak{m} \leq \inf(\mathfrak{m})$.

Define $g\colon \mathbf{ON} \to M$ recursively by

$$g(\alpha) = f(M \setminus \{g(\beta) \mid \beta < \alpha\})$$

for all $\alpha \in \mathbf{ON}$.

In order to justify the definition of g, we first prove the following claim.

Claim. $M \setminus \{g(\beta) \mid \beta < \alpha\}$ is infinite for all $\alpha \in \mathbf{ON}$.

If α is finite, we're done. Assume α is infinite. Suppose $M \setminus \{g(\beta) \mid \beta < \alpha\}$ is finite. Note that $|\{g(\beta) \mid \beta < \alpha\}| \leq |\alpha|$. By Corollary 2.3.13, $\mathfrak{m} = |M| = |M \setminus \{g(\beta) \mid \beta < \alpha\}| + |\{g(\beta) \mid \beta < \alpha\}| \leq |M \setminus \{g(\beta) \mid \beta < \alpha\}| + |\alpha| = |\alpha|$. Thus $\mathfrak{m}$ is an aleph and so, by Corollary 4.7, $2^{\mathfrak{m}} = \inf(\mathfrak{m})$. By the assumption, $\mathfrak{m} = \inf(\mathfrak{m}) = 2^{\mathfrak{m}}$. This contradicts Cantor's Theorem. Hence $M \setminus \{g(\beta) \mid \beta < \alpha\}$ is infinite.

Now we will prove by transfinite induction that g is one-to-one. Let $\alpha \in \mathbf{ON}$. Assume that for all ordinals β and γ such that $\gamma < \beta < \alpha$, $g(\beta) \neq g(\gamma)$. Fix $\beta < \alpha$. Then $g(\beta) \in M \setminus \{g(\gamma) \mid \gamma < \beta\}$ but $g(\beta) \notin M \setminus \{g(\gamma) \mid \gamma < \alpha\}$. Thus $M \setminus \{g(\gamma) \mid \gamma < \beta\} \neq M \setminus \{g(\gamma) \mid \gamma < \alpha\}$. Since

f is an injection, $g(\beta) = f(M \setminus \{g(\gamma) \mid \gamma < \beta\}) \neq f(M \setminus \{g(\gamma) \mid \gamma < \alpha\}) = g(\alpha)$. Thus g is one-to-one. $\square$

CHAPTER V

CONSISTENCY RESULTS FROM PERMUTATION MODELS

5.1 ZFA

The set theory with atoms, denoted by ZFA, is characterised by the fact that it admits objects other than sets.

Definition 5.1.1. *Atoms* or *urelements* are objects which do not have any elements and which are distinct from the empty set.

The language of ZFA is $\mathscr{L}_{\mathsf{ZFA}} = \{\in, \mathtt{A}\}$ where $\in$ is a binary relation symbol and $\mathtt{A}$ is a constant symbol representing a set of atoms. The axioms of ZFA are like the axioms of ZF, except the following.

Axiom of Empty Set (for ZFA):

$$\exists x\big(x \notin \mathtt{A} \wedge \forall z(z \notin x)\big).$$

Axiom of Extensionality (for ZFA):

$$\forall x \forall y\big((x \notin \mathtt{A} \wedge y \notin \mathtt{A}) \rightarrow (\forall z(z \in x \iff z \in y) \rightarrow x = y)\big).$$

Axiom of Atoms:

$$\forall x\big(x \in \mathtt{A} \leftrightarrow \big(x \neq \emptyset \wedge \neg\exists z(z \in x)\big)\big).$$

Theorem 5.1.2. [9] Con(ZF) *implies* Con(ZFA + $\mathtt{A}$ *is infinite*).

Definition 5.1.3. For any set S and ordinal α, we define

$$\begin{aligned} \mathcal{P}^0(S) &= S, \\ \mathcal{P}^{\alpha+1}(S) &= \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S)), \\ \mathcal{P}^\alpha(S) &= \bigcup_{\beta<\alpha} \mathcal{P}^\beta(S) \quad (\alpha \text{ is a limit ordinal}). \end{aligned}$$

Further let

$$\mathcal{P}^\infty(S) = \bigcup_{\alpha \in \mathbf{ON}} \mathcal{P}^\alpha(S).$$

Theorem 5.1.4. [4] *If $\mathcal{M}$ is a model of* ZFA *and A is the set of atoms of $\mathcal{M}$, then $\mathcal{M} = \mathcal{P}^\infty(A)$. The class $\mathcal{P}^\infty(\emptyset)$ which is a subclass of $\mathcal{M}$ is a model of* ZF.

Notation. Let $\hat{\mathbf{V}}$ denote $\mathcal{P}^\infty(\emptyset)$. We call $\hat{\mathbf{V}}$ the *kernel* or *pure part* and call members of $\hat{\mathbf{V}}$ *pure sets*.

Note that $\mathbf{ON} \subseteq \hat{\mathbf{V}}$.

5.2 Permutation models

In this section, A is a fixed but arbitrary set of atoms in a model $\mathcal{M}$ of ZFA.

Definition 5.2.1. Let $\mathcal{G}$ be a group of permutations of A. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *normal filter* on $\mathcal{G}$ if for all subgroups H and K of $\mathcal{G}$:

(i) $\mathcal{G} \in \mathcal{F}$,

(ii) if $H \in \mathcal{F}$ and $H \subseteq K$, then $K \in \mathcal{F}$,

(iii) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$,

(iv) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$,

(v) for each $a \in A$, $\{\pi \in \mathcal{G} \mid \pi(a) = a\} \in \mathcal{F}$.

Throughout this section, $\mathcal{G}$ is a group of permutations of A and $\mathcal{F}$ is a normal filter on $\mathcal{G}$.

