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ORTHOGONAL GRAPHS OVER FINITE COMMUTATIVE RINGS OF ODD CHARACTERISTIC

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ทรงพล ศรีวงค์ษา : กราฟเชิงตั้งฉากเหนือริงสลับที่จำกัดซึ่งมีแคแรกเทอริสติกคี่ (ORTHOGONAL GRAPHS OVER FINITE COMMUTATIVE RINGS OF ODD CHARACTERISTIC) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: รศ.ดร.ยศนันต์ มีมาก, 40 หน้า.

ให้ (V, β) เป็นปริภูมิตั้งฉากเหนือริงสลับที่จำกัด R ซึ่งมีแคแรกเทอริสติกคี่ เราศึกษาโครงสร้างของปริภูมิ V เมื่อ R เป็นริงเฉพาะที่จำกัด เรานิยามกราฟเชิงตั้งฉากสำหรับ V เราพิสูจน์ว่ากราฟเชิงตั้งฉากมีสมบัติถ่ายทอดบนจุดยอด และบนอาร์ก และ คำนวณ รงคเลขของกราฟนี้ ยิ่งกว่านั้นเมื่อ R เป็นริงเฉพาะที่จำกัด เราได้เงื่อนไขในการจำแนกกราฟออกเป็นกราฟปกติอย่างเข้มและ กราฟควอไซปกติอย่างเข้ม และ คำนวณกรุปอัตสัณฐาน

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Let (V, β) be an orthogonal space over a finite commutative ring R of odd characteristic. We determine the structure of V when R is a finite local ring. We define a graph for V called an orthogonal graph. We show that our graph is vertex and arc transitive and determine the chromatic number of the graph. Moreover, if R is a finite local ring, we can classify if it is a strongly regular or quasi-strongly regular graph and we obtain its automorphism group.

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CONTENTS

CHAPTER I
INTRODUCTION

Graphs arising from linear algebra over finite commutative rings have been widely studied. Meemark and Prinyasart [15] introduced the symplectic graphs over the ring of integers modulo p^n. Previously, the graphs were studied only over a finite field [18, 9]. There are several articles influenced by this definition such as [10], [11], [7], [6] and [5]. Meemark and Puirod [16, 17] extended this work to the symplectic graphs over finite local rings and finite commutative rings. Gu and Wan [5] defined and studied the orthogonal graphs over finite fields of odd characteristic. Recently, Li, Guo and Wang [12] studied the orthogonal graphs over Galois rings of odd characteristic using matrix theory over finite Galois rings. Their work depends on the structure of orthogonal spaces studied in [3] and the graphs are defined on orthogonal spaces similar to Meemark's graph which are defined on the symplectic spaces. Mostly, the works were on vertex and arc transitivities, strong and quasi-strong regularity, chromatic numbers, automorphism groups.

In this work, we define orthogonal graphs over finite commutative rings of odd characteristic and study their properties including strongly and quasi-strongly regular graphs, vertex and arc transitives, chromatic numbers and graph automorphisms.

The thesis is organized as follows. We determine the structure of orthogonal space (V, β) over a finite local ring R in Chapter II. In Chapter III, we study the orthogonal graph $\mathcal{G}_{\mathrm{O}_R(V)}$ when R is a finite local ring. We prove that this graph is vertex and arc transitive. Moreover, in Section 3.3, we obtain a classification for our graph to be strongly regular or to be a quasi-strongly regular graph. In the last section of Chapter III, we obtain results of orthogonal graph over finite commutative rings. Finally, we determine the chromatic number and obtain automorphism group of orthogonal graphs, in Chapter IV.

CHAPTER II
ORTHOGONAL SPACES

In this chapter, we determine the structure of orthogonal spaces over a finite local ring of odd characteristic. This generalizes the work of Cao [3]. The last section also contains basic properties of a Galois ring.

Let R be a commutative ring with unity. We shall denote its unit group by $R^\times$. Let V be a free R-module of rank n, where $n \geq 2$. Assume that we have a function $\beta : V \times V \to R$ which is R-bilinear, symmetric and the R-module morphism from V to $V^* = \hom_R(V, R)$ given by $\vec{x} \mapsto \beta(\cdot, \vec{x})$ is an isomorphism. For $\vec{x} \in V$, we call $\beta(\vec{x}, \vec{x})$ the **norm of** $\vec{x}$. The pair (V, β) is called an **orthogonal space**. Moreover, if $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis of (V, β), then we have the associated matrix $[\beta]_{\mathcal{B}} = [\beta(\vec{b}_i, \vec{b}_j)]_{n \times n}$. An **orthogonal basis** $\mathcal{B}' = \{\vec{e}_1, \ldots, \vec{e}_n\}$ of (V, β) is a basis satisfying $\beta(\vec{e}_i, \vec{e}_i) = u_i$ for some $u_i \in R^\times$ and $\beta(\vec{e}_i, \vec{e}_j) = 0$ for $i \neq j$.

2.1 Units and the square mapping

A **local ring** is a commutative ring which has a unique maximal ideal. Note that for a local ring R, its unique maximal ideal is given by $M = R \smallsetminus R^\times$(Proposition 1.2.11 of [2]) and we call the field R/M, the **residue field of** R.

Example 2.1.1. If p is a prime, then $\mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, is a local ring with maximal ideal $p\mathbb{Z}_{p^n}$ and residue field $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n}$ isomorphic to $\mathbb{Z}_p$. Moreover, every field is a local ring with maximal ideal $\{0\}$.

Recall a common theorem about local rings that:

Theorem 2.1.1. *Let R be a local ring with unique maximal ideal M. Then $1 + m$ is a unit of R for all $m \in M$. Furthermore, $u + m$ is a unit in R for all $m \in M$ and $u \in R^\times$.*

Proof. Suppose that $1+m$ is not a unit. Since R is local, $1+m \in M$. Hence, 1 must be in M, which is a contradiction. $\square$

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field k. From Theorem XVIII. 2 of [13] we have that the unit group of R, denoted by $R^\times$, is isomorphic to $(1+M) \times k^\times$. Consider the exact sequence of groups

$$1 \longrightarrow K_R \longrightarrow R^\times \longrightarrow (R^\times)^2 \longrightarrow 1$$

where $\theta : a \longmapsto a^2$ is the square mapping on $R^\times$ with kernel $K_R = \{a \in R^\times : a^2 = 1\}$ and $(R^\times)^2 = \{a^2 : a \in R^\times\}$. Note that K_R consists of the identity and all elements of order two in $R^\times$. Since R is of odd characteristic and $k^\times$ is cyclic, $K_R = \{\pm 1\}$. Hence, $[R^\times : (R^\times)^2] = |K_R| = 2$.

Proposition 2.1.2. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field k.*

(1) The image $(R^\times)^2$ is a subgroup of $R^\times$ with index $[R^\times : (R^\times)^2] = 2$.

(2) For $z \in R^\times \smallsetminus (R^\times)^2$, we have $R^\times \smallsetminus (R^\times)^2 = z(R^\times)^2$ and $|(R^\times)^2| = |z(R^\times)^2| = (1/2)|R^\times|$.

(3) For $u \in R^\times$ and $a \in M$, there exists $c \in R^\times$ such that $c^2(u+a) = u$.

(4) If $-1 \notin (R^\times)^2$ and $u \in R^\times$, then $1+u^2 \in R^\times$.

(5) If $-1 \notin (R^\times)^2$ and $z \in R^\times \smallsetminus (R^\times)^2$, then there exist $x, y \in R^\times$ such that $z = (1+x^2)y^2$.

Proof. We have proved (1) in the above discussion and (2) follows from (1). Let $u \in R^\times$ and $a \in M$. Then $u^{-1}(u+a) = 1+u^{-1}a \in 1+M$, so $(u^{-1}(u+a))^{|1+M|+1} = u^{-1}(u+a)$. Since $|1+M| = |M|$ is odd, $u^{-1}(u+a) = (c^{-1})^2$ for some $c \in R^\times$. Thus, $c^2(u+a) = u$ which proves (3).

For (4), assume that $-1 \notin (R^\times)^2$ and let $u \in R^\times$. Suppose that $1+u^2 = x \in M$. Then $u^2 = -(1-x)$. Since $|M|$ is odd and $1-x \in 1+M$, $(u^{|M|})^2 = (-(1-x))^{|M|} = (-1)^{|M|}(1-x)^{|M|} = (-1)(1) = -1$, which contradicts -1 is non-square. Hence, $1+u^2 \in R^\times$.

Finally, we observe that $|1+(R^\times)^2| = |(R^\times)^2|$ is finite. If $1+(R^\times)^2 \subseteq (R^\times)^2$, then they must be equal, so there exists $b \in (R^\times)^2$ such that $1+b = 1$, which forces $b = 0$, a contradiction. Hence, there exists an $x \in R^\times$ such that $1+x^2 \notin (R^\times)^2$. By (4), $1+x^2 \in R^\times$. Therefore, for a non-square unit z, we have $R^\times$ is a disjoint union of cosets $(R^\times)^2$ and $z(R^\times)^2$, so $1+x^2 = z(y^{-1})^2$ for some $y \in R^\times$ as desired. $\square$

2.2 Cogredient matrices

Throughout this section, we let R be a finite local ring of odd characteristic.

Notation. *For any $l \times n$ matrix A and $q \times r$ matrix B over R, we write*

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

which is an $(l+q) \times (n+r)$ matrix over R.

For any matrices $S_1, S_2 \in M_n(R)$, if there exists an invertible matrix P such that $PS_1P^T = S_2$, we say that S_1 is **cogredient** to S_2 over R and we write $S_1 \approx S_2$. Note that $S \approx c^2 S$ for all $c \in R^\times$ and $S \in M_n(R)$. The next lemma is a key for our structure theorem.

Lemma 2.2.1. *For a positive integer ν and $z \in R^\times \setminus (R^\times)^2$, $zI_{2\nu}$ is cogredient to $I_{2\nu}$.*

Proof. If $-1 = u^2$ for some $u \in R^\times$, we may choose $P = 2^{-1}\begin{pmatrix} (1+z) & u^{-1}(1-z) \\ u(1-z) & (1+z) \end{pmatrix}$ whose determinant is $z \in R^\times$. Note that our R of odd characteristic, so 2 is a unit. Hence, P is invertible and $PP^T = zI_2$. Next, we assume that -1 is non-square. Then, by Proposition 2.1.2 (5), $z = (1+x^2)y^2$ for some units x and y in $R^\times$.

Choose $Q = \begin{pmatrix} xy & y \\ -y & xy \end{pmatrix}$. Then $\det Q = (1+x^2)y^2 = z \in R^\times$, so Q is invertible and $QQ^T = \begin{pmatrix} (1+x^2)y^2 & 0 \\ 0 & (1+x^2)y^2 \end{pmatrix} = zI_2$. Therefore, $zI_{2\nu} = \overbrace{zI_2 \oplus \cdots \oplus zI_2}^{\nu' s}$ is cogredient to $I_{2\nu} = \overbrace{I_2 \oplus \cdots \oplus I_2}^{\nu' s}$. □

McDonald and Hershberger proved the following theorem.

