มุมมองบางประการของเกจซูเปอร์แกรวิตี

นายบุญพิทักษ์ งามวัฒนากุล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาฟิสิกส์ ภาควิชาฟิสิกส์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

Some Aspects of Gauged Supergravity

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Physics
Department of Physics
Faculty of Science
Chulalongkorn University
Academic Year 2014

Thesis Title	SOME ASPECTS OF GAUGED SUPERGRAVITY
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Field of Study	Physics
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บุญพิทักษ์ งามวัฒนากุล : มุมมองบางประการของเกจซูเปอร์แกรวิตี. (SOME ASPECTS OF GAUGED SUPERGRAVITY) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.อรรถกฤต ฉัตรภูติ, 140 หน้า.

เราได้ทำการศึกษาเกจซูเปอร์แกรวิตีแบบเชิร์น–ไซมอนในสามมิติที่มีซูเปอร์ซิมเมทรีเท่ากับห้าและหก รวมไปถึงการประยุกต์ใช้ผลลัพธ์ที่ได้ โดยเราได้แบ่งการศึกษาจำแนกตามเกจกรุ๊ปได้สามประเภท ได้แก่ เกจกรุ๊ปที่เป็นคอมแพ็ค, นอนคอมแพ็ค และ นอนเซมิซิมเปิ้ล ซึ่งกระบวนการเกจดำเนินการโดยใช้วิธีเอ็มเบดดิ้งเทนเซอร์ มานิโฟลด์ของเกจซูเปอร์แกรวิตีที่มีซูเปอร์ซิมเมทรีเท่ากับห้าและหกอยู่ในรูป $G/H = USp(4,k)/USp(4) \times USp(k)$ และ $G/H = SU(4,k)/S(U(4) \times U(k))$ ตามลำดับ โดยที่ k คือจำนวนซูเปอร์มัลติเพล็ตซึ่งเป็นจำนวนคู่สำหรับทฤษฎีซูเปอร์ซิมเมทรีเท่ากับห้าและเป็นจำนวนเต็มสำหรับทฤษฎีซูเปอร์ซิมเมทรีเท่ากับหก เราจำกัดการศึกษาไว้ที่ $k = 2,4$ สำหรับทฤษฎีซูเปอร์ซิมเมทรีเท่ากับห้าและ $k = 1,2,3,4$ สำหรับทฤษฎีซูเปอร์ซิมเมทรีเท่ากับหก สำหรับกรณีนอนคอมแพ็ค และ นอนเซมิซิมเปิ้ลนั้นเงื่อนไขของทฤษฎีได้รับการตรวจสอบเพิ่มเติมในงานนี้ซึ่งต่างจากกรณีของคอมแพ็คเกจกรุ๊ป ศักย์สเกลาร์ได้รับการคำนวนเพื่อประโยชน์ในการศึกษาอาร์จีโฟลว์ การพารามิไตซ์สเกลาร์มานิโฟลด์ในที่นี้มีทั้งแบบทั้งหมดและแบบบางส่วน และมีทั้งชนิดยูนิทารีเกจและแบบมุมออยเลอร์ จากศักย์สเกลาร์ที่ได้ทำให้หาจุดวิกฤตที่มีซูเปอร์ซิมเมทรีชนิด AdS_3 ได้จำนวนมากซึ่งจำเป็นในการศึกษาอาร์จีโฟลว์ในส่วนถัดมา สัดส่วนของเซ็นทรัลชาร์ต c_{UV}/c_{IR} ได้รับการคำนวนและสอดคล้องเป็นอย่างดีกับ ทฤษฎีบทเซ็นทรัลชาร์ตของซาโมลอดชิคอฟ กรณีนอนเซมิซิมเปิ้ลเกจกรุ๊ปจะอยู่ในรูปของ $SO(N) \ltimes \mathbf{T}^{\dim SO(N)}$ ความน่าสนใจของกรณีนอนเซมิซิมเปิ้ลเกจกรุ๊ปคือ มันสามารถเชื่อมโยงกับเกจซูเปอร์แกรวิตีในสี่มิติโดยการลดทอนมิติบนออร์บิโฟลด์ $S^1/\mathbb{Z}_2$ เราพบจุดวิกฤตที่มีซูเปอร์ซิมเมทรีแบบมากที่สุดชนิด AdS_3 ซึ่งมีซูเปอร์คอนฟอร์มอลกรุ๊ป $Osp(5|2,\mathbb{R}) \times Sp(2,\mathbb{R})$ สำหรับทฤษฎีซูเปอร์ซิมเมทรีเท่ากับห้าและผลเฉลยโดเมนวอลล์ที่มีซูเปอร์ซิมเมทรีแบบครึ่งสำหรับกรณีทฤษฎีซูเปอร์ซิมเมทรีเท่ากับหก

ภาควิชา .... ฟิสิกส์ .... ลายมือชื่อนิสิต ....

สาขาวิชา .... ฟิสิกส์ .... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....

ปีการศึกษา ..... 2557 ....

5273824423 : MAJOR PHYSICS
KEYWORDS : SUPERGRAVITY, HOLOGRAPHIC RG FLOW, ADS/CFT CORRESPONDENCE, GAUGE/GRAVITY CORRESPONDENCE

BOONPITHAK NGAMWATTHANAKUL : SOME ASPECTS OF GAUGED SUPERGRAVITY. ADVISOR : AUTTAKIT CHATRABHUTI, Ph.D., 140 pp.

We study $N = 5$ and $N = 6$ Chern-Simons gauged supergravity in three dimensions and applications thereof. The classification is determined by their gauged groups which are as follows: compact, non-compact and non-semisimple gauge groups. The gaugings are realized by the notion of embedding tensors. The manifolds of $N = 5$ and $N = 6$ theories are in the form of $G/H = USp(4, k)/USp(4) \times USp(k)$ and $G/H = SU(4, k)/S(U(4) \times U(k))$, respectively. The number of independent supermultiplets is specified by k which is an even integer in $N = 5$ theories and an integer in $N = 6$. For $N = 5$, we restrict ourselves to $k = 2, 4$ cases whereas for $N = 6$, we confine to $k = 1, 2, 3, 4$ cases. For non-compact and non-semisimple cases, the consistent condition is checked unlike the compact cases which are already classified. The scalar potentials which play an important role for the further analysis of grou nd states of theories and holographic RG flows are calculated. The potentials can be achieved by parametrization of scalar manifolds. We paramatrize full manifold for some cases and for the rest we simply parametrize submanifold thereof. The parametrizations are either unitary gauge or Euler angle parametrization. Many supersymmetric AdS_3 critical points are found and they are required for solving RG flows solutions interpolating between those critical points. The ratios of central charges c_{UV}/c_{IR} are calculated and they are in perfect agreement with the Zamolodchikov c-theorem. For non-semisimple gaugings, the gauge groups are in the form of $SO(N) \ltimes \mathbf{T}^{\dim SO(N)}$. We pay a special attention to the non-semisimple gaugings since they both are linked to four dimensional gauged supergravity via dimensional reduction on the orbifold $S^1/\mathbb{Z}_2$. We found maximally supersymmetric AdS_3 critical points with superconformal group $Osp(5|2, \mathbb{R}) \times Sp(2, \mathbb{R})$ in $N = 5$ non-semisimple gauging theory while a half-supersymmetric domain wall solution is found in $N = 6$ non-semisimple gauging theory.

Department : ......Physics...... Student's Signature ....

Field of Study : ......Physics...... Advisor's Signature ....

Academic Year : ......2014......

Acknowledgements

I would like to thank to my Ph.D. advisors, Asst. Prof. Dr. Auttakit Chatrabhuti, for teaching, mentoring and supporting me during the Ph.D. program. I am also very grateful to my co-worker Dr. Parinya Karndumri for his scientific knowledge and many insightful discussions. I also thank the members of my Ph.D. committee, Asst. Prof. Dr. Udomsilp Pinsook, Dr. Rujikorn Dhanawittayapol, Dr. Varagorn Hengpunya and Dr. Pichet Vanichchapongjaroen for their helpful advice and suggestions. I am thankful to my colleague Dr. Ekapong Hirunsirisawat for providing helpful advice during my Ph.D. program. I would like to thank staffs of the Physics department who have been very kind to me whenever I approached them.

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CHAPTER I

Introduction

Quantum electrodynamics (QED) is the most precise physical theory that human ever constructed. Another theory that is similarly constructed and embeds QED into its framework, the standard model, have been also constantly tested and in agreement with experiments over the past decades. These theories give us a much more advancing in understanding the fundamental laws of nature. Besides its precision and predictive power, the standard model explains three fundamental forces of nature: electromagnatic, weak and strong nuclear forces, into a single framework. This gives us hope to find the theory of everything – a single theory that can explain all fundamental laws of nature in one idea or framework. Unfortunately, the standard model does not include gravity. The obstacle is the unrenormalization of quantum field theory of gravity – the inability to get rid of the infinities during calculating scattering amplitudes systematically. During the success of the standard model, a new idea of physics had been being developed – it is supersymmetry. It was introduced as a mathematical idea and had no experimental guidance. It is customary that the theoretical physics in this century can drive by beauty of mathematics. This is quite distinctive when compare to the theories constructed before the early age of quantum mechanics. Supersymmetry is basically a symmetry that relates bosons to fermions. When apply this idea to particle physics, it suggests a new partner with spin difference by half integer. For example, a photon (a spin-1 particle) has a partner which is a fermion called photino (spin 1/2). The idea of supersymmetry became more interesting because it may solve the hierarchy problem. The hierarchy problem involves in the curiosity that why the gravitational force is 10^{32} times weaker than the weak nuclear force. Technically speaking, this problem is also referred to as the question of why the Higgs mass is so much lighter than the Plank mass. Supersymmetry can hopefully provide a solution by giving the explanation of how the Higgs mass is protected from the quantum correction. The simple answer is that supersymmetry can remove the divergence of some loop diagrams by the exact amount of the contribution from its superpartner. To the best of my knowledge, there are still no solid evidences for the existence of supersymmetry.

Around 1960s, another quantum theory was proposed as an alternative approach of field theory. It was called string theory. String theory abandons the notion of point-like particles and replaces them with string-like objects. The interactions can be visualized as joining and splitting of strings. It originally was constructed to explain strong nuclear force, since a number of mesons and hadrons were discovered in that period. The stringy concept was indeed succeed in describing the relation between masses and spin of particles. However, QCD (Quantum chromodynamics) turned out to be the correct framework for strong nuclear force. QCD is a non-abelian gauge theory with color gauge group $SU(3)$ and the mesons and hadrons in the description of QCD are made of quarks which carry color degrees of freedom. The string theory began to fade but it could not be totally abandoned because it has an very interesting feature that most physicists had not paid much attention in the first place. String theory can naturally explain massless spin-2 particles (graviton) and the only consistent spin two theory is gravity, so the string theory can be a good candidate for quantum gravity. Since graviton is a mode of the oscillation along with all particles known to us, string theory is also a perfect candidate for a theory of everything. However, there are still glitches. In order to get rid of inconsistency in the theory the spacetime have to be ten dimensional. The four dimensional spacetime theory can be achieved by compactifying the additional 6-dimensional spacetime. The low energy interactions are given by the geometrical profiles of the 6-dimensional compact manifold. Phenomenologically, Calabi-Yau manifold is the most promising, but still we cannot get the standard model at low energy limit.

At low energy limit, strings behave like point particle and we can rely on field theory framework. Supergravity can be considered as a low energy limit of string theory. For examples, the type IIB string theory can be approximated by the type IIB supergravity at low energy and the eleven dimensional supergravity is low energy limit of M theory. In order to get the lower D dimensional supergravity from the eleven dimensional supergravity, we need to compactify spacetime on $11 - D$ dimensional compact manifold. The shape of the compact manifold characterizes the supergravity theory. For examples, if the manifold is $11-D$-dimensional torus, we get the D-dimensional ungauged supergavity. If the manifold is $11 - D$-dimensional sphere, we get $SO(12 - D)$ gauged supergravity in D dimensions.

The correct prediction of mass-spin relation from string theory may not be an accident. There exists somehow the interrelation between string theory and gauge theory. It was t'Hooft who inspired the identification the large color N theory to a string theory [1]. To summarize: a string theory with some backgrounds which include gravity is dual to a quantum field theory. In 1997, it was Maldacena [2] who elaborated the idea and made a revolutionary conjecture of duality between the two theories. This conjecture was widely known as the AdS/CFT correspondence. Within a year, Witten [3] elaborated on this conjecture and proposed a prescription for connecting the partition function of the two worlds. Although the calculation was performed in the level of classical gravity, it paved the way to a large new field of research. The applications in AdS/CFT have grown tremendously in more than two decades, and seem likely to continue to grow at unreceding rate.

The AdS/CFT correspondence originally conjectured as a duality between the two very different theories, i.e. the $\mathcal{N} = 4$ super Yang-Mills conformal quantum field theory (SYM) in four dimensions and a type IIB string theory in $AdS_5 \times S_5$ background geometry. It relates two very different theories living in different spacetime dimensions, one lives in the bulk and the other lives in the boundary. The degree of validity may be divided into many levels; the lower the energy, the stronger the evidences we found. The conjecture allows us to calculate quantum phenomena in strongly coupled quantum field theory by performing calculation from classical gravitational theory. In some level of the conjecture, the calculations in gravity side give us the information of strong coupling limit of its dual in which the perturbation quantum field theory cannot be achievable. Up until now, the validity of the duality cannot be proved unconditionally. However, the convincing results are widely justified by their the low energy limits at which the string theory can be substituted by supergravity. It is quite bizarre that information of a quantum theory without gravity can be provided by a classical theory with gravity in one higher spacetime dimension. This also raises some philosophical questions whether it is a simply mathematical tool or new perspective of the universe. Excellent reviews on AdS/CFT can be found in [4, 5, 6, 7, 8, 9, 10, 11].

According to AdS/CFT correspondence, both theories shares the same symmetry. On the gravity side, the isometry group of AdS_5 and S^5 are $SO(4,2)$ and $SO(6) \sim SU(4)$, respectively. The superstring background has the superalgebra $SU(2,2|4)$ where $SO(4,2)$ and $SO(6) \sim SU(4)$ are its compact subgroups. On the field theory side, the conformal group is $SO(4,2)$ and the R-symmetry of $\mathcal{N}=4$ super Yang-Mills is also $SU(4)$; they can be combined into superconformal group $SU(2,2|4)$, so the symmetries on both side are equal. Not only the symmetries can be matched, but also some dynamics of theories can be related. One of the key ideas in the correspondence is matching between local fields in gravity side and composite gauge invariant operators of the boundary dual. The by-product of this correspondence is that masses of local field on the gravity side relate to scale dimension of the conformal ones. The other dynamical quantity such as n-points functions of the field theory can be calculated from the on-shell action of the bulk theory. Nowadays the term AdS/CFT correspondence is coined in a far more generalized fashion than in the old days. The correspondence can be extended to theories with less symmetries non-conformal theory. The relaxed condition of its dual for non-conformal is that the background space now merely asymtotically AdS. The non-conformal field theory associates with the bulk region and approaches the conformal theory at boundary. A well-known application of such non-conformal theory is the study of the so-called holographic renormalization group flow (RG flow).

An important concept of duality is that one theory can be explained by the other. Holographic renormalization group flow is an attempt to describe RG flow of field theory by a phenomenon from bulk theory. In conformal field theory, if the theory is deformed by an operator or the operator itself has non-vanishing expectation value, the conformal symmetry of the theory would be broken and the theory is driven to another conformal theory characterized by a different central charge. In the language of quantum field theory we can say that the theory is driven from a fixed point at higher energy to another fixed point at lower energy. If the theory is deformed by an operator, the conformal symmetry is explicitly broken because the effective Lagrangian no longer has conformal symmetry while non-vanishing expectation value case leads to spontaneously broken. According to AdS/CFT correspondence, this phenomenon has a dual picture and it can be described by a classical solution of the bulk theory. The solution is composed of a Poincaré invariant metric and kink-like solution of scalar field which is a function of solely radial coordinate. The metric must approach an AdS solution as radial coordinate goes to boundary and another AdS space characterized by different length as it goes to deep interior region. In order to identify different theories in distinct dimension of spacetime, the excess radial coordinate should be identified with a parameter of the boundary theory, so we identify the radial coordinate with energy scale of the field theory. Alternatively, we can say that each radial slice can portray a snapshot of a quantum field theory at specific energy scale. The RG flow solution can be either supersymmetric or non-supersymmetric. The supersymmetric flow can be obtained by solving the so-called BPS equations. As a result, a set of first order differential equations is given. On the other hand, if we pursue the other path which is more general, we have to tackle the second order differential equations derived from Euler-Lagrange equations. In this case, supersymmetric solutions are not guaranteed and the BPS solutions merely are sub-solutions.

Gauged supergravities necessitate the scalar potentials. They allow supersymmetric AdS backgrounds as implied by negative value of critical points of the potential, so the AdS/CFT correspondence can be established in the gauged supergravity framework. The very first attempt was the AdS_5/CFT_4 correspondence. It is strongly believed that $D = 5, N = 8$ supergravity is a consistent truncation[1] of $D = 10$ type IIB supergravity, then the study RG flows from $D = 5, N = 8$ supergravity is interesting in its own right. Some studies work directly in $D = 10$ type IIB supergravity which is more complex than those in $D = 5$ since some branes and some complicated geometries involve. The AdS_5 space can be achieved by critical solutions of $N = 8$ gauged supergravity in five dimensions with gauge group $SO(6)$ [12, 70, 93]. This theory can be embedded in ten dimensional type IIB supergravity [15]. Some AdS critical points were found by exploring a section of the manifold [71, 17]. The RG flows from those critical points are studied in [18, 19, 20, 21].

After they had reached the peak, the studies of RG flow from five dimensional supergravity began to decline. Meanwhile the studies in three dimensions were performed since the construction of gauged supergravity in three dimensions was fully developed and completely classified [37]. Superficially, three dimensional supergravity seems to be the last one picked since the higher dimensional counterpart had been already extensively explored. The reason why the theories in higher dimensions are quite popular is obvious, one of which is that they can obviously connect to four dimensional theories by dimensional reduction.

Since in this dissertation we focus on three dimensional gauged supergravities and their application on AdS/CFT. We allocate some space to elaborate some aspects of theories in three dimensions. Those include non-supersymmetric as well

[1]Consistent truncation means that a classical solution of lower dimensional theory can be uplifted to a solution of the higher one.

as non-gravitational theories. The AdS/CFT aspects are postponed to the end of the discussion.

Gravity as well as pure supergravity in three dimensions have a well-known feature – they are topological and their propagating degrees of freedom are zero. Consequently, there are no gravitational waves. The gravity coupling to point-particle was studied by Staruszkiewicz back in 1963 [23]. The source-free regions are flat as determined by Einstein equation. The localized sources can affect global geometry, as a result; the conserved quantities such as total energy-momentum is related to topological invariants [24]. Once the cosmological constant is taken into account, the curvature is simply constant outside the sources. Both positive and negative cosmological constant cases are discussed in [25]. All the classical system above can be exactly solved [26]. The three dimensional gravity can be alternatively formulated in the Chern-Simons formalism [46, 27]. Unlike higher dimensional gravity, this feature is exclusive for three dimensional case since it is topological. Note that this Chern-Simons formulation is not the same as those we discuss in this dissertation. The quantum version is also shown to be exactly solved [27]. A thorough review on classical as well as quantum theories in $2+1$ dimensions can be found in [28].

In the early period of the construction, two dimensional supersymmetric theories were exploited as prototype for building three dimensional supergravity theories since their structures are quite similar. The supersymmetric field theory in two dimensions was first constructed in [41] by superfield formalism. One of the theory constructed is $N=1$ supersymmetric non-linear sigma model with a scalar manifold parametrizing 2-sphere where all scalar fields are real. The $N=1$ two dimensional theories with Kähler manifold especially complex projective plane $\mathbb{C}P^{n-1}$ and Grassmann manifold $G_{p,q}=U(p+q)/U(p)\otimes U(q)$ are discussed in [42]. A classification of extended supersymmetric non-linear sigma models is given in [43]. The presence of almost complex structures is required to preserve supersymmetry. The scalar manifold of Kähler and hyperkähler type are extensively discussed as required manifolds for $N=2$ and $N=4$ theories, respectively. The gauge invariant supersymmetric non-linear sigma model in two dimensions is explored in [44]. Superfield formalism is used and gauge fields are introduced via vector superfield. The gauge groups are subgroup of isometry group of the Kähler manifold. Deser and Kay [47] constructed $N=1$ pure supergravity in three dimensions with Chern-Simons term added. The resulting theory contains a massive gravitino. The topological nature of the theory such as gauge invariant with the presence of mass is discussed.

A. Achúcarro and P.K. Townsend [46] proposed an arbitrary N supergravity in three dimensions which is composed of graviton, massive gravitini and Chern-Simons (CS) vectors. This theory described AdS ground state with supergroup $OSp(p|2;\mathbb{R})\otimes OSp(q|2;\mathbb{R})$ where $N=p+q$ and the variations of $p-q$ can give different theories for some fixed N. The CS vectors transform as gauge fields of gauge group $O(p)\otimes O(q)$. Since the whole theory is topological they can be recasted as a new CS theory with supergroup $OSp(p|2;\mathbb{R})\otimes OSp(q|2;\mathbb{R})$.

The ungauged supergravity coupled to matter fields for maximal and half-maximal supersymmetries was originally explored by N. Marcus and J. H. Schwarz [45]. The half-maximal supergravity has $N = 8$ local supersymmetries with $SO(8,n)$ global symmetry. Another theory is the maximal supergravity with $N = 16$ local supersymmetries and $E_{8,8}$ global symmetry. Later ungauged supergravity in three dimensions are systematically and exhausively catagorized by B. de Wit and A. K. Tollstén [36]. The supergravity theories for distinct N local supersymmetry are characterized by various class of scalar manifolds or alternatively called target space. The maximum supersymmetry is $N = 16$ which relied on mathematical rather than physical argument. For $N = 1$, the scalar manifold is Riemannian and for $N = 2$, it is Kähler manifold. For $N = 3$, it is quaternionic and for $N = 4$ it is a product of two quaternionic. For $N > 4$, they are symmetric space and written the coset space G/H. For $N = 5, 6, 8$, the theories can be composed of arbitrary k multiplets. For $N = 9, 10, 12$ and 16, the theories are very unique and can have at most only one multiplet. The details will be discussed in the content of this chapter.

In three spacetime dimensions a vector field can be dualized to a scalar via Hodge duality, so all bosonic degrees of freedom is realized by solely scalar fields and the theory can be constructed without vector fields. Since the supersymmetry is so restricted, no random vectors are allowed to be added without spoiling supermultiplet. This seems to be a huge problem when we try to gauge theory because there are no vector fields to become gauge fields. Unlike any other higher dimensional theories, fortunately, a class of vector fields can be added via Chern-Simons term with proper modifications on transformations and Lagrangian. This term is topological in the sense that the vectors contains no on-shell degrees of freedom and has vanishing field strength tensors.

Gauged supergravity can be constructed by promoting global symmetry to local symmetry while leaving supersymmetry unspoiled. The gauging is carried out by employing the embedding tensor method. This technique replaces the traditional gauging which is non-G-covariant and quite clumsy when we are dealing with many gaugings. It would not be a problem when we specifically study a particular gauge group. In order to classify all spectra of possible theories, it seems to be complicated and labour intensive tasks. However, group representation theory is intensively used compared to the traditional one. In the embedding tensor formalism, the consistent gaugings are implied by the so-called linear and quadratic constraints (on the embedding tensors). The linear constraint is originated from the closure of gauge algebra while supersymmetry necessitates the quadratic constraint.

In three dimensions, CS gauging provides a special class of theory beyond the common YM type. With a certain gauge group, both theories are shown on-shell equivalent. The YM gauged supergravity with semisimple gauge group G is equivalent to a CS gauged supergravity with non-semisimple gauge group $G \ltimes T$ where T is a translation group. The gauged supergravity with non-semisimple gauge groups can be obtained by dimensional reduction from higher dimensional gauged supergravity. However, this is not the case for other gauge groups. The existence of these theories may give us a clue for the new Physics beyond M-theory or at least a new variation thereof.

Some updates on gauged supergravity in three dimensions can be found in [38]. Here we summarize some important issues concerning gauged supergravity in three dimensions in general. Gauged supergravities in three dimensions are abundant since the global symmetry group can be very large. For example, $N = 16$ theory has E_8 global symmetry, so the number gauged theories with subgroups thereof can be large as well. The situation is reversed in higher dimensional theories such as for $D = 4$ the global symmetry is E_7 and for $D = 5$ the global symmetry is E_6. The higher the dimensions, the smaller the global symmetry group.

The first gauged supergravity in three dimensions is maximal supersymmetric ($N = 16$). The compact and non-compact gauge groups are classified and superconformal groups are identified in [48, 49]. The compact gauge groups are in the form of $SO(p, 8 - p) \times SO(p, 8 - p)$ and many the non-compact gauge groups are in the form of exceptional Lie groups. The extension of this work which covers the non-semisimple and complex gauge groups are done by [50]. They also discussed the connection to IIA/IIB supergravity in ten dimensions reduction of 7-sphere. The half-maximal theory $N = 8$ with coset manifold $SO(8, n)/(SO(8) \times SO(n))$ is gauged in [51]. The compact gauge groups are in the form of $SO(p, 4 - p) \times SO(q, 4 - q)$ where p and q are either 0 or 2. The various non-compact gauge groups are also specified. The authors also linked the relation between $N = (4, 4)$ theory with gauge group $SO(4) = SO(3)_L \times SO(3)_R$ to supergravity in six dimensions with supersymmetry $N = (2, 0)$ compactified on $AdS_3 \times S^3$.

Not so long after the construction of the theories and the classification of some admissible gauge groups, the series of studies started to focus on searching the critical points led by [52]. Vacua of $N = 16$ gauged supergravity are intensively studied. Many critical points from various gauge groups such as compact, non-compact and exceptional are found, and some of which admit AdS vacua; some admit Minkowskian and dS vacua. The mass spectra of bosonic and fermionic around the vacua are computed. The superconformal symmetry at critical points are also identified as well as their irreducible representation under subgroup of superconformal groups.

The gauged supergravity in three dimensions are completely categorized in [37]. The admissible gauge groups are dictated by a group theoretical argument; it is that it must not contain a particular representation in the T-tensor. For compact gauging, all the embedding tensors are given including $N = 5, 6$ which is the main subject of our study. Since this paper plays a crucial part for this dissertation, the details will be intensively explored in the content of this chapter. The connection between AdS_3 and $SCFT_2$ was traced back in 1986 before the Maldacena conjecture. J. D. Brown and Marc Henneaux [29] have shown that a gravitational theory in AdS_3 corresponds to a two dimensional conformal field theory with two commuting Virasoro algebra. After the year of Maldacena conjecture, the supersymmetric version of [29], the connection of $N = (1,1)$ AdS_3 supergravity and $SCFT_2$ was established and they [30] found the same central charge $c = 3L/2G$ as in non-supersymmetric case. The study for extended supergravity case can be found in [31].

A very first evidence for AdS_3/CFT_2 correspondence is the type IIB string theory on $AdS_3 \times S^3 \times M_4$ conjectured by Maldacena. It has been shown by [32, 33] that it is equivalent to $N = (4,4)$ superconformal field in two dimensions when exploring in the IR limit (near-horizon) of D1-D5 system for $M_4 = K3$ or T^4. In the supergravity realm, H. Nicolai and H. Samtleben [34] constructed the non-semisimple gauge group $SO(4) \times T_\infty$ from $N = 8$ supergravity with coset space $SO(8,\infty)/(SO(8) \times SO(\infty))$ where T_∞ is an infinite dimensional translational subgroup of $SO(8,\infty)$. This theory can be linked, like the theory mentioned above, to $N = (2,0)$ supergravity in six dimensions compactified on $AdS_3 \times S^3$ with Kaluza-Klien towers left untruncated.

In three dimensions, the very first analytic RG flow was studied in [53]. In the AdS side, it is described by $N = 8$ gauged supergravity with gauge group $SO(4) \times SO(4)$. The UV fixed points is $N = (4,4)$ supersymmetric; and the flow is driven by relevant operator of scale dimension $\Delta = 3/2$ and sends the theory to IR fixed point with residual supersymmetry $N = (1,1)$. The ratio of central charges $c_{UV}/c_{IR} = 2$. For $N = 2$, some RG flows are studied in [35]. The manifold of the theory is Kählerian, so there are some degrees of arbitrariness in choosing the target space manifold. This work focused on Kähler manifold of $SU(n+1)/SU(n) \times U(1)$ and $SU(n,1)/SU(n) \times U(1)$ type. In particular he consider only for $n = 1$ cases, so they are $S^2 = SU(2)/U(1)$ and $H^2 = SU(1,1)/U(1)$.

The vacua in $N = 10$ theory are provided by [54]. They are the ground states of gauged supergravity with the compact gauge groups : $SO(p) \times SO(10-p)$ for $p = 6, \cdots, 10$ and $SO(5) \times SO(5)$. For non-compact gauge groups, they are $SU(4,2) \times SU(2)$, $G_{2(-14)} \times SU(2,1)$ and $F_{4(-20)}$. Many trivial and non-trivial AdS_3 critical points are found and superconformal groups are identified at each critical point. Subsequent study for $N = 10$ is the study of Chern-Simons gauged supergravity with non-semisimple gauge group $SO(5) \ltimes \mathbf{T}^{10}$ [57]. This theory is on-shell equivalent to $SO(5)$ Yang-Mills gauged supergravity which can be derived from $N = 5$ gauged supergravity in four dimensions compactified on a circle. The author also found out that the theory admits $\frac{1}{4}$-BPS domain wall solution rather than $\frac{1}{2}$-BPS domain wall solution which is usually found in other cases.

For $N = 9$ theory, the vacua as well as their RG flow are investigated in [55]. They explored the compact gauge groups of the type $SO(p) \times SO(9-p)$ for $p = 0, 1, 2, 3$ and 4; for non-compact, they are as follows: $G_{2(-14)} \times SL(2)$ and $Sp(1,2) \times SU(2)$. Many supersymmetric and non-supersymmetric AdS ground states are found and some of which can be used to study RG flow. The RG flow solutions are obtained analytically and numerically for compact gauge groups: $SO(7) \times SO(2)$, $SO(6) \times SO(3)$, $SO(5) \times SO(5)$ and non-compact gauge group $G_{2(-14)} \times SL(2)$.

In $N = 8$ gauged supergravity, some supersymmetric AdS_3 vacua are found in [56]. They studied the theory with $(SO(4) \ltimes \mathbf{T}^6)^2$ non-semisimple gauge group in the context of Chern-Simons gauging. In that paper they also studied the vacua for $N = 4$ with the $SO(4) \ltimes \mathbf{T}^6$ gauge group and in particular the RG flow interpolates between two AdS_3 vacua. As examples, two flows are studied; both of which are associated with vacua that have the same amount supersymmetry. First, $N = (3, 1)$ supersymmetry flow, the analytical solutions are found whereas in another flow ($N = (2, 0)$) the solutions are numerically obtained. In both cases, the flows are driven by vacuum expectation values (v.e.v) of some operators. The authors also claimed that these flows were the very first examples of v.e.v flows in the context of gauged supergravity. There is a subsequent study for $N = 4$ and $N = 8$ [58]. Its objectives are to construct gauged supergravity for non-semisimple gauge group $N = 4$ and $N = 8$ supersymmetry. For $N = 4$ case, the gauge groups is $SO(3) \ltimes (\mathbf{T}^3, \hat{\mathbf{T}}^3)$ in the context of Chern-Simons gauging and it is equivalent to Yang-Mills type with gauge group $SO(3)$ coupled to three massive vector fields. It is also the resulting theory of $N = (1, 0)$ six dimensional supergravity reduced on an $SU(2)$ group manifold. For $N = 8$ case, the gauge group is $SO(8) \ltimes \mathbf{T}^{28}$ and it can be truncated from $SO(8) \ltimes \mathbf{T}^{28}$ gauged supergravity in $N = 16$ theory. Moreover, it also can be considered as a theory obtained by compactification of type I supergravity on 7-sphere. In application to Domain wall/QFT, the domain walls solutions of both theories are analytically found and they can be uplifted to the solutions of higher dimensional theories.