Definition 5.2.2. Let $\pi \in \mathcal{G}$. Using the hierarchy of $\mathcal{P}^{\alpha}(A)$'s, we can define $\pi(x)$ for every x in $\mathcal{M}$ as follows:

$$\pi(\emptyset) = \emptyset, \quad \pi(x) = \pi[x] = \{\pi(y) \mid y \in x\}.$$

Remarks.

1. We sometimes write πx for $\pi(x)$.
2. It can be proved by induction that π is one-to-one.

Lemma 5.2.3. *Let $\pi \in \mathcal{G}$. Then, for any x and y in $\mathcal{M}$,*

1. *$\pi\{x, y\} = \{\pi x, \pi y\}$ and $\pi\langle x, y\rangle = \langle \pi x, \pi y\rangle$.*
2. *if f is a function, then πf is a function and $(\pi f)(\pi x) = \pi(f(x))$.*
3. *$\pi x = x$ for all $x \in \hat{\mathbf{V}}$.*

Proof. Let x and y be any elements in $\mathcal{M}$.

1. Clearly $\pi\{x, y\} = \{\pi x, \pi y\}$. We have

$$\pi\langle x, y\rangle = \pi\{\{x\}, \{x, y\}\} = \{\pi\{x\}, \pi\{x, y\}\} = \{\{\pi x\}, \{\pi x, \pi y\}\} = \langle \pi x, \pi y\rangle.$$

2. Let f be a function. Then, by 1, $\pi f = \{\langle \pi x, \pi(f(x))\rangle \mid x \in \mathrm{dom}(f)\}$ is a relation. Since π is one-to-one, πf is a function and hence $(\pi f)(\pi x) = \pi(f(x))$.

3. This can be proved straightforwardly by induction on $\hat{\mathbf{V}}$. $\square$

Definition 5.2.4. For each x in $\mathcal{M}$, we define a *symmetric group* of x

$$\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} \mid \pi x = x\}.$$

Then $\mathrm{sym}_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}$. We say x is *symmetric* (with respect to $\mathcal{F}$) if $\mathrm{sym}_{\mathcal{G}}(x) \in \mathcal{F}$. Define a *permutation model*

$$\mathcal{V} = \{x \mid x \text{ is symmetric and } x \subseteq \mathcal{V}\}.$$

Theorem 5.2.5. [9] *$\mathcal{V}$ is a transitive model of* ZFA, $\hat{\mathbf{V}} \subseteq \mathcal{V}$*, and* $A \in \mathcal{V}$.

The *cardinality* of a set M, denoted by $\mathfrak{m}$, in the model $\mathcal{V}$, is defined by

$$\mathfrak{m} = \mathfrak{C}(M) \cap \mathcal{P}^{\alpha}(A) \cap \mathcal{V},$$

where $\mathfrak{C}(M) = \{x \in \mathcal{V} \mid x \approx M\}$ and α is the least ordinal such that $\mathfrak{C}(M) \cap \mathcal{P}^{\alpha}(A) \cap \mathcal{V} \neq \emptyset$.

We will work on the theory ZFA + AC (for the consistency, see [9]). Then we have that AC holds in the kernel $\hat{\mathbf{V}}$ (see [9]).

By Jech-Sochor Embedding Theorem (see [4]), we can embed a permutation model into a well-founded model, so every relation between cardinals in a permutation model also holds in a well-founded model. For example, if $\mathrm{fin}(\mathfrak{m}) < \mathrm{inf}(\mathfrak{m})$ holds in $\mathcal{V}$ for some cardinal $\mathfrak{m}$ in $\mathcal{V}$, then, by Jech-Sochor Embedding Theorem, there is a well-founded model $\mathbf{W}$ of ZF such that $\mathrm{fin}(\mathfrak{n}) < \mathrm{inf}(\mathfrak{n})$ holds for some cardinal $\mathfrak{n}$ in $\mathbf{W}$.

Hence, in order to prove that a relation between some cardinals is consistent with ZF, it is enough to find a permutation model for the statement.

Definition 5.2.6. A set I of subsets of A is a *normal ideal* if for all $E, F \subseteq A$:

(i) $\emptyset \in I$,

(ii) if $E \in I$ and $F \subseteq E$, then $F \in I$,

(iii) if $E \in I$ and $F \in I$, then $E \cup F \in I$,

(iv) if $\pi \in \mathcal{G}$ and $E \in I$, then $\pi(E) \in I$,

(v) for each $a \in A$, $\{a\} \in I$.

Remark. $\mathrm{fin}(A)$ is a normal ideal.

Definition 5.2.7. For each $S \subseteq A$, let

$$\mathrm{fix}_{\mathcal{G}}(S) = \{\pi \in \mathcal{G} \mid \pi(a) = a \text{ for all } a \in S\}.$$

Then $\mathrm{fix}_{\mathcal{G}}(S)$ is a subgroup of $\mathcal{G}$.

Theorem 5.2.8. [9] *Given a normal ideal I, then*

$$\mathcal{F} = \{H \mid H \text{ is a subgroup of } \mathcal{G} \text{ such that } \operatorname{fix}_{\mathcal{G}}(E) \subseteq H \textit{ for some } E \in I\}$$

is a normal filter.

Note that given a normal ideal I, there is a corresponding normal filter $\mathcal{F}$ as defined above and we say $\mathcal{V}$ is defined from I if $\mathcal{V}$ is the permutation model defined from such $\mathcal{F}$.

Definition 5.2.9. For each x and each $E \in I$, we say that E is a *support* of x if $\operatorname{fix}_{\mathcal{G}}(E) \subseteq \operatorname{sym}_{\mathcal{G}}(x)$.

Remarks.

1. x is symmetric iff there exists $E \in I$ such that E is a support of x. As a result, we have that

$$x \in \mathcal{V} \text{ iff } x \text{ has a support and } x \subseteq \mathcal{V}.$$

2. For each x and each $E, F \in I$, if E is a support of x and $E \subseteq F$, then F is also a support of x.

5.3 Some consistency results

5.3.1 The basic Fraenkel model

Definition 5.3.1.1. Let A be a countable infinite set and $\mathcal{G}$ be the group of all permutations of A. Let $\mathcal{V}_{F_0}$ be the permutation model defined from the normal ideal $\operatorname{fin}(A)$. We call $\mathcal{V}_{F_0}$ the *basic Fraenkel model*.