Theorem 2.2.2 (Theorem 3.2 of [14]). *Let (V, β) be an orthogonal space of rank $n \geq 2$. Then (V, β) processes an orthogonal basis $\mathcal{C}$ so that $[\beta]_{\mathcal{C}}$ is a diagonal matrix whose entries on the diagonal are units.*

Let (V, β) be an orthogonal space of rank $n \geq 2$. Let $\mathcal{C}$ be an orthogonal basis of V such that $[\beta]_{\mathcal{C}}$ is a diagonal matrix whose entries on the diagonal are units. From $[\beta]_{\mathcal{C}} = \operatorname{diag}(u_1, \ldots, u_n)$ and u_i are units for all i. Assume that $u_1, \ldots, u_r$ are squares and $u_{r+1}, \ldots, u_n$ are non-squares. Since $R^\times$ is a disjoint union of the cosets $(R^\times)^2$ and $z(R^\times)^2$ for some non-square unit z, we have $u_i = w_i^2$ for some $w_i \in R^\times$, $i = 1, \ldots, r$ and $u_j = zw_j^2$ for some $w_j \in R^\times$, $j = r+1, \ldots, n$. Thus, $[\beta]_{\mathcal{C}} = \operatorname{diag}(u_1, \ldots, u_r) \oplus z \operatorname{diag}(w_{r+1}, \ldots, w_n)$ which is cogredient to $I_r \oplus zI_{n-r}$. If $n - r$ is even, Lemma 2.2.1 implies that $[\beta]_{\mathcal{C}}$ is cogredient to I_n. If $n - r$ is odd, then $n - r - 1$ is even and so $[\beta]_{\mathcal{C}}$ is cogredient to $I_{n-1} \oplus (z)$ by the same lemma. Note that I_n and $I_{n-1} \oplus (z)$ are not cogredient since z is non-square. We record this result in the next theorem.

Theorem 2.2.3. *Let z be a non-square unit in R. Then $[\beta]_{\mathcal{C}}$ is cogredient to either I_n or $I_{n-1} \oplus (z)$.*

The next lemma follows by a simple calculation.

Lemma 2.2.4. *Let z be a non-square unit in R and ν a positive integer. Write $H_{2\nu} = \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix}$.*

(1) If $-1 \in (R^\times)^2$*, then* I_ν *is cogredient to* $H_{2\nu}$ *and* $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$.

(2) If $-1 \notin (R^\times)^2$*, then* $I_\nu \oplus zI_\nu$ *is cogredient to* $H_{2\nu}$ *and* $I_2 \approx \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$.

Proof. First we observe that if $-1 = u^2$ for some unit u, then

$$\begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

However, if -1 is non-square, then $-1 = zc^2$ for some unit $c \in R$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -zc^2 \end{pmatrix} = I_2.$$

Next, a simple calculation with $P = \frac{1}{2} \begin{pmatrix} I_\nu & -I_\nu \\ I_\nu & I_\nu \end{pmatrix}$ shows that $L = 2 \begin{pmatrix} I_\nu & 0 \\ 0 & -I_\nu \end{pmatrix}$ is cogredient to $H_{2\nu}$. Clearly, if -1 is square, L is cogredient to $I_{2\nu}$. Assume that -1 is non-square. By Proposition 2.1.2 (2), $-1 = zc^2$ for some unit c which also implies that 2 or -2 must be a square unit. If 2 is a square, then

$$L \approx I_\nu \oplus (-I_\nu) \approx I_\nu \oplus zc^2 I_\nu \approx I_\nu \oplus zI_\nu.$$

Similarly, if -2 is a square, then

$$L \approx (-I_\nu) \oplus I_\nu \approx zc^2 I_\nu \oplus I_\nu \approx I_\nu \oplus zI_\nu.$$

Therefore, $I_\nu \oplus zI_\nu$ is cogredient to $H_{2\nu}$. $\square$

Next, we apply Lemmas 2.2.1 and 2.2.4 in the following calculations. We distinguish three cases. Let z be a non-square unit and ν a positive integer.

1. Assume that -1 is square. Then

(a) $I_{2\nu} \approx H_{2\nu}$ and $I_{2\nu+1} \approx H_{2\nu} \oplus (1)$.

(b) $I_{2\nu} \oplus (z) \approx H_{2\nu} \oplus (z)$ and $I_{2\nu-1} \oplus (z) \approx I_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \approx H_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$.

2. Assume that -1 is non-square and ν is even. Then

(a) $I_{2\nu} \approx I_\nu \oplus I_\nu \approx I_\nu \oplus zI_\nu \approx H_{2\nu}$ and $I_{2\nu+1} \approx I_\nu \oplus I_\nu \oplus (1) \approx I_\nu \oplus zI_\nu \oplus (1) \approx H_{2\nu} \oplus (1)$.

(b) $I_{2\nu} \oplus (z) \approx I_\nu \oplus I_\nu \oplus (z) \approx I_\nu \oplus zI_\nu \oplus (z) \approx H_{2\nu} \oplus (z)$ and

$$\begin{aligned} I_{2\nu-1} \oplus (z) &\approx I_{\nu-2} \oplus I_{\nu-2} \oplus I_3 \oplus (z) \approx I_{\nu-2} \oplus zI_{\nu-2} \oplus I_3 \oplus (z) \\ &\approx I_{\nu-1} \oplus zI_{\nu-1} \oplus I_2 \approx H_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}. \end{aligned}$$

3. Assume that -1 is non-square and ν is odd. Then

(a) $I_{2\nu} \approx I_{\nu-1} \oplus I_{\nu-1} \oplus I_2 \approx I_{\nu-1} \oplus zI_{\nu-1} \oplus I_2 \approx H_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$ and $I_{2\nu+1} \approx I_{\nu-1} \oplus I_{\nu-1} \oplus I_2 \oplus (1) \approx I_{\nu-1} \oplus zI_{\nu-1} \oplus zI_2 \oplus (1) \approx I_\nu \oplus zI_\nu \oplus (z) \approx H_{2\nu} \oplus (z)$.

(b) $I_{2\nu} \oplus (z) \approx I_{\nu-1} \oplus I_{\nu-1} \oplus I_2 \oplus (z) \approx I_{\nu-1} \oplus zI_{\nu-1} \oplus I_2 \oplus (z) \approx I_\nu \oplus zI_\nu \oplus (1) \approx H_{2\nu} \oplus (1)$ and $I_{2\nu-1} \oplus (z) \approx I_{\nu-1} \oplus I_{\nu-1} \oplus (1) \oplus (z) \approx I_{\nu-1} \oplus I_{\nu-1} \oplus (1) \oplus (z) \approx I_\nu \oplus zI_\nu \approx H_{2\nu}$.

This proves a structure theorem of an orthogonal space over a finite local ring of odd characteristic.

Theorem 2.2.5. *Let R be a finite local ring of odd characteristic and let (V, β) be an orthogonal space where V is a free R-module of rank $n \geq 2$. Then there exists a $\delta \in$*

$\{0,1,2\}$ *such that* $\nu = \dfrac{n-\delta}{2} \geq 1$ *and the associating matrix of* β *is cogredient to*

$$S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I_\nu & \\ I_\nu & 0 & \\ & & \Delta \end{pmatrix},$$

where

$$\Delta = \begin{cases} \varnothing(\textit{disappear}) & \textit{if } \delta = 0, \\ (1) \textit{ or } (z) & \textit{if } \delta = 1, \\ \operatorname{diag}(1, -z) & \textit{if } \delta = 2, \end{cases}$$

and z *is a fixed non-square unit of* R.

We call ν a **hyperbolic rank of** V. Hence, when we work on an orthogonal space (V, β) over a finite local ring, we may assume that the associating matrix of β is in the above shape.

2.3 Galois rings

Let r, n be positive integers and p a prime. Then there exists a monic polynomial $f(x)$ in $\mathbb{Z}_{p^n}[x]$ of degree r such that the reduction $\bar{f}(x)$ in $\mathbb{Z}_p[x]$ is irreducible. Consider the ring extension $\mathbb{Z}_{p^n}[x]/(f(x))$ of $\mathbb{Z}_{p^n}$. This is given by

$$\{a_0 + a_1 x + \cdots + a_{r-1}x^{r-1} + (f(x)) : a_i \in \mathbb{Z}_{p^n} \text{ for all } i \in \{0, 1, \ldots, r-1\}\}.$$

It is called a **Galois extension**.

Theorem 2.3.1 (Theorem 5.1.8 of [2]). *Up to isomorphism the Galois extension with parameters* r, n *and* p *is unique.*

Hence, we may denote $\mathbb{Z}_{p^n}[x]/(f(x))$ by $GR(p^n, r)$, and call it the **Galois ring**.

Remark. $GR(p^n, 1) = \mathbb{Z}_{p^n}$ and $GR(p, r) = \mathbb{F}_{p^r}$, the field of p^r elements.

We record some properties of $GR(p^n, r)$ in the next theorem.

Theorem 2.3.2 (Section 6.2 of [2]). *Let n, r be positive integers and p a prime. Let $f(x) \in \mathbb{Z}_{p^n}[x]$ be a monic polynomial of degree r be such that the reduction $\bar{f}(x)$ in $\mathbb{Z}_p[x]$ is irreducible. Let $R = \mathbb{Z}_{p^n}[x]$. Then*

1. *R is a finite local ring of order p^{nr} with maximal ideal $M = p\mathbb{Z}_{p^n}[x]/(f(x))$ and residue field $R/M \cong \mathbb{F}_{p^r}$. Moreover, the characteristic of R is p^n.*

2. $R^\times = \big\{a_0 + a_1 x + \cdots + a_{r-1}x^{r-1} + (f(x)) : a_0, a_1, \ldots, a_{r-1} \in \mathbb{Z}_{p^n}$ *and* $a_i \in \mathbb{Z}_{p^n}^\times$ *for some* $i \in \{0, 1, \ldots, r-1\}\big\}$.

CHAPTER III
ORTHOGONAL GRAPHS

In this chapter, we first study the orthogonal graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ when R is a finite local ring of odd characteristic because we know the stucture of the orthogonal space over a finite local ring from the previous chapter. We prove that this graph is vertex and arc transitive. We also obtain a classification for our graph to be strongly regular or to be a quasi-strongly regular. When R is a finite commutative ring, it is well known that R is a product of finite local rings (Theorem 8.7 of [1]). We show how to use the decomposition of finite commutative rings into local rings and study basic properties of the orthogonal graphs.

3.1 Definitions and examples

Let R be a commutative ring and let (V, β) be an orthogonal space, where V is a free R-module of rank $n \geq 2$. A vector $\vec{x}$ in V is said to be **unimodular** if there is an f in $\hom_R(V, R)$ with $f(\vec{x}) = 1$; equivalently, if $\vec{x} = \alpha_1\vec{b}_1 + \ldots + \alpha_n\vec{b}_n$, where $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for V, then the ideal $(\alpha_1, \ldots, \alpha_n) = R$. If $\vec{x}$ is unimodular, then the **line** $R\vec{x}$ is a free R-direct summand of rank one. Moreover, it is easy to see that if $\vec{x}$ and $\vec{y}$ are unimodular vectors in V, then $R\vec{x} = R\vec{y}$ if and only if $\vec{x} = \lambda\vec{y}$ for some $\lambda \in R^\times$.

A **hyperbolic pair** $\{\vec{x}, \vec{y}\}$ is a pair of unimodular vectors in V with the property that $\beta(\vec{x}, \vec{x}) = \beta(\vec{y}, \vec{y}) = 0$ and $\beta(\vec{x}, \vec{y}) = 1$. The module $H = R\vec{x} \oplus R\vec{y}$ is called a **hyperbolic plane**. An R-module automorphism σ on V is an **isometry** on V if $\beta(\sigma(\vec{x}), \sigma(\vec{y})) = \beta(\vec{x}, \vec{y})$ for all $\vec{x}, \vec{y} \in V$. The group of isometries on V is called the **orthogonal group of** (V, β) **over** R and denoted by $\mathrm{O}_R(V)$.

Define the graph $\mathscr{G}_{\mathrm{O}_R(V)}$ whose vertex set $\mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V)})$ is the set of lines

$$\{R\vec{x} : \vec{x} \text{ is a unimodular vector in } V \text{ and } \beta(\vec{x}, \vec{x}) = 0\}$$

and its adjacency condition is given by

$$R\vec{x} \text{ is adjacent to } R\vec{y} \iff \beta(\vec{x}, \vec{y}) \in R^{\times} (\text{or equivalently}, \beta(\vec{x}, \vec{y}) = 1).$$

That is, $R\vec{x}$ is adjacent to $R\vec{y}$ if and only if $\{\vec{x}, \vec{y}\}$ is a hyperbolic pair. We call $\mathscr{G}_{\mathrm{O}_R(V)}$ the **orthogonal graph of** (V, β) **over** R.