The dissertation is based on the works in [39, 40]. We study $N = 5$ and $N = 6$ Chern-Simons gauged supergravities in three dimensions. They are classified into three categories: compact, non-compact and non-semisimple gauge groups. For $N = 5$ case, the manifold is in the form of $G/H = USp(4, k)/USp(4) \times USp(k)$, whereas we restrict to $k = 2, 4$. For $N = 6$ case, the manifold is in the form of $G/H = SU(4, k)/S(U(4) \times U(k))$, whereas we restrict to $k = 1, 2, 3, 4$. The gauge groups G_0 are subgroups of isometry group G. The gauging is executed by the technique of embedding tensors and the various gaugings are characterized by distinct embedding tensors. For the compact cases, the embedding tensors are already determined by [37]. For non-compact cases, the additional task is the consistency checking and it has to be done case by case. For non-semisimple gauging, the gauge group is in the form of $SO(N) \ltimes \mathbf{T}^{\dim SO(N)}$ where $SO(N)$ is R-symmetry group of N theory, so we consider $SO(5) \ltimes \mathbf{T}^{10}$ for $N = 5$ and $SO(6) \ltimes \mathbf{T}^{15}$ for $N = 6$. For $N = 5$ case, we study four RG flows, two of which are the flow between AdS vacua in $SO(5) \times USp(2)$ gauging. The other two are the flows between AdS vacua in non-compact gauging with gauge group $USp(2) \times USp(2, 2)$. For $N = 6$ case, all four RG flows we studied are of compact case because in non-compact gauging there is no non-trivial AdS critical point. The first two flows interpolate between supersymmetric critical points of the theory with gauge group $SO(6) \times SU(4) \times U(1)$ while the other two corresponding gauge group $SO(4) \times SO(2) \times SU(4) \times U(1)$.

We devote this section to discuss the outline for each chapter. In chapter 2, we review the structure gauged supergravity in three dimensions, holographic RG flows, Weyl anomaly and c-theorem. We start the chapter with a discussion on massless Poincaré supermultiplet. The supersymmetric algebra form a Clifford algebra and some mathematical properties thereof are discussed. The by-products are that the number of irreducible multiplets and centralizer for each supersymmetry N are specified. These ingredients play an important role in determining the shape of the scalar manifold. Next, we explore the ungauged supergravity in three dimensions. We first discuss the pure supergavity which is composed of graviton and gravitini and the theory exists for arbitrary supersymmetry N. Since in three dimensions both particles carry no on-shell degrees of freedom, the pure supergravity is simply topological. The matter supermultiplet can be in the form of supersymmetric non-linear sigma model and it exists for some value of N. The ungauged supergravity can be constructed by coupling the supersymmetric non-linear sigma model with the pure supergravity. The mathematical argument sets the bound of supersymmetry N to 16 as well as excludes some theories with $N = 7, 11, 13, 14, 15$. The supersymmetry puts severe constraint on scalar manifold, so it is fairly to say that supersymmetry shape up the scalar manifold. The more value of N, the more restricted the manifold is. For $N > 4$, the scalar manifolds are symmetric space and they can be specified in the coset G/H. Next, we discuss their symmetry beyond general coordinate transformation, local Lorentz and supersymmetry. The symmetry is isometries of scalar manifold generated by Killing vectors that take value in Lie algebra $\mathfrak{g}$. This symmetry is global because the parameters of transformation do not depend on spacetime. Then we move on to the discussion on equivalence between CS and YM gauging. In order to convert from YM to CS, a vector field in YM is exchanged for two CS vector fields and a scalar. In the next step, we will gauge this theory by promoting global to local symmetry. In doing so, we deviate from the traditional routine by introducing the notion of embedding tensor. In the gauging, the gauge fields are introduced via CS terms along with modification of some expressions and introduction of fermionic masslike term as well as a scalar potential. Not all subgroup of isometries can be gauged without sacrificing supersymmetry. In order to preserve supersymmetry the so-called T-tensor for a particular gauging must satisfies a constraint. Vacua of the theory are determined by the critical points of the scalar potential and we will discuss their symmetry at those points as well. The discussion on supergravity in three dimensions ends here and then we move on to holographic RG flows. We begin with the general concept of renormalization group in quantum field theory and then discuss RG flow in the context of AdS/CFT correspondence. We end this chapter with discussions on Weyl anomaly and c-theorem.

In the chapter 3, we study $N = 5$ Chern-Simons gauged supergravity in three dimensions. The gauge groups are classified into three classes: compact, non-compact and non-semisimple gauge groups. The scalar manifold is in the form of coset space $USp(4, k)/USp(4) \times USp(k)$ where, in general, k is even integer. Due to the complication in calculation, we restrict ourselves to only $k = 2, 4$ cases. We obtained many AdS_3 critical points via extremization the scalar potential. In each critical point we determine the unbroken supersymmetry as well as the residual gauge symmetry. Moreover, the scalar mass spectra are specified in the representation of unbroken gauge group. The non-semisimple group is in the form of $SO(5) \ltimes \mathbf{T}^{10}$ and it can be related to higher dimensional theories via dimensional reduction. In chapter 4, we study $N = 6$ CS gauged supergravity in three dimensions. The scalar manifold is in the form of coset $G/H = SU(4,k)/S(U(4)\times U(k))$, whereas we restrict to $k = 1, 2, 3, 4$. The non-semisimple case is in the form of $SO(6) \ltimes \mathbf{T}^{15}$. The format of this chapter is similar to the chapter 3. In the chapter 5, we study RG flow for $N = 5, 6$ using the results of chapter 3 and 4. For $N = 5$ theory, we explore four cases; the first two are of compact cases and the other two are of non-compact gaugings. For $N = 6$ theory, we consider four cases in compact gaugings. The solutions are analytical and exact. However, the approximated solutions also are required to determined some dynamical contents of the theory such as scale dimension. The ratio of central charges is also specified in each case and in agreement with c-theorem.

The conclusion and comments for this dissertation are given in the last chapter as well as open problems and future works are discussed. In the appendix, there are three sections. In the first section, we overview basic supersymmetry and supergravity. In the second section, we give the details of branching of T-tensors for $N = 5, 6$ theory. In the last section, we review the mathematical structure of Euler parametrization. Two examples are explicitly given, the first one is from parametrizing a coset in $N = 5$ theory and the other is in $N = 6$ theory.

CHAPTER II

3D Supergravity and RG Flow

This chapter is devoted to a review on the construction of three dimensional ungauged and gauged supergravity. We begin with an introduction to the subject and then we move on to massless Poincaré supermultiplets and Clifford modules. Next, we discuss the ungauged theory and present the ungauged Lagrangian as well as explore its target spaces. We then discuss the global symmetry of the theory. Before we jump into the gauged theory, we compare and contrast between Chern-Simons and traditional Yang-Mills gauging. Next, we explore the gauged theory which is achieved by the notion of embedding tensors. We also devote a section on studying the vacua and their symmetry. Moreover, we discuss the application of AdS/CFT on holographic renormalization group flow. For more information on ungauged and gauged supergravity, we refer the readers to the original papers [36, 37, 38] and a concise review on RG flow, Weyl anomaly and c-theorem, we refer to [5].

2.1 General aspects

In three dimensions, pure supergravity contains a graviton and a specific number of gravitini where the number of gravitini is equal to number of arbitrary N supersymmetries. The matter sector theory which is composed of scalar fields parametrizing a target space and spin one-half fermions can be described by a supersymmetric non-linear sigma model. For rigidly supersymmetric non-linear sigma model, the theories are bounded to $N \leq 4$, whereas in locally supersymmetric cases the bound does not exist. Once a non-linear sigma model is coupled to extended supergravity, the resulting theories are bounded to $N \leq 16$ and exist for some values of N. The argument that restricts the number of supersymmetries is purely mathematical and it makes the theories in three dimensions special compared higher dimensional theories. For example, in four dimensinal spacetime, the extended supergravity is bound to $N \leq 8$ because we require no massless particles with spin more than two since the consistent interacting theories does not exist. The physical argument above cannot apply to the three dimensional cases since helicity is not well-defined in three dimensions. The on-shell degrees of freedom of various particles in three dimensional spacetime is given in the table I. According to the table I graviton and gravitino have no propagating degrees of freedom. That agrees with the fact that pure supergravity is topological. Another unique feature in three dimensional case is that vector fields can be dualized to scalar fields, as a

Spin	Particle	On-shell d.o.f.
0	Scalar	1
$\frac{1}{2}$	Fermion	1
1	Vector (via CS term)	0
$\frac{3}{2}$	Gravitino	0
2	Graviton	0

Table I: On-shell degrees of freedom for each particle in three dimensions

result the bosonic degrees of freedom can be solely described by scalar fields. On the fermionic side, the degrees of freedom are counted solely from spin one-half fermions.

Local supersymmetry coupled to supersymmetric non-linear sigma model restricts the class of target space manifolds as shown in [36]. The argument is geometrical and can be briefly summarized in the diagram below.

$$\begin{aligned} \text{local SUSY} &\rightarrow \text{almost complex structure} \rightarrow \text{curvature tensor} \rightarrow \\ \text{holonomy} &\rightarrow \text{target space} \end{aligned}$$

The N extended local supersymmetry requires the existence of $N-1$ almost complex structure. Moreover, the local supersymmetry put constraints on the Riemann curvature tensor R_{ijkl} and this tensor is related to holonomy via Ambrose-Singer's theorem on holonomy [60]. If the holonomy group is determined, the target space manifold can be specified. The key results of [36] is the classification of target space for each supersymmetry N. We discuss them in detail in the section 2.4.

2.2 Massless $D = 3$ Poincaré supermultiplets

In three dimensional spacetime, irreducible representation of supercharges are Majorana with $SO(N)$ automorphism group or R-symmetry group. The anticommutator of supercharges of three dimensional rigid supersymmetry is

$$\big\{Q^I_\alpha\,,\bar{Q}^J_\beta\big\} = -2i\,\delta^{IJ}\gamma^\mu_{\alpha\beta}\,P_\mu\,, \qquad (I,J=1,\dots,N) \tag{2.2.1}$$

where Q^I_α are Majorana supercharges and P_μ is the translation generator. Recall Dirac conjugate $\bar{Q}^I_\alpha = i\,Q^{I\dagger}_\beta\,\gamma^0_{\beta\alpha}$ and Majorana condition, we pick the massless states in a particular frame of reference where $P^0 = P^1 = \omega$, $P^2 = 0$; the algebra now takes the form

$$\big\{Q^I_\alpha\,,Q^J_\beta\big\} = 2\omega\,\delta^{IJ}\big(\mathbf{1}+\sigma_3\big)_{\alpha\beta}. \tag{2.2.2}$$

The Q_2^I must annihilate states, $Q_2^I\,|p^\mu, s\rangle = 0$, so the leftover real supercharges Q_1^I now act as the creation operator acting on Hilbert space. The non-trivial part of the algebra now is

$$\{\frac{Q_1^I}{2\sqrt{\omega}}, \frac{Q_1^J}{2\sqrt{\omega}}\} = \delta^{IJ} \,. \tag{2.2.3}$$

It is N-dimensional Clifford algebra with signature (p,q) where $q = 0$. In order to construct a supermultiplet, fermion number operator $\mathbf{F}$ satisfying $\mathbf{F}^2 = \mathbf{1}$ and $\{\mathbf{F}, Q_1^I\} = 0$ is required. All in all a massless supermultiplet are representations of a real $(N+1)$-dimensional Clifford algebra with signature $(p,q) = (N+1, 0)$ or $Spin(N+1)$. Now we will use Γ^I instead of Q_α^I in discussing representation of Clifford algebra. The explicit form of Γ^I and $\mathbf{F}$ real representation are chosen to be

$$\Gamma^I = \begin{pmatrix} 0 & \Gamma^I_{A\dot{D}} \\ \Gamma^I_{\dot{B}C} & 0 \end{pmatrix} \,, \qquad \mathbf{F} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{2.2.4}$$

where Γ^I and $\mathbf{F}$ are $(2d \times 2d)$-dimensional matrices and $A, B, \ldots = 1, \ldots, d$ are bosonic indices and $\dot{A}, \dot{B}, \ldots = 1, \ldots, d$ are fermionic indices. As a result the Clifford algebra can be written as

$$\Gamma^I_{A\dot{C}}\,\Gamma^J_{B\dot{C}} + \Gamma^J_{A\dot{C}}\,\Gamma^I_{B\dot{C}} = 2\delta^{IJ}\delta_{AB} \,. \tag{2.2.5}$$

Another mathematical concept that is needed to be specified is *centralizer*. Roughly speaking, it can be defined as elements in G that commutes with a particular subset (or subgroup) S. In our case, they are a set of matrices that commute to Γ^I and fermion number operator $\mathbf{F}$. According to Shur's lemma, they form a division algebra[1] and by Frobenius theorem they are isomorphic to fields as follows: real numbers $\mathbb{R}$ with basis $\{I\}$ where I is an identity matrix, complex numbers $\mathbb{C}$ with basis $\{I, j\}$ where $j^2 = -1$ and quaternions $\mathbb{H}$ with basis $\{I, e_1, e_2, e_3\}$ where $e_i^2 = -1$ and $e_i\, e_j = -\delta_{ij} + \sum_{k=1}^{3} \epsilon_{ijk}\, e_k$ [61]. For $\mathbb{C}$, the centralizer corresponds to group $U(1)$ and for $\mathbb{H}$, the centralizer corresponds to group $SU(2)$. As an example, some constructions of representation has been worked out in [36]. It builds up from $N = 1$ to $N = 8$, i.e. $N = 1, 2, 4, 8$, by tensor product with Pauli matrices. The procedure requires no further analysis since the pattern repeats itself for $N > 8$. An intermediate values of N can be constructed by embedding in larger N representation.

Note that, at this stage, there is no limitation on the value of N as we mentioned earlier; however, the value of N is bounded and restricted to some values once we discuss local supersymmetry coupled to non-linear sigma model. So far we have discussed irreducible representation and it corresponds to a single multiplet. In order to extend the results to arbitrary k multiplets, the reducible representations are sufficient. The centralizers generate group $SO(k)$, $U(k)$ or $USp(k)$ corresponding to the division algebra of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively. The dimension d of manifold is related to number k of supermultiplets and number

[1] It requires the following properties : addition, subtraction, multiplication and division. Division requires a unique element of x that satisfies $a = b\,x$ where a, b and x are all elements in division algebra, for example, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

N	d_N	**Centralizer**
1	1	$\mathbb{R}$
2	2	$\mathbb{C}$
3	4	$\mathbb{H}$
4	4	$\mathbb{H}$
5	8	$\mathbb{H}$
6	8	$\mathbb{C}$
7	8	$\mathbb{R}$
8	8	$\mathbb{R}$
$8+n$	$16d_n$	same as n

Table II: Irreducible massless supermultiplets where d_N is the number of bosonic states and $n = 1, 2, \ldots, 8$.

d_N of bosonic states of an irreducible supermultiplet by $d = kd_N$. The content discussed above not only tells us about degrees of freedom for each multiplet, it also helps us to determine the holonomy group and then the shape of target space manifold as we will discuss later in the forthcoming sections.

2.3 The Ungauged Theory

Before we consider a pure supergravity coupled to supersymmetric non-linear sigma model, let us discuss them separately. Pure supergravity is composed of a graviton and gravitini whose fields are dreibein ${e_\mu}^a$ and Rarita-Schwinger fields ψ^I_μ. In three dimensions they are topological and both type of particles in theory have no on-shell degrees of freedom. It exists for any number of supersymmetries N. The resulting Lagrangian is composed of two sectors: the Einstein-Hilbert action written in an alternative form and Rarita-Schwinger Lagrangian modified to couple to graviton.

$$\mathcal{L}_{\text{s.g.}} = -\frac{1}{2} i\epsilon^{\mu\nu\rho}({e_\mu}^a\, R_{\nu\rho a}(\omega) + \bar{\psi}^I_\mu D_\nu(\omega)\psi^I_\rho). \qquad (2.3.1)$$

where ω^a_μ is the spacetime spin connection. For the whole dissertation, we use the Pauli-Källén metric with hermitean gamma matrices γ^a satisfying

$$\begin{aligned} \gamma_a\gamma_b &= \delta_{ab} + \mathrm{i}\varepsilon_{abc}\gamma^c\ , \\ \gamma_{[a}\gamma_b\gamma_{c]} &= i\varepsilon_{abc}\ , \\ \gamma_{ab} &\equiv \gamma_{[a}\gamma_{b]} = i\varepsilon_{abc}\,\gamma^c\ , \end{aligned} \qquad (2.3.2)$$

The covariant derivative acting on Rarita-Schwinger field is given by

$$D_\mu(\omega)\psi^I_\rho = (\partial_\mu + \frac{1}{2}\omega^a_\mu\gamma_a)\psi^I_\rho \qquad (2.3.3)$$

The Lagrangian (2.3.3) is locally supersymmetric and the supersymmetric transformations are as follows:

$$\begin{aligned} \delta e_\mu{}^a &= \tfrac{1}{2}\,\bar\epsilon^I \gamma^a\,\psi^I_\mu\,, \\ \delta\psi^I_\mu &= D_\mu(\omega)\epsilon^I\,, \end{aligned} \tag{2.3.4}$$

where ϵ^I is a supersymmetry transformation parameter. It also has other symmetry such as, spacetime diffeomorphism and local Lorentz invariance.

The Lagrangian of supersymmetric non-linear sigma model is composed of scalar fields ϕ_i acting as coordinates in target space and their superpartner spin 1/2 fermions χ_i with $i = 1, ..., d$. Now we are discussing the rigidly supersymmetric theory and the locally supersymmetric non-linear sigma model will be discussed later in this chapter. The rigidly supersymmetric non-linear sigma model sometimes is called matter sector and it is given by

$$\mathcal{L}_{\text{matter}} = -\frac{1}{2} g_{ij}(\phi)[\partial_\mu\phi^i\,\partial^\mu\phi^j + \bar\chi^i \not{D}(\Gamma)\chi^j] + \mathcal{L}_{\chi^4}\,, \tag{2.3.5}$$

where Γ is the Christoffel symbol for rigid supersymmetry but it can be arbitrary connection for locally symmetric theory. The covariant derivative is given by

$$D_\mu(\Gamma)\,\chi^i \equiv \partial_\mu\chi^i + \Gamma^i_{jk}(\phi)\,\partial_\mu\phi^j\,\chi^k\,; \tag{2.3.6}$$

and this definition can be applicable to any connection Γ. The four fermion term is quadratic in bilinear term of fermions and it is proportional to Riemann tensor of the target space,

$$\mathcal{L}_{\chi^4} = -\frac{1}{24} R_{ijkl}(\phi)\;\bar\chi^i\gamma_a\chi^j\;\bar\chi^k\gamma^a\chi^l\,. \tag{2.3.7}$$

A geometrical quantity which also plays an important role in analyzing manifold of the target space is the so-called almost complex structure[2] $f^{Pi}{}_j(\phi)$. It is originally introduced to the theories via rigid supersymmetric transformation ansatz [43]. In order to preserve supersymmetry, these tensors satisfy many identities so that they have a natural interpretation as almost complex structures. They are introduced to the theory when $N > 1$ therefore $P = 2, ..., N$ and $i, j = 1, .., d$ where d is the dimension of the target space. They are hermitean in the sense that

$$g_{ij}\,f^{Pj}{}_k + g_{kj}\,f^{Pj}{}_i = 0\,. \tag{2.3.8}$$

The supersymmetry algebra (2.2.1) implies

$$f^{Pi}{}_k\,f^{Qk}{}_j + f^{Qi}{}_k\,f^{Pk}{}_j = -2\,\delta^{PQ}\,\delta^i_j\,. \tag{2.3.9}$$

which can be considered as Clifford algebra. Consequently, we can construct $\frac{1}{2}N(N-1)$ SO(N) generators f^{IJ}_{ij} via the definition

$$f^{PQ} = f^{[P}\,f^{Q]}\,, \qquad f^{1P} = -f^{P1} = f^P\,. \tag{2.3.10}$$

[2] An almost complex structure f on a manifold M is a smooth tensor rank (1,1) with the propery $f_p^2 = -id_{T_pM}$. If this tensor can be defined globally, the manifold is a complex manifold.

They are anti-symmetric tensor for both pair indices i, j and I, J:

$$f_{ij}^{IJ} = -f_{ij}^{JI} = -f_{ji}^{IJ} \,. \tag{2.3.11}$$

The SO(N) commutation relation (in obvious matrix form) can be written as

$$f^{IJ}\, f^{KL} - f^{KL}\, f^{IJ} = 4\,\delta^{K[I}\, f^{J]L} - 4\,\delta^{L[I}\, f^{J]K} \,, \tag{2.3.12}$$

Moreover, they satisfy the following identities

$$\begin{aligned} f^{IJ}\, f^{KL} &= f^{[IJ}\, f^{KL]} - 4\,\delta^{[I[K}\, f^{L]J]} - 2\,\delta^{I[K}\,\delta^{L]J}\,\mathbf{1}\,, \\ f^{IJ\,ij}\, f^{KL}{}_{ij} &= 2d\,\delta^{I[K}\,\delta^{L]J} - \delta_{N,4}\,\varepsilon^{IJKL}\,\mathrm{Tr}(J)\,. \end{aligned} \tag{2.3.13}$$

J tensor is relevant for only $N = 4$ theory and its definition is $J = \frac{1}{6}\,\varepsilon_{PQR}\, f^P\, f^Q\, f^R$. Other identities and a discussion for J are mentioned in [36]. Once pure supergravity coupled to supersymmetric non-linear sigma model, the covariant derivative is modified by SO(N) target-space connection $Q_i^{IJ}(\phi)$ as will be given later. Its role is quite similar to a gauge field in gauge theory when IJ is replaced by gauge group index and i is traded for spacetime indices.

Note that the vanishing of variation on $\partial\phi\,\partial\phi\,\chi\,\epsilon$ and $\partial\phi\,\partial\phi\,\psi\,\epsilon$ requires that the $SO(N)$ curvature must satisfy the following condition

$$R_{ij}^{IJ}(Q) \equiv \partial_i\, Q_j^{IJ} - \partial_j Q_i^{IJ} + 2Q_i^{K[I} Q_j^{J]K} = \frac{1}{2} f_{ij}^{IJ} \,. \tag{2.3.14}$$

The equation above shows the relation between SO(N) connections $Q_i^{IJ}(\phi)$ and f_{ij}^{IJ} where $R_{ij}^{IJ}(Q)$ is SO(N) field strength tensor. The tensor f^{IJ} are covariantly constant in the sense that

$$D_i\,(\Gamma, Q)\, f_{jk}^{IJ} \equiv \partial_i f_{jk}^{IJ} - 2\,\Gamma_{i[k}{}^{l}\, f_{j]l}^{IJ} + 2\,Q_i^{K[I}\, f_{jk}^{J]K} = 0\,. \tag{2.3.15}$$

From (2.3.15) and (2.3.14) one can derive the integrability condition

$$R_{ijmk}\, f^{IJ\,m}{}_{l} - R_{ijml}\, f^{IJ\,m}{}_{k} = -f_{ij}^{K[I}\, f_{kl}^{J]K} \tag{2.3.16}$$

Contracting (2.3.16) with f^{MNkl}, one gets

$$R_{ijkl}\, f^{IJ\,kl} = \tfrac{1}{4} d\, f_{ij}^{IJ} \,. \tag{2.3.17}$$

Contracting (2.4.1) with g^{jl} and using cyclicity of Riemann tensor and (2.3.17), one can derive Ricci tensor as the following

$$R_{ij} \equiv R_{ikjl}\, g^{kl} = c\, g_{ij} \,, \tag{2.3.18}$$

with constant

$$c = N - 2 + \tfrac{1}{8} d > 0 \,, \tag{2.3.19}$$

so we are dealing with Einstein space ($R_{ij} \propto g_{ij}$). In order to solve for the Riemann tensor, we introduce antisymmetric tensor h_{ij}^{α} that commute with the complex structure, i.e.,

$$h_{ik}^{\alpha}\, f^{IJ\,k}{}_{j} - h_{jk}^{\alpha}\, f^{IJ\,k}{}_{i} = 0\,. \tag{2.3.20}$$

The tensor $h^{\alpha i}_j$ is called centralizer subgroup $\mathrm{H}' \subset \mathrm{SO}(d)$ which obviously commutes with the group $\mathrm{SO}(N)$. Let $f^{\alpha\beta}{}_\gamma$ is structure constant for H' subgroup, so the Lie algebra is the following

$$h^\alpha\, h^\beta - h^\beta\, h^\alpha = f^{\alpha\beta}{}_\gamma\, h^\gamma\ . \tag{2.3.21}$$

The normalization is chosen to be

$$h^\alpha_{ij} h^{\beta\, ij} = 2 d_N \delta^{\alpha\beta}\, . \tag{2.3.22}$$

They are covariantly constant in the following sense

$$D_i(\Gamma) h^\alpha_{jk} - \Omega_i^\alpha{}_\beta\, h^\beta_{jk} = 0\, , \tag{2.3.23}$$

where $\Omega_i^{\alpha\beta}$ is a connection associated with h^α_{ij}. After a some more analysis, one conclude that

$$R_{ijkl} \;=\; \frac{1}{8}\left(f^{IJ}_{ij}\, f^{IJ}_{kl} + C_{\alpha\beta}\, h^\alpha_{ij}\, h^\beta_{kl} \right)\, , \tag{2.3.24}$$

where $C_{\alpha\beta}$ is a symmetric tensor and it varies with N. $C_{\alpha\beta}$ plays an important role in analysis of manifold. A solution for $C_{\alpha\beta}$ is in the form of $C_{\alpha\beta} \propto \delta_{\alpha\beta}$. The detailed discussion about $C_{\alpha\beta}$ can be found in [36] and the solutions of $C_{\alpha\beta}$ for the orther cases are discussed in the appendix of the reference thereof.

Having obtained the explicit form of R_{ijkl} in term of f^{IJ}_{ij} and h^α_{ij}, many identities can be derived. Contracting (2.4.1) with metric as in (2.3.18) and using c in (2.3.19), one gets

$$C_{\alpha\beta}\, h_i^{\alpha\, k}\, h^\beta_{kj} = \big[N(N-1) - 8c \big]\, g_{ij}\, . \tag{2.3.25}$$

From above equation, we can derive the condition involving H'-invariant tensor [3] on $C_{\alpha\beta}$:

$$C_{\delta(\alpha}\, f^{\delta\gamma}_{\beta)}\, h^\alpha h^\beta = 0\, . \tag{2.3.26}$$

The constraint on cyclicity of Reimann curvature tensor (2.4.1) implies

$$f^{IJ}_{[ij}\, f^{IJ}_{kl]} + C_{\alpha\beta}\, h^\alpha_{[ij}\, h^\beta_{kl]} = 0\, . \tag{2.3.27}$$

Since the holonomy group can be determined from the Riemann tensor, equation (2.3.27) tells us that the holonomy group should be in $\mathrm{SO}(N) \times \mathrm{H}' \subset \mathrm{SO}(d)$ and also acts irreducibly on the target space manifold. Similar to $\mathrm{SO}(N)$ curvature tensor (2.3.14), the H' curvature tensor can be defined by

$$R^\alpha{}_{\beta ij} \equiv 2(\partial_{[i}\Omega_{j]}{}^\alpha{}_\beta - \Omega_{[i}{}^\alpha{}_\gamma\, \Omega_{j]}{}^\gamma{}_\beta) = \tfrac{1}{8} f^{\alpha\gamma}{}_\beta\, C_{\gamma\delta}\, h^\delta_{ij}\ , \tag{2.3.28}$$

We devote the rest of this section to discuss the ungauged Lagrangian and its supersymmetry transformations; in this theory no matter fields are charged under

[3] $C_{i_1,\cdots,i_m}$ is an invariant tensor if it satisfies $\sum_{s=1}^m f^\alpha_{\beta i_s}\, C_{i_1,\cdots,\hat{i}_s\alpha,\cdots,i_m} = 0$, where the hatted component is omitted.

some gauge groups. Before we proceed, it is useful to define a new set of matter fermions in an SO(N) covariant

$$\chi^{iI} = \left(\chi^i, f^{Pi}{}_j\,\chi^j\right)\ , \tag{2.3.29}$$

which is subject to projection constraint

$$\chi^{iI} \;=\; \mathbb{P}^{I\,i}_{Jj}\,\chi^{jJ} \;\equiv\; \frac{1}{N}\left(\delta^{IJ}\delta^i_j - f^{IJ\,i}{}_j\right)\chi^{jJ}\ , \tag{2.3.30}$$

where $\mathbb{P}^{Ii}_{Ii} = d$. The projection above reduces the independent part from dN to d so the number d of the fermion is left unchanged. This $SO(N)$ covariant notation helps us treat both matter fermions and gravitini on the same ground. It is a matter of convenience and it does not imply $SO(N)$ invariance of theory even though the $SO(N)$ tensors of Lagrangian itself appear in the contracted form.

The ungauged Lagrangian can be obtained by combining pure supergravity with supersymmetric non-linear sigma model and adding a Noether term which is required by supersymmetry. The covariant derivative is modified by $SO(N)$ target space connection Q_i^{IJ}. The target space Christoffel connection is replaced by a more general connection Γ. We give the final form of the ungauged Lagrangian here,

$$\begin{aligned}\mathcal{L}_0 \;=\; & -\frac{1}{2}\,\mathrm{i}\,\varepsilon^{\mu\nu\rho}\left(e_\mu{}^a\,R_{\nu\rho a} + \bar{\psi}^I_\mu D_\nu\psi^I_\rho\right) - \frac{1}{2}e\,g_{ij}\left(g^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\phi^j + N^{-1}\bar{\chi}^{iI}\not{D}\chi^{jI}\right)\\ & + \frac{1}{4}e\,g_{ij}\,\bar{\chi}^{iI}\gamma^\mu\gamma^\nu\psi^I_\mu\,(\partial_\nu\phi^j + \widehat{\partial}_\nu\phi^j) - \frac{1}{24}e\,N^{-2}R_{ijkl}\,\bar{\chi}^{iI}\gamma_a\chi^{jI}\,\bar{\chi}^{kJ}\gamma^a\chi^{lJ}\\ & + \frac{1}{48}e\,N^{-2}\left(3\,(g_{ij}\,\bar{\chi}^{iI}\chi^{jI})^2 - 2(N-2)\,(g_{ij}\,\bar{\chi}^{iI}\gamma^a\chi^{jJ})^2\right)\ . \end{aligned}\tag{2.3.31}$$

The covariant derivatives for gravitini and spin-1/2 fermions are given by

$$\begin{aligned} D_\mu\psi^I_\nu \;&=\; \left(\partial_\mu + \tfrac{1}{2}\omega^a_\mu\,\gamma_a\right)\psi^I_\nu + \partial_\mu\phi^i\,Q^{IJ}_i\psi^J_\nu\ ,\\ D_\mu\chi^{iI} \;&=\; \left(\partial_\mu + \tfrac{1}{2}\omega^a_\mu\,\gamma_a\right)\chi^{iI} + \partial_\mu\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q^{IJ}_j\chi^{iJ}\right)\ . \end{aligned}\tag{2.3.32}$$

Supersymmetry transformations for the ungauged lagrangian are

$$\begin{aligned} \delta e_\mu{}^a \;&=\; \tfrac{1}{2}\,\bar{\epsilon}^I\gamma^a\,\psi^I_\mu\ ,\\ \delta\psi^I_\mu \;&=\; D_\mu\epsilon^I - \tfrac{1}{8}g_{ij}\,\bar{\chi}^{iI}\gamma^\nu\chi^{jJ}\,\gamma_{\mu\nu}\,\epsilon^J - \delta\phi^i\,Q^{IJ}_i\psi^J_\mu\ ,\\ \delta\phi^i \;&=\; \tfrac{1}{2}\,\bar{\epsilon}^I\,\chi^{iI}\ ,\\ \delta\chi^{iI} \;&=\; \tfrac{1}{2}\left(\delta^{IJ}\mathbf{1} - f^{IJ}\right)^i{}_j\,\widehat{\not{\partial}}\phi^j\,\epsilon^J - \delta\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q^{IJ}_j\chi^{iJ}\right)\ , \end{aligned}\tag{2.3.33}$$

where supercovariant derivative $\widehat{\partial}_\mu\phi^i$ of scalar and covariant derivative $D_\mu(\omega, Q)\,\epsilon^I$ of supersymmetry parameter are defined by

$$\begin{aligned} \widehat{\partial}_\mu\phi^i \;&=\; \partial_\mu\phi^i - \tfrac{1}{2}\bar{\psi}^I_\mu\chi^{iI}\ ,\\ D_\mu\epsilon^I \;&=\; \left(\partial_\mu + \tfrac{1}{2}\omega^a_\mu\,\gamma_a\right)\epsilon^I + \partial_\mu\phi^i\,Q^{IJ}_i\,\epsilon^J\ . \end{aligned}\tag{2.3.34}$$

2.4 Target space geometry

In this section, we continue the analysis from the previous section and discuss some important geometrical constraints and the classes of solution of target space geometry. Since holonomy groups is related to curvature tensors by [60], the restriction governs the holonomy groups as well. For convenience, we recall Riemann curvature tensor

$$R_{ijkl} \;=\; \frac{1}{8}\left(f^{IJ}_{ij}\, f^{IJ}_{kl} + C_{\alpha\beta}\, h^{\alpha}_{ij}\, h^{\beta}_{kl}\right)\,, \tag{2.4.1}$$

with a proper of $C_{\alpha\beta}$; we can conclude that the holonomy group must be contained in $SO(N) \times H'$. If the target space is in the form of coset space G/H, Riemann curvature tensor (2.4.1) would associate with Lie algebra of subgroup H. Once H is specified along with N and number of scalar fields $d = kd_N = \text{dim}G/H$, $\text{dim}G$ and then coset G/H can be determined by looking it up in the categorized coset space [63, 62]. After a long discussion in [36] we get the condition that determine whether the theories should exist are the following equation

$$\frac{N(N-1)}{8c} = \begin{cases} \frac{d_N-1}{d_N+k-2} & \text{for } N = 7,8,9 \text{ mod } 8\ , \\ \frac{d_N^2-4}{d_N(d_N+2k)} & \text{for } N = 6 \text{ mod } 4\ , \\ \frac{d_N+2}{d_N+4k+4} & \text{for } N = 3,5,12 \text{ mod } 8\,. \end{cases} \tag{2.4.2}$$

The equations above imply that there is no theory beyond $N = 16$. As mentioned earlier, mathematical arguments govern the bound of supersymmetries rather than physical arguments like higher dimensional theory. The constraints above also imply that there is no theory with supersymmetry $N = 7, 11, 13, 14, 15$.