The following two lemmas are from [6].

Lemma 5.3.1.2. *In $\mathcal{V}_{F_0}$, for any $S \subseteq A$ with support E, S is either finite or co-finite, i.e., $A \setminus S$ is finite. Furthermore, if S is finite, then $S \subseteq E$, and if S is co-finite, then $(A \setminus S) \subseteq E$.*

Lemma 5.3.1.3. *Let* $\mathfrak{m} = |A|$. *Then* $\mathcal{V}_{F_0} \models \aleph_0 \nleq 2^{\mathfrak{m}}$.

All proofs below in this subsection are done in $\mathcal{V}_{F_0}$. Throughout this subsection, let $\mathfrak{m} = |A|$ unless otherwise stated.

Corollary 5.3.1.4. *A is weakly Dedekind-finite.*

Proof. Follows from Lemma 5.3.1.3 and Corollary 4.11. □

Theorem 5.3.1.5. *In $\mathcal{V}_{F_0}$ we have the following.*

1. $\operatorname{dfin}^*(\mathfrak{m}) = \operatorname{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}$.
2. $\operatorname{fin}(\mathfrak{m}) = \inf(\mathfrak{m}) = \operatorname{ded}(\mathfrak{m}) = \operatorname{ded}^*(\mathfrak{m})$.
3. $0 = \operatorname{dinf}(\mathfrak{m}) = \operatorname{dinf}^*(\mathfrak{m})$.
4. $0 < \mathfrak{m} < \operatorname{fin}(\mathfrak{m})$.

Proof. By Lemma 5.3.1.3, $\aleph_0 \not\leq 2^{\mathfrak{m}}$. By Corollary 4.11, $\aleph_0 \not\leq^* \mathfrak{m}$, i.e., $\omega \not\preceq^* A$. Thus $\omega \not\preceq^* S$ for all $S \subseteq A$, so $\mathcal{P}(A) \subseteq \operatorname{dfin}^*(A)$ and $\inf(A) \subseteq \operatorname{ded}^*(A)$. Since $\operatorname{dfin}^*(A) \subseteq \operatorname{dfin}(A) \subseteq \mathcal{P}(A)$ and $\operatorname{ded}^*(A) \subseteq \operatorname{ded}(A) \subseteq \inf(A)$, $\operatorname{dfin}^*(A) = \operatorname{dfin}(A) = \mathcal{P}(A)$, so $\emptyset = \operatorname{dinf}(A) = \operatorname{dinf}^*(A)$ and $\operatorname{ded}^*(A) = \operatorname{ded}(A) = \inf(A)$.

It remains to show that $\operatorname{fin}(\mathfrak{m}) = \inf(\mathfrak{m})$. Define $f\colon \operatorname{fin}(A) \to \inf(A)$ by $f(S) = A \setminus S$ for all $S \in \operatorname{fin}(A)$. Obviously f is one-to-one. Now let $S \in \inf(A)$. Since S has a support, by Lemma 5.3.1.2, S is cofinite. Thus $A \setminus S \in \operatorname{fin}(A)$ and $f(A \setminus S) = A \setminus (A \setminus S) = S$. Hence f is onto, and so $\operatorname{fin}(\mathfrak{m}) = \inf(\mathfrak{m})$.

By Lemma 5.3.1.3, $\omega \not\preceq \mathcal{P}(A)$, so $\omega \not\preceq \operatorname{fin}(A)$. Since the map $a \mapsto \{a\}$ is one-to-one and maps A onto a proper subset of $\operatorname{fin}(A)$ where $\operatorname{fin}(A)$ is Dedekind-finite, by Theorem 3.1.3, $A \prec \operatorname{fin}(A)$, i.e., $0 < \mathfrak{m} < \operatorname{fin}(\mathfrak{m})$. □

Remark. From the above theorem, we can conclude that

$$\begin{aligned}\mathcal{V}_{F_0} \models 0 = \operatorname{dinf}(\mathfrak{m}) = \operatorname{dinf}^*(\mathfrak{m}) < \mathfrak{m}& \\ &< \operatorname{fin}(\mathfrak{m}) = \inf(\mathfrak{m}) = \operatorname{ded}(\mathfrak{m}) = \operatorname{ded}^*(\mathfrak{m}) \\ &< \operatorname{dfin}^*(\mathfrak{m}) = \operatorname{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}.\end{aligned}$$

Corollary 5.3.1.6. $\mathcal{V}_{F_0} \models 2^{\mathfrak{m}} = 2 \cdot \operatorname{fin}(\mathfrak{m})$.

Proof. By Theorem 5.3.1.5, $\mathcal{V}_{F_0} \models \operatorname{fin}(\mathfrak{m}) = \inf(\mathfrak{m})$ and so $2^{\mathfrak{m}} = \inf(\mathfrak{m}) + \operatorname{fin}(\mathfrak{m}) = 2 \cdot \operatorname{fin}(\mathfrak{m})$. □

Theorem 5.3.1.7. $\mathcal{V}_{F_0} \models \exists \mathfrak{n}\big(0 < \operatorname{ded}^*(\mathfrak{n}) = \operatorname{ded}(\mathfrak{n}) < \operatorname{dinf}(\mathfrak{n}) = 2^{\mathfrak{n}}\big)$.

Proof. Let $\mathfrak{n} = \aleph_0 + \mathfrak{m}$ where $\mathfrak{m} = |A|$. By Lemma 5.3.1.3, $\aleph_0 \not\leq 2^{\mathfrak{m}}$. Thus $\aleph_0 \not\leq \inf(\mathfrak{m})$ and, by Corollary 4.11, $\aleph_0 \not\leq^* \mathfrak{m}$, i.e., $\omega \not\preceq^* A$. Then $A \in \operatorname{ded}^*(\omega \cup A)$ where $|\omega \cup A| = \mathfrak{n}$, and so $0 < \operatorname{ded}^*(\mathfrak{n})$.