To see that this adjacency condition is well defined, let $\vec{x}_1, \vec{x}_2, \vec{y}_1$ and $\vec{y}_2$ be unimodular vectors in V. Assume that $R\vec{x}_1 = R\vec{x}_2$ and $R\vec{y}_1 = R\vec{y}_2$. Then $\vec{x}_1 = \lambda\vec{y}_1$ and $\vec{x}_2 = \lambda'\vec{y}_2$ for some $\lambda, \lambda' \in R^{\times}$. Thus, we have

$$\beta(\vec{x}_1, \vec{y}_1) \in R^{\times} \Leftrightarrow \beta(\lambda\vec{x}_2, \lambda'\vec{y}_2) \in R^{\times} \Leftrightarrow \lambda\lambda'\beta(\vec{x}_2, \vec{y}_2) \in R^{\times} \Leftrightarrow \beta(\vec{x}_2, \vec{y}_2) \in R^{\times}.$$

Example 3.1.1. [5, 12] Let p be an odd prime number and let R be the ring of integers modulo p^n, $\mathbb{Z}_{p^n}$, or the field of p^n elements $\mathbb{F}_{p^n}$, or Galois ring $GR(p^n, r)$, where $n, r \in \mathbb{N}$. For $\nu \geq 1$ and $\delta = 0, 1$ or 2, let V_δ denote the set of $2\nu + \delta$- tuples $(a_1, \ldots, a_{2\nu+\delta})$ of elements in R. Define $\beta : V_\delta \times V_\delta \to R$ by

$$\begin{aligned} &\beta((a_1, a_2, \ldots, a_{2\nu+\delta}), (b_1, b_2, \ldots, b_{2\nu+\delta})) \\ &\qquad\qquad\qquad = (a_1, a_2, \ldots, a_{2\nu+\delta}) S_{2\nu+\delta,\Delta} (b_1, b_2, \ldots, b_{2\nu+\delta})^t, \end{aligned}$$

for all $(a_1, a_2, \ldots, a_{2\nu+\delta}), (b_1, b_2, \ldots, b_{2\nu+\delta}) \in V_\delta$, where

$$S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & \Delta \end{pmatrix}, \quad \Delta = \begin{cases} \varnothing(\text{disappear}) & \text{if } \delta = 0, \\ (1) \text{ or } (z) & \text{if } \delta = 1, \\ \operatorname{diag}(1, -z) & \text{if } \delta = 2, \end{cases}$$

and z is a fixed non-square element of $R^{\times}$ (as in Theorem 2.3.4). Then (V_δ, β) is an orthogonal space, and unimodular vectors in V_δ are those $(a_1, \ldots, a_{2\nu+\delta})$ of elements in R such that $a_i \in R^\times$ for some $i \in \{1, 2, \ldots, 2\nu + \delta\}$ because if all $a_i \in M$, then the ideal $(a_1, a_2, \ldots, a_{2\nu+\delta}) \subseteq M$ where M is the unique maximal ideal of R.

3.2 Vertex transitivity and arc transitivity

Let R be a finite local ring of odd characteristic with unique maximal ideal M and let (V_δ, β) be an orthogonal space of rank $2\nu + \delta$, where $\nu \geq 1, \delta \in \{0, 1, 2\}$. By Theorem 2.2.5, V_δ possesses a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{2\nu+\delta}\}$ in which $[\beta]_\mathcal{B} = S_{2\nu+\delta,\Delta}$. Therefore, if $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ and $\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2 + \cdots + y_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ are vectors in V, then

$$\begin{aligned}\beta(\vec{x}, \vec{y}) &= \begin{pmatrix} x_1 & x_2 & \cdots & x_{2\nu+\delta} \end{pmatrix} S_{2\nu+\delta,\Delta} \begin{pmatrix} y_1 & y_2 & \cdots & y_{2\nu+\delta} \end{pmatrix}^T \\ &= \begin{cases} \sum\limits_{i=1}^{\nu} (x_i y_{\nu+i} + x_{\nu+i} y_i) & \text{if } \delta = 0, \\ \sum\limits_{i=1}^{\nu} (x_i y_{\nu+i} + x_{\nu+i} y_i) + x_{2\nu+1} y_{2\nu+1} \Delta, & \text{if } \delta = 1, \\ \sum\limits_{i=1}^{\nu} (x_i y_{\nu+i} + x_{\nu+i} y_i) + x_{2\nu+1} y_{2\nu+1} - z x_{2\nu+2} y_{2\nu+2}, & \text{if } \delta = 2, \end{cases}\end{aligned}$$

where $\Delta = 1$ or z, for some non-square unit z. For convenience, we write $\tilde{x}_\delta = \varnothing$ (disappear), $-\frac{1}{2}\Delta x_{2\nu+1}^2$ or $-\frac{1}{2}(x_{2\nu+1}^2 - z x_{2\nu+2}^2)$ and $\tilde{x}\tilde{y}_\delta = \varnothing$ (disappear), $x_{2\nu+1} y_{2\nu+1} \Delta$ or $x_{2\nu+1} y_{2\nu+1} - z x_{2\nu+2} y_{2\nu+2}$ according to $\delta = 0, 1$ or 2, respectively.

We also have a criterion for determining whether a vector $\vec{x}$ in V is unimodular. It proof is as explained at the end of Example 3.1.1.

Theorem 3.2.1 (Theorem 2.2 of [16]). *An $\vec{x} = x_1\vec{b}_1 + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ in V is unimodular if and only if x_i is a unit of R for some $i \in \{1, \ldots, 2\nu + \delta\}$.*

Lemma 3.2.2. *Under the above set up, if $R\vec{x} \in \mathcal{V}(\mathscr{G}_{O_R(V_\delta)})$, then x_i is a unit of R for some $i \in \{1, \ldots, 2\nu\}$.*

Proof. Theorem 3.2.1 gives the result when $\delta = 0$. If $\delta = 1$ and $x_1, \ldots, x_{2\nu} \in M$, then $0 = \beta(\vec{x}, \vec{x}) = 2(x_1 x_{\nu+1} + \cdots + x_\nu x_{2\nu}) + \Delta x_{2\nu+1}^2$, where $\Delta = 1$ or z, implies

$x_{2\nu+1}$ is also in M which contradicts unimodularity of $\vec{x}$. Now assume $\delta = 2$ and $x_1, \ldots, x_{2\nu} \in M$. Again, $0 = \beta(\vec{x}, \vec{x}) = 2(x_1x_{\nu+1} + \cdots + x_\nu x_{2\nu}) + x_{2\nu+1}^2 - zx_{2\nu+2}^2$ implies $x_{2\nu+1}^2 - zx_{2\nu+2}^2 = m$ for some $m \in M$. This forces that $x_{2\nu+1}$ and $x_{2\nu+2}$ are units. By Proposition 2.1.2 (3), there exists $c \in R^\times$ such that $zx_{2\nu+2}^2 = x_{2\nu+1}^2 - m = c^2(x_{2\nu+1}^2 - m + m) = c^2x_{2\nu+1}^2$, contradicting z is non-square. This completes the proof of the lemma. $\square$

Let G and H be graphs. A function σ from G to H is a **homomorphism** from G to H if $\sigma(g_1)$ and $\sigma(g_2)$ are adjacent in H whenever g_1 and g_2 are adjacent in G. It is called an **isomorphism** if it is a bijection and σ^{-1} is a homomorphism from H to G. An isomorphism on G is also called an **automorphism**. The set of all automorphisms of a graph G is denoted by $\mathrm{Aut}(G)$. It is a group under composition, called the **automorphism group of a graph** G.

A graph G is **vertex transitive** if its automorphism group acts transitively on the vertex set. That is, for any two vertices of G, there is an automorphism carrying one to the other. An **arc** in G is an ordered pair of adjacent vertices, and G is **arc transitive** if its automorphism group acts transitively on its arcs. Note that an arc transitive graph is necessarily vertex transitive. More on transitive graphs can be found in Chapter 3 of Godsil's book [4]. We have the next result.

Lemma 3.2.3 (Theorem 4.2 of [14])**.** *Let R be a local ring having 2 a unit and let (V, β) be a free orthogonal space of hyperbolic rank ≥ 1. Then, $\mathrm{O}_R(V)$ acts transitively on unimodular vectors of the same norm.*

Observe that for any automorphism ϕ of V_δ, we have the induced automorphism T_ϕ on the vertex set of the orthogonal graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ given by

$$T_\phi : R\vec{x} \mapsto R\phi(\vec{x})$$

for all unimodular vectors $\vec{x} \in V_\delta$ and $\beta(\vec{x}, \vec{x}) = 0$. Let $\vec{x}$ and $\vec{y}$ be unimodular vectors in V_δ and $\beta(\vec{x}, \vec{x}) = \beta(\vec{y}, \vec{y}) = 0$. Since our ring is of odd characteristic, 2 is a unit in R. By Lemma 3.2.3, there is an automorphism $\phi \in \mathrm{O}_R(V_\delta)$ such that

$\phi(\vec{x}) = \vec{y}$. Thus, we have $T_\phi \in \operatorname{Aut} \mathscr{G}_{\mathrm{O}_R(V_\delta)}$ and $T_\phi : R\vec{x} \mapsto R\phi(\vec{x}) = R\vec{y}$. This proves vertex transitivity.

For arc transitivity, we shall need the following lemma.

Lemma 3.2.4. *Let R be a local ring having 2 a unit and let (V_δ, β) be a orthogonal space of hyperbolic rank ≥ 1. If $\{\vec{a}, \vec{b}\}$ and $\{\vec{c}, \vec{d}\}$ are hyberbolic pairs of unimodular vectors, then there exists an automorphism in $\mathrm{O}_R(V_\delta)$ which carries $\vec{a}$ to $\vec{c}$ and $\vec{b}$ to $\vec{d}$.*

Proof. We shall show that any ordered hyperbolic pair can be carried by an automorphism in $\mathrm{O}_R(V_\delta)$ to $\vec{e}_1$ and $\vec{e}_{\nu+1}$ and the result follows. Let $\{\vec{a}, \vec{b}\}$ be a hyberbolic pair of unimodular vectors. By Lemma 3.2.3, there exists an automorphism $\sigma \in \mathrm{O}_R(V)$ such that $\sigma(\vec{a}) = \vec{e}_1$. Since

$$1 = \beta(\vec{a}, \vec{b}) = \beta(\sigma(\vec{a}), \sigma(\vec{b})) = \beta(\vec{e}_1, \sigma(\vec{b})),$$

$\sigma(\vec{b})$ is of the form $(x_1, \ldots, x_\nu, 1, x_{\nu+2}, \ldots, x_{2\nu+2})$ for some $x_i \in R$. To finish the lemma, it suffices to show that there exists an automorphism in $\mathrm{O}_R(V_\delta)$ which leaves $\vec{e}_1$ invariant and carries $\sigma(\vec{b})$ to $\vec{e}_{\nu+1}$. If $\delta = 2$, then the map τ in $\mathrm{O}_R(V)$ given by

$$[\tau]_{\mathcal{B}} = \left(\begin{array}{cccc|cccccc}
1 & x_{\nu+2} & \cdots & x_{2\nu} & x_1 & x_2 & \cdots & x_\nu & x_{2\nu+1} & zx_{2\nu+2} \\
 & 1 & & & -x_2 & & & & & \\
 & & \ddots & & \vdots & & & & & \\
 & & & 1 & -x_\nu & & & & & \\
\hline
 & & & & 1 & & & & & \\
 & & & & -x_{\nu+2} & 1 & & & & \\
 & & & & \vdots & & \ddots & & & \\
 & & & & -x_{2\nu} & & & 1 & & \\
 & & & & -x_{2\nu+1} & & & & 1 & \\
 & & & & -x_{2\nu+2} & & & & & 1
\end{array}\right)$$

is a desired automorphism. The cases of $\delta = 0$ and $\delta = 1$ can be done in a similar way. $\square$

Let $R\vec{x}_1, R\vec{x}_2, R\vec{y}_1, R\vec{y}_2 \in \mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})$ be such that $R\vec{x}_1$ is adjacent to $R\vec{y}_1$ and $R\vec{x}_2$ is adjacent to $R\vec{y}_2$. That is, $\{\vec{x}_1, \vec{y}_1\}$ and $\{\vec{x}_2, \vec{y}_2\}$ are hyperbolic pairs. By Lemma 3.2.4, there exists an automorphism $\phi \in \mathrm{O}_R(V_\delta)$ such that $\phi(\vec{x}_1) = \vec{x}_2$ and $\phi(\vec{y}_1) = \vec{y}_2$. Hence, $T_\phi \in \mathrm{Aut}\,\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ carries $R\vec{x}_1$ to $R\vec{x}_2$ and $R\vec{y}_1$ to $R\vec{y}_2$. Therefore, we have shown:

Theorem 3.2.5. *Let R be a local ring of odd characteristic and let (V, β) be an orthogonal space of rank $n \geq 2$. Then the orthogonal graph $\mathscr{G}_{\mathrm{O}_R(V)}$ is vertex and arc transitive.*

3.3 Strong regularity

The number of vertices and degree of regularity of an orthogonal graph over finite local ring of odd characterisitc are presented in the next theorem.