To summarize: pure supergravity coupled to supersymmetric non-linear sigma model shapes up geometry of scalar manifolds. B. de Wit and A.K. Tollstén [36] show that for some supersymmetry N there exists theories and each supersymmetry N associates with a particular class of manifold. The exhaustive list for possible supersymmetry N is given in the table III. Here are some comments on the target space listed in the table. For $N = 1$, the target space is Riemannian with dimension d equals to number of matter supermultiplet and the holonomy group is $SO(d)$. For $N = 2$, the target space is a Kähler manifold with a complex structure f^{12} and vanishing of Nijenhuis tensors. The complex structure is also covariantly constant. Like any Käler manifold, number of dimension is even $d = 2k$ with holonomy group $U(k)$. For $N = 3$, the target space is a quaternion Kähler manifold with number of dimension mulipled of four $d = 4k$. Three covariantly constant almost complex structure f^{12}, f^{23} and f^{31} are required. The holonomy group is contained in $USp(2) \times USp(k)$. The $N = 4$ theory is special and requires a careful treatment. Since this dissertation focuses on $N = 5, 6$, so we skip the details and recommend the enthusiast to consult the original work. For theories $N > 4$, the target spaces are homogeneous spaces and can be written in the form of coset G/H. They are also symmetric spaces except for $N = 9$. For theories

N	**Target Space**	d	H'
1	Riemannian	k	$\mathbf{1}$
2	Kählerian	$2k$	$U(1)$
3	Quaternionic Kählerian	$4k$	$USp(k)$
4	Two Quaternionic Kählerian	$4(k_1+k_2)$	$USp(k_1)\times USp(k_2)$
5	$\frac{\mathrm{USp}(4,k)}{\mathrm{USp}(4)\times\mathrm{USp}(k)}$	$4k$	$USp(k)$
6	$\frac{\mathrm{SU}(4,k)}{\mathrm{SU}(k)\times\mathrm{SU}(4)\times\mathrm{U}(1)}$	$8k$	$U(k)$
8	$\frac{\mathrm{SO}(8,k)}{\mathrm{SO}(8)\times\mathrm{SO}(k)}$	$8k$	$SO(k)$
9	$\frac{\mathrm{F}_{4(-20)}}{\mathrm{SO}(9)}$	16	$\mathbf{1}$
10	$\frac{\mathrm{E}_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{U}(1)}$	32	$U(1)$
12	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{Sp}(1)}$	64	$USp(2)$
16	$\frac{\mathrm{E}_{8(8)}}{\mathrm{SO}(16)}$	128	$\mathbf{1}$

Table III: Target space or scalar manifold for $D = 3$ supergravities for each N. The number of independent supermultiplets is given by k. For $N = 5$, in our convention k is an even integer, so the number of independent supermultiplets is $k/2$.

with $4 < N \leq 8$, we are left with freedom in selecting number of supermultiplet k. For $N > 8$, the theory is very unique since only one supermultiplets is allowed.

Before we end this section, we would like to elaborate some arguments on target space behind the table III. It shows that the target space for high value of N must be homogenous and symmetric which are in the form of a coset space G/H. There are two theorems involving in classifying holonomy groups here, the first one is proposed by M. Berger [65] and another one belongs to Simon[64]. We recite only Simon's version as follows

Theorem 1(Simon's version): *Let $\mathcal{M}$ be an irreducible Riemannian manifold with dimension d. $\mathcal{M}$ is a symmetric space of rank ≥ 2, if the restricted holonomy group* $\mathrm{Hol}^0(\mathcal{M})$ *does not act transitively on the unit sphere S^{d-1} in $T_p\mathcal{M}$.*

As discussed earlier the holonomy group is contained in $SO(N) \otimes H'$, if $SO(N) \otimes H'$ does not act transitively on S^{d-1} so does the smaller group like holonomy group. On the other hand, if $SO(N) \otimes H'$ acts transitively on S^{d-1}, then $\mathcal{M}$ is not symmetric. Once H is specified together with N and number of scalar fields $d = kd_N = \mathrm{dim}G/H$, $\mathrm{dim}G$ and then the G/H can be determined by looking it up in the categorized symmetric coset space [63, 62]. This is how we apply the theorem. For $N \leq 4$, they are not symmetric space because $SO(N)\otimes H'$ acts transitively on S^{d-1}. For $N > 4$ except for $N = 9$, $SO(N) \otimes H'$ does not act transitively on S^{d-1}, so they are symmetric space and we can look them up to specify the exact form of coset G/H. For $N = 9$ case, $SO(9)$ acts transitively on S^{15} so it is either non-symmetric or symmetric rank 1. According to [62], $F_4/SO(9)$ is a symmetric space rank 1 so the theorem cannot directly apply; however, $F_4/SO(9)$ is still a valid solution.

Note that goup H can factorize into actual holonomy group $SO(N) \otimes \hat{H} \subset SO(d)$, where $\hat{H} \subset H'$. In our case the target space given in the table III, the factor group coincide with H', i.e. $\hat{H} \simeq H'$. We end this section here and the group-theoretical aspects on coset manifold will be discussed later in other section.

2.5 The Global Symmetry

Apart from local supersymmetry, local general coordinate transformations and local Lorentz transformations, there is a global symmetry associated with the isometries of target space manifold and $SO(N)$ R-symmetry rotations. The target space may have isometries – metric or distance preserving transformations. The question is: Can we extend the isometries to the symmetry of the Lagrangian? The answer is Yes; however, with the help of some tricks. The $SO(N)$ R-symmetry rotations solely cannot be an invariance of Lagrangian. The invariance of the Lagrangian can be established if isometries and $SO(N)$ R-symmetry rotations are combined and the parameters of transformation are properly compensated. The scenario is quite similar to comformal symmetry in bosonic string theory which is constituted by world sheet diffeomorphism compensated by Weyl transformation.

The fields that transform under $SO(N)$ R-symmetry rotations are ψ^I_μ, χ^{iI} and Q_i^{IJ}. The infinitesimal transformations are as follows

$$\delta\psi^I_\mu = \Lambda^{IJ}(\phi)\,\psi^J_\mu\,, \qquad \delta\chi^{iI} = \Lambda^{IJ}(\phi)\,\chi^{iJ}\,, \qquad \delta Q_i^{IJ} = -D_i\Lambda^{IJ}(\phi)\,. \tag{2.5.1}$$

The $\Lambda^{IJ}(\phi)$ is the field-dependent $SO(N)$ rotation parameter. Since f^{IJ}_{ij} depends on Q_i^{IJ}, one obtains

$$\delta f^{IJ} = 2\,\Lambda^{K[I}(\phi)\,f^{J]K}\,. \tag{2.5.2}$$

Those $SO(N)$ transformations above cannot establish the invariance of Lagrangian. Next, we will discuss isometries and then we will see that with those transformations combined the invariance can magically appear.

The isometries generated by Killing vectors denoted by $X(\phi)$ which are the solutions of

$$\mathcal{L}_X\, g_{ij} = 0\,. \tag{2.5.3}$$

They can be written as $X(\phi) = X^i(\phi)\partial_i$ and the component is $X^i(\phi) = X^{\mathcal{M}i}(\phi)\Lambda_{\mathcal{M}}$ where $\Lambda_{\mathcal{M}}$ is spacetime independent parameter of transformaion. The Calligraphic typeface style denotes element of isometry group G. In term of basis of generators, we have $X^{\mathcal{M}} = X^{\mathcal{M}i}(\phi)\partial_i$. The Killing vectors realizes Lie algebra $\mathfrak{g}$ which is

$$X^{\mathcal{M}i}\,\partial_i X^{\mathcal{N}} - X^{\mathcal{N}i}\,\partial_i X^{\mathcal{M}} \;=\; f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\,X^{\mathcal{K}}\,, \tag{2.5.4}$$

with structure constants $f^{\mathcal{MN}}{}_{\mathcal{K}}$.

The target space isometries acting on Q_i^{IJ} and f_{ij}^{IJ} via Lie derivative $\mathcal{L}_X Q_i^{IJ}$ and $\mathcal{L}_X f_{ij}^{IJ}$, respectively. In order to set up an invariance, the cancellation requires

$$\begin{aligned} \mathcal{L}_X Q_i^{IJ} &= -D_i \mathcal{S}^{IJ}(\phi, X) \,, \\ \mathcal{L}_X f_{ij}^{IJ} &= 2\, \mathcal{S}^{K[I}(\phi, X)\, f_{ij}^{J]K} \,. \end{aligned} \tag{2.5.5}$$

We now use $\mathcal{S}^{IJ}(\phi, X)$ to replace the old parameter $\Lambda^{IJ}(\phi)$ because it is also explicitly a function of Killing vectors X.

To summarize: the Lagrangian (2.3.31) is then invariant under the combined transformations,

$$\delta\phi^i = X^i(\phi) \,, \quad \delta\psi_\mu^I = \mathcal{S}^{IJ}(\phi, X)\, \psi_\mu^J \,, \quad \delta\chi^{iI} = \chi^{jI}\partial_j X^i + \mathcal{S}^{IJ}(\phi, X)\, \chi^{iJ} \,. \tag{2.5.6}$$

The fermion transformations can be covariantly rewritten as

$$\begin{aligned} \delta\psi_\mu^I &= \mathcal{V}^{IJ}(\phi, X)\, \psi_\mu^J - \delta\phi^i\, Q_i^{IJ} \psi_\mu^J \,, \\ \delta\chi^{iI} &= D_j X^i\, \chi^{jI} + \mathcal{V}^{IJ}(\phi, X)\, \chi^{iJ} - \delta\phi^j \left(\Gamma^i_{jk}\, \chi^{kI} + Q_j^{IJ} \chi^{iJ}\right) \,, \end{aligned} \tag{2.5.7}$$

where $\mathcal{V}^{IJ}(\phi, X) \equiv X^j Q_j^{IJ}(\phi) + \mathcal{S}^{IJ}(\phi, X)$. The merit of this definition will become clear later. From equations (2.3.14) and (2.3.15), one can show that the second equation of (2.5.5) can be written as,

$$D_i \mathcal{V}^{IJ}(\phi, X) = \tfrac{1}{2} f_{ij}^{IJ}(\phi)\, X^j(\phi) \,. \tag{2.5.8}$$

This equation has a natural interpretation as a the moment map associated with the isometry X^i. The second equation of (2.5.5) now become

$$f^{IJk}{}_{[i}(\phi)\; D_{j]} X_k(\phi) \;=\; f_{ij}^{K[I}(\phi)\; \mathcal{V}^{J]K}(\phi, X) \,. \tag{2.5.9}$$

This is merely the integrability condition of (2.5.8). If we contract (2.5.9) with f^{MNij}, we get

$$f^{IJ\,ij}\, D_i X_j \;=\; \begin{cases} \tfrac{1}{2} d\, \mathcal{V}^{IJ} \,, & \text{for } N \neq 2, 4 \\ (d_+\, \mathbb{P}_+^{IJ,KL} + d_-\, \mathbb{P}_-^{IJ,KL})\mathcal{V}^{KL} \,, & \text{for } N = 4 \end{cases} \tag{2.5.10}$$

One can derive

$$D^i D_i \mathcal{V}^{IJ} \;=\; \begin{cases} \tfrac{1}{2} d\, \mathcal{V}^{IJ} \,, & \text{for } N \neq 2, 4 \\ \tfrac{1}{2}(d_+\, \mathbb{P}_+^{IJ,KL} + d_-\, \mathbb{P}_-^{IJ,KL})\mathcal{V}^{KL} \,, & \text{for } N = 4 \end{cases} \tag{2.5.11}$$

According to the discussion above, we can extend an isometry of target space to a symmetry of the Lagrangian as long as equations above are satisfied. However, there is an exception for $N = 2$ where the discussion can be found in [37]. Let $\mathcal{S}^{\mathcal{M}\,IJ} \equiv \mathcal{S}^{IJ}(\phi, X^{\mathcal{M}})$ and $\mathcal{V}^{\mathcal{M}\,IJ} \equiv \mathcal{V}^{IJ}(\phi, X^{\mathcal{M}})$, the closure of the algebra implies

$$[\mathcal{S}^{\mathcal{M}}, \mathcal{S}^{\mathcal{N}}]^{IJ} = -f^{\mathcal{MN}}{}_{\mathcal{K}}\, \mathcal{S}^{\mathcal{K}\,IJ} + \left(X^{\mathcal{M}i}\, \partial_i \mathcal{S}^{\mathcal{N}\,IJ} - X^{\mathcal{N}i}\, \partial_i \mathcal{S}^{\mathcal{M}\,IJ}\right) \,. \tag{2.5.12}$$

One can derive

$$[\mathcal{V}^{\mathcal{M}},\, \mathcal{V}^{\mathcal{N}}]^{IJ} = -f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, \mathcal{V}^{\mathcal{K}\, IJ} + \tfrac{1}{2}\, f^{IJ}_{ij}\, X^{\mathcal{M}i} X^{\mathcal{N}j}\ , \tag{2.5.13}$$

Since the equation (2.5.9) implies that $D_i X_j - \frac{1}{4}\, f^{MN}_{ij}\, \mathcal{V}^{MN}$ commutes with the almost complex structures f^{IJ}_{ij}, combined with the fact that h^{α}_{ij} tensors commute with the almost complex structure , i.e. (2.3.20), this suggests that we can decompose as the following

$$D_i X^{\mathcal{M}}_j - \tfrac{1}{4} f^{IJ}_{ij}\, \mathcal{V}^{\mathcal{M}\, IJ} \quad \equiv \quad h^{\alpha}_{ij}\, \mathcal{V}^{\mathcal{M}}{}_{\alpha}\ . \tag{2.5.14}$$

Note that, for a generic Killing vector property, we have

$$D_i D_j X_k = R_{jkil}\, X^l\ , \tag{2.5.15}$$

together with $\mathcal{V}^{\mathcal{M}\, i} \equiv X^{\mathcal{M}\, i}$, we can derive a set of differential equations,

$$\begin{aligned} D_i \mathcal{V}^{\mathcal{M}\, IJ} &= \tfrac{1}{2}\, f^{IJ}_{ij}\, \mathcal{V}^{\mathcal{M}\, j}\ , \\ D_i \mathcal{V}^{\mathcal{M}}{}_{j} &= \tfrac{1}{4}\, f^{IJ}_{ij}\, \mathcal{V}^{\mathcal{M}\, IJ} + h^{\alpha}_{ij}\, \mathcal{V}^{\mathcal{M}}{}_{\alpha}\ , \\ D_i \mathcal{V}^{\mathcal{M}}{}_{\alpha} &= \tfrac{1}{8}\, C_{\alpha\beta}\, h^{\beta}_{ij}\, \mathcal{V}^{\mathcal{M}\, j}\ , \end{aligned} \tag{2.5.16}$$

where the covariant derivative contains the Christoffel connection and the SO(N)$\times$ H$'$ connections, i.e. $D_i(\Gamma, Q, \Omega)$. From equations above, we can derive two equations that is proved to be crucial in algebraic structure analysis:

$$\begin{aligned} f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, \mathcal{V}^{\mathcal{K}}{}_{i} &= \tfrac{1}{4} f^{IJ}_{ij} (\mathcal{V}^{\mathcal{M}\, IJ}\, \mathcal{V}^{\mathcal{N}\, j} - \mathcal{V}^{\mathcal{N}\, IJ}\, \mathcal{V}^{\mathcal{M}\, j}) + h^{\alpha}_{ij} (\mathcal{V}^{\mathcal{M}}{}_{\alpha}\, \mathcal{V}^{\mathcal{N}\, j} - \mathcal{V}^{\mathcal{N}}{}_{\alpha}\, \mathcal{V}^{\mathcal{M}\, j})\ , \\ f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, \mathcal{V}^{\mathcal{K}}{}_{\alpha} &= f^{\beta\gamma}{}_{\alpha}\, \mathcal{V}^{\mathcal{M}}{}_{\beta}\, \mathcal{V}^{\mathcal{N}}{}_{\gamma} + \tfrac{1}{8} C_{\alpha\beta}\, h^{\beta}_{ij}\, \mathcal{V}^{\mathcal{M}\, i}\ \mathcal{V}^{\mathcal{N}\, j}\ . \end{aligned} \tag{2.5.17}$$

In order to extract the algebraic structure from the equations (2.5.17), we first define the algebra $\mathfrak{a} \equiv \{t^{\mathcal{A}}\} \equiv \{t^{IJ}, t^{\alpha}, t^{i}\}$, as an extension of $\mathfrak{so}(N) \oplus \mathfrak{h}'$ with commutation relations,

$$\begin{aligned} \big[t^{IJ}, t^{KL}\big] &= -4\, \delta^{\,\overline{I[K}}\, t^{\overline{L]J}}\ , \qquad & \big[t^{\alpha}, t^{\beta}\big] &= f^{\alpha\beta}{}_{\gamma}\, t^{\gamma}\ , \qquad & \big[t^{IJ}, t^{i}\big] &= \tfrac{1}{2}\, f^{IJ}{}_{j}{}^{i}\, t^{j}\ , \\ \big[t^{\alpha}, t^{i}\big] &= h^{\alpha}{}_{j}{}^{i}\, t^{j}\ , & \big[t^{i}, t^{j}\big] &= \tfrac{1}{4}\, f^{ij}_{IJ}\, t^{IJ} + \tfrac{1}{8}\, C_{\alpha\beta} h^{\beta\, ij}\, t^{\alpha}\ . \end{aligned} \tag{2.5.18}$$

If $C_{\alpha\beta}$ is an H$'$-invariant tensor, the algebra $\mathfrak{a}$ is associative. In addition, we define the map

$$\mathcal{V}:\ \mathfrak{g} \rightarrow \mathfrak{a}\,,\ \mathcal{V}(X^{\mathcal{M}})\ :=\ \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\, t^{\mathcal{A}}\ =\ \tfrac{1}{2} \mathcal{V}^{\mathcal{M}}{}_{IJ}\, t^{IJ} + \mathcal{V}^{\mathcal{M}}{}_{\alpha}\, t^{\alpha} + \mathcal{V}^{\mathcal{M}}{}_{i}\, t^{i}\ . \tag{2.5.19}$$

We also define a Lie algebra homomorphism, *i.e.*

$$\mathcal{V}([X^{\mathcal{M}}, X^{\mathcal{N}}]) = [\mathcal{V}(X^{\mathcal{M}}), \mathcal{V}(X^{\mathcal{N}})]\ . \tag{2.5.20}$$

In general, the image of $\mathfrak{g}$ under $\mathcal{V}$ is an associative subalgebra of $\mathfrak{a}$. If the target space is symmetric space ($N > 4$), the algebra $\mathfrak{a}$ and $\mathfrak{g}$ coincide. Note that (2.5.16) can have a compact form by the new notation (2.5.19),

$$D_i \mathcal{V}(X^{\mathcal{M}}) \ = \ \big[g_{ij}\, t^{j}, \mathcal{V}(X^{\mathcal{M}})\big]\ . \tag{2.5.21}$$

The target space of $N > 4$ theories are a symmetric space represented by coset space G/H. The material discussed above work well for both symmetric and non-symmetric space, but the coset space formalism that we are about to discuss is more convenient when it comes to symmetric space. In this formalism, the scalar fields are described by G-valued matrix $L(\phi)$, so the kinetic term of the Lagrangian is in the trace of the derivative of the matrices thereof instead of the old-fashioned non-linear sigma model. The action of the scalar fields is invariant under a group operation that act globally from the left of $L(\phi)$ and act locally from the right. *Global* means that group elements $g \in G$ do not depend on the coordinates or in this case the scalar fields themselves. And *local* means that group elements $g \in G$ depend on the coordinates. The degrees of freedom of the G-valued matrix $L(\phi)$ could be redundant and the exceeding degrees of freedom can be eliminated by gauge fixing condition such as unitary gauge. Once the gauge is fixed, the G-valued matrix $L(\phi)$ is simply a coset representative $L(\phi)$. From now on we use $L(\phi)$ as a coset representative instead of generic G-valued matrix. Number of scalar fields corresponds to dimension of coset space $d = \dim(\mathrm{G/H}) = \dim \mathrm{G} - \dim \mathrm{H}$. We can decompose isometry group G into $\mathrm{H} = \mathrm{SO(N)} \times \mathrm{H}'$ and its complements; the generators can be written as $\{t^{\mathcal{M}} \,\varepsilon\, \mathfrak{g}\} = \{X^{IJ}, X^{\alpha}, Y^{A}\}$. The X^{IJ} generate SO(N) and the X^{α} generate the group H′. These generators are the basis of subalgebra $\mathfrak{h}$. The noncompact generators (or coset generators) Y^A transform in a spinor representation of SO(N).

Since we are dealing with symmetric space, we can replace Killing vectors which are differential operators that realize algebra with matrix representation of algebra G. The algebra (2.5.18) now become

$$\begin{aligned}
[X^{IJ}, X^{KL}] &= \delta^{JK} X^{IL} - \delta^{IK} X^{JL} - \delta^{JL} X^{IK} + \delta^{IL} X^{JK} = -4\,\delta^{\,I[K}\, X^{L]J}\ , \\
[X^{\alpha}, X^{\beta}] &= f^{\alpha\beta}{}_{\gamma}\, X^{\gamma}\ , \qquad [X^{IJ}, X^{\alpha}] = 0\ , \\
[X^{IJ}, Y^{A}] &= -\tfrac{1}{2}\Gamma^{IJ}_{AB}\, Y^{B}\ , \qquad [X^{\alpha}, Y^{A}] = -h^{\alpha}_{AB}\, Y^{B}\ , \\
[Y^{A}, Y^{B}] &= \tfrac{1}{4}\Gamma^{IJ}_{AB}\, X^{IJ} + \tfrac{1}{8} C_{\alpha\beta}\, h^{\alpha}_{AB}\, X^{\beta}\ ,
\end{aligned} \tag{2.5.22}$$

where $\Gamma^{IJ}_{AB} \equiv \Gamma^{[I}_{A\dot{A}}\Gamma^{J]}_{B\dot{A}}$. The matrices $\frac{1}{2}\Gamma^{IJ}_{AB}$ are generator of the spinor representation of SO(N). Analogous to the general case (not necessary symmetric), the H′ generators now satisfy

$$\begin{aligned}
h^{\alpha}_{AC}\Gamma^{I}_{C\dot{B}} + h^{\alpha}_{\dot{B}\dot{C}}\Gamma^{I}_{A\dot{C}} &= 0 \\
h^{\alpha}_{AC}\, h^{\beta}_{CB} - h^{\beta}_{AC}\, h^{\alpha}_{CB} &= f^{\alpha\beta}{}_{\gamma}\, h^{\gamma}_{AB}\,.
\end{aligned} \tag{2.5.23}$$

The tensor $C_{\alpha\beta}$ is the same as used previously. The Jacobian identity obtained from the commutator $[[Y^A, Y^B], Y^C]$ is given by

$$\Gamma^{IJ}_{[AB}\,\Gamma^{IJ}_{CD]} + C_{\alpha\beta}\, h^{\alpha}_{[AB}\, h^{\beta}_{CD]} = 0\,, \tag{2.5.24}$$

which is in the same form of (2.3.27) with f^{IJ}_{ij} and h^{α}_{ij} are replaced byΓ^{IJ}_{AB} and h^{α}_{AB}, respectively.

In the coset space formalism, we define a Lie-algebra-valued one-form $\mathfrak{g} \times T_p^* M$

$$L^{-1}\mathrm{d}L = L^{-1}\partial_i L \, \mathrm{d}\phi^i \tag{2.5.25}$$

Its components read

$$L^{-1}\partial_i L \;\; = \;\; \tfrac{1}{2}\, Q_i^{IJ}\, X^{IJ} + Q_i^{\alpha}\, X^{\alpha} + e_i{}^A\, Y^A \; , \tag{2.5.26}$$

where , Q_i^{IJ} and Q_i^{α} are SO(N) and H′ target space connections respectively. $e_i{}^A$ is vielbein for target space manifold. The relation to the metric is the following

$$g_{ij} \;\; = \;\; e_i{}^A\, e_j{}^B\, \delta_{AB} \; . \tag{2.5.27}$$

Note that in performing real calculation we usually operate in flat basis, the equations above can transform nicely from curved basis to flat basis by using vielbein. The h_{ij}^{α} tensor can be related to the tensor in orthonormal basis by

$$h_{ij}^{\alpha} = h_{AB}^{\alpha} e_i^A e_j^B \; . \tag{2.5.28}$$

Note that the curvature tensor on G/H is given by

$$R_{ijkl} = -e_k^A\, e_l^B \left(\tfrac{1}{4} R_{ij}^{IJ} \Gamma_{AB}^{IJ} + R_{ij}^{\alpha}\, h_{AB}^{\alpha} \right) \; , \tag{2.5.29}$$

which is an alternative form of (2.4.1).

The antisymmetric tensor that are derived from complex structure f_{ij}^{IJ} relates to SO(N) Γ-matrices Γ_{AB}^{IJ} by

$$f_{ij}^{IJ} \;\; = \;\; -\Gamma_{AB}^{IJ}\, e_i^A\, e_j^B \; , \tag{2.5.30}$$

Now the spin-1/2 fields can be defined as the conjugate spinor representation of SO(N)

$$\chi^{\dot{A}} \;\; \equiv \;\; \frac{1}{N}\, e_i^A\, \Gamma_{A\dot{A}}^I\, \chi^{iI} \; . \tag{2.5.31}$$

If one multiplies a constant element $g_0 \in G$ from the left of the coset representative $L(\phi)$, the coset representative changes and no longer is in the same form. In order to bring back coset representative in the same form but it is now written in a new coordinate system ϕ', we compensate them by multiplying a proper ϕ-dependent matrix which is an element in $\mathfrak{h}$, denoted $h(\phi)$, to the right. In explicit expression, we have

$$L(\phi) \;\longrightarrow\; g_0 L(\phi) = L(\phi')\, h(\phi) \tag{2.5.32}$$

The infinitesimal transformation is generated by Killing vectors as follows $\phi^i \to \phi'^i = \phi^i + X^i(\phi)$. The infinitesimal group element is $g_0 \approx \mathbf{1} + t$ and $h \approx \mathbf{1} + \mathcal{S}(\phi^i)$. In term of component $\mathcal{M}$, we obtain

$$X^{\mathcal{M}\,i}\, \partial_i L \;\; = \;\; t^{\mathcal{M}} L - L\, \mathcal{S}^{\mathcal{M}}(\phi^i) \; , \qquad \mathcal{S}^{\mathcal{M}}(\phi^i) \in \mathfrak{h} \; . \tag{2.5.33}$$

$\mathcal{S}^{\mathcal{M}}$ decomposes into $\mathcal{S}^{\mathcal{M}IJ}$ and $\mathcal{S}^{\mathcal{M}\alpha}$, so it is useful to define the quantity

$$\mathcal{V}^{\mathcal{M}} \equiv \mathcal{S}^{\mathcal{M}} + X^{\mathcal{M}i} Q_i \,, \tag{2.5.34}$$

which is the generalized version of the parameter previously given.

The map $\mathcal{V}$ mentioned in (2.5.19) can be written as

$$L^{-1} t^{\mathcal{M}} L \;\equiv\; \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\, t^{\mathcal{A}} \;=\; \tfrac{1}{2}\, \mathcal{V}^{\mathcal{M}\, IJ}\, X^{IJ} + \mathcal{V}^{\mathcal{M}}{}_{\alpha}\, X^{\alpha} + \mathcal{V}^{\mathcal{M}}{}_{A}\, Y^{A} \;. \tag{2.5.35}$$

where $\mathcal{V}^{\mathcal{M}\,i} = g^{ij} e_j{}^{A} \mathcal{V}^{\mathcal{M}}{}_{A}$. This equation allows us to obtain $\mathcal{V}^{\mathcal{M}}$'s once the coset representative L is chosen by exploiting orthogonality of the generators with some proper normalization. The trick is multiplying both side with one of the generators that associate with the desired component of $\mathcal{V}^{\mathcal{M}}$ and then take trace. The orthogonality leaves us only the desired component. For example,

$$\mathcal{V}^{\mathcal{M}}{}_{\alpha} = \tfrac{1}{k}\, \mathrm{tr}\, (L^{-1} t^{\mathcal{M}} L X^{\alpha}) \;, \tag{2.5.36}$$

where the normalization is defined by $\mathrm{tr}\,(X^{\alpha} X^{\beta}) = k\, \delta^{\alpha\beta}$ and k is a proper normalization constant.

2.6 Yang-Mills and Chern-Simons Gauging

The formalism used throughout this dissertation is called Chern-Simons gauging where vector fields are introduced to the theories via Chern-Simons (CS) terms instead of the traditional Yang-Mills (YM) terms. The YM gauging commonly emerges via dimensional reduction. On the other hand, the CS theory can be constructed by freely adding some CS terms to a well-established theory without changing the number of dynamic degrees of freedom. There is a possible conversion between those two if we manage the total degrees of freedom of both theories to be equal. We give an overview on this subject. Some serious discussions can be found in [66].

In the conversion from YM to CS, a vector field in YM is replaced by two CS vector fields and a new scalar field. Schematically, it reads

$$1\,\mathrm{YM\ Vector} \;\Rightarrow\; 2\,\mathrm{CS\ Vectors} + 1\,\mathrm{Scalar}. \tag{2.6.1}$$

Although there is a mismatch between field content of the two theories, the number of dynamic degrees of freedom of both theories are the same because the vectors in CS are topological and occupy zero degrees of freedom, that is left with the YM vectors and a new scalar which have the same number of degrees of freedom in three dimensions.