Let $S \in \operatorname{ded}(\omega \cup A)$. Then S is infinite and Dedekind-finite. Since $\omega \not\preceq S$ and every infinite subset of ω is equinumerous to ω, $S \cap \omega$ is finite and so $S \cap A$ is infinite. Since $\omega \not\preceq^* A$,

$\omega \not\leq^* S \cap A$. Hence $S = (S \cap \omega) \cup (S \cap A)$ is a finite union of weakly Dedekind-finite sets which is weakly Dedekind-infinite by Theorem 4.2. Then $S \in \text{ded}^*(\omega \cup A)$. Thus $\text{ded}(\omega \cup A) \subseteq \text{ded}^*(\omega \cup A)$, and so $\text{ded}(\omega \cup A) = \text{ded}^*(\omega \cup A)$. Hence $\text{ded}(\mathfrak{n}) = \text{ded}^*(\mathfrak{n})$.

Next claim that $\text{ded}(\mathfrak{n}) = \aleph_0 \cdot \text{inf}(\mathfrak{m})$.

Since the map $S \mapsto \langle S \cap \omega, S \cap A \rangle$ from $\text{ded}(\omega \cup A)$ onto $\text{fin}(\omega) \times \text{inf}(A)$ is a bijection, $\text{ded}(\omega \cup A) \approx \text{fin}(\omega) \times \text{inf}(A)$. By Lemma 4.6, $\omega \approx \text{fin}(\omega)$. Thus we have $\text{ded}(\mathfrak{n}) = \aleph_0 \cdot \text{fin}(\mathfrak{m})$.

Since AC holds in $\hat{\mathbf{V}}$, $2^{\aleph_0}$ is an aleph. Since $\aleph_0 < 2^{\aleph_0}$ and $\text{inf}(\mathfrak{m})$ is Dedekind-finite, by Theorem 3.1.7, $\aleph_0 \cdot \text{inf}(\mathfrak{m}) < 2^{\aleph_0} \cdot \text{inf}(\mathfrak{m}) \leq 2^{\aleph_0} \cdot 2^{\mathfrak{m}} = 2^{\aleph_0 + \mathfrak{m}} = 2^{\mathfrak{n}}$. Hence $\text{ded}(\mathfrak{n}) < 2^{\mathfrak{n}}$. It remains to show that $\text{dinf}(\mathfrak{n}) = 2^{\mathfrak{n}}$.

Let $S \in \text{dinf}(\omega \cup A)$. Since A is Dedekind-finite, so is $S \cap A$. Since S is the disjoint union of $S \cap \omega$ and $S \cap A$ where S is Dedekind-infinite, $S \cap \omega$ is infinite. Thus $S \cap \omega \in \text{inf}(\omega)$. Hence $\text{dinf}(\omega \cup A) \approx \text{inf}(\omega) \times \mathcal{P}(A)$ by the map $S \mapsto \langle S \cap \omega, S \cap A \rangle$, so $\text{dinf}(\mathfrak{n}) = \text{inf}(\aleph_0) \cdot 2^{\mathfrak{m}}$. By Corollary 4.7, $\text{dinf}(\mathfrak{n}) = 2^{\aleph_0} \cdot 2^{\mathfrak{m}} = 2^{\aleph_0 + \mathfrak{m}} = 2^{\mathfrak{n}}$. $\square$

5.3.2 The second Fraenkel model

Definition 5.3.2.1. Let A be a set consists of countably many mutually disjoint 2-element sets, i.e.,

$$A = \bigcup \{P_n \mid n \in \omega\},$$

where P_n is a 2-element set for $n \in \omega$ and all P_n's are mutually disjoint. Let $\mathcal{G}$ be the group of all those permutations of A which preserve the pairs P_n, i.e., $\pi(P_n) = P_n$ for all $n \in \omega$. Let $\mathcal{V}_{F_2}$ be the permutation model defined from the normal ideal $\text{fin}(A)$. We call $\mathcal{V}_{F_2}$ the *second Fraenkel model*.

Theorem 5.3.2.2. [9]

1. *For each $n \in \omega$, P_n belongs to $\mathcal{V}_{F_2}$.*
2. *The set $\{P_n \mid n \in \omega\}$ belongs to $\mathcal{V}_{F_2}$. In particular, $\{P_n \mid n \in \omega\}$ is countable in $\mathcal{V}_{F_2}$.*
3. *There is no choice function on $\{P_n \mid n \in \omega\}$.*

Throughout this subsection, all proofs are done in $\mathcal{V}_{F_2}$ and let $\mathfrak{m} = |A|$.

By modifying the proof of Theorem 5.3.2.2 (3) in [9], we have the following theorem.

Theorem 5.3.2.3. $\mathcal{V}_{F_2} \vDash \aleph_0 \not\leq \mathfrak{m}$.

Proof. Suppose there exists $f \in \mathcal{V}_{F_2}$ such that $f : \omega \to A$ is one-to-one. Let E be a support of f. Without loss of generality, we may assume that $E = \{a_0, b_0, \ldots, a_k, b_k\}$ for some $k \in \omega$

(this can be done since, if E is a support of f, then so is $\bigcup\{P_n \mid n \in \omega \text{ and } P_n \cap E \neq \emptyset\}$ since $E \subseteq \bigcup\{P_n \mid n \in \omega \text{ and } P_n \cap E \neq \emptyset\}$).