Theorem 3.3.1. *Let R be a finite local ring of odd characteristic and let (V_δ, β) be an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 1, \delta = 0, 1$ or 2. Then $\mathscr{G}_{\mathrm{O}_R(V)}$ is $|R|^{2\nu-2+\delta}$-regular on*

$$\frac{(|R|^\nu - |M|^\nu)(|R|^{\nu+\delta-1} + |M|^{\nu+\delta-1})}{|R| - |M|}$$

many vertices.

Proof. Since an orthogonal graph is vertex transitive, it is regular. For its degree of regularity, it suffices to calculate the number of neighbors for only one vertex. Observe that each vertex adjacent to $R\vec{b}_1$ is of the form $R(x_1\vec{b}_1 + \cdots + x_\nu\vec{b}_\nu + \vec{b}_{\nu+1} + x_{\nu+2}\vec{b}_{\nu+2} + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta})$, where $x_i \in F$ for all i and $x_1 = -x_2x_{\nu+2} - \cdots - x_\nu x_{2\nu} + \tilde{x}_\delta$. Thus, the degree of regularity is $|R|^{2\nu-2+\delta}$.

Next, we compute the number of vertices. Let $R\vec{x}$ be a vertex of the graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$. Write $\vec{x} = x_1\vec{b}_1 + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta}$. By Lemma 3.2.2, we may assume that one of $x_1, \ldots, x_{2\nu}$ is equal to 1. For $1 \leq i \leq 2\nu$, let Ω_i be the set of vertices of the form

$$R(x_1\vec{b}_1 + \cdots + x_{i-1}\vec{b}_{i-1} + \vec{b}_i + x_{i+1}\vec{b}_{i+1} + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta})$$

where $x_1, \ldots, x_{i-1} \in M$ and $x_{i+1}, \ldots, x_{2\nu+\delta} \in R$. Then $\{\Omega_1, \ldots, \Omega_{2\nu}\}$ is a partition of the vertex set $\mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})$. For $1 \leq i \leq \nu$, any vector $R(x_1\vec{b}_1 + \cdots + x_{i-1}\vec{b}_{i-1} + \vec{b}_i + x_{i+1}\vec{b}_{i+1} + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta})$ of Ω_i satisfies

$$x_{\nu+i} = -x_1x_{\nu+1} - \ldots - x_{i-1}x_{\nu+i-1} - x_{i+1}x_{\nu+i+1} - \ldots - x_\nu x_{2\nu} + \tilde{x}_\delta.$$

Then $|\Omega_i| = |M|^{i-1}|R|^{2\nu+\delta-i-1}$ for all $1 \leq i \leq \nu$. Also, for $1 \leq i \leq \nu$, any vector $x_1\vec{b}_1 + \cdots + x_{\nu+i-1}\vec{b}_{\nu+i-1} + \vec{b}_{\nu+i} + x_{\nu+i+1}\vec{b}_{\nu+i+1} + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ of $\Omega_{\nu+i}$ satisfies

$$x_i = -x_1x_{\nu+1} - \ldots - x_{i-1}x_{\nu+i-1} - x_{i+1}x_{\nu+i+1} - \ldots - x_\nu x_{2\nu} + \tilde{x}_\delta.$$

Note that $x_{2\nu+1} \in M$ if $\delta = 1$, and $x_{2\nu+1}, x_{2\nu+2} \in M$ if $\delta = 2$. Thus,

$$|\Omega_{\nu+i}| = |M|^{\nu+i-2+\delta}|R|^{\nu-i} \text{ for all } 1 \leq i \leq \nu.$$

Therefore, the number of vertices is the sum

$$\sum_{i=1}^{\nu}(|M|^{i-1}|R|^{2\nu+\delta-i-1}) + \sum_{i=1}^{\nu}(|M|^{\nu+i-2+\delta}|R|^{\nu-i}) = \frac{(|R|^\nu - |M|^\nu)(|R|^{\nu+\delta-1} + |M|^{\nu+\delta-1})}{|R| - |M|}$$

as desired. □

Arc transitivity also implies the next lemma.

Lemma 3.3.2. *Let R be a finite local ring of odd characteristic and let (V_δ, β) be an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 1, \delta = 0, 1$ or 2. Then*

(1) For any two adjacent vertices $R\vec{x}$ and $R\vec{y}$ of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$, the number of common neighbors of $R\vec{x}$ and $R\vec{y}$ is equal to the number of common neighbors of $R\vec{e}_1$ and $R\vec{e}_{\nu+1}$.

(2) For any two non-adjacent vertices $R\vec{x}$ and $R\vec{y}$ of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$, there is a vertex $R\vec{c}$ not adjacent to $R\vec{e}_1$ (i.e., $\beta(\vec{e}_1, \vec{c}) \in M$) such that the number of common neighbors of $R\vec{x}$ and $R\vec{y}$ is equal to the number of common neighbors of $R\vec{e}_1$ and $R\vec{c}$.

Proof. Let $R\vec{a}$, $R\vec{b}$ be two adjacent vertices. Since $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is arc transitive by Theorem 3.2.5, there exists an automorphism f on $\mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})$ such that $f(R\vec{e}_1) = R\vec{a}$ and $f(R\vec{e}_{\nu+1}) = R\vec{b}$. Then f maps each common neighbors of $R\vec{e}_1$ and $R\vec{e}_{\nu+1}$ to distinct common neighbors of $R\vec{a}$ and $R\vec{b}$. Now, let $R\vec{c'}$ be a common neighbors of $R\vec{a}$ and $R\vec{b}$. Since f is a graph automorphism, there is a vertex $R\vec{c}$ such that $f(R\vec{c}) = R\vec{c'}$ and $R\vec{c}$ is a common neighbor of $R\vec{e}_1$ and $R\vec{e}_{\nu+1}$. Hence, we have (1).

For (2), by using the vertex transitivity of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$, we can obtain the result by using the above technique. $\square$

To count the number of common neighbors of two distinct vertices, we need the next propositon.

Proposition 3.3.3. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and z a non-square unit. Then the number of the ordered pairs $(a, b) \in R \times R$ with $a^2 - zb^2 \in M$ is $|M|^2$.*

Proof. Let $(a, b) \in R \times R$ be such that $a^2 - zb^2 = m \in M$. Suppose that $a \in R^\times$ or $b \in R^\times$. Then, by Proposition 2.1.2 (3), $a^2 - m = zb^2 \in z(R^\times)^2 = R^\times \smallsetminus (R^\times)^2$ and so there exists a unit c in R such that $a^2 - m = c^2(a^2 - m + m) = c^2a^2 \in (R^\times)^2$ which contradicts c is non-square. Thus, $a, b \in M$ and hence the number is $|M|^2$. $\square$

A **complete graph** with parameters (n, k) is a k-regular graph on n vertices. A **strongly regular graph** with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that for every two adjacent vertices have λ common neighbors, and for every two non-adjacent vertices have μ common neighbors.

Gu and Wan [5] showed that:

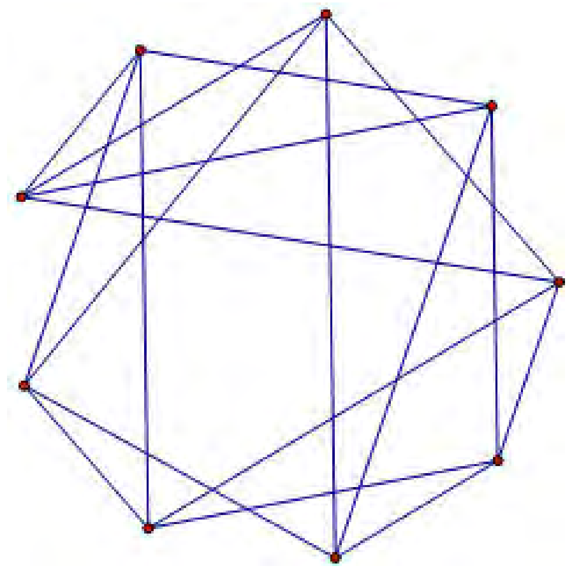
Theorem 3.3.4 (Theorem 2.1 of [5]). *Let p be an odd prime number and let $n, \nu \in \mathbb{N}$. Let R be a commutative ring of odd characteristic and let (V_δ, β) be an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 1, \delta = 0, 1$ or 2.*

1. *If $R = \mathbb{F}_{p^n}$ and $\nu = 1$, then the $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is a complete graph with parameters $(p^{n\delta}+1, p^{n\delta})$.*

2. *If $R = \mathbb{F}_{p^n}$ and $\nu \geq 2$, then the $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is a strongly regular graph with parameters*

$$\Big(\frac{(p^{n\nu}-1)(p^{n(\nu+\delta-1)}+1)}{p^n-1}, p^{n(2\nu+\delta-2)},$$
$$p^{n(2\nu+\delta-2)} - p^{n(2\nu+\delta-3)} - p^{n(\nu-1)} + p^{n(\nu+\delta-2)}, p^{n(2\nu+\delta-2)} - p^{n(2\nu+\delta-3)}\Big).$$

Example 3.3.1. The following figure shows the orthogonal graph $\mathscr{G}_{\mathrm{O}_{\mathbb{F}_2(V_0)}}$, where V_0 is the orthogonal space of dimension $2 \cdot 2 + 0$. It is strongly regular with parameters $(9, 4, 1, 2)$.



When R is a finite local ring of odd characteristic, we have the following result.

Theorem 3.3.5. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and let (V_δ, β) be an orthogonal space of dimension $2+\delta$, where $\delta = 0, 1$ or 2. Then $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is strongly regular with parameters (n, k, λ, μ), where n and k are studied in Theorem 3.3.1, $\lambda = |R|^\delta - |M|^\delta$ and $\mu = \lceil \delta/2 \rceil R^\delta$.*

Proof. Here, $\nu = 1$. To determine λ, by Lemma 3.3.2, it suffices to count the common neighbors of $R\vec{e}_1$ and $R\vec{e}_2$. It is easy to see that we have no common neighbor when $\delta = 0$. If $\delta = 1$ and $\Delta = 1$ or z, then a common neighbor of $R\vec{e}_1$ and $R\vec{e}_2$ is of the form $R(1, -\frac{1}{2}\Delta a^2, a)$ for some $a \in R^\times$, so $\lambda = |R| - |M|$. If $\delta = 2$, then a common neighbor of $R\vec{e}_1$ and $R\vec{e}_2$ is of the form $R(1, -\frac{1}{2}(a^2 - zb^2), a, b)$ for some $a, b \in R$ such that $a^2 - zb^2 \in R^\times$, and thus $\lambda = |R|^2 - |M|^2$ by Proposition 3.3.3.