In this analysis, we start with a YM kinetic term which the general form thereof can interact with some scalar fields Φ which transform in some representation of the gauge group G_{YM}. The Lagrangian reads

$$\begin{aligned} \mathcal{L} \;=\; & -\tfrac{1}{4}\sqrt{g}\, \big(F^{A}_{\mu\nu}(A) + O^{A}_{\mu\nu}(A,\Phi)\big)\, M_{AB}(\Phi)\, \big(F^{B\mu\nu}(A) + O^{B\mu\nu}(A,\Phi)\big) \\ & + \mathcal{L}'(A,\Phi) \;. \end{aligned} \tag{2.6.2}$$

$O^A_{\mu\nu}$ is a gauge covariant object which is depends on both vector and scalar field forming the so-called moment interaction. A^A_μ is the non abelian gauge field and $F^A_{\mu\nu}$ is its field strength tensor. Gauge group indices are $A, B, C, \ldots$. The definition of field strength is conventional and $f_{AB}{}^C$ denotes structure constant, so it reads

$$F^A_{\mu\nu}(A) = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f_{BC}{}^A\, A^B_\mu A^C_\nu \ .$$

The scalar symmetric tensor $M_{AB}(\Phi)$ is a function of only scalar fields. $\mathcal{L}'(A,\Phi)$ is an allowed term that is separately gauge invariant. We can avoid discussing the exact form of this term here, since it plays a minor role in our discussion. The equation of motion and the Bianchi identity are the following:

$$D_\mu \tilde{F}^{A\,\mu}(A) = 0\ ,\ D_{[\mu}\Big(M_{AB}(\Phi)\,(\tilde{F}^B_{\nu]}(A) + \tilde{O}^B_{\nu]}(A,\Phi))\Big) - J_{A\,\mu\nu}(A,\Phi) = 0\ , \quad (2.6.3)$$

where

$$J_{A\,\mu\nu}(A,\Phi) \ = \ \tfrac{1}{2} i\sqrt{g}\,\varepsilon_{\mu\nu\rho}\,\frac{\partial \mathcal{L}'(A,\Phi)}{\partial A^A_\rho} . \quad (2.6.4)$$

The dual field strength take the form

$$\begin{aligned}\tilde{F}^A_\mu(A) &= \tfrac{1}{2} i\sqrt{g}\,\varepsilon_{\mu\nu\rho}\,F^{A\,\nu\rho}(A)\ ,\\ \tilde{O}^A_\mu(A,\Phi) &= \tfrac{1}{2} i\sqrt{g}\,\varepsilon_{\mu\nu\rho}\,O^{A\,\nu\rho}(A,\Phi)\ .\end{aligned}$$

Before we proceed, we first denote the final form of the new Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm n} \ = \ & -\tfrac{1}{2}\sqrt{g}\,\hat{D}_\mu\phi_A\,M^{AB}(\Phi)\hat{D}^\mu\phi_B + \tfrac{1}{2}\,\mathrm{i}\,\varepsilon^{\mu\nu\rho}\,(F^A_{\mu\nu}B_{A\,\rho} - O^A_{\mu\nu}\,\hat{D}_\rho\phi_A) \quad (2.6.5)\\ & + \mathcal{L}'(A,\Phi)\ ,\end{aligned}$$

where the covariant derivative is given by

$$\hat{D}_\mu\phi_A \equiv D_\mu\phi_A - B_{A\mu} = \partial_\mu\phi_A - f_{AB}{}^C\,A^B_\mu\phi_C - B_{A\mu}\ . \quad (2.6.6)$$

With this goal in mind, we can walk through the analysis a lot easier. In this scenario, we write the field strength in term of a new vector field $B_{A\,\mu}$ and the compensating scalar field ϕ_A, and all of which transform in adjoint representation of the gauge group. This setting also has an interpretation as a field equation of the new Lagrangian. Such field equation reads

$$\tfrac{1}{2}\,\mathrm{i}\,\sqrt{g}\,\varepsilon_{\mu\nu\rho}(F^{A\,\nu\rho}(A) + O^{A\,\nu\rho}(A,\Phi)) \ = \ M^{AB}(B_{B\,\mu} - D_\mu\phi_B)\ , \quad (2.6.7)$$

From equation of motion (2.6.7), one observes that they have additional abelian gauge symmetry,

$$\delta B_{A\,\mu} = D_\mu\Lambda_A\ , \qquad \delta\phi_A = \Lambda_A\ . \quad (2.6.8)$$

We call this abelian group $\mathcal{T}$. It contains nilpotent generators ($A \neq 0, A^2 = 0$) and transform in adjoint representation of gauge group $\mathrm{G_{YM}}$. The full gauge symmetry now becomes a semidirect product $\mathrm{G_{YM}} \ltimes \mathcal{T}$ which its dimension is doubled compared to the original. In order to restore the original Lagrangian (2.6.2), we impose gauge fixing condition $\phi_A = 0$ and integrates out the vector fields $B_{A\,\mu}$.

2.7 Gauged Theory, Embedding Tensors and Constraints

In this section, we construct a gauged supergravity from the ungauged from the previous section. A subgroup of the isometries group $G_0 \subset G$ can be gauged by promoting global symmetry (spacetime independent parameters) to local symmetry (spacetime dependent parameters). The gauging is not trivial. Only some gauge groups are allowed because of the constraints due to supersymmetries. The technique we use here is the notion of embedding tensor originally introduced in theories in three dimensions [37]. Theories with different gauge groups are characterized by their unique embedding tensors. The embedding tensor method allows us to threat theories G-covariantly. Each theory with different gauge groups share the same structure such as Lagrangian and supersymmetry transformations, but they have different embedding tensor. The applications to higher dimensions such as four dimensional spacetime can be found in [67] and five dimensional spacetime are presented by the same author in [68].

The embedding tensors $\Theta_{\mathcal{M}\mathcal{N}}$ are constant rank-2 symmetric G tensors. They can be considered as a projection operator on the isometry group G to the gauge group G_0. If the gauge group is the isometry group itself, the embedding tensor is simply identity operator. The Killing vectors that generate the gauge group are defined by

$$X^i = g\, \Theta_{\mathcal{M}\mathcal{N}}\, \Lambda^{\mathcal{M}}(x)\, X^{\mathcal{N}i}\,, \tag{2.7.1}$$

with spacetime dependent parameters $\Lambda^{\mathcal{N}}(x)$ and a gauge coupling constant g. Dimension of gauge group is rank of matrix constructed from embedding tensor, i.e.,

$$\dim \mathfrak{g}_0 = \operatorname{rank} \Theta\ . \tag{2.7.2}$$

Subset of Killing vectors of a gauged theory must generate a group, the embedding tensors must satisfy the following condition,

$$\Theta_{\mathcal{M}\mathcal{P}}\, \Theta_{\mathcal{N}\mathcal{Q}}\, f^{\mathcal{P}\mathcal{Q}}{}_{\mathcal{R}} = \hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\, \Theta_{\mathcal{P}\mathcal{R}}\,, \tag{2.7.3}$$

where constants $\hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ are the structure constants of the gauge group. The constraint above is derived by requiring that the gauge generators $J_{\mathcal{M}}$ form an algebra $\mathfrak{g}_0$

$$[J_{\mathcal{M}}, J_{\mathcal{N}}] = \hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} J_{\mathcal{P}}\ . \tag{2.7.4}$$

Note that the G algebra $\mathfrak{g}$ is

$$[t^{\mathcal{M}}, t^{\mathcal{N}}] = f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}}\, t^{\mathcal{P}}\,, \tag{2.7.5}$$

and the gauge generators are the projection under the embedding tensor of generators in $\mathfrak{g}$:

$$J_{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}}\, t^{\mathcal{N}}\ . \tag{2.7.6}$$

The embedding tensors are gauge invariant, i.e. $\delta_{\text{gauge}}\Theta_{\mathcal{MN}} = 0$. It gives rise to the following constraint

$$\hat{f}_{\mathcal{MP}}{}^{\mathcal{Q}}\,\Theta_{\mathcal{QN}} + \hat{f}_{\mathcal{NP}}{}^{\mathcal{Q}}\,\Theta_{\mathcal{MQ}} = 0\,. \tag{2.7.7}$$

Using equation (2.7.3), it can be written in G-covariant form as

$$\Theta_{\mathcal{PL}}\left(f^{\mathcal{KL}}{}_{\mathcal{M}}\Theta_{\mathcal{NK}} + f^{\mathcal{KL}}{}_{\mathcal{N}}\Theta_{\mathcal{MK}}\right) \;=\; 0\,. \tag{2.7.8}$$

The Jacobi identity for the gauge group structure constants are also satisfied.

The gauge fields $A_\mu^{\mathcal{M}}$ are introduced to Lagrangian via two channels: modifying covariant derivative and adding Chern-Simons term to the Lagrangian in order to restore supersymmetry. In the new definition of covariant derivatives, the gauge fields attach to embedding tensors. For example, for scalar field we define

$$\mathcal{D}_\mu\phi^i = \partial_\mu\phi^i + g\,\Theta_{\mathcal{MN}}\,A_\mu^{\mathcal{M}}\,X^{\mathcal{N}\,i}\,. \tag{2.7.9}$$

The transformation of gauge fields is traditional but valid when attach to embedding tensors

$$\Theta_{\mathcal{MN}}\,\delta_{\text{gauge}}A_\mu^{\mathcal{M}} = \Theta_{\mathcal{MN}}\left(-\partial_\mu\Lambda^{\mathcal{M}} + g\,\hat{f}_{\mathcal{PQ}}{}^{\mathcal{M}}\,A_\mu^{\mathcal{P}}\,\Lambda^{\mathcal{Q}}\right)\,. \tag{2.7.10}$$

So is the field strength tensor

$$\Theta_{\mathcal{MN}}\,F_{\mu\nu}^{\mathcal{M}} = \Theta_{\mathcal{MN}}\left(\partial_\mu A_\nu^{\mathcal{M}} - \partial_\nu A_\mu^{\mathcal{M}} - g\,\hat{f}_{\mathcal{PQ}}{}^{\mathcal{M}}\,A_\mu^{\mathcal{P}}A_\nu^{\mathcal{Q}}\right)\,. \tag{2.7.11}$$

The commutator of two covariant derivatives is

$$\left[\mathcal{D}_\mu,\mathcal{D}_\nu\right]\phi^i = g\,\Theta_{\mathcal{MN}}\,F_{\mu\nu}^{\mathcal{M}}\,X^{\mathcal{N}i}\,. \tag{2.7.12}$$

The covariant derivatives for fermions are found to be

$$\begin{aligned}
\mathcal{D}_\mu\psi_\nu^I &= \left(\partial_\mu + \tfrac{1}{2}\omega_\mu^a\gamma_a\right)\psi_\nu^I + \partial_\mu\phi^i Q_i^{IJ}\,\psi_\nu^J + g\,\Theta_{\mathcal{MN}}A_\mu^{\mathcal{M}}\,\mathcal{V}^{\mathcal{N}\,IJ}\,\psi_\nu^J\,,\\
\mathcal{D}_\mu\epsilon^I &= \left(\partial_\mu + \tfrac{1}{2}\omega_\mu^a\,\gamma_a\right)\epsilon^I + \partial_\mu\phi^i\,Q_i^{IJ}\,\epsilon^J + g\,\Theta_{\mathcal{MN}}\,A_\mu^{\mathcal{M}}\,\mathcal{V}^{\mathcal{N}\,IJ}\,\epsilon^J\,,\\
\mathcal{D}_\mu\chi^{iI} &= \left(\partial_\mu + \tfrac{1}{2}\omega_\mu^a\gamma_a\right)\chi^{iI} + \partial_\mu\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\chi^{iJ}\right)\\
&\quad + g\,\Theta_{\mathcal{MN}}A_\mu^{\mathcal{M}}\left(\delta^i_j\,\mathcal{V}^{\mathcal{N}\,IJ} - \delta^{IJ}g^{ik}\,D_k\mathcal{V}^{\mathcal{N}}{}_j\right)\chi^{jJ}\,.
\end{aligned} \tag{2.7.13}$$

Modifying covariant derivatives with gauge fields is a cause of supersymmetry breaking. One of terms that leads to this violation is proportional to the field strength tensors. It takes the form

$$(\delta\mathcal{L})_1 \;=\; -\tfrac{1}{2}\mathrm{i}\,g\,\Theta_{\mathcal{MN}}F_{\nu\rho}^{\mathcal{N}}\,\epsilon^{\mu\nu\rho}\,\left(\mathcal{V}^{\mathcal{N}\,IJ}\,\bar{\psi}_\mu^I\epsilon^J + \tfrac{1}{2}\,\mathcal{V}^{\mathcal{N}}{}_i\,\bar{\chi}^{iI}\gamma_\mu\epsilon^I\right)\,. \tag{2.7.14}$$

It can be cancelled by introducing a Chern-Simons term [69] which is topological[4]

$$\mathcal{L}_{\text{CS}} \;=\; \tfrac{1}{4}\,\mathrm{i}g\,\varepsilon^{\mu\nu\rho}\,A_\mu^{\mathcal{M}}\,\Theta_{\mathcal{MN}}\left(\partial_\nu A_\rho^{\mathcal{N}} - \tfrac{1}{3}g\,\hat{f}_{\mathcal{PQ}}{}^{\mathcal{N}}\,A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}}\right)\,, \tag{2.7.15}$$

[4]In the general context, it means that it does not rely on notions of geometrical structure of manifold, i.e. metric independent.

Its supersymmetry transformation is,

$$\Theta_{\mathcal{M}\mathcal{N}}\, \delta A^{\mathcal{M}}_{\mu} \;=\; \Theta_{\mathcal{M}\mathcal{N}} \left[2\, \mathcal{V}^{\mathcal{M}\, IJ}\, \bar{\psi}^{I}_{\mu}\epsilon^{J} + \mathcal{V}^{\mathcal{M}}{}_{i}\, \bar{\chi}^{iI}\gamma_{\mu}\epsilon^{I}\right] . \qquad (2.7.16)$$

According to many higher dimensional gauged supergravities studied previously, it is useful to define the so-called T-tensors even though the embedding tensors technique did not emerge. In our case, we can define T-tensors as the images of the map $\mathcal{V}\times\Theta\times\mathcal{V}\rightarrow T$

$$\begin{aligned} T^{IJ,KL} &\equiv \mathcal{V}^{\mathcal{M}\,IJ}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}\,KL} , & T^{IJi} &\equiv \mathcal{V}^{\mathcal{M}\,IJ}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}\,i} , \\ T^{ij} &\equiv \mathcal{V}^{\mathcal{M}\,i}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}\,j} , & T_{\alpha}{}^{i} &\equiv \mathcal{V}^{\mathcal{M}}{}_{\alpha}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}\,i} , \\ T_{\alpha\beta} &\equiv \mathcal{V}^{\mathcal{M}}{}_{\alpha}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_{\beta} , & T^{IJ}{}_{\alpha} &\equiv \mathcal{V}^{\mathcal{M}\,IJ}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_{\alpha} . \end{aligned} \qquad (2.7.17)$$

The virtues of these definitions will become clear later when we discuss about the constraints on T-tensors forced by supersymmetry.

Even though we can get rid off term (2.7.14), still there is another exceeding term that is a consequence of adding vector fields. It emerges from introducing supersymmetry variation of vector fields. It is in the first order of the coupling constant g:

$$(\delta\mathcal{L})_2 \;=\; -eg\, \Theta_{\mathcal{M}\mathcal{N}} \left(2\, \mathcal{V}^{\mathcal{M}\, IJ}\, \bar{\psi}^{I}_{\mu}\epsilon^{J} + \mathcal{V}^{\mathcal{M}}{}_{i}\, \bar{\chi}^{iI}\gamma_{\mu}\epsilon^{I}\right) \mathcal{V}^{\mathcal{N}}{}_{j}\, \mathcal{D}^{\mu}\phi^{j} . \qquad (2.7.18)$$

In order to cancel this term, we postulate mass-like terms

$$\mathcal{L}_g \;=\; eg\Big\{ \tfrac{1}{2} A_1^{IJ}\, \bar{\psi}^{I}_{\mu}\, \gamma^{\mu\nu}\, \psi^{J}_{\nu} + A_{2j}^{IJ}\, \bar{\psi}^{I}_{\mu}\, \gamma^{\mu}\chi^{jJ} + \tfrac{1}{2} A_{3}{}^{IJ}_{ij}\, \bar{\chi}^{iI}\chi^{jJ} \Big\} , \qquad (2.7.19)$$

together with additional modifications of the supersymmetry transformation rules of fermions

$$\delta_g\psi^{I}_{\mu} = g\, A_1^{IJ}\gamma_{\mu}\,\epsilon^{J} , \qquad \delta_g\chi^{iI} = -gN\, A_2^{JiI}\,\epsilon^{J} . \qquad (2.7.20)$$

The unknowns now reside in the tensors A_1, A_2 and A_3. The shape of A's tensors should be revealed as we continue to construct the supersymmetric Lagrangian. Note that tensors A_1 and A_3 are symmetric,

$$A_1^{IJ} = A_1^{JI} , \qquad A_{3}{}^{IJ}_{ij} = A_{3}{}^{JI}_{ji} . \qquad (2.7.21)$$

After some calculations, for $N>2$, the tensors A_1, A_2 and A_3 are written in terms of the T-tensor and tensor f^{IJ}_{ij}

$$\begin{aligned} A_1^{IJ} &= -\frac{4}{N-2}\, T^{IM,JM} + \frac{2}{(N-1)(N-2)}\,\delta^{IJ}\, T^{MN,MN} , \\ A_{2j}^{IJ} &= \frac{2}{N}\, T^{IJ}{}_{j} + \frac{4}{N(N-2)}\, f^{M(I}{}_{j}{}^{m}\, T^{J)M}{}_{m} + \frac{2\,\delta^{IJ}}{N(N-1)(N-2)}\, f^{KL}{}_{j}{}^{m}\, T^{KL}{}_{m} , \\ A_{3}{}^{IJ}_{ij} &= \frac{1}{N^2}\Big\{ -2\, D_{(i}D_{j)}A_1^{IJ} + g_{ij}\, A_1^{IJ} + A_1^{K[I}\, f^{J]K}_{ij} \\ &\qquad +2\, T_{ij}\,\delta^{IJ} - 4\, D_{[i}T^{IJ}{}_{j]} - 2\, T_{k[i}\, f^{IJk}{}_{j]} \Big\} . \end{aligned} \qquad (2.7.22)$$

It is obvious that A_1, A_2 and A_3 become undetermined when $N = 1$ or $N = 2$, so $N = 1, 2$ cases need special treatment as discussed in [37].

The analysis above is valid up to the first order in coupling constant g. In order to preserve supersymmetry to order g^2 we need to add more term to the Lagrangian, it reads

$$\mathcal{L}_{g^2} \;=\; -\,eV \;\equiv\; \frac{4\,eg^2}{N}\left(A_1^{IJ}A_1^{IJ} - \tfrac{1}{2}N\,g^{ij}\,A_{2i}^{I\,J}A_{2j}^{I\,J}\right)\;. \tag{2.7.23}$$

It is proportional to g^2 and depends only on scalar field so it is called the scalar potential. Not only we add scalar potential, we also need more constraints on A's tensors to eliminate the unwanted terms coming from varying the scalar potential. They read

$$\begin{aligned} 2\,A_1^{IK}A_1^{KJ} - N\,A_2^{IiK}\,A_{2i}^{J\,K} \;&=\; \frac{1}{N}\,\delta^{IJ}\left(2\,A_1^{KL}\,A_1^{KL} - N\,A_2^{KiL}A_{2i}^{KL}\right)\;,\\ 3\,A_1^{IK}\,A_{2j}^{K\,J} + N\,g^{kl}\,A_{2k}^{IK}\,A_{3lj}^{\;KJ} \;&=\; \mathbb{P}^{I\,i}_{Jj}\left(3\,A_1^{KL}\,A_{2i}^{KL} + N\,g^{kl}\,A_{2k}^{LK}\,A_{3li}^{\;KL}\right)\;. \end{aligned} \tag{2.7.24}$$

There are still many constraints that we do not mention, all of which can be found in [37]. Setting supersymmetry as a main goal and manipulating many constraints, it boils down to one master equation that justifies validity of gauging without spoiling supersymmetry,

$$T^{IJ,KL} - T^{[IJ,KL]} - \frac{4}{N-2}\,\delta^{\,\overline{I[K}\;\overline{T^{L]M,MJ}}} - \frac{2\,\delta^{I[K}\delta^{L]J}}{(N-1)(N-2)}\,T^{MN,MN} \;=\; 0\;. \tag{2.7.25}$$

There is a more compact way to represent the constraint (2.7.25) by exploiting group representation theory:

$$\mathbb{P}_{\boxplus}\,T^{IJ,KL} \;=\; 0\;. \tag{2.7.26}$$

The symbol $\boxplus$ denotes the Young tableaux for a vector representation of SO(N). This notation comes from an observation that

$$\boxminus \times_{\text{sym}} \boxminus \;=\; 1 \;+\; \square\!\square \;+\; \begin{smallmatrix}\square\\ \square\\ \square\\ \square\end{smallmatrix} \;+\; \boxplus\;, \tag{2.7.27}$$

Note that the dimension of each Young tableaux for $SO(N)$ is as follows , $\frac{1}{2}N(N+1) - 1$, $\frac{1}{12}N(N-3)(N+1)(N+2)$, and $\binom{N}{4}$.

For symmetric space ($N > 4$), the constraint on T-tensor (an $SO(N)$ tensors) can be uplifted to constraint on embedding tensor (a group G tensors). As a result, we deal with constraint on constant tensor instead of field-dependent constraint. In order to uplift $SO(N)$ to G constraint we simply project out the representation $\boxplus$ in $SO(N)$ that contain in a representation R_0 in G of T-tensor and the projection is a G-covariant condition, so it also work for the embedding tensor.

$$\mathbb{P}_{R_0}\, T_{\mathcal{AB}} = 0 \longrightarrow \mathbb{P}_{R_0}\, \Theta_{\mathcal{MN}} = 0 \; . \tag{2.7.28}$$

The T-tensor is decomposed under G as symmetrized tensor product of the adjoint representation of G as follows

$$R_{\text{adj}} \times_{\text{sym}} R_{\text{adj}} = \mathbf{1} \oplus \left[\textstyle\bigoplus_i R_i\right] \; , \tag{2.7.29}$$

where $\mathbf{1}$ and R_{adj} are the singlet and the adjoint representation of G, respectively, and "$\times_{\text{sym}}$" denotes the symmetrized tensor product. According to the branching above, it can be written as

$$T_{\mathcal{AB}} = g_1\theta\, \eta_{\mathcal{AB}} + \textstyle\sum_i g_{i+1}\, \mathbb{P}_{R_i}\, T_{\mathcal{AB}} \; , \tag{2.7.30}$$

where $\eta_{\mathcal{AB}}$ is the Cartan-Killing form of G, θ is constant that we will identify later , g_1 and g_{i+1} are the gauge coupling constants and $\mathbb{P}_{R_i}$ denotes the G-invariant projector onto the representation R_i.

To summarize: the key equation whose solutions are called admissible gauging G_0 is

$$\mathbb{P}_{R_0}\, \Theta_{\mathcal{MN}} = 0 \; . \tag{2.7.31}$$

Analogous to T-tensor, the embedding tensor can be decomposed as

$$\Theta_{\mathcal{MN}} = g_1\theta\, \eta_{\mathcal{MN}} + \textstyle\sum_i g_{i+1}\, \mathbb{P}_{R_i}\, \Theta_{\mathcal{MN}} \; . \tag{2.7.32}$$

Finding exhaustive solutions of equation (2.7.31) may give us a complete classification of gauged theories for $N > 4$; therefore, it is purely group-theoretical argument which distinguish itself from higher dimensional theories. The trivial solution is the pure singlet which corresponds to gauging full isometry group G. According to (2.7.32) the solution is simple Cartan-Killing form of G. The non-trivial solutions are classified into four classes: compact gauge group, non-compact gauge groups, non-semisimple gauge groups and complex gauge groups. Only the last one that is excluded from the study in this dissertation.

For compact gauge groups, the solutions are provided by [37] for arbitrary $N > 5$ and $N \neq 8$; we simply give the results here

$$\begin{aligned} \Theta_{IJ\,A} &= 0 \\ \Theta_{IJ\,\alpha} &= 0 \\ \Theta_{AB} &= 0 \\ \Theta_{IJ,KL} &= \theta\, \delta_{IJ}^{KL} + \delta_{\underline{I[K}\, \Xi_{L]\underline{J}}} \; . \end{aligned} \tag{2.7.33}$$

The traceless symmetric tensor Ξ_{IJ} and θ are chosen to be

$$\Xi_{IJ} = \begin{cases} 2(1-\frac{p}{N})\delta_{IJ} & \text{for } I \leq p \\ -2\frac{p}{N} & \text{for } I > p \end{cases} \; , \theta = \frac{2p-N}{N} \; . \tag{2.7.34}$$

This embedding tensor corresponds to the gauge group of the type

$$\mathrm{SO}(p) \times \mathrm{SO}(N-p) \subset \mathrm{SO}(N) \,, \tag{2.7.35}$$

with relatively opposite charges between two subgroups; the embedding tensor reads

$$\Theta = \Theta^{\mathrm{SO}(p)} - \Theta^{\mathrm{SO}(N-p)} \,. \tag{2.7.36}$$

Note that the overall coupling constant is omitted. In this dissertation, for the one coupling theories, the coupling appear as g_1. For some cases with more than two couplings the embedding tensor is written in the explicit form with couplings g_1, g_2,.... Not all compact subgroup of G can be gauged; for example, the $SO(p_1) \times SO(p_2) \times SO(p_3) \times \ldots$ gauge groups with more than two factors are excluded from admissible gauge groups because they are not solutions of (2.7.31).

Since this dissertation studies only on $N = 5, 6$ cases, we limit ourselves discussing only those two. For $N = 5$ with $G = USp(4, k)$, the adjoint representation R_{adj} is $(2, 0, \ldots)$ and

$$R_{\mathrm{adj}} \times_{\mathrm{sym}} R_{\mathrm{adj}} \longrightarrow (0, \ldots) + (0, 1, \ldots) + (0, 2, \ldots) + \underline{(4, 0, \ldots)} \tag{2.7.37}$$

Note that the underlined representation contains R_0 and the three dots $\ldots$ is filled with zeros where the number of zero depends on the rank of the group. We use Dynkin label instead of conventional notation since they share the same form for arbitrary k.

For $N = 6$ with $G = SU(4, k)$, the adjoint representation R_{adj} is $(1, \ldots, 1)$ and

$$R_{\mathrm{adj}} \times_{\mathrm{sym}} R_{\mathrm{adj}} \longrightarrow (0, \ldots, 0) + (1, \ldots, 1) + (0, 1, \ldots, 1, 0) + \underline{(2, \ldots, 2)} \tag{2.7.38}$$

To summarize: we list the full Lagrangian, covariant derivatives, supersymmetry transformation rules and gauge transformations below. The full Lagrangian reads

$$\begin{aligned}
\mathcal{L} \;=\; & -\tfrac{1}{2}\,\mathrm{i}\,\varepsilon^{\mu\nu\rho}\left(e_\mu{}^a\,R_{\nu\rho a} + \bar\psi^I_\mu \mathcal{D}_\nu \psi^I_\rho\right) - \tfrac{1}{2} e\, g_{ij}\left(g^{\mu\nu}\,\mathcal{D}_\mu\phi^i\,\mathcal{D}_\nu\phi^j + N^{-1}\bar\chi^{iI}\not{\mathcal{D}}\chi^{jI}\right) \\
& + \tfrac{1}{4}\,\mathrm{i}g\,\varepsilon^{\mu\nu\rho}\,A^{\mathcal{M}}_\mu\,\Theta_{\mathcal{M}\mathcal{N}}\Big(\partial_\nu A^{\mathcal{N}}_\rho - \tfrac{1}{3} g\,\hat f_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{N}}\,A^{\mathcal{P}}_\nu A^{\mathcal{Q}}_\rho\Big) \\
& + \tfrac{1}{4} e\, g_{ij}\,\bar\chi^{iI}\gamma^\mu\gamma^\nu\psi^I_\mu\,(\mathcal{D}_\nu\phi^j + \widehat{\mathcal{D}}_\nu\phi^j) - \tfrac{1}{24} e\,N^{-2} R_{ijkl}\,\bar\chi^{iI}\gamma_a\chi^{jI}\,\bar\chi^{kJ}\gamma^a\chi^{lJ} \\
& + \tfrac{1}{48} e\,N^{-2}\left(3\,(g_{ij}\,\bar\chi^{iI}\chi^{jI})^2 - 2(N-2)\,(g_{ij}\,\bar\chi^{iI}\gamma^a\chi^{jJ})^2\right) \\
& + eg\Big(\tfrac{1}{2} A_1^{IJ}\,\bar\psi^I_\mu\,\gamma^{\mu\nu}\,\psi^J_\nu + A_{2j}^{IJ}\,\bar\psi^I_\mu\,\gamma^\mu\chi^{jJ} + \tfrac{1}{2} A_3{}^{IJ}_{ij}\,\bar\chi^{iI}\chi^{jJ}\Big) \\
& - 2\,eg^2\left(g^{ij}\,A_{2i}^{IJ} A_{2j}^{IJ} - 2N^{-1} A_1^{IJ} A_1^{IJ}\right) \,.
\end{aligned} \tag{2.7.39}$$

The covariant derivatives are given by

$$
\begin{aligned}
\mathcal{D}_\mu\phi^i &= \partial_\mu\phi^i + g\,\Theta_{\mathcal{M}\mathcal{N}}\,A_\mu^{\mathcal{M}}\,X^{\mathcal{N}\,i}\;, \qquad \widehat{\mathcal{D}}_\mu\phi^i \;=\; \mathcal{D}_\mu\phi^i - \tfrac{1}{2}\bar{\psi}^I_\mu\chi^{iI}\;,\\
\mathcal{D}_\mu\psi^I_\nu &= \left(\partial_\mu + \tfrac{1}{2}\omega_\mu^a\gamma_a\right)\psi^I_\nu + \partial_\mu\phi^i Q_i^{IJ}\,\psi^J_\nu + g\,\Theta_{\mathcal{M}\mathcal{N}}A_\mu^{\mathcal{M}}\,\mathcal{V}^{\mathcal{N}\,IJ}\,\psi^J_\nu\;,\\
\mathcal{D}_\mu\chi^{iI} &= \left(\partial_\mu + \tfrac{1}{2}\omega_\mu^a\gamma_a\right)\chi^{iI} + \partial_\mu\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\chi^{iJ}\right)\\
&\quad + g\,\Theta_{\mathcal{M}\mathcal{N}}A_\mu^{\mathcal{M}}\left(\delta^i_j\,\mathcal{V}^{\mathcal{N}\,IJ} - \delta^{IJ}g^{ik}\,D_k\mathcal{V}^{\mathcal{N}}{}_j\right)\chi^{jJ}\;,\\
\mathcal{D}_\mu\epsilon^I &= \left(\partial_\mu + \tfrac{1}{2}\omega_\mu^a\gamma_a\right)\epsilon^I + \partial_\mu\phi^i Q_i^{IJ}\,\epsilon^J + g\,\Theta_{\mathcal{M}\mathcal{N}}A_\mu^{\mathcal{M}}\,\mathcal{V}^{\mathcal{N}\,IJ}\,\epsilon^J\;.
\end{aligned}
\tag{2.7.40}
$$

The supersymmetry transformations are

$$
\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\,\bar{\epsilon}^I\gamma^a\,\psi^I_\mu\;,\\
\delta A_\mu^{\mathcal{M}} &= 2\,\mathcal{V}^{\mathcal{M}\,IJ}\,\bar{\psi}^I_\mu\epsilon^J + \mathcal{V}^{\mathcal{M}}{}_i\,\bar{\chi}^{iI}\gamma_\mu\epsilon^I\;,\\
\delta\psi^I_\mu &= \mathcal{D}_\mu\epsilon^I - \tfrac{1}{8}g_{ij}\,\bar{\chi}^{iI}\gamma^\nu\chi^{jJ}\,\gamma_{\mu\nu}\,\epsilon^J - \delta\phi^i\,Q_i^{IJ}\psi^J_\mu + g\,A_1^{IJ}\gamma_\mu\,\epsilon^J\;,\\
\delta\phi^i &= \tfrac{1}{2}\,\bar{\epsilon}^I\,\chi^{iI}\\
\delta\chi^{iI} &= \tfrac{1}{2}\left(\delta^{IJ}\mathbf{1} - f^{IJ}\right)^i{}_j\,\widehat{\mathcal{D}}\phi^j\,\epsilon^J - \delta\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\chi^{iJ}\right) - gN\,A_2^{JiI}\,\epsilon^J\;,
\end{aligned}
\tag{2.7.41}
$$

The gauge transformations read

$$
\begin{aligned}
\delta_{\text{gauge}}\phi^i &= g\,\Theta_{\mathcal{M}\mathcal{N}}\,\Lambda^{\mathcal{M}}X^{\mathcal{N}i}\;,\\
\delta_{\text{gauge}}\psi^I_\mu &= g\,\Theta_{\mathcal{M}\mathcal{N}}\,\Lambda^{\mathcal{M}}\mathcal{V}^{\mathcal{N}IJ}\,\psi^J_\mu - \delta\phi^i\,Q_i^{IJ}\psi^J_\mu\;,\\
\delta_{\text{gauge}}\chi^{iI} &= g\,\Theta_{\mathcal{M}\mathcal{N}}\,\Lambda^{\mathcal{M}}(\chi^{jI}\,D_j\mathcal{V}^{\mathcal{N}i} + \mathcal{V}^{\mathcal{N}IJ}\,\chi^{iJ}) - \delta\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\chi^{iJ}\right)\;,\\
\Theta_{\mathcal{M}\mathcal{N}}\,\delta_{\text{gauge}}A_\mu^{\mathcal{M}} &= \Theta_{\mathcal{M}\mathcal{N}}(-\partial_\mu\Lambda^{\mathcal{M}} + g\,\hat{f}_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{M}}\,A_\mu^{\mathcal{P}}\,\Lambda^{\mathcal{Q}})\,.
\end{aligned}
\tag{2.7.42}
$$

2.8 Vacua and their symmetries

In the previous section, we have discussed the construction of theory and the constraints. In this section, we discuss how to obtain the vacua of theories as well as their symmetries. At this stage, the dynamical quantity that concerns us the most is the scalar potential $V(\phi)$. Once being obtained, many things can be drawn from such as critical points or vacua. The computational codes are written and run by the computer application *Mathematica* whereby the scalar potentials and many quantities are symbolically solved. The very first task in the computation is to identify the explicit form of various generators such as isometry group, $SO(N)$ R-symmetry group, etc. Next, the gauge group in question is chosen and we construct coset representative accordingly. The coset representatives represent the manifold of the scalars and we construct them under either unitary gauge or Euler's angle parametrization. A short review on Euler's angle parametrization is given in the appendix. The constraints on T-tensor are also checked for non-predefined gaugings. And finally, the scalar potential is produced for further analysis. The detailed discussions for $N = 5$ and $N = 6$ theories will be given in chapter 3 and 4, respectively.