Since $f[\omega]$ is infinite and E is finite, there exists $a \in f[\omega] \setminus E$. Then $a \in P_l$ for some $l \in \omega$. Let $b \in P_l \setminus \{a\}$ and $\pi \in \text{fix}_{\mathcal{G}}(E)$ be such that $\pi(a) = b$. Since $a \in f[\omega]$, $a = f(m)$ for some $m \in \omega$. Since $m \in \hat{\mathbf{V}}$, $\pi(m) = m$ and so, by Lemma 5.2.3(2), $(\pi f)(m) = (\pi f)(\pi m) = \pi(f(m)) = \pi(a) = b \neq a = f(m)$. Hence $\pi(f) \neq f$. This contradicts the fact that E is a support of f. Hence there is no such f in $\mathcal{V}_{F_2}$, so $\omega \not\preceq A$, i.e., $\aleph_0 \not\leq \mathfrak{m}$. □

Theorem 5.3.2.4. $\mathcal{V}_{F_2} \models \aleph_0 \leq 2^{\mathfrak{m}}$.

Proof. Note that $A = \bigcup\{P_n \mid n \in \omega\}$ where all P_n's are pairwise disjoint 2-element sets. Define $f\colon \omega \to \{P_n \mid n \in \omega\}$ by $n \mapsto P_n$. Clearly, f is one-to-one. Next we will show that $f \in \mathcal{V}_{F_2}$. Since ω and $\{P_n \mid n \in \omega\}$ are in $\mathcal{V}_{F_2}$ where $\mathcal{V}_{F_2}$ is transitive and satisfies the Axiom of Paring, $f \subseteq \mathcal{V}_{F_2}$. It remains to show that f is symmetric. Let $\pi \in \mathcal{G}$. Then $(\pi f)(n) = (\pi f)(\pi n) = \pi(f(n)) = \pi(P_n) = P_n = f(n)$ for all $n \in \omega$. Thus $\pi(f) = f$, so $\text{sym}_{\mathcal{G}}(f) = \mathcal{G}$ where $\mathcal{G}$ belongs to any normal filter. Then f is symmetric. Hence f belongs to $\mathcal{V}_{F_2}$. Since $\{P_n \mid n \in \omega\} \subseteq \mathcal{P}(A)$, we can conclude that $\mathcal{V}_{F_2} \models \aleph_0 \leq 2^{\mathfrak{m}}$. □

Corollary 5.3.2.5. $\mathcal{V}_{F_2} \models \aleph_0 \leq^* \mathfrak{m}$ *and* $\mathcal{V}_{F_2} \models \text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Follows from Theorem 5.3.2.4 and Corollary 4.11. □

Lemma 5.3.2.6. *For any subset S of A in $\mathcal{V}_{F_2}$,*

$$S = \bigcup_{n \in N} \{c_n\} \cup \bigcup_{m \in M} P_m$$

for some subsets M and N of ω such that N is finite and $M \cap N = \emptyset$ where $c_n \in P_n$ for all $n \in N$.

Proof. It is easy to see that for any $S \subseteq A$, $S = \bigcup_{n \in N} \{c_n\} \cup \bigcup_{m \in M} P_m$ for some $N \subseteq \omega$ and $M \subseteq \omega \setminus N$ where $c_n \in P_n$ for all $n \in N$.

Since $\mathcal{V}_{F_2} \models \aleph_0 \not\leq \mathfrak{m}$ by Theorem 5.3.2.3 and $N \preceq A$ by the map $n \mapsto c_n$, N must be finite. □

Theorem 5.3.2.7. *In $\mathcal{V}_{F_2}$ we have the following.*

1. $\mathfrak{m} < \text{fin}(\mathfrak{m}) = \text{dfin}^*(\mathfrak{m}) < \text{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}$.
2. $\text{ded}(\mathfrak{m}) = \text{inf}(\mathfrak{m}) = \text{dinf}^*(\mathfrak{m})$.
3. $0 = \text{dinf}(\mathfrak{m}) = \text{ded}^*(\mathfrak{m})$.

Proof. By Theorem 5.3.2.3, $\mathcal{V}_{F_2} \vDash \aleph_0 \not\leq \mathfrak{m}$. Then for any subset S of A, $\omega \not\preceq S$. Thus $\operatorname{dfin}(A) = \mathcal{P}(A)$ and $\emptyset = \operatorname{dinf}(A)$, i.e., $\mathcal{V}_{F_2} \vDash \operatorname{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}$ and $\mathcal{V}_{F_2} \vDash 0 = \operatorname{dinf}(\mathfrak{m})$. Since every subset of A is Dedekind-finite, $\inf(A) = \operatorname{ded}(A)$.

Note that $a \mapsto \{a\}$ maps A onto a proper subset of $\operatorname{fin}(A)$. If $\operatorname{fin}(A) \approx A$, then, by Theorem 3.1.3, $\omega \preceq \operatorname{fin}(A) \approx A$, i.e., $\aleph_0 \leq \mathfrak{m}$, a contradiction. Hence $\operatorname{fin}(A) \not\approx A$ and so $\mathfrak{m} < \operatorname{fin}(\mathfrak{m})$.

Now we will show that $\inf(A) = \operatorname{dinf}^*(A)$. Since $\operatorname{dinf}^*(A) \subseteq \inf(A)$, it remains to show that $\inf(A) \subseteq \operatorname{dinf}^*(A)$.

Let $S \in \inf(A)$. By Lemma 5.3.2.6, $S = \bigcup_{n\in N} \{c_n\} \cup \bigcup_{m\in M} P_m$ for some finite $N \subseteq \omega$ and some infinite $M \subseteq \omega \setminus N$ where $c_n \in P_n$ for all $n \in N$. Thus $|N \cup M| = \aleph_0$. Define $g\colon S \to N \cup M$ by $g(a) = k$ if $a \in P_k$. It is easy to see that g is onto. Thus $\aleph_0 \leq^* |S|$, so $S \in \operatorname{dinf}^*(A)$. Hence $\operatorname{dinf}^*(A) = \inf(A)$ and so $\operatorname{dfin}^*(A) = \operatorname{fin}(A)$ and $\operatorname{ded}^*(A) = \emptyset$.