Next, let $R\vec{x}$ and $R\vec{y}$ be two non-adjacent vertices of the graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$. By Lemma 3.3.2, we may choose a vertex $R\vec{c}$ not adjacent to $R\vec{e}_1$ and count the common neighbors of $R\vec{e}_1$ and $R\vec{c}$. Again, it is clear that we have no common neighbor when $\delta = 0$. If $\delta = 1$, then a common neighbor of $R\vec{e}_1$ and $R\vec{c}$ is of the form $R(-\frac{1}{2}\Delta a^2, 1, a)$, where $a \in R$, so $\mu = |R|$. Finally, if $\delta = 2$, then by Proposition 2.1.2 (3), a common neighbor of $R\vec{e}_1$ and $R\vec{c}$ is of the form $R(-\frac{1}{2}(a^2 - zb^2), 1, a, b)$ for some $a, b \in R$, and hence $\mu = |R|^2$. □

A connected graph of valency k on n vertices is **quasi-strongly regular** of grade d with parameters $(n, k, \lambda; c_1, c_2, \ldots, c_d)$ if any two adjacent vertices have λ common neighbors, and any two non-adjacent vertices have c_i common neighbors for some i.

Li, Guo and Wang [12] had the next theorem for othogonal graphs over a Galois ring.

Theorem 3.3.6 (Theorems 3.3 and 3.4 of [12])**.** *Let p be an odd prime number and let n, ν and $r \in \mathbb{N}$. Let R be a commutative ring of odd characteristic and let (V_δ, β) be an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 1, \delta = 0, 1$ or 2.*

1. *If $R = \mathbb{Z}_{p^n}$ and $\nu = 1$, then the $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is strongly regular with parameters*

$$\left((p^\delta + 1)p^{(n-1)\delta}, p^{n\delta}, (p^\delta - 1)p^{\delta(n-1)}, \lceil \delta/2 \rceil\, p^{n\delta}\right).$$

2. *If $R = \mathbb{Z}_{p^n}$ and $\nu \geq 2$, then the $\mathscr{G}_{\mathrm{O}_{R(V_\delta)}}$ is quasi-strongly regular with parameters*

$$\Big(\frac{(p^\nu - 1)(p^{(\nu+\delta-1)} + 1)p^{(n-1)(2\nu-2+\delta)}}{p-1}, p^{n(2\nu-2+\delta)},$$
$$(p-1)(1 + (\delta - 1)p^{(1-\nu-\frac{\delta}{2})})p^{(2n\nu-2n-1+n\delta)}; (p-1)p^{(n(2\nu-2+\delta)-1)}, p^{n(2\nu-2+\delta)}\Big).$$

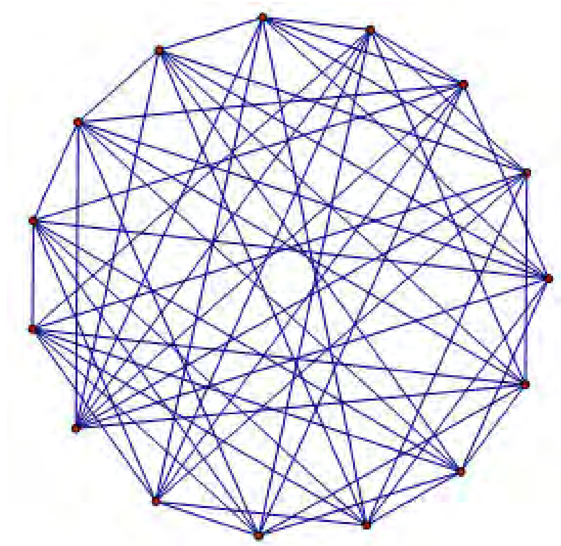
3. *If $R = GR(p^{mr}, r)$ and $\nu = 1$, then the $\mathscr{G}_{\mathrm{O}_{R(V_\delta)}}$ is strongly regular with parameters*

$$\Big(p^{m\delta} + 1)p^{m(r-1)\delta}, p^{mr\delta}, (p^{m\delta} - 1)p^{m\delta(r-1)}, \lceil \delta/2 \rceil\, p^{mr\delta}\Big).$$

4. *If $R = GR(p^{mr}, r)$ and $\nu \geq 2$, then the $\mathscr{G}_{\mathrm{O}_{R(V_\delta)}}$ is quasi-strongly regular with parameters*

$$\Big(\frac{(p^{m\nu} - 1)(p^{m(\nu+\delta-1)} + 1)p^{m(s-1)(2\nu-2+\delta)}}{p^m-1}, p^{mr(2\nu-2+\delta)},$$
$$(p^m-1)(1 + (\delta - 1)p^{m(1-\nu-\frac{\delta}{2})})p^{m(2r\nu-2r-1+r\delta)}; (p^m-1)p^{m(r(2\nu-2+\delta)-1)},$$
$$p^{mr(2\nu-2+\delta)}\Big).$$

Example 3.3.2. The following figure shows the orthogonal graph $\mathscr{G}_{\mathrm{O}_{\mathbb{Z}_2(V_1)}}$, where V_1 is the orthogonal space of dimension $2 \cdot 2 + 1$. which is quasi-strongly regular with parameters $(15, 8, 1; 4, 8)$.



We can prove more general results for a local ring as follows.

Theorem 3.3.7. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and let (V_δ, β) be an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 2, \delta = 0, 1$ or 2. Then $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is quasi-strongly regular with $(n, k, \lambda; c_1, c_2)$, where n, k are given by Theorem 3.3.1, $\lambda = (|R| - |M|)(|R|^{\nu+\frac{\delta}{2}-1} + (\delta - 1)|M|^{\nu+\frac{\delta}{2}-1})|R|^{\nu-2+\frac{\delta}{2}}$, $c_1 = (|R| - |M|)|R|^{2\nu-3+\delta}$ and $c_2 = |R|^{2\nu-2+\delta}$.*

Proof. Again, to determine λ, by Lemma 3.3.2, it suffices to count the common neighbors of $R\vec{e}_1$ and $R\vec{e}_2$. A common neighbor of $R\vec{e}_1$ and $R\vec{e}_2$ is of the form $R\vec{a} = R(1, a_2, \ldots, a_{2\nu+\delta})$, where $a_{\nu+1} \in R^\times$ and $a_{\nu+1} = -a_2a_{\nu+2} - \ldots - a_\nu a_{2\nu} + \tilde{a}_\delta$. Note that there exists a unit among $a_2, \ldots, a_\nu, \tilde{a}_\delta$. If a_i is a unit for some $i \in \{2, \ldots, \nu\}$, then

$$a_{\nu+i} = -a_i^{-1}(a_{\nu+1} + a_2a_{\nu+2} + \cdots + a_{i-1}a_{\nu+i-1} + a_{i+1}a_{\nu+i+1}, \ldots, a_\nu a_{2\nu} - \tilde{a}_\delta),$$

and so there are $(|R| - |M|)(|R|^{\nu-1} - |M|^{\nu-1})|R|^{\nu-2+\delta}$ possible vectors. Next, we assume that $a_i \in M$ for all $i \in \{2, \ldots, \nu\}$. This forces $\tilde{a}_\delta$ a unit. By Proposition 3.3.3, there are $(|R|^\delta - |M|^2)|M|^{\nu-1}|R|^{\nu-1}$ possible vectors. Hence,

$$\lambda = (|R| - |M|)(|R|^{\nu+\frac{\delta}{2}-1} + (\delta - 1)|M|^{\nu+\frac{\delta}{2}-1})|R|^{\nu-2+\frac{\delta}{2}}.$$

Now we compute c_1 and c_2. Let $R\vec{x}$ and $R\vec{y}$ be two non-adjacent vertices of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$. By Lemma 3.3.2, we may choose a vertex $R\vec{c}$ not adjacent to $R\vec{e_1}$ and count the common neighbors of $R\vec{e_1}$ and $R\vec{c}$. Since $\beta(\vec{e_1},\vec{c})$ is in M, $y_{\nu+1} \in M$ and so y_j is a unit for some $j \in \{1,2,\ldots,2\nu\} \smallsetminus \{\nu+1\}$ by Lemma 3.2.2. A common neighbor of $R\vec{e_1}$ and $R\vec{c}$ is of the form $R\vec{a} = R(a_1,\ldots,a_\nu,1,a_{\nu+2},\ldots,a_{2\nu+\delta})$ where $a_1 = -\sum\limits_{l=2}^{\nu} a_l a_{\nu+l} + \tilde{a}_\delta$ and also $(a_1 y_{\nu+1} + y_1) + \sum\limits_{l=2}^{\nu}(a_l y_{\nu+l} + a_{\nu+l} y_l) + (\tilde{a}\tilde{y})_\delta = r \in R^\times$. If $j \neq 1$, then we can assume that $2 \leq j \leq \nu$. Then

$$
\begin{aligned}
a_{\nu+j} = \frac{y_j^{-1}}{1 - y_j^{-1} a_j y_{\nu+1}} \Big(r + \Big(\sum_{l=2, l\neq j}^{\nu} a_l a_{\nu+l} - \tilde{a}_\delta \Big) y_{\nu+1} \\
- y_1 - \sum_{l=2, l\neq j}^{\nu} (a_l y_{\nu+l} + a_{\nu+l} y_l) - a_j y_{\nu+j} - (\tilde{a}\tilde{y})_\delta \Big),
\end{aligned}
$$

so we have $c_1 = (|R| - |M|)|R|^{2\nu-3+\delta}$. If $j = 1$ and $y_l \in M$ for all $l \in \{2,\ldots,2\nu\} \smallsetminus \{\nu+1\}$, then by Proposition 2.1.2 (3), $\tilde{y}_\delta$ must be in M, and so there are $|R|^{2\nu-2+\delta}$ possible vectors from the relation $a_1 = -\sum\limits_{l=2}^{\nu} a_l a_{\nu+l} + \tilde{a}_\delta$. $\square$

Remark. Gu and Wan [5] worked on the orthogonal graphs over finite fields. These graphs are either complete or strongly regular depending on the dimension of their othogonal spaces. More generally, our orthogonal graphs are either strongly regular or quasi-strongly regular and our proof uses combinatorial method which is similar to Li, Guo and Wang's over Galois rings [12].

3.4 Results over finite commutative rings

In the previous sections, we have studied orthogonal graphs over a finite local ring of odd characteristic. Now, we let R be a finite commutative ring of odd characteristic.