The jargon *critical point* is mentioned constantly throughout this dissertation, so it is crucial that we should elaborate this term. The critical points are the loci where $\partial V(\phi)/\partial\phi_i = 0$. In general, they may contain maxima, minima or saddle points. They can be normally obtained via traditional multi variables calculus; or other branches of Mathematics, if they are available. If the critical points admit the AdS vacua and there are at least two supersymmetric points, an RG flow can be calculated as we will encounter in chapter 5.

In some cases, we can parametrize full manifold without any problems. However, in some cases, to deal with many scalar fields (more than 8) all at once seems to be a daunting task. Armed with the brilliant trick by Warner [71, 72], it allows us search for critical points by analyzing much smaller manifold. The trick originally was applied to $D = 5, N = 8$ gauged supergravity case. Because it is based on group theoretical argument, there are no problems if we apply it to other dimensional theories. The strategy can be break down as follows: first, we pick a subgroup H_0 of the group G. Next, we find the set of generators that are singlets of H_0. This means that some scalars ψ form singlets while the other scalars ξ transform non-trivially under group H_0. The expansion around critical point should take the form $V(\psi, \xi) = V_0(\psi) + V_2(\psi)\xi^2 + (O)(\xi^3)$. One can see that if ψ_0 is a critical point of $V_0(\psi)$, then $\psi = \psi_0$ and $\xi = 0$ is also the critical point of the potential.

In order to study supersymmetry of the vacua, we consider the so-called BPS solution. It can be determined by considering differential equations originated from vanishing of supersymmetry transformation of fermionic fields together with setting those fields to zero. Schematically, the equation reads

$$D_\mu \,\epsilon(x) = 0\,. \tag{2.8.1}$$

These are necessary and sufficient condition for invariant of spacetime under supersymmetry [59]. The discussion on BPS solutions is closely related to studying RG Flow which will be explored in more details in the chapter 5. According to the proof in [49], the unbroken supersymmetries are determined by the equation

$$A_1^{IK} A_1^{KJ} \epsilon^J = -\frac{V_0}{4g^2}\epsilon^I = \frac{1}{N}(A_1^{KJ} A_1^{KJ} - \frac{1}{2} N g^{ij} A_{2i}^{KJ} A_{2i}^{KJ})\epsilon^I\,. \tag{2.8.2}$$

which is derived from the BPS equation mentioned above. The number of residual symmetries corresponds to the number of eigenvalue of A_1^{IJ}. The positive and the negative eigenvalues correspond to left n_L and right n_R chirality of supersymmetry, respectively. The total supersymmetry is then $n_L + n_R$. In practice, we simply check whether eigenvalue of A_1^{IJ} at critical points equals to $\sqrt{-\frac{V_0}{4}}$ up to sign.

It is also pointed out in [49] that at the origin where all scalars are turn-off, the supersymmetry is maximal and the residual gauge symmetry is the maximal compact subgroup of gauge group G_0. The argument is originally developed for

$N = 16$ theory in three dimensions [48, 49]. However, for the other value of N, the same argument is equivalently applicable. At the origin, the vacua undergo the Brout-Englert-Higgs mechanisms and the non-propagating vector fields from Chern-Simons theory in non-compact direction acquire mass. Subsequently, the vector fields split into two sector which are $\dim(G_0/H_0) = d_0$ massive self-dual vectors and $\dim H_0$ non-propagating vector fields from Chern-Simons theory.

Next, we discuss supergroup at critical points. The supergroup at vacua is the superextension of the product of maximal gauge groups and isometry group of AdS_3, i.e. $H_0 \times SO(2,2)$. Note that the group $SO(2,2)$ is homomorphic to $SU(1,1)_L \times SU(1,1)_R$ and sometimes we often use this group as an isometry group since it explicitly splits left and right chirality. Subsequenctly, the extended supergroup can be written as $G_L \times G_R$; H_0 can be factorized into $H_L \times H_R$ and the supersymmetry generators are split into $N = (n_L, n_R)$. According to classification in [70], the supergroup G_L and G_R is given by one of the following

1. $OSp(N|2,\mathbb{R}) \supset O(N) \times Sp(2,\mathbb{R})$
2. $SU(N|1,1) \supset U(N) \times SU(1,1)$ for $N \neq 2$,
   $SU(2|1,1) \supset SU(2) \times SU(1,1)$ for $N = 2$
3. $OSp(4^*|2N) \supset O^*(4) \times USp(2N) \simeq SU(2) \times USp(2N) \times SU(1,1)$
4. $G(3) \supset G_2 \times SU(1,1)$
5. $F(4) \supset Spin(7) \times SU(1,1)$
6. $D^1(2,1,\alpha) \supset SU(2) \times SU(2) \times SU(1,1)$

Note that the fermionic generators of the supergroups listed above transform as $(\mathbf{2},0) \oplus (0,\mathbf{2})$ of isometry group $SO(2,2) \simeq SU(1,1) \times SU(1,1)$.

The discussion on the stability of gauged supergravity can be found in [92, 59]. The stability of gauged supergravity with AdS background is required for a sensible quantum field theory. The stability in this context roughly means the positivity of energy of fluctuations around background. Even though the condition is proved in the context of gauged supergravity in four dimensions but it also works in three dimensional cases or even non-supersymmetric theories with scalar fields coupled to gravity.

2.9 Holographic RG flow

The AdS/CFT correspondence is more attractive when we go beyond conformal world since the physical world we live in is non-conformal. *Holograpic RG flow* is coined to describe the duality between a gravitational theory with scalar potential and renormalization group flows of deformed comformal field theories. Note that

RG flows throughout this paper mostly means holographic RG flows. The word *holographic* occasionally is dropped for the sake of brevity.

Physical systems depend on scale. We describe nature defined at a different energy scale or a distance (sometimes both are called a scale) by distinct theories with a different dynamics and degrees of freedom. We will consider this phenomenon in the context of quantum field theory. In quantum field theory, renormalization groups play an important role in describing the change in the behavior of the physical system when the scale is changed. Different physics at different scales emerge because a scale parameter is inevitably introduced via mass scale parameter attached to coupling constant in dimensional regularization scheme or cut-off momentum parameter. We call this parameter a scale μ. Most often, the prescription for renormalizations is arbitrary. Obviously Physics must not depend on those prescriptions, so two distinct discriptions are related and share the same Physics. However, their relations somehow might form abelian group (actually this is not necessary true), so they are called renormalization group due to partially historical reason.

The spirit of renormalization group is captured by *renormalization group equation (RGE)*. It is constructed by an observation that the bare n-points Green's functions Γ do not depend on scale parameter. Consequently, we get a differential equation describing how renormalized quantities change with scale parameter. It is given by

$$\mu\,\frac{d\Gamma_R[\mu, g]}{d\mu} = \mu\,\frac{\partial\Gamma_R[\mu, g]}{\partial\mu} + \beta(g(\mu))\frac{\partial\Gamma_R[\mu, g]}{\partial g} = 0\,. \tag{2.9.1}$$

According to equation above, a quantity that play a major role in the study of RG flow is called beta function defined by

$$\beta(g(\mu)) \equiv \mu\frac{dg(\mu)}{d\mu}\,, \tag{2.9.2}$$

where g is coupling constant of the theory. The $g^* \in g$ when $\beta(g^*) = 0$ is called *fixed points*. At $g^* \neq 0$, it is called non-trivial fixed point. Quantum field theories defined at fixed point do not depend on scale, in other words they are conformal. g^* is called an ultra-violet (UV) fixed point when the coupling is driven to this point when increasing in μ. On the other hand, we call g^* an infra-red (IR) fixed point when the coupling is driven to this point when decreasing in μ. According to flow equation, the theories would be driven along a trajectory in the coupling space changing in energy scale μ visualized as a flow, so this process deserves the name. Note that there is a theory whose beta function vanishes identically, it is $\mathcal{N} = 4$ SYM and of course it is conformal. There is no RG flow for such theory; however, the scenario is changed once it is perturbed by some operators with specific scale dimension. As a result, the conformal symmetry would break and RG flow can be triggered.

We continue by elaborating the discussion above with a conformal field theory in d dimensions. If the original action of a conformal theory is deformed by an operator with scale dimension Δ such that:

$$S_{CFT} \to S_{QFT} = S_{CFT} + \int d^d x\, \phi(x) O_\Delta(x). \tag{2.9.3}$$

The conformal symmetry could be broken and the theory would be driven along a trajectory in the coupling space changing in energy scale μ. This process is commonly called RG flow as we discussed earlier.

The operators above are classified into three classes:

- *relevant operator* : an operator that increasingly deforms the theory as μ flow to IR.
- *maginal operator* : an operator that leaves the theory remain conformal.
- *irrelevant operator*: an operator that decreasingly deforms the theory as μ flow to IR.

The theory at UV contains more information than at IR. If a theory makes a transition from UV to IR, some degrees of freedom are discarded or integrated out. Once the process undergoes the transitions, it cannot retrieve all the lost information back. This is referred to as irreversibility. Special to two dimensional spacetime, there is a theorem called Zamolodchikov c-theorem that can guarantee the irreversibility. We will discuss this issue again in the section 2.11.

2.9.1 RG flow via Supergravity

In the AdS/CFT context the holographic renormalization group flow is the renormalized group flow of the boundary theory narrated by a particular class of classical solutions of the bulk theory. The boundary theory in this context is a conformal field theory perturbed by specific operators or the operators themselves acquiring non-vanishing expectation value. In the perturbing operator scenario, the additional terms may cause conformal symmetry breaking. As a result the deformed theory induces RG flow between fixed points. In our study we narrow down to a class of RG flow that is a flow between two conformal fixed points identified as UV and IR fixed points. In the bulk theory this can be interpreted as a class of solution of gauged supergravity which contains a scalar potential. The solution contains a metric of domain wall type and a kink-like solution of scalars. The critical points of scalar potential are identified with specific AdS solutions characterized by distinct AdS radii. For simplicity, we are now considering flows associating with two critical points, so there are AdS spaces of two regions in the bulk; the first one is at the boundary and the second one is in the deep interior region identified by radial coordinate r associated with limit $r \to +\infty$ and $r \to -\infty$. In our convention the two AdS radii characterized by L_{UV} and L_{IR} correspond to $r \to +\infty$ and $r \to -\infty$, respectively. In general, there are various classes of flows, for example, the flow can also undergo from a UV fixed point to a new non-conformal field theory that associates with non-AdS geometry inside the bulk. Unfortunately, in this dissertation we will not discuss this case.

The flow solutions can be either analytical or numerical. They can be either approximated or exact. Even though the flow solution is exact, the asymptotic analysis of the flow solutions is still very useful in determining the scale dimension Δ of perturbed operator. They are considered in two limit at which AdS spaces are specified, so the AdS with radii L_{UV} and L_{IR} correspond to scale dimensions Δ_{UV} and Δ_{IR}, respectively.

One of key concepts of holographic renormalized group flow is that the radial coordinate r in the bulk theory has a dual picture as an energy scale μ in QFT. At intermediate radius r, it can be interpreted as a specific quantum field theory defined at a particular energy scale. This interpretation gives us a wonderful discription between a single theory in higher dimensions and a spectrum of theories defined in a range of energy scales formulated in lower dimensions.

The other dynamical quantity such as central charges can be obtained as well. According to AdS/CFT, the AdS with radii L_{UV} and L_{IR} are identified with conformal field theories characterized by central charge c_{UV} and c_{IR}, respectively. The detailed discussion is mention in the section 2.10

There are many ways to study flow solutions. For example, one can completely abandons fermionic sector and obtains flows from Euler-Lagrange equation like a toy model we about to discuss or one can solve for equations of motion via Cartan structure equations. In this dissertation we only interested in supersymmetric flow whereby a sub-solution of full Euler-Lagrange equations is considered. It can be obtained by solving the so-called BPS solutions which are set up by vanishing of supersymmetry transformations on fermions with vanishing of fermions themselves. The equations are first order derivative rather than second order derivative which obtained from full Euler-Lagrange equations and they preserve some supersymmetries of the original Lagrangian. The detailed discussion on BPS solutions for 3D gauged supergravity will be addressed in the end of this section.

In order to obtain RG flow manifestation from gravity dual, it is easier to consider a toy model of a supergravity. It simplifies the gauged supergravity by considering only gravity and a single scalar with a scalar potential. The vectors, matter fermions and gravitini are neglected. It is easier if we work in $d+1$ dimensional Euclidean space instead of Minkowskian. The results in this section are in the form of general $d+1$ dimensions; therefore, in three dimensions we simply set $d = 2$. The action is given by

$$S = \frac{1}{4\pi G} \int d^{d+1}x\sqrt{g}\left(-\frac{1}{4}R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right) \tag{2.9.4}$$

This simple model still serves our purpose quite well since throughout this dissertation we study flows from a single scalar. We explore only the supersymmetric flows associated two different AdS vacua. The dual picture is the UV CFT is perturbed by a particular operator O and drive a flow to another IR CFT. That amounts to the scalar potential $V(\phi)$ should have at least two critical points, i.e. solution of $V'(\phi) = 0$ and we label the distinct critical points by the abbreviated notation ϕ_i. The scalar potential may have a local maximum and a local minimum critical points as shown in figure 2.1. In order to get AdS geometry profile, we identify the value of the scalar potential as follows

$$\Lambda_i \sim V(\phi_i) \sim -\frac{1}{L_i^2}, \tag{2.9.5}$$

so another requirement is that $V(\phi_i) < 0$. The sketch of the profile of scalar potential is shown in figure 2.1. Note that we may shift the maximum to $\phi = 0$

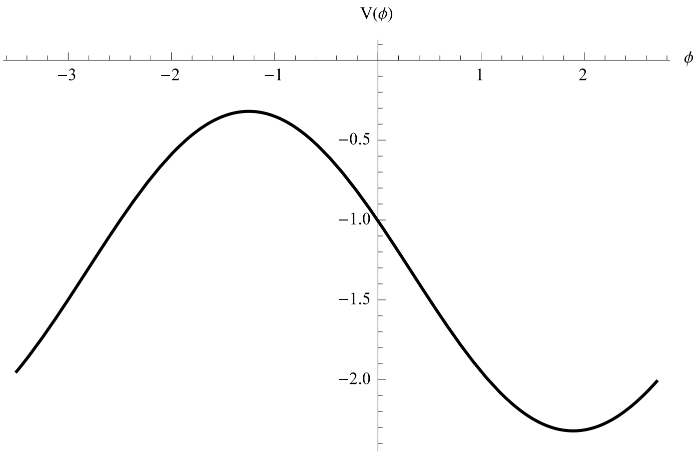

Figure 2.1: Scalar Potential $V(\phi)$.

without loss of generality.

The equation of motions obtained from extremizing the action (2.9.4) or Euler-Lagrange are given by

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - V'(\phi) = 0 \tag{2.9.6}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\left[\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right)\right] = 2T_{\mu\nu} \tag{2.9.7}$$

At critical points ϕ_i, the first equation trivially satisfies. The second equation reduces to Einstein equation with cosmological constant. Using the identification in (2.9.5), we get the connection between the critical points and AdS radii or cosmological constant. We now need a proper ansatz to study renormalized group flow. Since conformal symmetry would break down to the non-conformal one and the symmetry should match on both side, so AdS geometry now become less symmetric, i.e. Poincaré invariance. The generalized Poincaré invariant metric in some literature is called domain wall solution. It reads

$$ds^2 = e^{2A(r)}\delta_{ij}dx^i dx^j + dr^2, \tag{2.9.8}$$

$$\phi = \phi(r)\,, \tag{2.9.9}$$

where r is radial coordinate and x_i are transverse coordinates. Note that the scalar field is a function of radial coordinate only. The role played by the radial coordinate of the bulk field is crucial. It can practically be identified as an energy scale of the boundary theory. This seems to be a perfect way to interpret the additional coordinate in the context of lower dimensional theory. We are looking for solutions that interpolate between two conformal fixed points of the dual theory. The boundary region of AdS ($r \to +\infty$) corresponds to the UV regime of conformal field theory, whereas the deep interior region ($r \to -\infty$) is associated with the IR. So we must have

$$\lim_{r\to\infty} A(r) = \frac{r}{L_{UV}}\,, \qquad \lim_{r\to-\infty} A(r) = \frac{r}{L_{IR}} \tag{2.9.10}$$

As explained in [5], the solution can be alternatively solved by Cartan structure equations. Having equipped with the ansatz above, we finally get three coupled second order differential equations and they are not independent. They read

$$A'^2 = \frac{2}{d(d-1)}\left(\phi'^2 - 2V(\phi)\right), \tag{2.9.11}$$

$$\phi'' + dA'\phi' = \frac{dV(\phi)}{d\phi}, \tag{2.9.12}$$

$$A'' = -\frac{2}{d-1}\phi'^2. \tag{2.9.13}$$

The differential equations above are more interesting when we evaluate at critical points, i.e. $\phi(r) = \phi_i$. Using (2.9.5), the solution from the A'^2 equation is

$$A(r) = \pm\frac{r}{L_i} + a_0 \tag{2.9.14}$$

The integration constant is not important here since we can always relabel the coordinates. The positivity of solution is purely conventional, and we choose the positive signature. The solution above now corresponds to the AdS space as we have identified in (2.9.10).

We have discussed only the solution at the boundary and the deep interior region. The next task is to discuss the interpolated solution between the two limits. The solution may be exact; however, it is also useful to obtain the expansion around the critical points themselves. As mentioned earlier, this section is crucial in determining the scale dimension Δ. At critical points, the scalar field $\phi(r)$ and the warped factor of the metric $A(r)$ can be written as

$$\phi(r) = \phi_i + h(r)\,, \qquad A' = \frac{1}{L_i} + a'(r)\,. \tag{2.9.15}$$

The equation of motion reduces from non-linear and coupled differential equation of motion to a second order homogeneous ordinary differential equation :

$$h'' + \frac{d}{L_i}h' - \frac{m_i^2}{L_i^2}h = 0 \tag{2.9.16}$$

The general solution is given by

$$h(r) = B\, e^{(\Delta_i - d)r/L_i} + C\, e^{-\Delta_i r/L_i}\,. \tag{2.9.17}$$

From Witten's prescription [3], scale dimension Δ_i is identified with scale dimension and it relates with mass of bulk field and bulk dimension by

$$\Delta_i L_i = \frac{1}{2}\left(d + \sqrt{d^2 + 4m_i^2}\right) \text{ or} \tag{2.9.18}$$

$$m_i^2 L_i^2 = \Delta_i(\Delta_i - d) \tag{2.9.19}$$

Note that a' is in order $\mathcal{O}(h^2)$, so the contribution from this factor is neglected since we consider only linear order. The asymptotic solutions in the limit $r \to \pm\infty$ are given by

$$\phi(r \to +\infty) \approx \phi_1 + B_1\, e^{(\Delta_1 - d)r/L_1} + C_1\, e^{-\Delta_1 r/L_1}\,, \tag{2.9.20}$$

$$\phi(r \to -\infty) \approx \phi_2 + B_2\, e^{(\Delta_2 - d)r/L_2} + C_2\, e^{-\Delta_2 r/L_2}\,. \tag{2.9.21}$$

Note that the scalar potential around the critical points can be expanded such that

$$V(\phi) \approx V(\phi_i) + \frac{1}{2}\frac{m_i^2}{L_i^2}h^2 + \mathcal{O}(h^3) \tag{2.9.22}$$

In the context of AdS/CFT, h is the fluctuation of the bulk field which is dual to some operator O_Δ where the mass of the bulk field m_i is given by

$$m_i^2 = L_i^2 V''(\phi_i) \tag{2.9.23}$$

which is equal to (2.9.19).

As mentioned earlier, the solution approaching critical points (constant values) in the limit $r \to \pm\infty$ is a kink-like solution and its sketch is given by figure 2.2. Some constants in (2.9.21) need to be specified to get sensible solutions. The

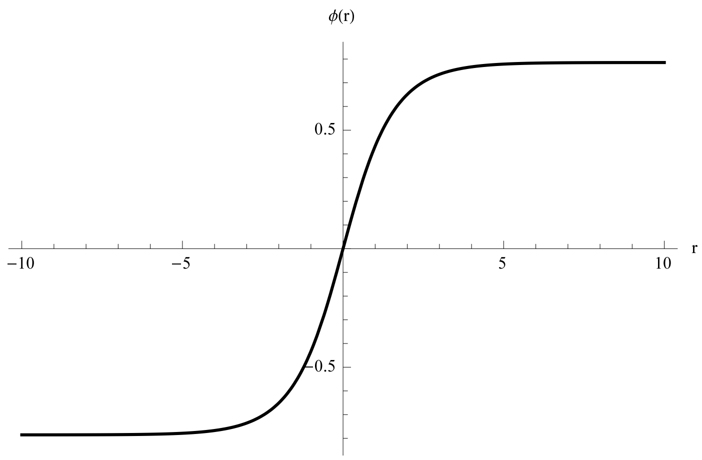

Figure 2.2: A sketch of a kink solution.

first boundary condition is that the fluctuation field must vanish in both limit. Using the fact that the scale dimension Δ_i is always positive and negative mass squared is allowed, one can obtain the valid range of the scale dimension. For $r \to +\infty$, B_1 and C_1 can be non-vanishing but the scale dimension must take value within some domain. From the solution, it requires $\Delta_1 < d$ and (2.9.19) implies $\frac{d}{2} < \Delta_1$ with $m_1^2 < 0$, so it should take value within the range

$$\frac{d}{2} < \Delta_1 < d\,. \tag{2.9.24}$$

Since the mass squared is negative and the local maximum has $V'' < 0$, so this critical point is at local maximum. The generic solution with $C_1 = 0$ corresponds to deformation of relevant operator. On the other hand, the solution with $C_1 \neq 0$ and $B_1 = 0$ corresponds to the deformation of conformal field theory (in UV) by a non-vanishing vacuum expectation value, $\langle O_{\Delta_1} \rangle \sim C_1 \neq 0$. In the context of holographic RG flow this critical point corresponds to a conformal field theory (UV fixed point) perturbed by relevant operator of dimension Δ_1 and it induces the flow to drive to other field theory. This interpretation fits perfectly well for identifying radial coordinate to energy scale in field theory, i.e., the larger the r, the higher the energy in dual theory. We may set L_1 to L_{UV} including other parameters with subscript 1.

For $r \to -\infty$, C_2 must vanish. According to the asymptotic solution, it requires $\Delta_2 > d$ and from (2.9.19) we get $m_2^2 > 0$ which is consistent to the fact that this critical point should be local minimum. In view of dual theory, this can be interpreted as an IR fixed point deformed by irrelevant operator with scale dimension $\Delta_2 > d$. We may set L_2 to L_{IR} including other parameters with subscript 2.

There are alternative ways to obtain equation of motion, many of which are in the form of a collection of coupled first order differential equations. One way to achieve this is that we exploit Hamilton-Jacobi formalism instead of Lagrange-Euler formalism. It requires introduction of auxiliary function such as Hamilton-Jacobi function $W(\phi)$ where its explicit form may reveal once we solve the system of equations. In some literatures $W(\phi)$ may be called superpotential. In our example, two second order flow equations are replaced by three first order flow equations with introducing superpotential $W(\phi)$. They read

$$\frac{1}{2}\left(\frac{dW}{d\phi}\right)^2 - \frac{d}{d-1}W^2 = V(\phi) \tag{2.9.25}$$

$$\frac{d\phi}{dr} = \frac{dW(\phi)}{d\phi} \tag{2.9.26}$$

$$\frac{dA}{dr} = -\frac{2}{d-1}W(\phi(r))\,. \tag{2.9.27}$$

The first order equations are also originated from BPS equations sometimes called Killing spinor equations. Since their mathematical structures are quite similar, so two names can be used interchangeably throughout the literatures. As mentioned earlier, Killing spinors also tell us about the residual supersymmetry of the solution via eigenvalue of A_1 tensor. In order to get the idea how equations for solving Killing spinors are set up, consider a variation of a generic supergravity action

$$\delta S = \int \left(\frac{\delta S}{\delta B}\delta B + \frac{\delta S}{\delta F}\delta F \right) \equiv 0, \tag{2.9.28}$$

where B are collections of bosonic fields in the theory and F are collections of bosonic fields. It satisfies equations of motion if $\delta S = 0$. Since $\delta B \sim F$ and classical background requires $F = 0$, so it is left with vanishing of fermions equation.

From this point forward, we discuss RG flows from gauged supergravity in three dimensions via BPS equations. This material is crucial in obtaining all flows in chapter 5. For convenience, we present the transformations of the fermions $\delta\psi^I_\mu$ and $\delta\chi^{iI}$ here (they appears once when we discussed supersymmetry transformation of the Lagrangian in the beginning of chapter 2).

$$\begin{aligned} \delta\psi^I_\mu &= \mathcal{D}_\mu \epsilon^I + gA_1^{IJ}\gamma_\mu \epsilon^J = 0 \\ \delta\chi^{iI} &= \frac{1}{2}(\delta^{IJ}\mathbf{1} - f^{IJ})^i{}_j \not{\mathcal{D}}\phi^j \epsilon^J - gNA_2^{JIi}\epsilon^J = 0 \end{aligned} \tag{2.9.29}$$

where $\mathcal{D}_\mu \epsilon^I = \left(\partial_\mu + \frac{1}{2}\omega^a_\mu \gamma_a\right)\epsilon^I$ for vanishing vector fields. In three dimension, the domain wall ansatz is of the form

$$ds^2 = e^{2A(r)}dx^2_{1,1} + dr^2 \tag{2.9.30}$$

where $dx^2_{1,1}$ is the compact form of $-dx_0^2 + dx_1^2$. It obviously preserves Poincaré symmetry in two dimensions. The ansatz for scalar is

$$\phi = \phi(r)\,, \tag{2.9.31}$$

which depends only on radial coordinate r. According to AdS/CFT correspondence, the radial coordinate r is identified to energy scale of the dual theory. The ansatz for Killing spinor is

$$\epsilon^I = e^{\frac{A(r)}{2}}\epsilon^I_0\,, \tag{2.9.32}$$

with constant spinor satisfying projection condition $\gamma_r \epsilon^I_0 = \epsilon^I_0$ which reduces the supersymmetry by half. From the domain wall metric given above, the spin connection is defined by

$$\omega_{\hat{\mu}}{}^{\hat{\nu}\hat{r}} = A'\delta^\nu_\mu \tag{2.9.33}$$

where $\hat{\mu}, \hat{\nu} = 0, 1$ which are flat basis indices. The covariant derivative for scalar reduce to ordinary derivative with respect to r. Note that, given coset representative, we can obtain the derivative of scalar field by

$$\frac{d\phi^A}{dr} = \mathrm{Tr}(T^A L^{-1} L')\,. \tag{2.9.34}$$

The analysis above can give us analytical solution, then we can expand solution around critical point to obtain scale dimension Δ as previously discussed.

2.10 Weyl anomaly

The calculation of conformal anomaly or Weyl anomaly from gravity side is one of the successful prediction from AdS/CFT correspondence. The very first evidence is the agreement in calculations of conformal anomaly of $D = 5$ AdS supergravity dual to $D = 4$ SYM. The conformal anomaly reveals the failure of conformal symmetry at quantum level. Note that the criterion for conformal symmetry of classical system is

$$T^i_i = g_{ij}\frac{\delta S}{\delta g_{ij}} = 0 \tag{2.10.1}$$

The conformal anomaly can be expressed as the non-vanishing of the trace of energy-momentum tensor, i.e.

$$\langle T^i_i \rangle \neq 0\,. \tag{2.10.2}$$

A very first example of conformal anomaly from a gauge theory coupled to non-dynamical metric was done by [74] for $D = 4$ and in an appendix for $D = 2$ where in those theories the scale is inevitably introduced via dimensional regularization. Soon after string theory had become a mainstream, much efforts were paid for calculation of anomaly in string world sheet $D = 2$. Polyakov [76] showed that Weyl anomaly-free condition for bosonic string corresponds to 26 dimensional spacetime. For superstring, he proved that critical spacetime dimensions is 10 [77]. A nostalgic review with a slightly sarcastic tone on Weyl anomaly can be found in [75]. An approach in the AdS/CFT context called the holographic Weyl anomaly and the paper itself goes by this name is done by Henningson and Skenderis [73]. The calculation gives a correct values of the central charges and proposes a possible monotonic c-function [5].

Let us first consider the conformal anomaly for $D = 4$ since it is widely discussed among many literatures. In quantum field theory the conformal anomaly in general is given by

$$\langle T^i_i \rangle = \frac{c}{16\pi^2}W^2_{ijkl} - \frac{a}{16\pi^2}\tilde{R}^2_{ijkl} \tag{2.10.3}$$

where a and c are central charges which characterize the theories and of course they are model-dependent. The W_{ijkl} is Weyl tensor and it relates to Reimann tensor and Ricci tensor via

$$W^2_{ijkl} = R^2_{ijkl} - 2R^2_{ij} + \frac{1}{3}R^2 \tag{2.10.4}$$

and the second contribution is Euler density

$$\tilde{R}^2_{ijkl} = \left(\frac{1}{2}\epsilon^{mn}_{ij}R^2_{mnkl}\right)^2 = R^2_{ijkl} - 4R^2_{ij} + R^2 \tag{2.10.5}$$

whose integral gives rise to a topological term which is Gauss-Bonnet term and it is proportional to Euler characteristic χ of manifold, i.e. $\int d^4x\sqrt{g}\tilde{R}^2_{ijkl} = 4\pi\chi$.

[5]The monotonic function is a function which is either increasing or decreasing.

We have discussed the general structure of $D = 4$ field theory. Now we discuss the special case which plays an important role in justifying AdS/CFT correspondence and it is of course $\mathcal{N} = 4$ SYM. We simply give the important results since the detailed discussions can be found in many reviews on AdS/CFT correspondence. The central charges a and c relate to the number of degrees of freedom via N in gauge group $SU(N)$ as follows

$$a = c = \frac{1}{4}(N^2 - 1) \xrightarrow[N\to\infty]{} \frac{1}{4}N^2 \,. \tag{2.10.6}$$

The equality of a and c is a unique feature of conformal field theory that is involving in AdS/CFT correspondence. The conformal anomaly is given by

$$\langle T_i^i \rangle = \frac{c}{8\pi^2}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right) \tag{2.10.7}$$

This anomaly can be derived in the context of AdS/CFT correspondence as we will discuss in the next topic.

The next task is calculating the conformal anomaly via AdS/CFT correspondence. For $D = 2, 4, 6$ the holographic Weyl anomalies are calculated in [73]. We discuss the $D = 5$ AdS dual to $D = 4$ SYM case. First, one considers the action of pure AdS gravity in five dimensions,

$$S = \frac{-1}{16\pi G}\left[\int d^5z\sqrt{g}(R + \frac{12}{L^2}) + \int d^4z 2\sqrt{\gamma}K\right] \,. \tag{2.10.8}$$

Note that the surface term called Gibbons-Hawking is added.

Replacing the radial coordinate $\rho = e^{-\frac{2r}{L}}$, the metric for asymptotic AdS (AAdS) becomes

$$ds^2 = L^2(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j) \tag{2.10.9}$$

Note that the boundary is replaced from $r \to \infty$ to $\rho = 0$ and the boundary metric is given by

$$g_{ij}(x,\rho) \quad \xrightarrow[\rho\to 0]{} \quad \bar{g}_{ij}(x) \,. \tag{2.10.10}$$

The expansion around flat background can be written as $\bar{g}_{ij}(x) = \delta_{ij} + h_{ij}$. The flat metric associates with background metric for conformal theory at boundary and h_{ij} can be interpreted as the source term for the energy-momentum tensor T_{ij}.