Since $\operatorname{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$, $\mathcal{V}_{F_2} \vDash \operatorname{fin}(\mathfrak{m}) = \operatorname{dfin}^*(\mathfrak{m}) < \operatorname{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}$. □

5.3.3 The ordered Mostowski model

Definition 5.3.3.1. Let A be a countable infinite set with a linear order $<^M$ on A such that A is densely ordered and does not have a smallest or greatest element, i.e., A is isomorphic to the set of rationals $\mathbb{Q}$. Let $\mathcal{G}$ be the group of all order-preserving permutations of A. Let $\mathcal{V}_M$ be the permutation model defined from the normal ideal $\operatorname{fin}(A)$. We call $\mathcal{V}_M$ the *ordered Mostowski model*.

Lemma 5.3.3.2. [4] *$<^M$ belongs to $\mathcal{V}_M$. Then for any atoms a_1 and a_2, we can decide in $\mathcal{V}_M$ whether $a_1 <^M a_2$ or $a_2 <^M a_1$.*

All proofs in this subsection are done in $\mathcal{V}_M$ and let $\mathfrak{m} = |A|$.

The next lemma is modified from Lemma 7.12 in [4].

Lemma 5.3.3.3. *Let $E \in \operatorname{fin}(A)$, say $|E| = n$ for some $n \in \omega$. Then there are exactly 2^{2n+1} subsets of A with support E where 2^n of them are finite.*

Proof. Let $S \subseteq A$ be such that E is a support of S. Assume $E = \{a_1, a_2, \ldots, a_n\}$ where $a_1 <^M a_2 <^M \ldots <^M a_n$. Let

$$I_0 = \{a \in A \mid a <^M a_1\},$$
$$I_k = \{a \in A \mid a_k <^M a <^M a_{k+1}\} \quad \text{for } 1 \leq k < n, \text{ and}$$
$$I_n = \{a \in A \mid a_n <^M a\}.$$

We will show that for all $k \leq n$, either $I_k \cap S = \emptyset$ or $I_k \subseteq S$. Let $1 \leq k < n$. Suppose $I_k \cap S \neq \emptyset$, say $s_0 \in I_k \cap S$. Then $a_k <^M s_0 <^M a_{k+1}$. Let $s \in I_k$. Then $a_k <^M s <^M a_{k+1}$. Let $\pi \in \mathrm{fix}_{\mathcal{G}}(E)$ such that $\pi(s_0) = s$. Thus $s = \pi(s_0) \in \pi(S) = S$, so $I_k \subseteq S$. The cases $k = 0$ and $k = n$ are similar.

Let $N = \{i \mid a_i \in S\}$ and $K = \{j \mid I_j \subseteq S\}$. Then

$$S = \bigcup_{i \in N} \{a_i\} \cup \bigcup_{j \in K} I_j.$$

Since $N \subseteq \{1, 2, \ldots, n\}$ and $K \subseteq \{0, 1, 2, \ldots, n\}$, there are 2^n and 2^{n+1} possible forms of N and K, respectively. It follows that there are $2^{n+(n+1)} = 2^{2n+1}$ possible forms of S where, when $K = \emptyset$, 2^n of them are finite. □

Theorem 5.3.3.4. [6] $\mathcal{V}_M \models \aleph_0 \not\leq 2^{\mathfrak{m}}$

Lemma 5.3.3.5. *For any cardinals* $\mathfrak{m}$ *and* $\mathfrak{n}$*, if* $\mathfrak{m} \leq^* \mathfrak{n}$*, then* $2^{\mathfrak{m}} \leq 2^{\mathfrak{n}}$.

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinals, say $\mathfrak{m} = |M|$ and $\mathfrak{n} = |N|$. Assume $\mathfrak{m} \leq^* \mathfrak{n}$, i.e., there is $f : N \to M$ which is onto. Define $g : \mathcal{P}(M) \to \mathcal{P}(N)$ by

$$g(X) = f^{-1}[X] \quad \text{for all } X \subseteq M.$$

Since f is onto, g is one-to-one. Hence $2^{\mathfrak{m}} \leq 2^{\mathfrak{n}}$. □

Corollary 5.3.3.6. $\mathcal{V}_M \models 2^{\mathrm{fin}(\mathfrak{m})} = 2^{2^{\mathfrak{m}}}$.

Proof. Follows from Theorem 5.3.3.4(3) and Lemma 5.3.3.5. □

Theorem 5.3.3.7. *In* $\mathcal{V}_M$ *we have the following.*

1. $\mathrm{dfin}^*(\mathfrak{m}) = \mathrm{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}$.
2. $\inf(\mathfrak{m}) = \mathrm{ded}(\mathfrak{m}) = \mathrm{ded}^*(\mathfrak{m})$.
3. $0 = \mathrm{dinf}(\mathfrak{m}) = \mathrm{dinf}^*(\mathfrak{m})$.

Proof. Since $\mathcal{V}_M \models \aleph_0 \not\leq 2^{\mathfrak{m}}$, the proof is similar to the proof of Theorem 5.3.1.5. □

Theorem 5.3.3.8. $\mathcal{V}_M \models 0 < \mathfrak{m} < \mathrm{fin}(\mathfrak{m}) < \inf(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Since $\mathcal{V}_M \models \aleph_0 \not\leq 2^{\mathfrak{m}}$ and $\inf(A) \subset \mathcal{P}(A)$, by Corollary 3.1.4, $\mathcal{V}_M \models \inf(\mathfrak{m}) < 2^{\mathfrak{m}}$. By Theorem 5.3.3.4, $0 < \mathfrak{m} < \mathrm{fin}(\mathfrak{m})$.

Now we will show that $\mathcal{V}_M \models \mathrm{fin}(\mathfrak{m}) < \inf(\mathfrak{m})$. Since A is infinite, by Theorem 3.2.1 (5), $\mathrm{fin}(\mathfrak{m}) \leq \inf(\mathfrak{m})$.