It is well known that:

Theorem 3.4.1 (Theorem 3.1.4 of [2]). *Let R be a finite commutative ring of odd characteristic. Then R is a product of finite local rings of odd characteristic.*

Write

$$R = R_1 \times R_2 \times \cdots \times R_t$$

as a direct product of finite local rings of odd characteristic R_i, $i = 1, 2, \ldots, t$. Consider $V_\delta = R^{2\nu+\delta}$, a free R-module of rank $2\nu + \delta$, where $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$. We have the canonical 1-1 correspondence

$$\vec{x} = (x_1, x_2, \ldots, x_{2\nu+\delta}) \overset{\varphi}{\mapsto} ((x_1^{(j)})_{j=1}^t, (x_2^{(j)})_{j=1}^t, \ldots, (x_{2\nu+\delta}^{(j)})_{j=1}^t).$$

Note that if $\vec{x}, \vec{y} \in V_\delta$, then this correspondence induces the orthogonal map β on V_δ by

$$\begin{aligned}\beta(\vec{x}, \vec{y}) &= \beta(((x_1^{(j)})_{j=1}^t, (x_2^{(j)})_{j=1}^t, \ldots, (x_{2\nu+\delta}^{(j)})_{j=1}^t), ((y_1^{(j)})_{j=1}^t, (y_2^{(j)})_{j=1}^t, \ldots, (y_{2\nu+\delta}^{(j)})_{j=1}^t)) \\ &= (\beta_1(\vec{x}^{(1)}, \vec{y}^{(1)}), \beta_2(\vec{x}^{(2)}, \vec{y}^{(2)}), \ldots, \beta_t(\vec{x}^{(t)}, \vec{y}^{(t)}))\end{aligned}$$

where $\vec{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots, x_{2\nu+\delta}^{(j)}) \in V_\delta^{(j)} := R_j^{(2\nu+\delta)}$ and $(V_\delta^{(j)}, \beta_j)$ is an orthogonal space of R_j of rank $2\nu + \delta$ associated with the matrix $S_{2\nu+\delta,\Delta_j}$ as in Theorem 2.2.5, for all $j \in \{1, 2, \ldots, t\}$. Since $R^\times = R_1^\times \times R_2^\times \times \cdots \times R_t^\times$, we have

$$\beta(\vec{x}, \vec{y}) \in R^\times \Leftrightarrow \beta_j(\vec{x}^{(j)}, \vec{y}^{(j)}) \in R_j^\times \text{ for all } j \in \{1, 2, \ldots, t\}.$$

Therefore, it follows that

$$\mathscr{G}_{\mathrm{O}_R(V_\delta)} \cong \mathscr{G}_{\mathrm{O}_{R_1}(V_\delta^{(1)})} \otimes \mathscr{G}_{\mathrm{O}_{R_2}(V_\delta^{(2)})} \otimes \cdots \otimes \mathscr{G}_{\mathrm{O}_{R_t}(V_\delta^{(t)})},$$

as a graph isomorphism. Here, for two graphs G and H, we define their **tensor product** $G \otimes H$ to be the graph with vertex set $\mathcal{V}(G) \times \mathcal{V}(H)$, where (u, v) is adjacent to (u', v') if and only if u is adjacent to u' and v is adjacent to v'.

From Theorem 3.3.1 and the above isomorphism, we have the number of vertices of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is equal to

$$|\mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})| = \prod_{j=1}^{t} |\mathcal{V}(\mathscr{G}_{\mathrm{O}_{R_j}(V_\delta)})| = \prod_{j=1}^{t} \frac{(|R_j|^\nu - |M_j|^\nu)(|R_j|^{\nu+\delta-1} + |M_j|^{\nu+\delta-1})}{|R_j| - |M_j|}$$

and $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is regular of degree $|R_1|^{2\nu-2+\delta}|R_2|^{2\nu-2+\delta} \cdots |R_t|^{2\nu-2+\delta} = |R|^{2\nu-2+\delta}$. Moreover, every two adjacent vertices of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ has $|R|^\delta - |M|^\delta$ common neighbors if $\nu = 1$ by Theorem 3.3.5 or $(|R|-|M|)(|R|^{\nu+\frac{\delta}{2}-1}+(\delta-1)|M|^{\nu+\frac{\delta}{2}-1})|R|^{\nu-2+\frac{\delta}{2}}$ common neighbors if $\nu \geq 2$ by Theorem 3.3.7. We record these results in the next theorem.

Theorem 3.4.2. *Let R be a finite commutative ring and (V_δ, β) be the induced orthogonal space of rank $2\nu + \delta$, $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$, discussed above.*

(1) The orthogonal graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is a $|R|^{2\nu-2+\delta}$-regular and isomorphic to the graph

$$\mathscr{G}_{\mathrm{O}_{R_1}(V_\delta^{(1)})} \otimes \mathscr{G}_{\mathrm{O}_{R_2}(V_\delta^{(2)})} \otimes \cdots \otimes \mathscr{G}_{\mathrm{O}_{R_t}(V_\delta^{(t)})}.$$

(2) Every two adjacent vertices of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ has $\prod_{j=1}^{t} |R_j|^\delta - |M_j|^\delta$ common neighbors if $\nu = 1$ and $\prod_{j=1}^{t}(|R_j| - |M_j|)(|R_j|^{\nu+\frac{\delta}{2}-1} + (\delta - 1)|M_j|^{\nu+\frac{\delta}{2}-1})|R_j|^{\nu-2+\frac{\delta}{2}}$ common neighbors if $\nu \geq 2$.

Let G and H be two graphs. If ρ and τ are automorphisms of G and of H, respectively, then it is easy to see that the map

$$\rho : (g, h) \mapsto (\rho(g), \tau(h)) \text{ for all } g \in \mathcal{V}(G), h \in \mathcal{V}(H),$$

is an automorphism of $G \otimes H$. Thus, we have shown:

Theorem 3.4.3. *For two graphs G and H, $\mathrm{Aut}(G) \times \mathrm{Aut}(H) \subseteq \mathrm{Aut}(G \otimes H)$.*

Recall from Theorem 3.2.5 that for each i, we have $\mathscr{G}_{\mathrm{O}_{R_i}(V_\delta^{(i)})}$ is vertex and arc transitive. Then it follows from Theorem 3.4.2 (1) and Theorem 3.4.3 that

$$\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_{R_1}(V_\delta^{(1)})}) \times \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_{R_2}(V_\delta^{(2)})}) \times \cdots \times \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_{R_t}(V_\delta^{(t)})}) \subseteq \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_R(V_\delta)}).$$

Thus, $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is also vertex and arc transitive. Hence, we have proved:

Theorem 3.4.4. *If (V,β) is an orthogonal space over a finite commutative ring R, then the orthogonal graph $\mathscr{G}_{\mathrm{O}_R(V)}$ is vertex and arc transitive.*

CHAPTER IV

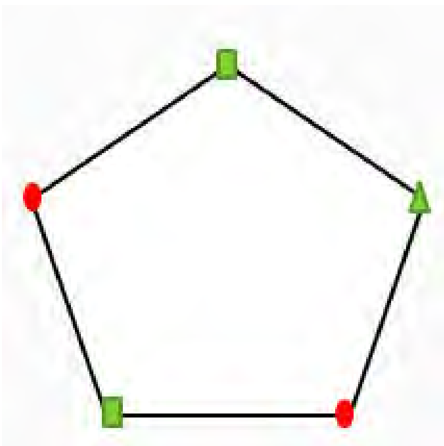
CHROMATIC NUMBERS AND AUTOMORPHISM GROUPS

This final chapter contains results on chromatic numbers and fractional coloring of a graphs. Moreover, we determine automorphism groups of our orthogonal graphs over a finite local ring of odd characteristic using the results of orthogonal graph over a finite field [5].

4.1 Chromatic numbers

The **chromatic number of a graph** G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. The chromotic number of a graph G is commonly denoted by $\chi(G)$.

Example 4.1.1. The chromatic number of this cycle of length five below is 3.



If k is a finite field and V_δ is an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$, then Gu and Wan [5] computed the chromatic number of $\mathscr{G}_{\mathrm{O}_k(V_\delta)}$ in the following Proposition.

Proposition 4.1.1 (Propostion 2.4 and Theorem 2.5 of [5]). *If k is a field of q elements and V_δ is the orthogonal space of dimension $2\nu+\delta$, there exist subsets $Y_1,\dots,Y_\kappa$ of $\mathcal{V}(\mathscr{G}_{\mathrm{O}_k}(V_\delta))$, where $\kappa = q^{\nu+\delta-1}+1$, such that*

$$\mathcal{V}(\mathscr{G}_{\mathrm{O}_k}(V_\delta)) = Y_1 \cup \cdots \cup Y_\kappa$$

and $Y_i \cap Y_j = \varnothing$ for all $i \neq j$, and $|Y_i| = \dfrac{q^\nu - 1}{q-1}$ for all $i \in \{1,\dots,\kappa\}$, and there exists no edge of $\mathscr{G}_{\mathrm{O}_{k(V_\delta)}}$ joining two vertices of the same subset. Moreover, $\chi(\mathscr{G}_{\mathrm{O}_{k(V_\delta)}}) = q^{\nu+\delta-1}+1$.

Example 4.1.2. From the above proposition, the orthogonal graph $\mathscr{G}_{\mathrm{O}_{\mathbb{F}_2(V_0)}}$, where V_0 is the orthogonal space of dimension $2\cdot 2+0$ has the chromatic number equal to $2^{2+0-1}+1=3$.

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue $k = R/M$. Let (V_δ,β) be an orthogonal space of rank $2\nu+\delta$, where $\nu \geq 1$ and $\delta \in \{0,1,2\}$. This orthogonal space induces a $2\nu+\delta$ dimensional vector space (V'_δ,β'), where β' is given via the canonical map $\pi : R \mapsto k$ by

$$\beta'(\pi(\vec{a},\vec{b})) = \pi(\beta(\vec{a},\vec{b}))$$

for all $\vec{a},\vec{b} \in V_\delta$. Here, we write $\pi(\vec{a}) = (\pi(a_1),\dots,\pi(a_{2\nu+\delta}))$ for all $\vec{a} = (a_1,\dots,$ $a_{2\nu+\delta}) \in V_\delta$. It also follows that

$$\beta'(\pi(\vec{a}),\pi(\vec{b})) \in k^\times \Leftrightarrow \beta(\vec{a},\vec{b}) \in R^\times$$

for all $\vec{a},\vec{b} \in V_\delta$. This gives (3) of the next theorem. Moreover, we can see that (4) is an immediate consequence of (3).

Theorem 4.1.2. *Under the above set up, let $\kappa = |k|^{\nu+\delta-1}+1$ and $l = \frac{|k|^\nu-1}{|k|-1}$. For each $i \in \{1,\dots,\kappa\}$, let $X_i = \{\vec{x}_{i_1},\dots,\vec{x}_{i_l}\}$ be the set of unimodular vectors in V_δ with zero norm such that $\{\{k\pi(\vec{x}_{i_s}) : s = 1,\dots,l\} : i = 1,\dots,\kappa\}$ is a partition of $\mathcal{V}(\mathscr{G}_{\mathrm{O}_{k(V'_\delta)}})$ satisfying $k\pi(\vec{x}_{i_s})$ and $k\pi(\vec{x}_{i_t})$ are non-adjacent vertices for all $s \neq t$. Then* *the following statements hold.*

(1) *The set* $\Pi = \{R(X_1 + M^{2\nu+\delta}), \ldots, R(X_\kappa + M^{2\nu+\delta}\}$ *is a partition of the vertex set* $\mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})$*, where* $R(X_i + M^{2\nu+\delta}) = \{R(\vec{x}_{i_s} + \vec{m}) : s = 1, \ldots, l, \vec{m} \in M^{2\nu+\delta}$ *and* $\beta(\vec{x}_{i_s} + \vec{m}, \vec{x}_{i_s} + \vec{m}) = 0\}$ *for all* $i \in \{1, \ldots, \kappa\}$*. Moreover, for each* $i \in \{1, \ldots, \kappa\}$*, any two distinct vertices in* $R(X_i + M^{2\nu+\delta})$ *are non-adjacent vertices. Hence,* $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ *is a* κ*-partitie graph.*

(2) $|R(X_i + M^{2\nu+\delta})| = l|M|^{2\nu+\delta-2}$ *for all* $i \in \{1, \ldots, \kappa\}$*.*

(3) *For unimodular vectors with zero norm* $\vec{a}, \vec{b} \in V_\delta$*, we have* $R\vec{a}$ *and* $R\vec{b}$ *are adjacent vertices in* $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ *if and only if* $k\pi(\vec{a})$ *and* $k\pi(\vec{b})$ *are adjacent vertices in* $\mathscr{G}_{\mathrm{O}_k(V'_\delta)}$*.*

(4) *For* $i, j \in \{1, \ldots, \kappa\}$*,* $s, t \in \{1, \ldots, l\}$ *and* $s \neq t$*, if* $k\pi(\vec{x}_{i_s})$ *and* $k\pi(\vec{x}_{i_t})$ *are adjacent vertices, then* $R(\vec{x}_{i_s} + \vec{m}_1)$ *and* $R(\vec{x}_{i_t} + \vec{m}_2)$ *are adjacent vertices in the graph* $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ *for all* $\vec{m}_1, \vec{m}_2 \in M^{2\nu+\delta}$ *such that* $\beta(\vec{x}_{i_s} + \vec{m}_1, \vec{x}_{i_s} + \vec{m}_1) = \beta(\vec{x}_{i_t} + \vec{m}_2, \vec{x}_{i_t} + \vec{m}_2) = 0$*.*