The metric is divergent at boundary $\rho = 0$. In order to regularize it we need to introduce the cutoff at $\rho = \epsilon$. The integration of the action above is bounded to the range $\rho > \epsilon$ and we evaluate the boundary integral at $\rho = \epsilon$. The induced metric at $\rho = \epsilon$ is given by

$$\gamma_{ij} = \frac{g_{ij}(x, \rho = \epsilon)}{\epsilon} \,. \tag{2.10.11}$$

The K in the surface term in the action is given by

$$K = \gamma^{ij} K_{ij} - g^{ij}\rho \frac{\partial}{\partial \rho}\left(\frac{g_{ij}(x,\rho)}{\epsilon}\right)|_{\rho=\epsilon} \tag{2.10.12}$$

The Weyl transformation at boundary is given by

$$\bar{g}_{ij}(x) \quad \to \quad \bar{g}'_{ij}(x) = (1+2\sigma(x))\bar{g}_{ij}(x)\,. \tag{2.10.13}$$

It induces the diffeomorphism called Penrose-Brown-Henneaux transformation in the bulk, i.e.

$$\rho = \rho'(1-2\sigma(x')) \tag{2.10.14}$$
$$x^i = x'^i + a^i(x',\rho')\,, \tag{2.10.15}$$

where

$$a^i(x,\rho) = \frac{L^2}{2}\int_0^\rho d\hat{\rho}\, g^{ij}(x,\hat{\rho})\partial_j\sigma(x)\,. \tag{2.10.16}$$

The next task is to derive on-shell renormalized action $S_{ren}[\bar{g}]$ from which the anomaly can be derive via

$$\left\langle T_i^i \right\rangle = \bar{g}^{ij}\left\langle T_{ij}\right\rangle = -\frac{\delta S_{ren}[\bar{g}]}{\delta\sigma}\,, \tag{2.10.17}$$

where Weyl transformation is given by $\delta\bar{g}^{ij} = -2\bar{g}^{ij}\delta\sigma$. The renormalized on-shell action is defined by

$$S_{ren}[\bar{g}] \equiv \lim_{\epsilon\to 0}\left(S_\epsilon[\bar{g}] - S_\epsilon[\bar{g}]_{ct}\right)\,, \tag{2.10.18}$$

where $S_\epsilon[\bar{g}]$ is the on-shell action with cutoff ϵ and $S_\epsilon[\bar{g}]_{ct}$ is the counter term constructed to cancel the divergent terms as the $\epsilon \to 0$. In order to evaluate the $S_\epsilon[\bar{g}]_{ct}$, we need the data near boundary. According to Fefferman-Graham theorem [79], we can expand the metric near the boundary as

$$g_{ij}(x,\rho) = \bar{g}^{ij} + \rho g_{(2)ij} + \rho^2 g_{(4)ij} + \rho^2 \ln\rho\, h_{(4)ij} + \cdots \tag{2.10.19}$$

Note that if we are working in odd dimensions, the expansion above is defined differently, for instance; there is no $\ln\rho$ term. We substitute (2.10.19) to the action (2.10.8) and identify the divergent terms. At the lowest order, we simply give the result here,

$$S_\epsilon[\bar{g}]_{ct} = \frac{1}{4\pi G}\int d^4x\,\sqrt{\gamma}\left(\frac{3}{2L^2} - \frac{\hat{R}}{8} - \frac{L^2\ln\epsilon}{32}(\hat{R}^{ij}\hat{R}_{ij} - \frac{1}{3}\hat{R}^2)\right)\,. \tag{2.10.20}$$

where $\hat{R}_{ij}$ is the Ricci tensor and $\hat{R}$ is the Ricci scalar of induced metric γ_{ij}. We can associate divergences above to its UV divergences of its dual field theory if we identify ϵ with $1/\Lambda^2$ in field theory. Finally, we can calculate the conformal anomaly,

$$\left\langle T_i^i \right\rangle = \frac{L^3}{8\pi G}(\frac{1}{8}R^{ij}R_{ij} - \frac{1}{24}R^2) \tag{2.10.21}$$

In order to make AdS/CFT a correct framework, we compare the result with (2.10.7) and identify central charge as follows

$$c = \frac{\pi L^3}{8G} \tag{2.10.22}$$

Note that the identification above depends on dimensions of spacetime, so it is valid for bulk theory in $D = 5$. For $D = 3$ case, the central charge is given by

$$c = \frac{3L}{2G}\,. \tag{2.10.23}$$

The equation (2.10.23) is crucial in justifying c-theorem in our context from RG flows analysis in chapter 5.

2.11 c-theorem

It is worth mentioning the relation between central charges and degrees of freedom of a system. The insight is influenced by Wilson approach of renormalization. The central charges c measure the degrees of freedom of the conformal field theory. This is shown by the explicit form of central charges which can be written in term of number of particles as discussed in the previous section. As a matter of fact, not only it applies to conformal fixed points, it can be used to measure the degrees of freedom of the intermediate states with particular energy scales. The constant central charges are replaced by a monotonic function which approaches central charge in a particular limit. As it undergoes the transition to long distance physics, the heavy particles will decouple from the low energy dynamics in some literatures sometimes is referred to as integrating out the heavy particles degrees of freedom. If the theory has two fixed points in short distance and long distance identified by two conformal theories with central charges c_{UV} and c_{IR}, respectively, a direct outcome is that $c_{UV} > c_{IR}$. This inequality plays an important role of this section, since this observation help establish the so-called c-theorem.

The Zamolodchikov c-theorem [90] states that there exists a monotonic function $c(g)$ of the coupling constant g in a two dimensional renormalizable field theory. This function has a constant value only at fixed points and the $c(g^*)$ at that point is the central charge of the conformal theory in two dimensions which coincides with central charge in Virasoro algebra. Alternatively, Zamolodchikov c-theorem can also be encapsulated by three properties as follows

- $c(g)$ is a monotonic function along renormalized group flow.
- $c(g)$ is exactly a central charge of a conformal field theory at fixed point.
- $c(g)$ is stationary at fixed points $g = g^*$ where $\partial c(g)/\partial g|_{g^*} = 0$.

For a theory with two conformal fixed points, a major consequence of Zamolodchikov c-theorem is that

$$c_{UV} > c_{IR} \tag{2.11.1}$$

which is in agreement with the argument from Wilsonion renormalization discussed in the beginning of this section.

The question of generalization to higher dimensions was firstly proposed by Cardy [82] and also discussed the difficulties in the extending. This paper was titled *Is there a c-theorem in Four Dimensions?* and around that time short answer is *Probably No.* However, until 2011, the proof for four dimensional case was proposed by [83] and it is widely accepted among physicists. For a general proof for arbitrary dimensions, to the best of my knowledge, it may not exist.

It is important to note that the monotonic function $c(g)$ has an interpretation as central charges only at fixed points. At UV conformal fixed point and IR conformal fixed point, they are $c(g_1^*) = c_{UV}$ and $c(g_2^*) = c_{IR}$, respectively. The intermediate value of c along the flow depends on what regularization scheme we choose. This interpretation concurs with c function calculated from AdS/CFT correspondence, since there is generally no unique way to identify a monotonic function interpolating between two AdS spaces characterized by L_{UV} and L_{IR}. In our case for four dimensions, the obvious candidate for holographic c-function is given by $c(r) = \pi/8GA'^3$. However, this does not mean that we have only this choice.

The next task is to find a description for c-theorem from gravitational theory side via AdS/CFT correspondence. The c-theorem obtained in this way is called the holographic c-theorem. According to the identification (2.10.22) for four dimensinal conformal fields, the central charges in UV and IR relate to radius of AdS_{UV} and AdS_{IR} by

$$c_{UV} = \frac{\pi L_{UV}^3}{8G}\,, \qquad c_{IR} = \frac{\pi L_{IR}^3}{8G}\,, \tag{2.11.2}$$

so we have $L_{UV} > L_{IR}$. It corresponds to label local maximum as V_{UV} and the minimum as V_{IR},

$$V_{UV} = -\frac{d(d-1)}{4L_{UV}^2} > V_{IR} = -\frac{d(d-1)}{4L_{IR}^2}\,. \tag{2.11.3}$$

Recall that the scale factor $A(r)$ is concave downward because $A''(r) < 0$ as sketched in the figure 2.3 and the slopes $A'(r)$ at the two fixed points are related by $1/L_{IR} > 1/L_{UV}$, so as the deeper we go, the steeper $A(r)$ we get.

Let us consider a candidate for c-function in four dimensional case,

$$c(r) = \frac{\pi}{8GA'^3}\,. \tag{2.11.4}$$

it gives the correct value of central charges at fixed points. The first order derivative $c'(r) \geq 0$ since $A''(r) \leq 0$. Note that for arbitrary dimension d, it is given by

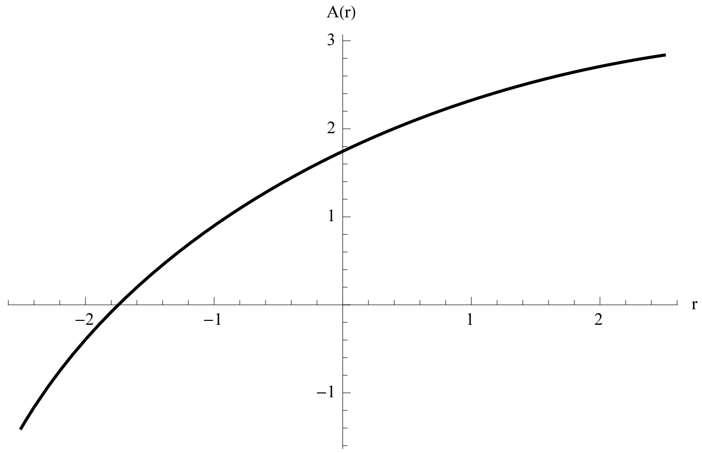

Figure 2.3: Profile of the scale factor $A(r)$.

$c(r) = \frac{c_0}{A'^{D-2}}$ and for $D = 3$, it is given by

$$c(r) = \frac{3}{2GA'}\,. \tag{2.11.5}$$

The function $c(r)$ is suitable for c-function since it satisfies the properties listed above.

Before we end this section we note that the RG flows studied in this dissertation are all in agreement of c-theorem, i.e. $c_{UV} > c_{IR}$. We will discuss the results in details in chapter 5.

CHAPTER III

$N = 5$ Theory

In this chapter, we study $N = 5$ gauged supergravity in three dimensions [39]. They are categorized by gauge group type: compact, non-compact and non-semisimple groups. The scalar manifold is a symmetric space with coset manifold $USp(4, k)/USp(4) \times USp(k)$ where k is an even integer. For compact and non-compact gaugings, we explore $k = 2$ and $k = 4$ cases which are the subgroup of $USp(4, k)$. And for non-semisimple case, the gauge group we studied is $SO(5) \ltimes \mathbf{T}^{10}$ corresponding to $USp(4, 4)/USp(4) \times USp(4)$ scalar manifold. Among all the gaugings we considered, non-semisimple case has a special aspect, since it can be described by dimensional reduction on an orbifold $S^1/\mathbb{Z}_2$ from four spacetime dimensional theory as previously pointed out in [57]. Moreover, according to Yang-Mills and Chern-Simons equivalency [66], this theory is equivalent to $SO(5)$ gauged supergravity of Yang-Mills type.

The dimension of scalar manifold of $k = 2$ and $k = 4$ case correspond to 8 and 16 respectively. For compact gauging in $k = 2$ case, we can parametrize full 8-dimensional manifold by Euler angles which is discussed in the appendix. On the other hand, we cannot parametrize full manifold for $k = 4$ in compact case since it is too complex for personal computer to handle. We have to parametrize submanifold which is invariant under a certain subgroup of the gauge group instead. According to argument in [71, 72] and review in chapter 2, a critical point found from this submanifold is also a critical point of the full manifold.

Many supersymmetric critical points for each case are reported together with their unbroken supersymmetry and unbroken gauge symmetry. The scalar mass spectra are identified as a representation of unbroken gauge group. Breitenlohner-Freedman stability condition (BF bound)[92] is checked and no violation is found if the gauge couplings belong to specific domain. Moreover, we determine the mass spectrum of the scalar around background at which we expand and present their irreducible representation of supergroup extension of AdS isometry group $SO(2, 2)$. At each critical point, the A_1 tensors are reported. The study of RG flows from discovered vacua are postponed to chapter 5.

This chapter is organized as follows. We discuss some theoretical as well as technical aspects for $N = 5$ theory. We identify group generators for $USp(4, k)$ and R-symmetry group $SO(5)$. In exploring vacua we start with compact gauging of $k = 2$ case which is split into three subcases:

- $SO(5) \times USp(2)$

• $SO(4) \times USp(2)$

• $SO(3) \times SO(2) \times USp(2)$.

Next, for compact gauge group of $k = 4$ case we consider:

• $SO(5) \times USp(4)$

• $SO(4) \times USp(4)$

• $SO(3) \times SO(2) \times USp(4)$.

Then we move on to the non-compact gauge groups. For $k = 2$, we study gauge group $USp(2,2)$. For $k = 4$, we study gauge group $USp(2,2) \times USp(2,2)$. Finally, non-semisimple $SO(5) \ltimes \mathbf{T}^{10}$ case is explored.

Before we proceed, it is important to identify the generators used in $N = 5$ theory. We define $USp(8)$ generators from Gell-Mann matrices for $SU(8)$ labelled by λ_i. For analysis of $k = 2$ case we need $USp(6)$ subgroup which are identified by the first 21 generators given below.

$$\begin{aligned}
J_1 &= \frac{i\lambda_1}{\sqrt{2}}, \qquad J_2 = \frac{i\lambda_2}{\sqrt{2}}, \qquad J_3 = \frac{i\lambda_3}{\sqrt{2}}, \\
J_4 &= \frac{i\lambda_{13}}{\sqrt{2}}, \quad J_5 = \frac{i\lambda_{14}}{\sqrt{2}}, \qquad J_6 = -\frac{i\lambda_8}{\sqrt{6}} + \frac{i\lambda_{15}}{\sqrt{3}}, \\
J_7 &= \frac{i\lambda_6}{2} + \frac{i\lambda_9}{2}, \qquad J_8 = -\frac{i\lambda_7}{2} + \frac{i\lambda_{10}}{2}, \qquad J_9 = \frac{i\lambda_4}{2} - \frac{i\lambda_{11}}{2},
\end{aligned}$$

$$\begin{aligned}
J_{10} &= -\frac{i\lambda_5}{2} - \frac{i\lambda_{12}}{2}, \quad J_{11} = \frac{i\lambda_{33}}{\sqrt{2}}, \quad J_{12} = \frac{i\lambda_{34}}{\sqrt{2}}, \\
J_{13} &= -\frac{i\lambda_{24}}{\sqrt{5}} + \sqrt{\frac{3}{10}}i\lambda_{35}, \quad J_{14} = \frac{i\lambda_{18}}{2} + \frac{i\lambda_{25}}{2}, \quad J_{15} = -\frac{i\lambda_{19}}{2} + \frac{i\lambda_{26}}{2}, \\
J_{16} &= \frac{i\lambda_{16}}{2} - \frac{i\lambda_{27}}{2}, \quad J_{17} = \frac{i\lambda_{22}}{2} + \frac{i\lambda_{29}}{2}, \quad J_{18} = -\frac{i\lambda_{23}}{2} + \frac{i\lambda_{30}}{2}, \\
J_{19} &= \frac{i\lambda_{20}}{2} - \frac{i\lambda_{31}}{2}, \quad J_{20} = -\frac{i\lambda_{17}}{2} - \frac{i\lambda_{28}}{2}, \quad J_{21} = -\frac{i\lambda_{21}}{2} - \frac{i\lambda_{32}}{2},
\end{aligned}$$

$$\begin{aligned}
J_{22} &= \frac{i\lambda_{61}}{\sqrt{2}}, \quad J_{23} = \frac{i\lambda_{62}}{\sqrt{2}}, \quad J_{24} = -\sqrt{\frac{3}{14}}i\lambda_{48} + \sqrt{\frac{2}{7}}i\lambda_{63}, \\
J_{25} &= \frac{i\lambda_{38}}{2} + \frac{i\lambda_{49}}{2}, \quad J_{26} = -\frac{i\lambda_{39}}{2} + \frac{i\lambda_{50}}{2}, \quad J_{27} = \frac{i\lambda_{36}}{2} - \frac{i\lambda_{51}}{2}, \\
J_{28} &= \frac{i\lambda_{42}}{2} + \frac{i\lambda_{53}}{2}, \quad J_{29} = -\frac{i\lambda_{43}}{2} + \frac{i\lambda_{54}}{2}, \quad J_{30} = \frac{i\lambda_{40}}{2} - \frac{i\lambda_{55}}{2},
\end{aligned}$$

$$\begin{aligned}
J_{31} &= \frac{i\lambda_{46}}{2} + \frac{i\lambda_{57}}{2}, \quad J_{32} = -\frac{i\lambda_{47}}{2} + \frac{i\lambda_{58}}{2}, \quad J_{33} = \frac{i\lambda_{44}}{2} - \frac{i\lambda_{59}}{2}, \\
J_{34} &= -\frac{i\lambda_{37}}{2} - \frac{i\lambda_{52}}{2}, \quad J_{35} = -\frac{i\lambda_{41}}{2} - \frac{i\lambda_{56}}{2}, \quad J_{36} = -\frac{i\lambda_{45}}{2} - \frac{i\lambda_{60}}{2}. \qquad (3.0.1)
\end{aligned}$$

The $SO(5)_R$ R-symmetry generators, labeled by a pair of anti-symmetric indices $T^{IJ} = -T^{JI}$, can be identified as follow

$$\begin{aligned}
T^{12} &= \frac{1}{\sqrt{2}}(J_3 - J_6)\,, \qquad T^{13} = -\frac{1}{\sqrt{2}}(J_1 + J_4)\,, \qquad T^{23} = \frac{1}{\sqrt{2}}(J_2 - J_5)\,,\\
T^{34} &= \frac{1}{\sqrt{2}}(J_3 + J_6)\,, \qquad T^{14} = \frac{1}{\sqrt{2}}(J_2 + J_5)\,, \qquad T^{24} = \frac{1}{\sqrt{2}}(J_1 - J_4)\,,\\
T^{15} &= -J_9, \qquad T^{25} = -J_{10}, \qquad T^{35} = J_8, \qquad T^{45} = J_7\,. \qquad (3.0.2)
\end{aligned}$$

The non-compact generators Y^A are identified by

$$\begin{aligned}
Y^1 &= iJ_{14}, \qquad Y^2 = iJ_{15}, \qquad Y^3 = iJ_{16}, \qquad Y^4 = iJ_{17},\\
Y^5 &= iJ_{18}, \qquad Y^6 = iJ_{19}, \qquad Y^7 = iJ_{20}, \qquad Y^8 = iJ_{21},\\
Y^9 &= iJ_{25}, \qquad Y^{10} = iJ_{26}, \qquad Y^{11} = iJ_{27}, \qquad Y^{12} = iJ_{28},\\
Y^{13} &= iJ_{29}, \qquad Y^{14} = iJ_{30}, \qquad Y^{15} = iJ_{31}, \qquad Y^{16} = iJ_{32}\,. \qquad (3.0.3)
\end{aligned}$$

For $k = 2$ case corresponding to 8 scalars, the associated non-compact generators are given by the first 8 generators, Y^A with $A = 1, \ldots, 8$.

3.1 Compact gauge groups

In this section, we study $N = 5$ gauged supergravity with compact gauge groups. The gauge groups are subgroups of maximal subgroup $G_0 \subset H \subset G$. We classify them into three cases: $SO(p) \times SO(5-p) \times USp(k)$ where $p = 5, 4, 3$. Since $SO(0)$ and $SO(1)$ are ill-defined, so they can explicitly be written as the following: $SO(5) \times USp(k)$, $SO(4) \times USp(k)$ and $SO(3) \times SO(2) \times USp(k)$. The sector $SO(p) \times SO(5-p)$ can be embedded in $USp(4) \sim SO(5)_R$ as in the representation $\mathbf{5} \to (\mathbf{p}, \mathbf{1}) + (\mathbf{1}, \mathbf{5} - \mathbf{p})$. For convenience, we repeat the embedding tensor below

$$\Theta_{IJ,KL} = \theta \delta^{KL}_{IJ} + \delta_{[I[K}\Xi_{L]J]} \qquad (3.1.1)$$

where

$$\Xi_{IJ} = \begin{cases} 2\left(1 - \frac{p}{5}\right)\delta_{IJ}, & I \le p \\ -\frac{2p}{5}\delta_{IJ}, & I > p \end{cases}\,, \qquad \theta = \frac{2p-5}{5}\,. \qquad (3.1.2)$$

Then the full embedding tensor for $SO(p) \times SO(5-p) \times USp(k)$ schematically written as

$$\Theta = g_1 \Theta_{SO(p)\times SO(5-p)} + g_2 \Theta_{USp(k)} \qquad (3.1.3)$$

where g_1 and g_2 are gauged coupling constants. At this stage, we treat them independently but in the study of mass spectrum and RG flow, they are related and bounded to some domain. $\Theta_{SO(p)\times SO(5-p)}$ and $\Theta_{USp(k)}$ are the Killing forms of $SO(p) \times SO(5-p)$ and $USp(k)$ respectively.

3.1.1 The $k = 2$ case

As mentioned earlier, the number of the scalar of the theory is $4k$. For $k = 2$, the scalar sector of the theory is parametrized by 8 scalars of coset space $USp(4,2)/USp(4) \times USp(2)$. We choose Euler angle parametrization with full 8-dimensional manifold for $SO(5) \times USp(2)$ and $SO(4) \times USp(2)$. For $SO(3) \times SO(2) \times USp(2)$, we parametrize merely submanifold as we will discuss this issue in $SO(3) \times SO(2) \times USp(2)$ gauging section.

In this case, the theory contains 8 scalars parametrized by $USp(4,2)/USp(4) \times USp(2)$ coset space. The full 8-dimensional manifold can be conveniently parametrized by the Euler angles of $SO(5) \times USp(2) \sim USp(4) \times USp(2)$. The details of the parametrization can be found in the appendix.

$SO(5) \times USp(2)$ **gauging**

In this case, we use Euler angle parametrization with $USp(4) \times USp(2)$ Euler angles. The full $USp(4,2)/USp(4) \times USp(2)$ coset can be parametrized by the coset representative

$$L = e^{a_1 X_1} e^{a_2 X_2} e^{a_3 X_3} e^{a_4 J_7} e^{a_5 J_8} e^{a_6 J_9} e^{a_7 J_{10}} e^{b Y^7} \tag{3.1.4}$$

where X_i's are defined by

$$X_1 = \frac{1}{\sqrt{2}}(J_1 - J_{11}), \qquad X_2 = \frac{1}{\sqrt{2}}(J_2 - J_{12}), \qquad X_3 = \frac{1}{\sqrt{2}}(J_3 - J_{13}). \tag{3.1.5}$$

According to L we have 7 scalar fields associated with generators which are elements in compact generators and a scalar associated to non-compact generator. The scalar potential is found to be

$$\begin{aligned} V &= \frac{1}{32}\left[64\left(g_2^2 - 12g_1^2 + 4g_1g_2\right)\cosh b - 1076g_1^2 - 180g_1g_2 - 45g_2^2 \right. \\ &\quad \left. -4\left(52g_1^2 + 20g_1g_2 + 5g_2^2\right)\cosh(2b) + (2g_1 + g_2)^2\cosh(4b)\right]. \end{aligned} \tag{3.1.6}$$

The scalar potential V depends only on scalar field b associated with non-compact generator Y^7. This feature is a result of gauge invariance. Note that the scalar potential should not depend on how we parametrize manifold, we expect the same result as in unitary gauge parametrization.

The exhaustive list of critical points is given in the table I. We found three critical points, they are labelled by roman number in the first column of the table. Note that b in the second column represents the value of scalar at their critical point and V_0 is the value of the potential associated with each critical point. Unbroken supersymmetries at critical points are listed in the forth column. We use two dimensional dual CFT to represent the number of supersymmetry; i.e., (n_-, n_+) where n_- and n_+ are the number of supersymmetry of different chirality of supercharge. This also can be realized in three dimensional theory via the

	b	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-64g_1^2$	$(5,0)$	$SO(5)\times USp(2)$
II	$\cosh^{-1}\left[\frac{g_2-2g_1}{2g_1+g_2}\right]$	$-\frac{64g_1^2(g_1+g_2)^2}{(2g_1+g_2)^2}$	$(4,0)$	$USp(2)\times USp(2)$
III	$\cosh^{-1}\left[\frac{6g_1+g_2}{2g_1+g_2}\right]$	$-\frac{64g_1^2(3g_1+g_2)^2}{(2g_1+g_2)^2}$	$(1,0)$	$USp(2)\times USp(2)$

Table I: Critical points of $SO(5)\times USp(2)$ gauging.

numbers of negative and positive eigenvalues of A_1^{IJ} tensor where the eigenvalues of A_1 tensor is given by $\pm\sqrt{-\frac{V_0}{4}}$.

It is well-known that at $L=\mathbf{I}$ (all scalars are turned off) is a trivial critical point and automatically preserves full supersymmetries and full gauge symmetry. The other two traditionally called non-trivial critical points preserve $USp(2)\times USp(2)$ guage symmetry and preserve only a subset of full supersymmetry. Note that we will consider RG flow in this case later in next chapter. Since A_1 tensor also plays many roles in the analysis such as counting supersymmetry, so we give the A_1 tensor explicitly at each critical point below

$$\begin{aligned}
A_1^{(\mathrm{I})} &= -4g_1\mathbf{I}_{5\times5}, \\
A_1^{(\mathrm{II})} &= \mathrm{diag}\left(\alpha,\alpha,\alpha,\alpha,\frac{4g_1(g_1-g_2)}{2g_1+g_2}\right), \\
A_1^{(\mathrm{III})} &= \mathrm{diag}\left(\beta,\beta,\beta,\beta,\frac{-4g_1(3g_1+g_2)}{2g_1+g_2}\right),
\end{aligned} \tag{3.1.7}$$

where

$$\alpha=\frac{-4g_1(g_1+g_2)}{2g_1+g_2}, \qquad \beta=\frac{-4g_1(5g_1+g_2)}{2g_1+g_2}. \tag{3.1.8}$$

The superscript of A_1 tensor denote the critical point at which it is calculated. Note that α and β are local parameters which are defined independently for each gauging.

In order to compute the mass spectrum at each critical point, we simply perform Taylor expansion around the critical point and set vector fields to zero. The quadratic term implies mass squared of the scalar particle. The coset representative must be calculated in unitary gauge, i.e.,

$$L=\prod_{i=1}^{8}e^{a_iY^i}. \tag{3.1.9}$$

The scalar mass spectrum at the trivial critical point (I) is given in the table below.

m^2L^2	$SO(5)\times USp(2)$
$-\frac{3}{4}$	$(\mathbf{4},\mathbf{2})$

All scalars have the same mass $m^2L^2 = -\frac{3}{4}$ with L being the AdS_3 radius at this critical point. There are no massless scalars or Goldstone bosons, since the ground states preserves full symmetry. The full symmetry of the background corresponds to $Osp(5|2,\mathbb{R}) \times Sp(2,\mathbb{R})$ superconformal group.

The mass spectrum at $(4,0)$ critical point is shown below. We have one massive particle in the singlet and seven massless scalars corresponding to $(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3})$. Those massless scalar are called Goldstone bosons. The number of massless scalars is confirmed by $\dim G_0 - \dim\hat{G} = \dim(SO(5)\times USp(2)) - \dim(USp(2)\times USp(2))$ corresponding to symmetry breaking $SO(5)\times USp(2) \rightarrow USp(2)\times USp(2)$. Since we set the vector fields to zero and consider only composite symmetry, therefore no vector fields eat up scalar and become massive. The example for non-vanishing vectors whereas the traditional Higgs mechanism can happen can be found in [93].

m^2L^2	$USp(2)\times USp(2)$
$\frac{g_2(2g_1+3g_2)}{(g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1})$
0	$(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3})$

The mass spectrum at $(1,0)$ critical point are as follow. The result is similar to the (4,0) case except for different massive scalar. We have the same number of Goldstone bosons corresponding to the same symmetry breaking.

m^2L^2	$USp(2)\times USp(2)$
$\frac{(4g_1+g_2)(10g_1+3g_2)}{(3g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1})$
0	$(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3})$

This $SO(5)\times USp(2)$ gauging case has two non-equal-supersymmetric vacua, so we are capable of discussing their RG flow in the chapter 5.

$SO(4)\times USp(2)$ gauging

We parametrize full manifold and the coset representative is the same as $SO(5)\times USp(2)$ case. The smaller the gauge group, the more complicated the calculation. The scalar potential in this case turns out to be very complex even though it depends only 5 from 8 scalars; i.e., they are a_4, a_5, a_6, a_7 and b. The scalar potential for compact gauge group $SO(4) \times USp(2)$ is given by

$$\begin{aligned}
V &= 2g_2^2(3+\cosh b)\sinh^6\frac{b}{2} + \frac{1}{16}g_1g_2\,[68+4\cos(2a_4)+2\cos(2(a_4-a_5)) \\
&+4\cos(2a_5)+2\cos(2(a_4+a_5))+2\cos(2(a_4-a_6))+\cos(2(a_4-a_5-a_6)) \\
&+2\cos(2(a_5-a_6))+\cos(2(a_4+a_5-a_6))+4\cos(2a_6)+2\cos(2(a_4+a_6)) \\
&+\cos(2(a_4-a_5+a_6))+2\cos(2(a_5+a_6))+\cos(2(a_4+a_5+a_6)) \\
&+32\cos^2 a_4\cos^2 a_5\cos^2 a_6\cos(2a_7)\big]\,(3+\cosh b)\sinh^6\frac{b}{2} \\
&-4g_1^2\Bigg[\cos^2 a_5\cos^2 a_6\cos^2 a_7\cosh^2\frac{b}{2}(3+\cosh b)^2\sin^2(2a_4) \\
&+64\cos^2 a_4\cosh^4\frac{b}{2}\sin^2 a_4\sin^2 a_5+64\cos^2 a_4\cos^2 a_5\cosh^4\frac{b}{2} \\
&\sin^2 a_4\sin^2 a_6+64\cos^2 a_4\cos^2 a_5\cos^2 a_6\cosh^4\frac{b}{2}\sin^2 a_4\sin^2 a_7 \\
&+\frac{1}{16384}\Bigg[51+259\cos(2a_4)+4(-17+63\cos(2a_4))\cosh b+(17+\cos(2a_4))\times \\
&\cosh(2b)+16\cos^2 a_4\cos(2a_5)\sinh^4\frac{b}{2}+32\cos^2 a_4\cos^2 a_5\cos(2a_6)\sinh^4\frac{b}{2} \\
&+64\cos^2 a_4\cos^2 a_5\cos^2 a_6\cos(2a_7)\sinh^4\frac{b}{2}\Bigg]^2+\frac{1}{2}\Bigg[-4\cos^4 a_4\cos^2 a_5\cos^2 a_6 \\
&\cos^2 a_7\sin^2 a_5\sinh^6\frac{b}{2}-4\cos^4 a_4\cos^4 a_5\cos^2 a_6\cos^2 a_7\sin^2 a_6\sinh^6\frac{b}{2} \\
&-4\cos^4 a_4\cos^4 a_5\cos^4 a_6\cos^2 a_7\sin^2 a_7\sinh^6\frac{b}{2}-4\sin^2(2a_4)\sin^2 a_5\sinh^2 b \\
&-16\cos^2 a_4\cos^2 a_5\sin^2 a_4\sin^2 a_6\sinh^2 b-16\cos^2 a_4\cos^2 a_5\cos^2 a_6\sin^2 a_4 \\
&\sin^2 a_7\sinh^2 b-\frac{1}{16}\cos^2 a_5\cos^2 a_6\cos^2 a_7\sin^2(2a_4)\left[7\sinh\frac{b}{2}+3\sinh\frac{3b}{2}\right]^2 \\
&-\frac{1}{4096}\Bigg[16\cos^2 a_4\left[\cos(2a_5)+2\cos^2 a_5\left(\cos(2a_6)+2\cos^2 a_6\cos(2a_7)\right)\right]\times \\
&\cosh\frac{b}{2}\sinh^3\frac{b}{2}+2[63\cos(2a_4)+17\cosh b-17]\sinh b \\
&+\cos(2a_4)\sinh(2b)\Bigg]^2\Bigg]\Bigg]. \qquad (3.1.10)
\end{aligned}$$

The trivial critical point is traditional which is defined by requiring all scalar fields to vanish. Because of the complication of the computed scalar potential, we need to put some scalars to zero for keeping it manageable. For non-trivial critical points, for simplicity we pick $a_4 = a_5 = a_6 = a_7 = 0$, therefore the final potential depends only on b. All critical points are listed in the table II along with their unbroken supersymmetry and unbroken gauge symmetry. The corresponding superconformal symmetry is $Osp(4|2,\mathbb{R}) \times Osp(1|2,\mathbb{R})$. The A_1 tensor for trivial

	b	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-64g_1^2$	$(4,1)$	$SO(4)\times USp(2)$
II	$\cosh^{-1}\left[\frac{g_2-2g_1}{2g_1+g_2}\right]$	$-\frac{64g_1^2(g_1+g_2)^2}{(2g_1+g_2)^2}$	$(4,1)$	$USp(2)\times USp(2)$
III	$\cosh^{-1}\left[\frac{6g_1+g_2}{2g_1+g_2}\right]$	$-\frac{64g_1^2(3g_1+g_2)^2}{(2g_1+g_2)^2}$	$(0,0)$	$USp(2)\times USp(2)$

Table II: Critical points of $SO(4)\times USp(2)$ gauging.

critical point is given by

$$A_1^{(\mathrm{I})} = -4g_1 \mathrm{diag}\,(1,1,1,1,-1)\,, \tag{3.1.11}$$

The mass spectrum for trivial critical point are given below

m^2L^2	$SO(4)\times USp(2)\sim SU(2)\times SU(2)\times USp(2)$
$-\frac{3}{4}$	$(\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{2})$

The trivial critical point preserves full gauge symmetry. On the other hand, critical points II and III preserve $USp(2)_{\mathrm{diag}}\times USp(2)$ gauge group. Note that in the table shown below the subscript 'diag' is omitted. The $USp(2)_{\mathrm{diag}}$ is a diagonal subgroup which is a sector in $USp(2)\times USp(2)\sim SO(4)$. The other $USp(2)$ is generated by the linear combination J_1+J_{11},J_2+J_{12} and J_3+J_{13}. The A_1 tensor of the critical point II is the following

$$A_1^{(\mathrm{II})} = -\frac{4g_1(g_1+g_2)}{2g_1+g_2}\mathrm{diag}\,(1,1,1,1,-1)\,. \tag{3.1.12}$$

At critical point II, we have three massless scalars, a massive scalar with mass $\frac{g_2(2g_1+3g_2)}{(g_1+g_2)^2}$, and another four massive scalars with mass $-\frac{g_1g_2(g_1+2g_2)}{(g_1+g_2)^2(2g_1+g_2)}$ as shown in the table below.

m^2L^2	$USp(2)\times USp(2)$
0	$(\mathbf{1},\mathbf{3})$
$\frac{g_2(2g_1+3g_2)}{(g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1})$
$-\frac{g_1g_2(g_1+2g_2)}{(g_1+g_2)^2(2g_1+g_2)}$	$(\mathbf{2},\mathbf{2})$

Critical point III is non-supersymmetric with residual $USp(2)_{\mathrm{diag}}\times USp(2)$ gauge symmetry. The mass spectrum is given by

m^2L^2	$USp(2)\times USp(2)$
0	$(\mathbf{1},\mathbf{3})$
$\frac{(4g_1+g_2)(10g_1+3g_2)}{(3g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1})$
$-\frac{g_1(4g_1+g_2)(5g_1+2g_2)}{(2g_1+g_2)(3g_1+g_2)^2}$	$(\mathbf{2},\mathbf{2})$

.