Suppose there exists $f \in \mathcal{V}_M$ such that $f\colon \inf(A) \to \operatorname{fin}(A)$ is one-to-one. Let $E \in \operatorname{fin}(A)$ be a support of f, say $|E| = n$. By Lemma 5.3.3.3, there are 2^{2n+1} subsets of A with a support E and there are 2^n of them which are finite. Thus the remaining $2^{2n+1} - 2^n$ are infinite subsets of A with support E. Since there are 2^n subsets of E and $2^{2n+1} - 2^n > 2^n = |\mathcal{P}(E)|$, there exists an infinite subset S of A with support E such that $f(S) \nsubseteq E$.

Let $a \in f(S) \setminus E$. Since $f(S)$ and E are finite, there is $\pi \in \operatorname{fix}_G(E)$ such that $\pi(a) \in A \setminus (f(S) \cup E)$. Since $\pi(a) \in \pi(f(S))$, $\pi(f(S)) \neq f(S)$. Since E is a support of S, $\pi(S) = S$, so $(\pi f)(S) = (\pi f)(\pi S) = \pi(f(S)) \neq f(S)$. This contradicts the fact that $\pi(f) = f$. Hence there is no such f in $\mathcal{V}_M$, so $\mathcal{V}_M \vDash \operatorname{fin}(\mathfrak{m}) < \inf(\mathfrak{m})$. □

Remark. From the above theorems, we can conclude that

$$\begin{aligned}\mathcal{V}_M \vDash 0 = \operatorname{dinf}(\mathfrak{m}) = \operatorname{dinf}^*(\mathfrak{m}) &< \mathfrak{m} \\ &< \operatorname{fin}(\mathfrak{m}) \\ &< \inf(\mathfrak{m}) = \operatorname{ded}(\mathfrak{m}) = \operatorname{ded}^*(\mathfrak{m}) \\ &< \operatorname{dfin}^*(\mathfrak{m}) = \operatorname{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}.\end{aligned}$$

CHAPTER VI

CONCLUSIONS

In this chapter, we summarize all results that we have shown in Chapters IV and V.

The following are new results that can be proved from ZF.

1. For any set A, if there exists a surjection from a proper subset of A onto A, then A is weakly Dedekind-infinite (Theorem 4.1).
2. A weakly Dedekind-finite union of weakly Dedekind-finite sets is weakly Dedekind-finite (Theorem 4.2).
3. If $\mathfrak{m}$ and $\mathfrak{n}$ are weakly Dedekind-finite cardinals, then so are $\mathfrak{m} + \mathfrak{n}$ and $\mathfrak{m} \cdot \mathfrak{n}$ (Corollary 4.3).
4. Let $\mathfrak{m}$ be a cardinal and $n \in \omega$. If $\mathfrak{m}$ is (weakly) Dedekind-finite, then so is $\mathfrak{m}^n$. This statement also holds if we replace "(weakly) Dedekind-finite" by "(weakly) Dedekind-infinite" for the case $n \neq 0$ (Corollary 4.4).

For results concerning cardinal relations, we have shown that for any infinite cardinal $\mathfrak{m}$,

5. $\aleph_0 \leq^* \mathfrak{m} \leftrightarrow \text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$ (Theorem 4.9).
6. $\aleph \leq \text{dfin}^*(\mathfrak{m}) \rightarrow 2^{\aleph} \leq 2^{\mathfrak{m}}$ for any aleph $\aleph$ (Theorem 4.10).
7. $\aleph_0 \leq \text{dfin}^*(\mathfrak{m}) \rightarrow \text{dfin}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$ (Corollary 4.12).
8. $\text{ded}^*(\mathfrak{m}) < 2^{\mathfrak{m}}$ (Corollary 4.14).
9. $\mathfrak{m} \geq 5 \rightarrow \text{dfin}^*(\mathfrak{m}) < \text{part}(\mathfrak{m})$ (Corollary 4.15).
10. $\mathfrak{m} < \text{inf}(\mathfrak{m})$ (Theorem 4.16).

Next, we will list new consistency results concerning cardinal relations.

First, note that, in ZF, for any infinite cardinal $\mathfrak{m}$,

(a) $0 < \mathfrak{m} \leq \text{fin}(\mathfrak{m}) \leq \text{dfin}^*(\mathfrak{m}) \leq \text{dfin}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

(b) $0 \leq \text{ded}^*(\mathfrak{m}) \leq \text{ded}(\mathfrak{m}) \leq \text{inf}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

(c) $0 \leq \text{dinf}(\mathfrak{m}) \leq \text{dinf}^*(\mathfrak{m}) \leq \text{inf}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

(d) $0 < \mathfrak{m} \leq \text{fin}(\mathfrak{m}) \leq \text{inf}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$.

(e) $\text{ded}^*(\mathfrak{m}) \leq \text{dfin}^*(\mathfrak{m})$ and $\text{ded}(\mathfrak{m}) \leq \text{dfin}(\mathfrak{m})$.

The following statements are consistent with ZF, provided that ZF is consistent.