(5) *The chromatic number of the orthogonal graph* $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ *is* $|k|^{\nu+\delta-1} + 1$*.*

Proof. The first part of (1) follows from the fact that $\mathscr{G}_{\mathrm{O}_k(V'_\delta)}$ is a κ-partite and so Π consists of the inverse image of each partite set of the canonical map π. Note that

$$\beta(\vec{x}_{i_s} + \vec{m}_1, \vec{x}_{i_t} + \vec{m}_2) = \beta(\vec{x}_{i_s}, \vec{x}_{i_t}) + \beta(\vec{x}_{i_s}, \vec{m}_2) + \beta(\vec{m}_1, \vec{x}_{i_t}) + \beta(\vec{m}_1, \vec{m}_2) \in M$$

for all $i, j \in \{1, \ldots, \kappa\}$ and $s, t \in \{1, \ldots, l\}$ and $\vec{m}_1, \vec{m}_2 \in M^{2\nu+\delta}$. This proves the second part of (1). Next, let $i \in \{1, \ldots, \kappa\}$. observe that if $s, t \in \{1, \ldots, l\}$ and $s \neq t$, then $k\pi(\vec{x}_{i_s}) \neq k\pi(\vec{x}_{i_t})$, so $R(\vec{x}_{i_s} + \vec{m}_1) \neq R(\vec{x}_{i_t} + \vec{m}_2)$ for all $\vec{m}_1, \vec{m}_2 \in M^{2\nu+\delta}$. Now, we shall fix $s \in \{1, 2, \ldots, l\}$ and show that the number of distinct vertices in $\{R(\vec{x}_{i_s} + \vec{m}) : \vec{m} \in M^{2\nu+\delta}$ and $\beta(\vec{x}_{i_s} + \vec{m}, \vec{x}_{i_s} + \vec{m}) = 0\}$ is $|M|^{2\nu+\delta-2}$.

Let $s \in \{1, 2, \ldots, l\}$, $\vec{x}_{i_s} \in X_i$ and assume that $R(\vec{x}_{i_s} + \vec{m}_1) = R(\vec{x}_{i_s} + \vec{m}_2)$ in $R(X_i + M^{2\nu+\delta})$. Then $\vec{x}_{i_s} + \vec{m}_1 = \lambda(\vec{x}_{i_s} + \vec{m}_2)$ for some $\lambda \in R^\times$. Thus, $(1 - \lambda)\vec{x}_{i_s} = \lambda\vec{m}_2 - \vec{m}_1 \in M^{2\nu+\delta}$. Since $\vec{x}_{i_s}$ is unimodular, $1 - \lambda \in M$, so $\lambda = 1 + \mu$ for some

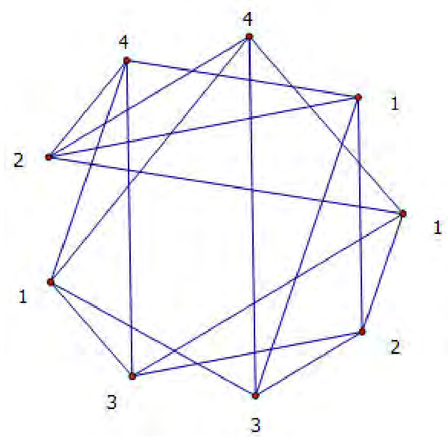
$\mu \in M$. Hence, $\vec{x}_{i_s} + \vec{m}_1 = (1+\mu)(\vec{x}_{i_s} + \vec{m}_2)$. Next, we show that $R(1+\mu)(\vec{x}+\vec{m}) = R(\vec{x}+\vec{m})$ for all $\mu \in M$, $\vec{x} \in V$ and $\vec{m} \in M^{2\nu+\delta}$. Clearly, $R(1+\mu)(\vec{x}+\vec{m}) \subseteq R(\vec{x}+\vec{m})$. Since $\mu \in M$, $1+\mu \in R^{\times}$. Then $r(\vec{x}+\vec{m}) = (r(1+\mu)^{-1})(1+\mu)(\vec{x}+\vec{m})$ for all $r \in R$ which gives another inclusion. Therefore, the number of elements in the set $\{R(\vec{x}_{i_s}+\vec{m}) : \vec{m} \in M^{2\nu+\delta}\}$ is $|M|^{2\nu+\delta-1}$. However, the $\vec{x}_{i_s}+\vec{m}$ is also required to have norm zero. Write $\vec{x}_{i_s} = (x_1, x_2, \ldots, x_{2\nu+\delta})$. By Lemma 3.2.2, x_j is a unit for some $j \in \{1, 2, \ldots, x_{2\nu}\}$. The requirement $\beta(\vec{x}+\vec{m}, \vec{x}+\vec{m}) = 0$ and x_j is a unit allow us to count the number of possible vectors $\vec{m}$ and see that $|\{R(\vec{x}_{i_s}+\vec{m}) : \vec{m} \in M^{2\nu+\delta}$ and $\beta(\vec{x}_{i_s}+\vec{m}, \vec{x}_{i_s}+\vec{m}) = 0\}| = |M|^{2\nu+\delta-2}$ as desired. This proves (2).

Finally, we determine the chromatic number. Since our graph is $|k|^{\nu+\delta-1}+1$-partite, $\chi(\mathscr{G}_{\mathrm{O}_R(V_\delta)})$ is at most $|k|^{\nu+\delta-1}+1$. To prove the reverse inequality, we consider the induced subgraph of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ on the vertex set $\{R(\vec{x}_{i_s}) : i = 1, \ldots, \kappa$ and $s = 1, \ldots, l\}$. By (3), this subgraph is isomorphic to the orthogonal graph $\mathscr{G}_{\mathrm{O}_k(V'_\delta)}$ with chromatic number $|k|^{\nu+\delta-1}+1$. Hence, the chromatic number of the graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is $|k|^{\nu+\delta-1}+1$. $\square$

A set I of vertices of a connected graph G is called an **independent set** if no two distinct vertices of I are adjacent. Write $\alpha(G)$ for the size of largest independent set of G.

Example 4.1.3. If R is a finite local ring of odd characteristic with unique maximal ideal M and residue field $k = R/M$, Theorem 4.1.2 implies that the sets $R(X_i + M^{2\nu+\delta})$, $i \in \{1, 2, \ldots, \kappa\}$, are independent sets in the orthogonal graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$.

Example 4.1.4. The following figure shows the orthogonal graph $\mathscr{G}_{\mathrm{O}_{\mathbb{F}_2(V_0)}}$, where V_0 is the orthogonal space of dimension $2\cdot 2+0$. The size of largest independent set of $\mathscr{G}_{\mathrm{O}_{\mathbb{F}_2(V_0)}}$ is $\alpha(\mathscr{G}_{\mathrm{O}_{\mathbb{F}_2(V_0)}}) = 3$ (vertices labelled 1).



Gu and Wan [5] showed that $\alpha(\mathscr{G}_{O_k(V'_\delta)}) \le \frac{|k|^\nu-1}{|k|-1}$. Together with Theorem 4.1.2 (2), we have $\alpha(\mathscr{G}_{O_k(V'_\delta)}) = \frac{|k|^\nu-1}{|k|-1}$. Then, for any two distinct indices i and j, every $\vec{x} \in X_i$, $k\pi(\vec{x})$ is adjacent to $k\pi(\vec{y})$, for some $\vec{y} \in X_j$. By Theorem 4.1.2 (4), for any two distinct indices i and j and every $\vec{x} \in X_i + M^{2\nu+\delta}$, the vertex $R\vec{x}$ is adjacent to a vertex $R\vec{y}$, for some $\vec{y} \in X_j + M^{2\nu+\delta}$. Hence, we obtain the following theorem.

Theorem 4.1.3. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $k = R/M$ and let (V_δ, β) be an orthogonal space of rank $2\nu+\delta$, $\nu \ge 1$ and $\delta \in \{0,1,2\}$. Then*

$$\alpha(\mathscr{G}_{O_R(V_\delta)}) = \left(\frac{|k|^\nu - 1}{|k| - 1}\right) |M|^{2\nu+\delta-2}.$$

A **fractional coloring of a graph** G is a mapping f which assigns to each independent set I of G a real number $f(I) \in [0,1]$ such that for any vertex v, $\sum_{v\in I} f(I) = 1$. The **total weight** $w(f)$ **of a fractional coloring** f of G is the sum of $f(I)$ over all the independent sets I of G. The **fractional chromatic number of** G, denoted by $\chi^*(G)$, is the minimum total weight of a fractional coloring of G.

Example 4.1.5. Consider the 5-cycle C_5. It has exactly five independent set of size two, and each vertex lies in two of them. Thus, if we define f to take the value $\frac{1}{2}$ on each of these independent sets and 0 on all others, then the fractional chromatic number of a cycle graph C_5 is $\chi^*(C_5) = \frac{5}{2}$.

The color classes of a proper l-coloring of G form a collection of l pairwise disjoint independent sets $I_1, I_2, \ldots, I_l$ whose union is $\mathcal{V}(G)$. The function f such that $f(I_j) = 1$ for all $j \in \{1, 2, \ldots, l\}$ and $f(S) = 0$ for all other independent sets S is a fractional coloring of weight l. Therefore, $\chi^*(G) \leq \chi(G)$. Moreover, when G is vertex transitive, we have the following proposition.

Proposition 4.1.4 (Corollary 7.5.2 of [4]). *If G is a vertex transitive graph, then*

$$\chi^*(G) = \frac{|\mathcal{V}(G)|}{\alpha(G)}.$$

Let R be a finite local ring of odd characteristic and let (V_δ, β) be an orthogonal space of rank $2\nu + \delta$, $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$. By Theorems 3.3.1 and 4.1.3, we have

$$|\mathcal{V}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})| = \frac{(|R|^\nu - |M|^\nu)(|R|^{\nu+\delta-1} + |M|^{\nu+\delta-1})}{|R| - |M|}$$

and

$$\alpha(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = \left(\frac{|k|^\nu - 1}{|k| - 1}\right) |M|^{2\nu+\delta-2},$$

respectively. Since the graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ is vertex transitive, it follows from Proposition 4.1.4 that

$$\chi^*(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = \frac{\dfrac{(|R|^\nu - |M|^\nu)(|R|^{\nu+\delta-1} + |M|^{\nu+\delta-1})}{|R| - |M|}}{\left(\dfrac{|k|^\nu - 1}{|k| - 1}\right) |M|^{2\nu+\delta-2}} = |k|^{\nu+\delta-1} + 1,$$

which is equal to the chromatic number of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$. We record this result in the next theorem.

Theorem 4.1.5. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $k = R/M$ and let (V_δ, β) be an orthogonal space of rank $2\nu+\delta$,*

$\nu \geq 1$ *and* $\delta \in \{0, 1, 2\}$. *Then*

$$\chi^*(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = |k|^{\nu+\delta-1} + 1 = \chi(\mathscr{G}_{\mathrm{O}_R(V_\delta)}).$$

It is easy to see that if there is a homomorphism from a graph X to a graph Y, then $\chi(X) \leq \chi(Y)$. Let G and H be graphs. Since both G and H are homomorphic images of $G \otimes H$ (using the projection homomorphisms), we have that

$$\chi(G \otimes H) \leq \min\{\chi(G), \chi(H)\}.$$

Hedetniemi [8] has conjectured that for all graphs G and H equality occurs in the above bound. This conjecture is still open. However, Zhu [19] showed that Hedetniemi's conjecture is true for fractional chromatic numbers.