The stability of the critical point III can be checked by verifying with Breitenlohner-Freedman (BF) bound $m^2L^2 \geq -1$. According to the table II, and using $x > 1$ for $\cosh^{-1}$x, we have $g_1 > 0$ and $g_2 > -2g_1$ or $g_1 < 0$ and $g_2 < -2g_1$. We pick the condition that $g_1 > 0$. Consequently, the BF bound for $(\mathbf{1}, \mathbf{1})$ scalar becomes $g_1 > 0$ and $g_2 > -3g_1$. The mass of $(\mathbf{2}, \mathbf{2})$ scalars satifies the BF bound when $g_2 > 0.21432g_1$ and $g_1 > 0$. All in all, the condition for stability of the critical point II is $g_1 > 0$ and $g_2 > 0.21432g_1$. Since the residual gauge symmetry of the critical points II and III are the same; therefore, they both have three goldstone bosons associated with symmetry breaking $SO(4) \times USp(2) \rightarrow USp(2) \times USp(2)$.

$SO(3) \times SO(2) \times USp(2)$ **gauging**

In this case, it requires too much computer resources to tackle the scalar potential for full coset manifold $USp(4, 2)/USp(4) \times USp(2)$. The complication is not lessened even we employ Euler angle parametrization and the situation is even worse in the unitary gauge due to the fact that it has many uncancelled variables. The situation is opposite to the $SO(5) \times USp(2)$ case, recall that it has only one scalar left as a result of a larger gauge symmetry group. However, the obstacle does not refrain us from searching for the scalar potential. Equipped with a powerful technique discussed in the chapter 2, we can study the potential but from sub manifold.

The technique employs the fact that the critical points of submanifold are also critical points of the full one. In this case, we consider a submanifold of $USp(4, 2)/USp(4) \times USp(2)$ that is invariant under $U(1)_{\text{diag}}$ which is a linear combination $T^{12} + T^{45}$. In order to obtain the generators for constructing the coset representative, we have to find all possible of linear combination of non-compact generators that commute with $U(1)_{\text{diag}}$. Four singlets are found and given by

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{2}}(Y^1 + Y^6), \qquad X_2 = \frac{1}{\sqrt{2}}(Y^2 + Y^8), \\ X_3 &= \frac{1}{\sqrt{2}}(Y^4 - Y^3), \qquad X_4 = \frac{1}{\sqrt{2}}(Y^7 - Y^5). \end{aligned} \tag{3.1.13}$$

We now have four generators that associate with four scalars. A manifold with four scalar is manageable, so the unitary gauged coset representative is adequate and it is given by

$$L = e^{a_1 X_1} e^{a_2 X_2} e^{a_3 X_3} e^{a_4 X_4}. \tag{3.1.14}$$

The potential is a function of four scalar fields and it is not too long, so we present

	a_1	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-64g_1^2$	$(3,2)$	$SO(3)\times SO(2)\times USp(2)$
II	$\frac{1}{2}\ln\left[\frac{g_2-8g_1-4\sqrt{g_1(4g_1-g_2)}}{g_2}\right]$	$-\frac{64g_1^2(g_1-g_2)^2}{g_2^2}$	$(2,0)$	$U(1)\times U(1)$
III	$\frac{1}{2}\ln\left[\frac{g_2+8g_1-4\sqrt{g_1(4g_1+g_2)}}{g_2}\right]$	$-\frac{64g_1^2(g_1+g_2)^2}{g_2^2}$	$(1,2)$	$U(1)\times U(1)$

Table III: Critical points of $SO(3)\times SO(2)\times USp(2)$ gauging.

it here,

$$\begin{aligned} V &= \frac{1}{128}\left[3+\cosh a_1\cosh a_2\cosh a_3\cosh a_4\right]\Big[-2\left(512g_1^2+19g_2^2\right) \\ &+\left(99g_2^2-1024g_1^2\right)\cosh a_1\cosh a_2\cosh a_3\cosh a_4+3g_2^2\cosh(2a_1)\times \\ &(\cosh a_1\cosh a_2\cosh a_3\cosh a_4)-2-12g_2^2\cosh^2 a_1\big[\cosh(2a_2) \\ &+2\cosh^2 a_2\left(\cosh(2a_3)+2\cosh^2 a_3\cosh(2a_4)\right)\big]+2g_2^2\cosh^3 a_1\times \\ &\cosh a_2\cosh a_3\left(3\left(\cosh(2a_2)+2\cosh^2 a_2\cosh(2a_3)\right)\cosh a_4\right. \\ &\left.+4\cosh^2 a_2\cosh^2 a_3\cosh(3a_4)\right)\Big]\,. \end{aligned} \tag{3.1.15}$$

Even though we are dealing with four variables function, it is not easy to find all critical points without additional conditions In order to make the potential more tractable, we simply set $a_2 = a_1$ and $a_3 = a_4 = 0$. Consequently, the problem reduced to one variable function of a_1. The critical points are given in the table III and the value at critical point labeled uniquely by value of a_1. The trivial critical point is expectedly found with $V_0 = -64g_1^2$ which is equal to the other cases. It has a maximal supersymmetric critical point with $N = (3,2)$ and $SO(3)\times SO(2)\times USp(2)$ unbroken gauge symmetry. The eigenvalues of A_1 tensor at a critical point imply number of its unbroken supersymmetry and it is

$$A_1^{(\mathrm{I})} = -4g_1\mathrm{diag}\,(1,1,1,-1,-1)\,. \tag{3.1.16}$$

The superconformal group $Osp(3|2,\mathbb{R})\times Osp(2|2,\mathbb{R})$ is a symmetry group of this ground state. The mass spectrum at trivial critical point is presented below. In the group representation column, the regular typeface denotes $U(1)\sim SO(2)$ charge.

m^2L^2	$SO(2)\times SO(3)\times USp(2)$
$-\frac{3}{4}$	$(1,\mathbf{2},\mathbf{2})+(-1,\mathbf{2},\mathbf{2})$

The critical point II and III preserve the same residual gauge symmetry. The unbroken supersymmetry corresponds to eigenvalue $\sqrt{\frac{-V_0}{4}}$ of A_1 tensor. The critical points and the A_1 tensors are the following:

$$\begin{aligned} A_1^{(\mathrm{II})} &= = \mathrm{diag}\,(\alpha,\alpha,\beta,-\beta,-\beta)\,, \\ A_1^{(\mathrm{III})} &= \mathrm{diag}\,(\gamma,\gamma,-\delta,\delta,\delta)\,, \end{aligned} \tag{3.1.17}$$

where

$$\begin{aligned}
\alpha &= \frac{4g_1(g_1-g_2)}{g_2}, \qquad \beta = -\frac{4g_1(g_2-3g_1)}{g_2}, \\
\gamma &= -\frac{4g_1(3g_1+g_2)}{g_2}, \qquad \delta = \frac{4g_1(g_1+g_2)}{g_2}.
\end{aligned} \tag{3.1.18}$$

For $A_1^{(II)}$, α is equal to $\sqrt{\frac{-V_0}{4}}$ while β is not, so it corresponds to (2,0) supersymmetry. For $A_1^{(III)}$, γ is equal to $\sqrt{\frac{-V_0}{4}}$ while δ is not, so it corresponds to (1,2) supersymmetry. The mass spectra are shown in the table below with properly normalized $U(1)$ charge.

The other two critical points preserve $U(1)\times U(1)$ symmetry. The corresponding A_1 tensor at these points is given by

$$\begin{aligned}
A_1^{(II)} &= = \mathrm{diag}\,(\alpha,\alpha,\beta,-\beta,-\beta)\,, \\
A_1^{(III)} &= \mathrm{diag}\,(\gamma,\gamma,-\delta,\delta,\delta)\,,
\end{aligned} \tag{3.1.19}$$

where

$$\begin{aligned}
\alpha &= \frac{4g_1(g_1-g_2)}{g_2}, \qquad \beta = -\frac{4g_1(g_2-3g_1)}{g_2}, \\
\gamma &= -\frac{4g_1(3g_1+g_2)}{g_2}, \qquad \delta = \frac{4g_1(g_1+g_2)}{g_2}.
\end{aligned} \tag{3.1.20}$$

Having normalizd the $U(1)$ charges, the scalar mass spectra can be computed as shown in the tables below. The original four singlets under $U(1)_{\mathrm{diag}}$ correspond to one massless and three massive modes in the tables. The $U(1)_{\mathrm{diag}}$ is given by a combination of the two $U(1)$'s in the unbroken symmetry $U(1)\times U(1)$. Therefore, the $(0,\pm4)$ and $(\pm4,0)$ modes, which are singlets under one of the two $U(1)$'s, will not be invariant under $U(1)_{\mathrm{diag}}$.

- $(2,0)$ point:

m^2L^2	$U(1)\times U(1)$
0	$(0,4)+(0,-4)+(4,0)+(-4,0)+(0,0)$
$\frac{32g_1^2-32g_1g_2+6g_2^2}{(g_1-g_2)^2}$	$(0,0)$
$-\frac{2g_1(g_1-2g_2)}{(g_1-g_2)^2}$	$(-2,-2)+(2,2)$

- $(1,2)$ point:

m^2L^2	$U(1)\times U(1)$
0	$(0,4)+(0,-4)+(4,0)+(-4,0)+(0,0)$
$\frac{32g_1^2+32g_1g_2+6g_2^2}{(g_1+g_2)^2}$	$(0,0)$
$\frac{2g_1(3g_1+2g_2)}{(g_1+g_2)^2}$	$(-2,-2)+(2,2)$

	b	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-64g_1^2$	$(5,0)$	$SO(5)\times USp(4)$
II	$\cosh^{-1}\left[\frac{g_2-2g_1}{2g_1+g_2}\right]$	$-\frac{64g_1^2(g_1+g_2)^2}{(2g_1+g_2)^2}$	$(4,0)$	$USp(2)^3$
III	$\cosh^{-1}\left[\frac{6g_1+g_2}{2g_1+g_2}\right]$	$-\frac{64g_1^2(3g_1+g_2)^2}{(2g_1+g_2)^2}$	$(1,0)$	$USp(2)^3$

Table IV: Critical points of $SO(5)\times USp(4)$ gauging.

3.1.2 The $k=4$ case

In this section, we study $k=4$ case of $\frac{USp(4,k)}{USp(4)\times USp(k)}$ coset manifold. The $\frac{USp(4,4)}{USp(4)\times USp(4)}$ is parametrized by 16 scalars. The dimension of the manifold is twice the size of the $k=2$ case. This can lead to be more problematic in practical computation. It is that we cannot directly analyze full 16 scalars theory. The solution is the same as the previous cases, we simply study their submanifold and apply the powerful technique mentioned earlier. Compact gauge groups in this case are as follows: $SO(5)\times USp(4)$, $SO(4)\times USp(4)$ and $SO(3)\times SO(2)\times USp(4)$ and we study them in order. The methodology is similar to the $k=2$ case, so many steps will be omitted including A_1 tensors.

$SO(5)\times USp(4)$ **gauging**

The submanifold we chose is invariant under $USp(2)\subset USp(4)$. We found eight singlets under this invariant. The corresponding non-compact generators reside in $USp(4,2)\subset USp(4,4)$. We employ Euler angle paremetrization, so the coset representative now becomes

$$L = e^{a_1\tilde{X}_1}e^{a_2\tilde{X}_2}e^{a_3\tilde{X}_3}e^{a_4K_1}e^{a_5K_2}e^{a_6K_3}e^{a_7K_4}e^{bY^8} \tag{3.1.21}$$

where

$$\begin{aligned}\tilde{X}_1 &= \frac{1}{\sqrt{2}}(J_4-J_{11}), \qquad \tilde{X}_2=\frac{1}{\sqrt{2}}(J_5-J_{12}), \qquad \tilde{X}_3=\frac{1}{\sqrt{2}}(J_6-J_{13}),\\ K_1 &= J_{31}, \qquad K_2=J_{32}, \qquad K_3=J_{33}, \qquad K_4=J_{36}\,. \end{aligned}\tag{3.1.22}$$

Even though we do not study the full manifold and the gauge group is larger than the $k=2$ $SO(5)\times USp(2)$ case, we found out that the scalar potential is same as in (3.1.6). The critical points are shown in table IV. The critical points also have the same structure but the residual symmetries are larger. The scalar mass spectrum for each critical points are as follows

- $(5,0)$ point:

m^2L^2	$SO(5)\times USp(4)$
$-\frac{3}{4}$	$(\mathbf{4},\mathbf{4})$

	b	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-64g_1^2$	$(4,1)$	$SO(4)\times USp(4)$
II	$\cosh^{-1}\left[\frac{g_2-2g_1}{2g_1+g_2}\right]$	$-\frac{64g_1^2(g_1+g_2)^2}{(2g_1+g_2)^2}$	$(4,1)$	$USp(2)^3$
III	$\cosh^{-1}\left[\frac{6g_1+g_2}{2g_1+g_2}\right]$	$-\frac{64g_1^2(3g_1+g_2)^2}{(2g_1+g_2)^2}$	$(0,0)$	$USp(2)^3$

Table V: Critical points of $SO(4)\times USp(4)$ gauging.

- $(4,0)$ point:

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{2},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3},\mathbf{1})$
$\frac{g_2(2g_1+3g_2)}{(g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{4g_1^2+8g_1g_2+3g_2^2}{4(g_1+g_2)^2}$	$(\mathbf{2},\mathbf{1},\mathbf{2})$

- $(1,0)$ point:

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{2},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3},\mathbf{1})$
$\frac{40g_1^2+22g_1g_2+3g_2^2}{(3g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{3\left(12g_1^2+8g_1g_2+g_2^2\right)}{4(3g_1+g_2)^2}$	$(\mathbf{2},\mathbf{1},\mathbf{2})$

It can be easily checked that the number of Goldstone bosons corresponding to symmetry breaking $SO(5)\times USp(4)\to USp(2)\times USp(2)\times USp(2)$ for (4,0) and (1,0) critical points.

$SO(4)\times USp(4)$ **gauging**

The coset representative is the same as in the previous case $k=5$ $SO(5)\times USp(4)$. The scalar potential is exactly the same for $k=2$ $SO(4)\times USp(4)$ case as well as the structure of the critical points, except for the unbroken gauge symmetry as listed in the table V. The trivial critical point and critical point II preserve full supersymmetry while critical point III is non-supersymmetric. According to BF bound, the non-supersymmetric critical point III is stable if and only if $g_1>0$ and $g_2>0.21432g_1$. The scalar mass spectra are given below.

- $(4,1)$ point:

m^2L^2	$SO(4)\times USp(2)\sim SU(2)\times SU(2)\times USp(4)$
$-\frac{3}{4}$	$(\mathbf{2},\mathbf{1},\mathbf{4})+(\mathbf{1},\mathbf{2},\mathbf{4})$

- $(4,1)$ point:

	a_1	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-64g_1^2$	(3, 2)	$SO(3)\times SO(2)\times USp(4)$
II	$\frac{1}{2}\ln\left[\frac{g_2-8g_1-4\sqrt{g_1(4g_1-g_2)}}{g_2}\right]$	$-\frac{64g_1^2(g_1-g_2)^2}{g_2^2}$	(2, 0)	$U(1)\times U(1)\times USp(2)$
III	$\frac{1}{2}\ln\left[\frac{g_2+8g_1-4\sqrt{g_1(4g_1+g_2)}}{g_2}\right]$	$-\frac{64g_1^2(g_1+g_2)^2}{g_2^2}$	(1, 2)	$U(1)\times U(1)\times USp(2)$

Table VI: Critical points of $SO(3)\times SO(2)\times USp(4)$ gauging.

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3},\mathbf{1})$
$\frac{g_2(2g_1+3g_2)}{(g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{g_1g_2(g_1+2g_2)}{(g_1+g_2)^2(2g_1+g_2)}$	$(\mathbf{2},\mathbf{1},\mathbf{2})$
$-\frac{(2g_1+g_2)(2g_1+3g_2)}{4(g_1+g_2)^2}$	$(\mathbf{2},\mathbf{2},\mathbf{1})$

- Non-supersymmetric point:

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{3},\mathbf{1})$
$\frac{40g_1^2+22g_1g_2+3g_2^2}{(3g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{3(2g_1+g_2)(6g_1+g_2)}{4(3g_1+g_2)^2}$	$(\mathbf{2},\mathbf{1},\mathbf{2})$
$-\frac{g_1\left(20g_1^2+13g_1g_2+2g_2^2\right)}{(2g_1+g_2)(3g_1+g_2)^2}$	$(\mathbf{2},\mathbf{2},\mathbf{1})$

$SO(3)\times SO(2)\times USp(4)$ **gauging**

In this case, we choose $USp(2)\times U(1)_{\text{diag}}$-singlet submanifold, so the coset representative is the same as $k=2$ $SO(3)\times SO(2)\times USp(2)$ case which is (3.1.14). The dimension of submanifold is four which corresponds to four singlets of $USp(2)\times U(1)_{\text{diag}}$. The scalar potential is the same as in $k=2$ $SO(3)\times SO(2)\times USp(2)$ case. The critical points and their unbroken supersymmetry and gauge symmtery are given in the table VI. The scalar mass spectra are given in the following tables.

- (3, 2) point:

m^2L^2	$SO(3)\times USp(4)$
$-\frac{3}{4}$	$(\mathbf{2},\mathbf{4})+(\mathbf{2},\mathbf{4})$

- (2, 0) point:

m^2L^2	$U(1)\times U(1)\times USp(2)$
0	$(4,0,\mathbf{1})+(-4,0,\mathbf{1})+(0,4,\mathbf{1})+(0,-4,\mathbf{1})+(0,0,\mathbf{1})$ $+(1,-1,\mathbf{2})+(-1,1,\mathbf{2})$
$\frac{32g_1^2-32g_1g_2+6g_2^2}{(g_1-g_2)^2}$	$(0,0,\mathbf{1})$
$-\frac{2g_1(g_1-2g_2)}{(g_1-g_2)^2}$	$(-2,-2,\mathbf{1})+(2,2,\mathbf{1})$
$-\frac{4g_1^2-8g_1g_2+3g_2^2}{4(g_1-g_2)^2}$	$(-1,-1,\mathbf{2})+(1,1,\mathbf{2})$

- $(1,2)$ point:

m^2L^2	$U(1)\times U(1)\times USp(2)$
0	$(4,0,\mathbf{1})+(-4,0,\mathbf{1})+(0,4,\mathbf{1})+(0,-4,\mathbf{1})+(0,0,\mathbf{1})$ $+(1,-1,\mathbf{2})+(-1,1,\mathbf{2})$
$\frac{32g_1^2+32g_1g_2+6g_2^2}{(g_1+g_2)^2}$	$(0,0,\mathbf{1})$
$-\frac{2g_1(3g_1+2g_2)}{(g_1+g_2)^2}$	$(-2,-2,\mathbf{1})+(2,2,\mathbf{1})$
$-\frac{4g_1^2+8g_1g_2+3g_2^2}{4(g_1+g_2)^2}$	$(-1,-1,\mathbf{2})+(1,1,\mathbf{2})$

Note that critical points in this case is similar to the $k=2$ case because $USp(4,2)/USp(4)\times USp(2)$ scalar manifold can be embedded in the theory with $USp(4,4)/USp(4)\times USp(4)$ scalar manifold. This is confirned by the fact that $USp(2)$ singlets are parametrized by non-compact generators of $USp(4,2)$ which is a subgroup of $USp(4,4)$.

3.2 Non-compact gauge groups

In the previous section, compact gaugings are all possible because they satisfied two constraints on T-tensor mentioned earlier. On the other hand, viable non-compact gaugings are not priori classified, so the additional task here is to test them with constraint on T-tensors. Like compact cases, the embedding tensors are defined by Cartan-Killing form and will be elaborated in subsections. It is well known that at critical points the unbroken gauge groups are compact subgroups, especially at trivial critical points the unbroken gauge groups are their maximal gauge groups. The unbroken gauge symmetry and the unbroken supersymmetry are presented in the tables for each case as well as their A_1 tensor. As in the compact case, we study two categories: $k=2$ and $k=4$ and the studies here are similar to the compact cases. For $k=2$ case, we will study its RG flow in the chapter 5.

The $k=2$ case

In this case, we have two possible gauge groups namely the whole isometry group $USp(2)\times USp(4,2)$ itself and $USp(2)\times USp(2,2)$. However, the potential of $USp(2)\times USp(4,2)$ gauging simply gives rise to a cosmological constant. As

	b	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-4(g_1+g_2)^2$	$(4,1)$	$USp(2)^3$
II	$\cosh^{-1}\left(\frac{g_2-g_1}{g_1+g_2}\right)$	$-\frac{4g_1^2(2g_1+g_2)^2}{(g_1+g_2)^2}$	$(4,0)$	$USp(2)\times USp(2)$
III	$\cosh^{-1}\left(-\frac{g_1+3g_2}{g_1+g_2}\right)$	$-\frac{4g_1^2(2g_1+3g_2)^2}{(g_1+g_2)^2}$	$(1,0)$	$USp(2)\times USp(2)$
IV	$\ln(2+\sqrt{3})$	$-\frac{1}{4}(27g_1^2+54g_1g_2+19g_2^2)$	$(0,0)$	$USp(2)\times USp(2)$

Table VII: Critical points of $USp(2)\times USp(2,2)$ gauging.

a result, the only non-trivial admissible gauging is $USp(2)\times USp(2,2)$. The embedding tensor reads

$$\Theta = g_1\Theta_{USp(2)} + g_2\Theta_{USp(2,2)} \tag{3.2.1}$$

where g_1 and g_2 are two independent coupling constants. The factor $\Theta_{USp(2,2)}$ and $\Theta_{USp(2)}$ are the Cartan-Killing forms of $USp(2,2)$ and $USp(2)$, respectively.

The overlapped non-compact generators of the coset representative with the generators of gauge groups do not appear in the final form of scalar potential. It allows us to neglect this part in the coset representative. Consequently, this leaves us with four scalars associated with non-compact directions of another $USp(2,2)$ in $USp(4,2)$. The submanifold now becomes $USp(2,2)/USp(2)\times USp(2)$. With Euler angles of $USp(2)\times USp(2)$, the coset representative reads

$$L = e^{a_1X_1}e^{a_2X_2}e^{a_3X_3}e^{bY^7} \tag{3.2.2}$$

where X_i are given in (3.1.5). The scalar potential depends only on b and it reads

$$\begin{aligned} V &= \frac{1}{16}\left[8(g_1-g_2+(g_1+g_2)\cosh(b))^2\sinh^2 b \right. \\ &\quad \left. -(3g_1+11g_2+4(g_1-g_2)\cosh b+(g_1+g_2)\cosh(2b))^2\right]. \end{aligned} \tag{3.2.3}$$

Some of the critical points are shown in table VII. The A_1 tensor at each symmetric critical point is given by

$$\begin{aligned} A_1^{(\mathrm{I})} &= (g_1+g_2)\mathrm{diag}\,(-1,-1,-1,-1,1)\,, \\ A_1^{(\mathrm{II})} &= \mathrm{diag}\left(\beta,\beta,\beta,\beta,\frac{g_2(-2g_1+g_2)}{g_1+g_2}\right), \\ A_1^{(\mathrm{III})} &= \mathrm{diag}\left(\gamma,\gamma,\gamma,\gamma,-\frac{g_2(2g_1+3g_2)}{g_1+g_2}\right) \end{aligned} \tag{3.2.4}$$

where

$$\beta = -\frac{g_2(2g_1+g_2)}{g_1+g_2}, \qquad \gamma = -\frac{g_2(2g_1+5g_2)}{g_1+g_2}\,. \tag{3.2.5}$$

We found four critical points and three of which preserve (4,1), (4,0) and (1,0) supersymmetry, respectively. The last critical IV is non-supersymmetric with the value of scalar b independent of coupling constants. Note that at critical point I, if $g_2 = -g_1$, then $V_0 = 0$ and we have Minkowskian vacuum; otherwise, we get AdS groundstate. As in the other cases, trivial critical point I preserves full supersymmetry, in this case, $N = (4,1)$. However, the gauge symmetry is not fully preserved as in those compact cases. The unbroken gauge symmetry is its maximal compact subgroup $USp(2)\times USp(2)\times USp(2)$. The number of massless Goldstone bosons is four corresponding to non-compact generators of gauge group. The superconformal group at trivial critical point is $Osp(4|2,\mathbb{R}) \times Osp(1|2,\mathbb{R})$. This comes from the fact that the supercharges transform under $USp(2) \times USp(2) \subset SO(5)_R$ as $(\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1})$.

Scalar mass spectra at all four critical points are given below.

- $(4,1)$ point:

m^2L^2	$USp(2) \times USp(2) \times USp(2)$
0	$(\mathbf{1},\mathbf{2},\mathbf{2})$
$-\frac{g_1(g_1+2g_2)}{(g_1+g_2)^2}$	$(\mathbf{2},\mathbf{1},\mathbf{2})$

- $(4,0)$ point:

m^2L^2	$USp(2) \times USp(2)$
0	$(\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3})$
$\frac{4g_1(3g_1+g_2)}{(2g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1})$

- $(1,0)$ point:

m^2L^2	$USp(2) \times USp(2)$
0	$(\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3})$
$\frac{4(g_1+2g_2)(3g_1+5g_2)}{(2g_1+3g_2)^2}$	$(\mathbf{1},\mathbf{1})$

- Non-supersymmetric point:

m^2L^2	$USp(2) \times USp(2)$
0	$(\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3})$
$\frac{12(3g_1+g_2)(3g_1+5g_2)}{27g_1^2+54g_1g_2+19g_2^2}$	$(\mathbf{1},\mathbf{1})$

At supersymmetric critical point II and III, massless Goldstone bosons corresponding to symmetry breaking $USp(2) \times USp(2) \rightarrow USp(2)_{\text{diag}}$. The non-supersymmetric critical point IV satisfies BF bound if $g_2 > \frac{3}{79}(2\sqrt{210} - 45)g_1$.

The $k = 4$ case

Roughly speaking, there are three possible cases which are subgroups of $USp(4,4)$: $USp(2,2)\times USp(2,2)$, $USp(2)\times USp(4,2)$ and $USp(2)\times USp(2)\times USp(2,2)$, but two from all three fail the T-tensor constraint test. The only admissible gauge group is $USp(2,2)\times USp(2,2)$, and its embedding tensors reads

$$\Theta = g_1\Theta_{USp(2,2)} + g_2\Theta_{USp(2,2)} \,, \tag{3.2.6}$$

where g_1 and g_2 are independent coupling constants. $\Theta_{USp(2,2)}$ is given by the Cartan-Killing form of $USp(2,2)$. The manifold is parametrized by $\left(\frac{USp(2,2)}{USp(2)\times USp(2)}\right)^2$ coset space with the two $USp(2,2)$ non-compact groups which are different from those in the gauge group. We use Euler angle parametrization, the coset representative reads

$$L = e^{a_1X_1}e^{a_2X_2}e^{a_3X_3}e^{b_1Y^7}e^{a_4X_4}e^{a_5X_5}e^{a_6X_6}e^{b_2Y^{16}} \tag{3.2.7}$$

where

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{2}}(J_1 - J_{11}), \qquad X_2 = \frac{1}{\sqrt{2}}(J_2 - J_{12}), \qquad X_3 = \frac{1}{\sqrt{2}}(J_3 - J_{13}), \\ X_4 &= \frac{1}{\sqrt{2}}(J_4 - J_{22}), \qquad X_5 = \frac{1}{\sqrt{2}}(J_5 - J_{23}), \qquad X_6 = \frac{1}{\sqrt{2}}(J_6 - J_{24}) \end{aligned} \tag{3.2.8}$$

The scalar potential depends on two scalar and is given by

$$\begin{aligned} V &= \frac{1}{16}\left[(g_1+g_2)(6+\cosh(2b_1)) - (4(g_1-g_2)\cosh b_1 + 4(g_2-g_1)\cosh b_2\right. \\ &\quad +(g_1+g_2)\cosh(2b_2))^2 + 8(g_1-g_2+(g_1+g_2)\cosh(b_1))^2\sinh^2 b_1 \\ &\quad \left.+8(g_2-g_1+(g_1+g_2)\cosh b_2)^2\sinh^2 b_2\right] . \end{aligned} \tag{3.2.9}$$

The calculation of critical points can be simplified if we simply turn off scalar b_2. As a result, the critical points are solely determined by the value of b_1. We found four critical points; three of which are supersymmetric with (4,1), (4,0) and (1,0). The critical point IV is non-supersymmetric and b_1 is independent of couplings. As in the previous case, if $g_2 = -g_1$, we simply get $V_0 = 0$ or Minkowskian vacuum for trivial critical point. They are shown in the table VIII below. The scalar masses at all critical points are also given below.

- $(4,1)$ point:

m^2L^2	$USp(2)\times USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$
$-\frac{g_2(2g_1+g_2)}{(g_1+g_2)^2}$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$
$-\frac{g_1(g_1+2g_2)}{(g_1+g_2)^2}$	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$

- $(4,0)$ point:

	b_1	V_0	unbroken SUSY	unbroken gauge symmetry
I	0	$-4(g_1+g_2)^2$	$(4,1)$	$USp(2)^4$
II	$\cosh^{-1}\left(\frac{-g_1+g_2}{g_1+g_2}\right)$	$-\frac{4g_1^2(2g_1+g_2)^2}{(g_1+g_2)^2}$	$(4,0)$	$USp(2)^3$
III	$\cosh^{-1}\left(\frac{-g_1-3g_2}{g_1+g_2}\right)$	$-\frac{4g_1^2(2g_1+3g_2)^2}{(g_1+g_2)^2}$	$(1,0)$	$USp(2)^3$
IV	$\cosh^{-1}2$	$-\frac{1}{4}(27g_1^2+54g_1g_2+19g_2^2)$	$(0,0)$	$USp(2)^3$

Table VIII: Critical points of $USp(2,2)\times USp(2,2)$ gauging.