1. $\exists\mathfrak{m}\big(0 < \mathfrak{m} = \mathrm{fin}(\mathfrak{m}) = \mathrm{dfin}^*(\mathfrak{m}) = \mathrm{dfin}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Corollary 4.7).
2. $\exists\mathfrak{m}\big(0 < \mathfrak{m} < \mathrm{fin}(\mathfrak{m}) < \mathrm{dfin}^*(\mathfrak{m}) = \mathrm{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.1.5).
3. $\exists\mathfrak{m}\big(0 < \mathfrak{m} < \mathrm{fin}(\mathfrak{m}) = \mathrm{dfin}^*(\mathfrak{m}) < \mathrm{dfin}(\mathfrak{m}) = 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.2.7).
4. $\exists\mathfrak{m}\big(0 = \mathrm{ded}^*(\mathfrak{m}) = \mathrm{ded}(\mathfrak{m}) < \mathrm{inf}(\mathfrak{m}) = 2^{\mathfrak{m}}\big)$ (see Corollary 4.7).
5. $\exists\mathfrak{m}\big(0 < \mathrm{ded}^*(\mathfrak{m}) = \mathrm{ded}(\mathfrak{m}) = \mathrm{inf}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.1.5).
6. $\exists\mathfrak{m}\big(0 = \mathrm{ded}^*(\mathfrak{m}) < \mathrm{ded}(\mathfrak{m}) = \mathrm{inf}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.2.7).
7. $\exists\mathfrak{m}\big(0 < \mathrm{dinf}(\mathfrak{m}) = \mathrm{dinf}^*(\mathfrak{m}) = \mathrm{inf}(\mathfrak{m}) = 2^{\mathfrak{m}}\big)$ (see Corollary 4.7).
8. $\exists\mathfrak{m}\big(0 = \mathrm{dinf}(\mathfrak{m}) = \mathrm{dinf}^*(\mathfrak{m}) < \mathrm{inf}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.1.5).
9. $\exists\mathfrak{m}\big(0 = \mathrm{dinf}(\mathfrak{m}) < \mathrm{dinf}^*(\mathfrak{m}) = \mathrm{inf}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.2.7).
10. $\exists\mathfrak{m}\big(0 < \mathfrak{m} = \mathrm{fin}(\mathfrak{m}) < \mathrm{inf}(\mathfrak{m}) = 2^{\mathfrak{m}}\big)$ (see Corollary 4.7).
11. $\exists\mathfrak{m}\big(0 < \mathfrak{m} < \mathrm{fin}(\mathfrak{m}) = \mathrm{inf}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.1.5).
12. $\exists\mathfrak{m}\big(0 < \mathfrak{m} < \mathrm{fin}(\mathfrak{m}) < \mathrm{inf}(\mathfrak{m}) < 2^{\mathfrak{m}}\big)$ (see Theorem 5.3.3.8).
13. $\exists\mathfrak{m}\big(0 < \mathrm{ded}^*(\mathfrak{m}) = \mathrm{ded}(\mathfrak{m}) < \mathrm{dfin}^*(\mathfrak{m}) = \mathrm{dfin}(\mathfrak{m})\big)$ (see Theorem 5.3.1.5).

The consistency results from (1)–(12) tell us that the relation $\leq$ between each pair of cardinals in (a)–(d) cannot be replaced by neither $=$ nor $<$.

For (13), it is consistent with ZF that there is a cardinal $\mathfrak{m}$ such that $\mathrm{ded}^*(\mathfrak{m}) < \mathrm{dfin}^*(\mathfrak{m})$ and $\mathrm{ded}(\mathfrak{m}) < \mathrm{dfin}(\mathfrak{m})$. Furthermore, the relations are not trivial since $\mathrm{ded}^*(\mathfrak{m})$ and $\mathrm{ded}(\mathfrak{m})$ are not 0. Hence we cannot show in ZF that "for all infinite cardinal $\mathfrak{n}$, $\mathrm{ded}^*(\mathfrak{n}) = \mathrm{dfin}^*(\mathfrak{n})$" or "for all infinite cardinal $\mathfrak{n}$, $\mathrm{ded}(\mathfrak{n}) = \mathrm{dfin}(\mathfrak{n})$". However, we do not know whether it can be proved from ZF that "for all infinite cardinals $\mathfrak{n}$, $\mathrm{ded}^*(\mathfrak{n}) < \mathrm{dfin}^*(\mathfrak{n})$" or "for all infinite cardinals $\mathfrak{n}$, $\mathrm{ded}(\mathfrak{n}) < \mathrm{dfin}(\mathfrak{n})$" or not.

We summarize the results we got by listing all the possible relationships between $\mathfrak{m}$, $\mathrm{fin}(\mathfrak{m})$, $\mathrm{dfin}^*(\mathfrak{m})$, $\mathrm{dinf}(\mathfrak{m})$, $\mathrm{dinf}^*(\mathfrak{m})$, $\mathrm{ded}^*(\mathfrak{m})$, $\mathrm{ded}(\mathfrak{m})$, $\mathrm{inf}(\mathfrak{m})$, $2^{\mathfrak{m}}$, and $\mathrm{part}(\mathfrak{m})$, where $\mathfrak{m}$ is some infinite cardinal, in the following table.

One has to read the table from the left to the right and upwards.

	$\mathrm{fin}(\mathfrak{m})$	$\mathrm{dfin}^*(\mathfrak{m})$	$\mathrm{dfin}(\mathfrak{m})$	$\mathrm{dinf}(\mathfrak{m})$	$\mathrm{dinf}^*(\mathfrak{m})$	$\mathrm{ded}^*(\mathfrak{m})$	$\mathrm{ded}(\mathfrak{m})$	$\mathrm{inf}(\mathfrak{m})$	$2^{\mathfrak{m}}$	$\mathrm{part}(\mathfrak{m})$
$\mathfrak{m}$	$= <$	$= <$	$= <$	$> <$	$> <$	$> <$	$> <$	$<$	$<$	$<$
$\mathrm{fin}(\mathfrak{m})$		$= <$	$= <$	$> <$	$> <$	$> = <$	$> = <$	$= <$	$<$	$<$
$\mathrm{dfin}^*(\mathfrak{m})$			$= <$	$> <$	$> <$	$>$	$>$	$> <$	$= <$	$<$
$\mathrm{dfin}(\mathfrak{m})$				$> <$	$> <$	$>$	$>$	$> <$	$= <$	
$\mathrm{dinf}(\mathfrak{m})$					$= <$	$> = <$	$> <$	$= <$	$= <$	
$\mathrm{dinf}^*(\mathfrak{m})$						$= <$	$> = <$	$= <$	$= <$	
$\mathrm{ded}^*(\mathfrak{m})$							$= <$	$= <$	$<$	$<$
$\mathrm{ded}(\mathfrak{m})$								$= <$	$<$	
$\mathrm{inf}(\mathfrak{m})$									$= <$	

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