Proposition 4.1.6. (Theorem 2 of [19]) *For graphs G and H,*

$$\chi^*(G \otimes H) = \min\{\chi^*(G), \chi^*(H)\}.$$

Now, let R be a finite commutative ring of odd characteristic decomposed as $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a finite local ring of odd characteristic for all $i = 1, 2, \ldots, t$ same as in Section 3.4. Then

$$\mathscr{G}_{\mathrm{O}_R(V)} \cong \mathscr{G}_{\mathrm{O}_{R_1}(V^{(1)})} \otimes \mathscr{G}_{\mathrm{O}_{R_2}(V^{(2)})} \otimes \cdots \otimes \mathscr{G}_{\mathrm{O}_{R_t}(V^{(t)})},$$

as we have seen earlier. By Proposition 4.1.6 and the above discussion,

$$\min_{1 \leq i \leq t} \chi^*(\mathscr{G}_{\mathrm{O}_{R_i}(V^{(i)})}) = \chi^*(\mathscr{G}_{\mathrm{O}_R(V)}) \leq \chi(\mathscr{G}_{\mathrm{O}_R(V)}) \leq \min_{1 \leq i \leq t} \chi(\mathscr{G}_{\mathrm{O}_{R_i}(V^{(i)})}).$$

Since $\chi^*(\mathscr{G}_{\mathrm{O}_{R_i}(V^{(i)})}) = \chi(\mathscr{G}_{\mathrm{O}_{R_i}(V^{(i)})})$ for al $i = 1, 2, \ldots, t$, it forces that

$$\chi^*(\mathscr{G}_{\mathrm{O}_R(V)}) = \chi(\mathscr{G}_{\mathrm{O}_R(V)}) = \min_{1 \leq i \leq t} \chi(\mathscr{G}_{\mathrm{O}_{R_i}(V^{(i)})}).$$

Together with Theorem 4.1.2 (5), we obtain our desired chromatic number.

Theorem 4.1.7. *Let R be a finite commutative ring of odd characteristic decomposed as $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a finite local ring of odd characteristic and k_i is its residue field, for all $i = 1, 2, \ldots, t$. If (V_δ, β) is an orthonal space over R of rank $2\nu + \delta$, $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$, then*

$$\chi^*(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = \chi(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = \min_{1 \leq i \leq t} |k_i|^{\nu+\delta-1} + 1.$$

Corollary 4.1.8. *Let $m > 1$ and $R = \mathbb{Z}_m \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_t^{n_t}}$, where $n_i \in \mathbb{N}$ and p_i are primes such that $p_1 < p_2 < \cdots < p_t$. For the orthogonal space V_δ over R of dimension $2\nu + \delta$, $\nu \geq 1$ and $\delta = 0, 1$ or 2, we have the chromatic number of the graph $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ given by*

$$\chi^*(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = \chi(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) = |p_1|^{\nu+\delta-1} + 1.$$

4.2 Automorphisms

In Section 3.2, we have defined an automorphism of a graph G and the set of all automorphisms of G forms a group under composition, denoted by $\mathrm{Aut}(G)$. We shall close this work by describing the automorphism group of orthogonal graphs over a finite local ring.

We begin by recalling the work over a finite field. Let k be a field and write $\mathrm{Aut}\, k$ for the group of automorphism of k. Let φ be the natural action of $\mathrm{Aut}(k)$ on the group $(k^\times)^{(\nu)} = k^\times \times \cdots \times k^\times$ ($\nu \geq 1$ copies) defined by

$$\varphi(\phi)((a_1, \ldots, a_\nu)) = (\phi(a_1)), \ldots, \phi(a_\nu),$$

for all $\phi \in \mathrm{Aut}(k)$ and $a_1, \ldots, a_\nu \in k^\times$. Then the semidirect product of $(k^\times)^{(\nu)}$ by $\mathrm{Aut}(k)$ corresponding to φ, denoted by $(k^\times)^{(\nu)} \rtimes_\varphi \mathrm{Aut}(k)$, is the group consisting of all elements of the form $((a_1, \ldots, a_\nu), \phi)$, where $a_1, \ldots, a_\nu \in k^\times$ and

$\phi \in \mathrm{Aut}(k)$, with the multiplication defined by

$$((a_1, \ldots, a_\nu), \phi)((a'_1, \ldots, a'_\nu), \phi') = ((a_1, \ldots, a_\nu)(\varphi(\phi)(a'_1, \ldots, a'_\nu)), \phi \circ \phi')$$
$$= ((a_1\phi(a'_1), \ldots, a_\nu\phi(a'_\nu)), \phi \circ \phi').$$

Also, we have $\big((k^\times)^2 \times (k^\times)^{(\nu-1)} \times \{\pm 1\}\big) \rtimes_\varphi \mathrm{Aut}(k)$ which isa similarly defined as above. The set of all permutations of a set S is denoted by $\mathrm{Sym}(S)$ or just $\mathrm{Sym}(n)$ if $|S| = n$. Note that $|\mathrm{Sym}(n)| = n!$. Gu and Wan [5] determined the automorphism groups of orthogonal graphs for every case except $\Delta = z$ and $\delta = 1$. However, that case can be proved in a similar manner. Hence, the automorphism group of orthogonal graphs over a finite field can be described as follows.

Theorem 4.2.1. *Let R be a commutative ring and (V_δ, β) an orthogonal space over R. For each $\sigma \in \mathrm{O}_R(V_\delta)$, σ can be considered as an automorphism of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$. That is, we have the imbedding $\mathrm{O}_R(V_\delta) \hookrightarrow \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_R(V_\delta)})$.*

Proof. Let $\sigma \in \mathrm{O}_R(V_\delta)$. Define the map $\bar{\sigma}$ on $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ by $\bar{\sigma} : R(\vec{x}) \mapsto R\sigma(\vec{x})$ for all unimodular vectors $\vec{x} \in V_\delta$ with zero norm. Since σ is an isometry, $\bar{\sigma}$ is a bijection and $\beta(\vec{x}, \vec{y}) = \beta(\sigma(\vec{x}), \sigma(\vec{y}))$ for all unimodular vectors $\vec{x}, \vec{y} \in V_\delta$ with zeor norm. Thus,

$$\beta(\vec{x}, \vec{y}) \in R^\times \Longleftrightarrow \beta(\sigma(\vec{x}), \sigma(\vec{y})) \in R^\times$$

for all unimodular vectors $\vec{x}, \vec{y} \in V_\delta$. Hence, $\bar{\sigma} \in \mathrm{Aut}\left(\mathscr{G}_{\mathrm{O}_R(V_\delta)}\right)$. $\square$

Theorem 4.2.2 (Theorems 3.3, 4.1 and 5.1 of [5])**.** *Let k be a finite field and V_δ be a orthogonal space over k of dimension $2\nu+\delta$, $\nu \geq 1, \delta \in \{0, 1, 2\}$. Regard the orthogonal group $\mathrm{O}_k(V_\delta)/\{\pm I_{2\nu+\delta}\}$ as a subgroup of $\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V_\delta)})$ (as shown in Theorem 4.2.1) and let E_δ be the subgroup of $\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V_\delta)})$ defined by*

$$E_\delta = \{\sigma \in \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V_\delta)}) : \sigma(k\vec{e}_i) = k\vec{e}_i \textit{ for all } i = 1, \ldots, 2\nu\},$$

where $\vec{e}_i$ has 1 at the ith position and the rest is zero for all $i = 1, \ldots, 2\nu$. If $\nu = 1$, then $\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V_\delta)}) = \mathrm{Sym}(|k|^\delta + 1)$. *Assume $\nu \geq 2$. Then*

$$\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V_\delta)}) = (\mathrm{O}_k(V)/\{\pm I_{2\nu}\}) \cdot E_\delta,$$

where

$$E_0 \cong \begin{cases} \mathrm{Sym}(k^\times) \times \mathrm{Sym}(k^\times) & \text{if } \nu = 2, \\ (k^\times)^{(\nu)} \rtimes_\varphi \mathrm{Aut}(k) & \text{if } \nu \geq 3, \end{cases}$$

$$E_1 \cong \big((k^\times)^2 \times (k^\times)^{(\nu-1)} \times \{\pm 1\}\big) \rtimes_\varphi \mathrm{Aut}(k),$$

and E_2 consists of all σ depending on $u_1, u_2, \cdots, u_\nu \in k^\times$, $\tau \in \mathrm{Aut}(k)$, $x_1, x_2, y_1, y_2 \in k$, which maps a vertex $k(a_1, a_2 \cdots, a_{2\nu+2})$ of $\mathscr{G}_{\mathrm{O}_k(V)}$ to

$$k(\tau(a_1), u_2\tau(a_2) \cdots, u_\nu\tau(a_\nu), u_1\tau(a_{\nu+1}), u_1u_2^{-1}\tau(a_{2\nu}), a'_{2\nu+1}, a'_{2\nu+2}),$$

where $x_1^2 - zx_2^2 = u_1$, $y_1^2 - zy_2^2 = u_1\tau(-z)$ and

$$\begin{aligned} a'_{2\nu+1} &= u_1(y_2\tau(a_{2\nu+1}) + x_2\tau(za_{2\nu+2}))/(x_1y_2 - x_2y_1), \\ a'_{2\nu+2} &= u_1z^{-1}(y_1\tau(a_{2\nu+1}) + x_1\tau(za_{2\nu+2}))/(x_1y_2 - x_2y_1). \end{aligned}$$

For a finite local ring, the automorphism group of an orthogonal graph is the direct product of the automorphism group of the graph over its residue field and a symmetric group.

Theorem 4.2.3. *Let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V_δ, β) be an orthogonal space of rank $2\nu+\delta$, $\nu \geq 1, \delta \in \{0, 1, 2\}$. Then*

$$\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) \cong \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V'_\delta)}) \times (\mathrm{Sym}(|M|^{2\nu+\delta-2}))^{l\kappa},$$

where $\kappa = |k|^{\nu+\delta-1} + 1$, $l = \frac{|k|^\nu - 1}{|k| - 1}$, V'_δ is the $2\nu + \delta$-dimensional orthogonal space over k induced from V_δ and $\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V'_\delta)})$ is presented in Theorem 4.2.2.

Proof. For each $i \in \{1, \ldots, \kappa\}$, let $X_i = \{\vec{x}_{i_1}, \ldots, \vec{x}_{i_l}\}$ be the set of unimodular vectors in V_δ with zero norm such that $\{\{k\pi(\vec{x}_{i_s}) : s = 1, \ldots, l\} : i = 1, \ldots, \kappa\}$ is a partition of $\mathcal{V}(\mathscr{G}_{\mathrm{O}_k(V'_\delta)})$. Theorem 4.1.2 shows that the subgraph of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ induced from the vertex set $\{R(\vec{x}_{i_s}) : i = 1, \ldots, \kappa$ and $s = 1, \ldots, l\}$ is isomorphic to the orthogonal graph $\mathscr{G}_{\mathrm{O}_k(V'_\delta)}$. Moreover, each automorphism of $\mathscr{G}_{\mathrm{O}_R(V_\delta)}$ corresponds with an automorphism of the graph $\mathscr{G}_{\mathrm{O}_k(V'_\delta)}$ and a permutation of vertices in the set $R(\vec{x}_{i_s} + M^{2\nu+\delta})$ for all $i \in \{1, \ldots, \kappa\}$ and $s \in \{1, \ldots, l\}$. Thus,

$$\begin{aligned}\mathrm{Aut}(\mathscr{G}_{\mathrm{O}_R(V_\delta)}) &\cong \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V'_\delta)}) \times \prod_{i=1}^{\kappa}\prod_{s=1}^{l} \mathrm{Sym}(|R(\vec{x}_{i_s} + M^{2\nu+\delta})|) \\ &= \mathrm{Aut}(\mathscr{G}_{\mathrm{O}_k(V'_\delta)}) \times (\mathrm{Sym}(|M|)^{2\nu+\delta-2})^{l\kappa}\end{aligned}$$

because $|R(X_i + M^{2\nu+\delta})| = l|M|^{2\nu+\delta-2}$ for all $i \in \{1, \ldots, \kappa\}$. $\square$

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