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{2},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{3},\mathbf{1},\mathbf{1})$
$\frac{4g_1(3g_1+g_2)}{(2g_1+g_2)^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{(g_1+g_2)(3g_1+g_2)}{(2g_1+g_2)^2}$	$(\mathbf{1},\mathbf{2},\mathbf{2})$

- $(1,0)$ point:

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{2},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{3},\mathbf{1},\mathbf{1})$
$\frac{4\left(3g_1^2+11g_1g_2+10g_2^2\right)}{(2g_1+3g_2)^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{3\left(g_1^2+4g_1g_2+3g_2^2\right)}{(2g_1+3g_2)^2}$	$(\mathbf{1},\mathbf{2},\mathbf{2})$

- Non-supersymmetry point:

m^2L^2	$USp(2)\times USp(2)\times USp(2)$
0	$(\mathbf{2},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{3},\mathbf{1},\mathbf{1})$
$\frac{12(3g_1+g_2)(3g_1+5g_2)}{27g_1^2+54g_1g_2+19g_2^2}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$-\frac{24g_2(3g_1+g_2)}{27g_1^2+54g_1g_2+19g_2^2}$	$(\mathbf{1},\mathbf{2},\mathbf{2})$

Note that at trivial critical point I, the $SO(5)_R$ R-symmetry is broken to $SU(2)\times SU(2)\sim USp(2)\times USp(2)$. The supercharges transform under this subgroup as $(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{1})$. The corresponding superconformal group is $Osp(4|2,\mathbb{R})\times Osp(1|2,\mathbb{R})$. The stability condition for non-supersymmetric point is $g_2>\frac{3}{79}(2\sqrt{210}-45)g_1$ which is the same condition as in the previous case.

3.3 Non-semisimple $N=5$, $SO(5)\ltimes\mathbf{T}^{10}$ gauged supergravity

The aspect that makes non-semisimple gaugings outshine the others is that it can obviously be identified as a dimensional reduction theory from higher dimensional sibling on an orbifold. In this section, we study non-semisimple gauge group that is in the form of $G_0 \ltimes \mathbf{T}^{\dim G_0}$ where G_0 is a semisimple group and $\mathbf{T}^{\dim G_0}$ is translational group with dimension of $\dim G_0$. Additionally, $\mathbf{T}^{\dim G_0}$ transform in the adjoint representation of G_0 and mutually commute. Specifically, we consider $k = 4$ case with non-semisimple group $SO(5) \ltimes \mathbf{T}^{10}$ which is a subgroup of global symmetry group $USp(4,4)$. The embedding tensor can be defined by

$$\Theta = g_1\Theta_{\mathrm{ab}} + g_2\Theta_{\mathrm{bb}} . \tag{3.3.1}$$

where g_1 and g_2 are, at this stage, the independent coupling constants. The roman indices a and b denotes semisimple $SO(5)$ and translation sections $\mathbf{T}^{10}$, respectively. The supersymmetric condition is checked by verifying constraint on T-tensors. We now discuss the explicit form of the generators $SO(5) \ltimes \mathbf{T}^{10}$. Note that the semisimple $SO(5)$ subgroup is a diagonal subgroup of $SO(5) \times SO(5) \sim USp(4) \times USp(4) \subset USp(4,4)$. For brevity, we simply neglect subscript "diag". Their generators, labelled J^{ij}, can be defined by summation of the generic $SO(5)$ R-symmetry generators T^{ij} and another $USp(4)$ generators that can be identified as of replica of T^{ij} generators which live in orthogonal subspace, i.e., $\tilde{T}^{ij}$. They read

$$J^{ij} = T^{ij} + \tilde{T}^{ij}, \qquad i,j = 1,2,...,5 . \tag{3.3.2}$$

The translational generators are built from a linear combination of $T^{ij} - \tilde{T}^{ij}$ and some non-compact generators. The generators of $\mathbf{T}^{10}$ are given by

$$t^{ij} = T^{ij} - \tilde{T}^{ij} + \tilde{Y}^{ij}, \qquad i,j = 1,2,...,5 . \tag{3.3.3}$$

The explicit form of the $SO(5)_{\mathrm{diag}}$ are given by $T^{ij} + \tilde{T}^{ij}$ in which

$$\begin{aligned}
\tilde{T}^{12} &= \frac{1}{\sqrt{2}}(J_{13} - J_{24}), \qquad \tilde{T}^{13} = -\frac{1}{\sqrt{2}}(J_{11} + J_{22}), \qquad \tilde{T}^{23} = \frac{1}{\sqrt{2}}(J_{12} - J_{23}), \\
\tilde{T}^{34} &= \frac{1}{\sqrt{2}}(J_{13} + J_{24}), \qquad \tilde{T}^{14} = \frac{1}{\sqrt{2}}(J_{12} + J_{23}), \qquad \tilde{T}^{24} = \frac{1}{\sqrt{2}}(J_{11} - J_{22}), \\
\tilde{T}^{45} &= J_{31}, \qquad \tilde{T}^{15} = -J_{33}, \qquad \tilde{T}^{25} = -J_{36}, \qquad \tilde{T}^{35} = J_{32} .
\end{aligned} \tag{3.3.4}$$

The generators $\tilde{Y}^{ij}$ in $\mathbf{T}^{10}$ are given by

$$\begin{aligned}
\tilde{Y}^{12} &= i(J_{16} - J_{30}), \qquad \tilde{Y}^{13} = -i(J_{14} + J_{28}), \qquad \tilde{Y}^{23} = i(J_{15} + J_{29}), \\
\tilde{Y}^{34} &= i(J_{16} + J_{30}), \qquad \tilde{Y}^{14} = i(J_{15} + J_{29}), \qquad \tilde{Y}^{24} = i(J_{14} - J_{28}), \\
\tilde{Y}^{45} &= i(J_{17} + J_{25}), \qquad \tilde{Y}^{15} = -i(J_{19} + J_{27}), \qquad \tilde{Y}^{25} = i(J_{21} - J_{34}), \\
\tilde{Y}^{35} &= i(J_{18} + J_{26}).
\end{aligned} \tag{3.3.5}$$

Note that in this $k = 4$ theory the 16 scalar fields transform as $(\mathbf{4},\mathbf{4})$ under $SO(5) \times SO(5)$. On the other hand, under $SO(5)_{\mathrm{diag}}$, they transform as $\mathbf{1}+\mathbf{5}+\mathbf{10}$. The scalars in the $\mathbf{10}$ representation are associated with $\mathbf{T}^{10}$ generators.

We parametrize submanifold which is $SO(5)_{\mathrm{diag}}$ singlet. Only one scalar corresponds to this singlet and the linear combination of the generators is Y^7+Y^{16}, so the coset representative is

$$L = e^{a(Y^7+Y^{16})} . \tag{3.3.6}$$

The scalar potential is found to be

$$V = -64 g_1 e^{-3a} \left(3e^a g_1 + 2g_2\right) . \tag{3.3.7}$$

As mentioned earlier, at the level of Lagrangian, g_1 and g_2 are arbitrary and independent to preserve supersymmetry. However, at trivial critical point, to preserve maximally supersymmetry this relation must be satisfied, i.e., $g_2 = -g_1$. The same condition occurs in $N = 4, 8$ gauged supergravity [34, 94]. Applying this condition and relabelling g_1 to g, now we have the potential

$$V = -64 g^2 e^{-3a} \left(3e^a - 2\right) . \tag{3.3.8}$$

The scalar potential is plot in Figure 3.1 with $g = 1$ and the critical point is easy to spot. There is only trivial critical point with $a = 0$, $V_0 = -64g^2$ and $N = (5, 0)$ supersymmetry. Note that the vacuum is similar to the ground state in $N = 16$ gauged supergravity with $SO(4) \times SO(4) \ltimes (\mathbf{T}^{12}, \hat{\mathbf{T}}^{34})$ gauge group [95]. The mass spectra and their representation are given below

m^2L^2	$SO(5)$
3	**1**
3	**5**
0	**10**

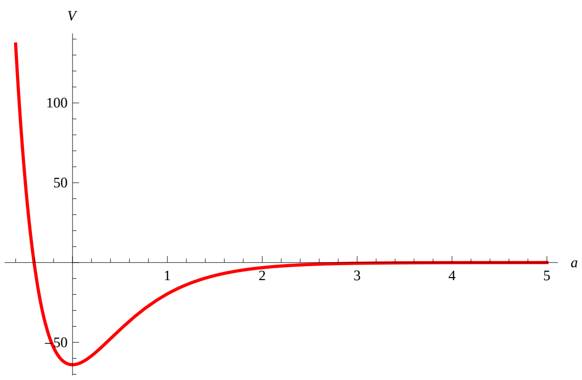

Figure 3.1: The scalar potential of $N = 5$, $SO(5) \ltimes \mathbf{T}^{10}$ gauged supergravity for $SO(5)$ singlet scalar with $g = 1$.

The superconformal group is $Osp(5|2, \mathbb{R}) \times Sp(2, \mathbb{R})$. There are ten Goldstone bosons associated with the symmetry breaking $SO(5) \ltimes \mathbf{T}^{10} \to SO(5)$ at trivial critical point. In order to find more critical points, we explore a larger manifold. We choose the manifold that is invariant under $SO(3)_{\text{diag}}$. The 16 scalars transform under this group as $(\mathbf{2}+\mathbf{2}) \times (\mathbf{2}+\mathbf{2}) = 4 \times (\mathbf{1}+\mathbf{3})$. Four singlets are found, so the coset representative is given by

$$L = e^{a_1 Y^4} e^{a_2 Y^7} e^{a_3 Y^9} e^{a_4 Y^{16}} . \tag{3.3.9}$$

The scalar potential is given by

$$
\begin{aligned}
V \;=\; & -8g_1^2\left(\cosh\left(\frac{a_3}{2}\right)\cosh\left(\frac{a_4}{2}\right)\sinh\left(\frac{a_2}{2}\right)-\cosh\left(\frac{a_4}{2}\right)\sinh\left(\frac{a_1}{2}\right)\sinh\left(\frac{a_3}{2}\right)\right.\\
& +\cosh\left(\frac{a_2}{2}\right)\sinh\left(\frac{a_4}{2}\right)-\cosh\left(\frac{a_1}{2}\right)\left(\cosh\left(\frac{a_2}{2}\right)\cosh\left(\frac{a_3}{2}\right)\cosh\left(\frac{a_4}{2}\right)\right.\\
& \left.\left.+\sinh\left(\frac{a_2}{2}\right)\sinh\left(\frac{a_4}{2}\right)\right)\right)^2(13-\cosh(a_3)+\cosh(a_4)-\cosh(a_3)\cosh(a_4)\\
& +\cosh(a_2)(-1+3\cosh(a_4)+\cosh(a_3)(1+\cosh(a_4)))\\
& -4\cosh\left(\frac{a_2}{2}\right)\sinh(a_1)\sinh(a_3)-4\cosh\left(\frac{a_2}{2}\right)\cosh(a_4)\sinh(a_1)\sinh(a_3)\\
& -4\cosh\left(\frac{a_3}{2}\right)\sinh(a_2)\sinh(a_4)+8\sinh(a_1)\sinh\left(\frac{a_2}{2}\right)\sinh\left(\frac{a_3}{2}\right)\sinh(a_4)\\
& +\cosh(a_1)\,(1-3\cosh(a_4)\,+3\cosh(a_3)(1+\cosh(a_4))\\
& +\cosh(a_2)(-1+3\cosh(a_4)+\cosh(a_3)(1+\cosh(a_4)))\\
& \left.\left.-4\cosh\left(\frac{a_3}{2}\right)\sinh(a_2)\sinh(a_4)\right)\right). \qquad (3.3.10)
\end{aligned}
$$

The potential is too complicated to find critical points, so the study of this potential will not be further analyzed in this work.

CHAPTER IV

$N = 6$ Theory

In this chapter we study the vacua of $N = 6$ gauged supergravity in three dimensions whose target space is in the form of $\frac{SU(4,k)}{S(U(4)\times U(k))}$[1] where $k = 1, 2, 3, 4$. The discussion is based on paper in [40]. As in the $N = 5$ theory, we classify the studied gauge groups into: compact, non-compact and non-semisimple. The compact gauge groups are in the form of $SO(p) \times SO(6-p) \times SU(k) \times U(1)$ for $p = 3, 4, 5, 6$. For non-compact gauging, the admissible gauge groups are non-compact subgroups of $SU(4, k)$. It is in the form of $SO(6) \ltimes \mathbf{T}^{15}$ where $\mathbf{T}^{15}$ is translational group with 15 generators and the scalar manifold is $\frac{SU(4,4)}{S(U(4)\times U(4))}$. We also obtain the domain wall solution for this theory and present it in the last section.

The analysis and format is quite similar to $N = 5$ case. We start the chapter by identifying the generators such as R-symmetry group $SO(6)$. Then we parametrize the coset manifold. Some cases we parametrize full manifold and the others we paremetrize submanifold thereof. In parametrization, we employ Euler angle parametrization interchangeably with the traditional unitary gauge for some cases. The complicated scalar potential with too many variables can be simplified by turning some of them off or setting them to be the same variable. The scalar potential can be computed by coding and running the computer application *Mathematica*. Next, we identify critical points, unbroken supersymmetry and residual gauge symmetry at that points. The mass spectra at trivial critical points are given along with their representation under residual gauge group.

Now we present the generators we use for this chapter. Note that this convention is local, i.e. the non-compact generators Y^As are not identical to those of $N = 5$ theory. However, for some generators such as generalized Gell-Mann matrices λ_i, we intentionally connect between those two. The non-compact generators constructed by Weyl unitary trick of $SU(4, 4)$ in terms of generalized Gell-Mann matrices, λ_i, $i = 1, \ldots, 63$, are given by

$$Y^A = \begin{cases} \frac{1}{\sqrt{2}} c_{A+15}, & A = 1, \ldots, 8 \\ \frac{1}{\sqrt{2}} c_{A+16}, & A = 9, \ldots, 16 \\ \frac{1}{\sqrt{2}} c_{A+19}, & A = 17, \ldots, 24 \\ \frac{1}{\sqrt{2}} c_{A+24}, & A = 25, \ldots, 32 \end{cases} . \tag{4.0.1}$$

In our convention, c_is are given by

$$c_i = -i\lambda_i \tag{4.0.2}$$

[1] $S(U(4) \times U(k))$ is isomorphic to $SU(4) \times SU(k) \times U(1)$

The $SO(6)_R$ R-symmetry generators are identified to be

$$
\begin{aligned}
\bar{T}^{12} &= \frac{1}{2}c_3 + \frac{1}{2\sqrt{3}}c_8 - \frac{1}{\sqrt{6}}c_{15}, \qquad \bar{T}^{13} = -\frac{1}{2}(c_2 + c_{14}), \qquad \bar{T}^{23} = \frac{1}{2}(c_1 - c_{13}), \\
\bar{T}^{34} &= \frac{1}{2}c_3 - \frac{1}{2\sqrt{3}}c_8 + \frac{1}{\sqrt{6}}c_{15}, \qquad \bar{T}^{14} = \frac{1}{2}(c_1 + c_{13}), \qquad \bar{T}^{35} = -\frac{1}{2}(c_6 + c_9), \\
\bar{T}^{56} &= \frac{1}{\sqrt{3}}c_8 + \frac{1}{\sqrt{6}}c_{15}, \qquad \bar{T}^{36} = -\frac{1}{2}(c_7 + c_{10}), \qquad \bar{T}^{24} = \frac{1}{2}(c_2 - c_{14}), \\
\bar{T}^{45} &= \frac{1}{2}(c_7 - c_{10}), \qquad \bar{T}^{46} = \frac{1}{2}(c_9 - c_6), \qquad \bar{T}^{15} = \frac{1}{2}(c_4 - c_{11}), \\
\bar{T}^{16} &= \frac{1}{2}(c_5 - c_{12}), \qquad \bar{T}^{25} = \frac{1}{2}(c_5 + c_{12}), \qquad \bar{T}^{26} = -\frac{1}{2}(c_4 + c_{11}) \, . \qquad (4.0.3)
\end{aligned}
$$

The generators used for constructing non-semisimple gauge group $SO(6) \ltimes \mathbf{T}^{15}$ will be discussed in the non-semisimple gauging section.

4.1 Compact gauge groups and their vacua

In this section, we study $N = 6$ theories with compact gauge group of the form $SO(p) \times SO(6-p) \times SU(k) \times U(1)$. The integer p helps us classify subgroup of R-symmetry group $SO(6)$. It has four possible choices which are 0, 1, 2 and 3. Note that when $p = 1$, the $SO(p) \times SO(6-p)$ factor is simply $SO(5)$ and for $p = 0$ is $SO(6)$. The integer k denotes the number of independent matter multiplets. It also labels the group $SU(k)$ it can range from 1 to infinity but in this work we consider only for $k = 1, 2, 3$ and 4. All in all, we consider 16 cases.

In each section we study for a specific k and the subsections thereof we explore for a particular number p starting from $p = 1$ all the way to $p = 3$. We start by parametrizing the full manifold or, in some cases, submanifolds. The technique mentioned in the $N = 5$ theories allows us to deal with submanifold instead of the larger full manifold. Then we calculate the scalar potential along with some important quantities such as A_1 tensor. The critical points can be calculated by optimization problem in one or multi variables calculus. The unbroken supersymmetry and residual gauge symmetry are specified at each critical point. The mass spectrum at trivial critical point are presented along with their residual gauge group representation. Note that we will not study mass spectrum for non-trivial critical points since the calculations are much more complicated than $N = 5$ theories.

The pre-classified admissible gauge groups of compact type are discussed in [37]. The embedding tensors for this case is given by

$$
\Theta = \Theta_{SO(p)} - \Theta_{SO(6-p)} + \alpha\Theta_{SU(k)} - \frac{4\alpha(k-1) + k(p-3)}{4+k}\Theta_{U(1)} \qquad (4.1.1)
$$

where the free parameter α is a relative coupling constant between the factor

$SU(k)$ and the factor $SO(p) \times SO(6-p)$. The $\Theta_{SO(p)} - \Theta_{SO(6-p)}$ is given by

$$\Theta_{IJ,KL} = \theta\delta^{KL}_{IJ} + \delta_{[I[K}\Xi_{L]J]}, \tag{4.1.2}$$

$$\Xi_{IJ} = \begin{cases} 2\left(1-\frac{p}{N}\right)\delta_{IJ}, & I \le p \\ -\frac{2p}{N}\delta_{IJ}, & I > p \end{cases}, \qquad \theta = \frac{2p-N}{N} \tag{4.1.3}$$

with $N = 6$.

4.1.1 The $k = 1$ case

This case is the simplest since we are dealing with only one matter multiplet. The manifold is in the form of coset $\frac{SU(4,1)}{SU(4)\times U(1)}$ parametrized by eight scalars. In the traditional unitary gauge parametrization, the calculation might be intractable for some computer. With the help of Euler angle parametrization, this problem can be partly resolved. In this case, we choose the Euler angle parametrization and the coset representative reads

$$L = e^{a_1 c_3} e^{a_2 c_2} e^{a_3 c_3} e^{a_4 c_5} e^{a_5 c_8} e^{a_6 c_{10}} e^{a_7 c_{15}} e^{\frac{i}{\sqrt{2}} b_1 c_{17}}. \tag{4.1.4}$$

The gauge group in the general form $SO(p) \times SO(6-p) \times SU(k) \times U(1)$ may not straightforwardly apply to $k = 1$ case, since the actual gauge group for $k = 1$ is $SO(p) \times SO(6-p) \times U(1)$. The exhaustive list of possible gauging is the following

- $SO(6) \times U(1)$
- $SO(5) \times U(1)$
- $SO(3) \times SO(3)$

Note that there is no $U(1)$ factor for $k = 3$ case. The $SU(k)$ factor for $k = 1$ is neglected. As a result, no free parameter α in $k = 1$ theory. Together with simple manifold structure, we get a simple result for all $p = 0, 1, 2, 3$ cases with the same coset representative (4.1.4). The potential depends on one scalar field and it reads

$$V = -8g^2[5 + 3\cosh(\sqrt{2}b_1)]. \tag{4.1.5}$$

The reason why the other seven scalars a_i's do not appear in the final form of the scalar potential is that the potential is gauge invariant, and the generators of parametrization (4.1.4) overlapped with some gauge groups. Consequently, the cancelation occur during the calculation process.

Obviously, we have only one critical point and it is of course trivial. At trivial critical point $b_1 = 0$ the value of potential is $V_0 = V(b_1 = 0) = -64g^2$. It preserves full gauge symmetry and full supersymmetry. The preserved supersymmetry at this point is given by two number $(p, 6-p)$ originated from $SO(p)\times SO(6-p)$. That number can also be written with notation (n_-, n_+) as in the previous chapter where

n_- and n_+ are number of negative and positive eigenvalues of the corresponding A_1 tensor at that critical point. This is the same number of supersymmetries as the dual two dimensional conformal field theory where n_- and n_+ represent the chirality of spinor in two dimensions. We do not present the explicit value of A_1 tensors from now on, since only number of negative and positive eigen values are used to determine type of supersymmetry.

The mass spectrum at trivial critical point can be computed by using unitary gauge, so the coset representative is given by

$$L = e^{\phi_1 Y_1} e^{\phi_2 Y_2} e^{\phi_3 Y_3} e^{\phi_4 Y_4} e^{\phi_5 Y_5} e^{\phi_6 Y_6} e^{\phi_7 Y_7} e^{\phi_8 Y_8}. \tag{4.1.6}$$

The unitary gauge is preferred to Euler angle parametrization since it give the correct interpretation of mass via canonically normalization. The scalar potential computed from this coset is quite messy compared to (4.1.5); therefore, in other gaugings in $N = 5$ and $N = 6$, we omitted the results. It depends on full eight scalars instead of one and it is given by

$$\begin{aligned} V_M \;=\; & -g^2 \Big[19 + 3\cosh(\sqrt{2}\phi_5)\cosh(\sqrt{2}\phi_6) + 3\cosh(\sqrt{2}\phi_7)\cosh(\sqrt{2}\phi_8) \\ & +3\cosh(\sqrt{2}\phi_5)\cosh(\sqrt{2}\phi_6)\cosh(\sqrt{2}\phi_7)\cosh(\sqrt{2}\phi_8) + 3\cosh(\sqrt{2}\phi_3)\times \\ & \cosh(\sqrt{2}\phi_4)\Big[1+\cosh(\sqrt{2}\phi_5)\cosh(\sqrt{2}\phi_6)\Big]\Big[1+\cosh(\sqrt{2}\phi_7)\cosh(\sqrt{2}\phi_8)\Big] \\ & +3\cosh(\sqrt{2}\phi_1)\cosh(\sqrt{2}\phi_2)\Big[1+\cosh(\sqrt{2}\phi_3)\cosh(\sqrt{2}\phi_4)\Big]\times \\ & \Big[1+\cosh(\sqrt{2}\phi_5)\cosh(\sqrt{2}\phi_6)\Big]\Big[1+\cosh(\sqrt{2}\phi_7)\cosh(\sqrt{2}\phi_8)\Big]\Big]. \end{aligned} \tag{4.1.7}$$

The mass spectrum is given in table below. In the table we present m^2L^2, conformal group $SO(2, 2)$ representation and residual gauge symmetry representation.

m^2L^2	$(h, \bar{h})$	$SO(6)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$\mathbf{4}$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$\bar{\mathbf{4}}$

The relations between scale dimension Δ, $(h, \bar{h})$ and m^2L^2 is given by the well-known equation

$$\begin{aligned} \Delta &= h + \bar{h}, \\ m^2L^2 &= \Delta(\Delta - 2). \end{aligned} \tag{4.1.8}$$

Note that scalar field has spin $s = 0$, so we have $h = \bar{h}$.

Recall that the second order derivative of potential is related to the mass of a scalar. The scalar potential (4.1.5) seems correspond to mass of one scalar while keeping the other seven scalar massless. The caveat is that the scalar potential from coset (4.1.5) does not give rise to the correct interpretation of mass since the scalar kinetic term are not canonically normalized as mentioned earlier. The explanation is quite simple in term of cartesian and polar coordinates analog.

The scalar fields in unitary gauge ϕ_i are metaphorically cartesian coordinates and the scalar field b is analogous to radial coordinate and seven angle variables θ_i for $i = 1, \ldots, 7$. This allows us to define relation between fields by the relation $\sum_{i=1}^{8} \phi_i^2 = b^2$ and $\phi_i(\theta_i, b)$.

Note that all compact gauge groups for $k = 1$ have the same scalar potential even in the unitary gauge. In general it is not necessarily the case; however, this mystery is still not solved even at the time of writing. The scalar mass spectrum for the other three cases are given below

- For $SO(5) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(5)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$\mathbf{4}$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$\mathbf{4}$

- For $SO(4) \times SO(2) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(4) \sim SU(2) \times SU(2)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$

- For $SO(3) \times SO(3)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(3) \times SO(3)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$(\mathbf{2}, \mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$(\mathbf{2}, \mathbf{2})$

To summarize: the $k = 1$ case is the simplest since they are parametrized by eight scalars which much smaller compared to higher k (number of scalars is $8k$). They all share the exact form of scalar potential. There is exactly one critical point which is, of course, trivial. No Goldstone bosons are found, since full gauge symmetry is preserved at that critical point.

4.1.2 The $k = 2$ case

The $k = 1$ theories studied in the previous section have only one coupling constant. The situation is slightly different for $k > 1$ case, they all have another coupling for the $SU(k)$ factor. In our convention it is relative coupling the gauge group $SO(p) \times SO(6-p)$ and $SU(k)$. This is different from the $N = 5$ case since we keep both couplings g_1 and g_2 unmodified. To tackle all 16 scalars is beyond our capability. Fortunately, some of the critical can be found by the help of the method discussed in the previous chapter. This technique reduces the complexity by considering submanifold instead of full manifold $\frac{SU(4,2)}{SU(4)\times SU(2)\times U(1)}$.

The submanifold we used is invariant under $U(1)_{\text{diag}}$ group. It can be embedded in submanifold $\frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)}$, so the number of scalars reduce to eight which is as small as the full manifold of $k = 1$ cases. The possible gauge groups we will study are the following

- $SO(6) \times SU(2) \times U(1)$
- $SO(5) \times SU(2) \times U(1)$
- $SO(4) \times SO(2) \times SU(2) \times U(1)$
- $SO(3) \times SO(3) \times SU(2) \times U(1)$

The $U(1)_{\text{diag}}$ is generated by $X_{12} + X_{56}$ where X_{IJ} are generators of $SO(6)$. With the Euler angle parametrization, the coset representative is given by

$$L = e^{a_1 c_{33}} e^{a_2 c_{34}} e^{a_3 K_3} e^{a_5 M_1} e^{a_6 M_2} e^{a_4 M_3} e^{\frac{i}{\sqrt{2}} b_1 c_{18}} e^{\frac{i}{\sqrt{2}} b_2 c_{31}} \tag{4.1.9}$$

where

$$\begin{aligned} K_3 &= \frac{1}{\sqrt{2}} [c_{33}, c_{34}] , \qquad M_1 = -\frac{1}{2\sqrt{2}} [c_{18}, c_{22}] , \\ M_2 &= -\frac{1}{2\sqrt{2}} [c_{19}, c_{22}] , \qquad M_3 = \frac{1}{\sqrt{2}} [M_1, M_2] . \end{aligned} \tag{4.1.10}$$

Note that the $SO(5) \times SU(2) \times U(1)$ case need a special treatment. The generators X_{56} is not an element in $SO(5)$; therefore, the coset representative (4.1.9) cannot be used. However, we can parametrize $U(1)_{\text{diag}}$ invariant manifold by replacing the generators to $X_{12} + X_{34}$. Consequently, the coset representative exclusively to this case is given by

$$L = e^{a_1 \kappa} e^{a_2 c_{14}} e^{a_3 \kappa} e^{a_4 c_{33}} e^{a_5 c_{34}} e^{a_6 \lambda} e^{\frac{i}{\sqrt{2}} b_1 c_{20}} e^{\frac{i}{\sqrt{2}} b_2 c_{31}} \tag{4.1.11}$$

where

$$\kappa = \frac{1}{\sqrt{2}} [c_{13}, c_{14}] , \qquad \lambda = \frac{2}{\sqrt{10}} c_{24} - \frac{3}{\sqrt{15}} c_{35} . \tag{4.1.12}$$

Having established the explicit form of various generators and coset representative, we now ready to find the scalar potential and their critical points starting from $SO(6) \times SU(2) \times U(1)$ gauging.

	b	V_0	unbroken SUSY
I	0	$-64g^2$	$(6,0)$
II	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha-1}{\alpha+1}\right),\ \alpha<-1$	$-\frac{16g^2(1+2\alpha)^2}{(1+\alpha)^2}$	$(4,0)$
III	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha+3}{\alpha+1}\right),\ \alpha>-1$	$-\frac{16g^2(3+2\alpha)^2}{(1+\alpha)^2}$	$(2,0)$

Table I: Critical points of $SO(6)\times SU(2)\times U(1)$ gauging for the $k=2$ case.

$SO(6)\times SU(2)\times U(1)$ **gauging**

Recall that we parametrize $U(1)_{\text{diag}}$ invariant submanifold with the coset representative (4.1.9). The scalar potential V is a function of two variables reduced from eight and is given by

$$\begin{aligned} V \;=\; & \frac{1}{8}g^2\Big[-222+32(-3+2\alpha+\alpha^2)\cosh(\sqrt{2}b_1)-2(1+\alpha)^2\cosh(2\sqrt{2}b_1) \\ & -48\cosh[\sqrt{2}(b_1-b_2)]-32\alpha\cosh[\sqrt{2}(b_1-b_2)]-16\alpha^2\cosh[\sqrt{2}(b_1-b_2)] \\ & +\cosh[2\sqrt{2}(b_1-b_2)]+2\alpha\cosh[2\sqrt{2}(b_1-b_2)]+\alpha^2\cosh[2\sqrt{2}(b_1-b_2)] \\ & -96\cosh(\sqrt{2}b_2)+64\alpha\cosh(\sqrt{2}b_2)+32\alpha^2\cosh(\sqrt{2}b_2)-2\cosh(2\sqrt{2}b_2) \\ & -4\alpha\cosh(2\sqrt{2}b_2)-2\alpha^2\cosh(2\sqrt{2}b_2)-48\cosh[\sqrt{2}(b_1+b_2)] \\ & -32\alpha\cosh[\sqrt{2}(b_1+b_2)]-16\alpha^2\cosh[\sqrt{2}(b_1+b_2)]+\cosh[2\sqrt{2}(b_1+b_2)] \\ & +2\alpha\cosh[2\sqrt{2}(b_1+b_2)]+\alpha^2\cosh[2\sqrt{2}(b_1+b_2)]-60\alpha-30\alpha^2\Big]. \end{aligned} \tag{4.1.13}$$

There are three critical points found and they are shown in Table I. We use the roman number to label different critical point. The number I in this case and the others to come is always tagged as trivial critical point. In order to simplify problem, we put constraint $b_1=b_2=b$ on the potential and we simply give the value of b at each critical point in the second column. In the third column V_0 is the value of the potential at the corresponding critical point. The next column is the unbroken supersymmetries at that particular critical point labelled by two number which is N_+ and N_- for different chirality of supercharges in its dual theory in two dimensions. The unbroken gauge symmetries at each critical point are as follows

- Critical point I : $SO(6)\times SU(2)\times U(1)$
- Critical point II : $SU(2)\times SU(2)\times U(1)_{\text{diag}}$
- Critical point III : $SU(2)\times SU(2)\times U(1)_{\text{diag}}$

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2L^2	$(h,\bar{h})$	$SO(6)\times SU(2)$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$(\mathbf{4},\mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(\bar{\mathbf{4}},\mathbf{2})$

	b	V_0	unbroken SUSY
I	0	$-64g^2$	$(5,1)$
II	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha-1}{\alpha+1}\right)$, $\alpha<-1$	$-\frac{16g^2(1+2\alpha)^2}{(1+\alpha)^2}$	$(4,1)$
III	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha+3}{\alpha+1}\right)$, $\alpha>-1$	$-\frac{16g^2(3+2\alpha)^2}{(1+\alpha)^2}$	$(1,0)$

Table II: Critical points of $SO(5)\times SU(2)\times U(1)$ gauging for the $k=2$ case.

$SO(5)\times SU(2)\times U(1)$ **gauging**

As mentioned before, this case is special and we use different coset from the others (4.1.11). The scalar potential is exactly the same as in $SO(6)\times SU(2)\times U(1)$ case (4.1.13). The critical points resembles the previous case except for unbroken supersymmetry as shown in the table II.

The unbroken gauge symmetries at each critical point are as follows

- Critical point I : $SO(5)\times SU(2)\times U(1)$
- Critical point II : $SU(2)_{\text{diag}}\times SU(2)$
- Critical point III : $SU(2)_{\text{diag}}\times SU(2)$

The $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)$ from $SU(k)$ and one of the $SU(2)$ in the $SO(4)\subset SO(5)$ subgroup of R-symmetry group. The scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h,\bar{h})$	$SO(5)\times SU(2)$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$(\mathbf{4},\mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(\mathbf{4},\mathbf{2})$

$SO(4)\times SO(2)\times SU(2)\times U(1)$ **gauging**

The coset representative for this case is the same as $SO(6)\times SO(2)\times SU(2)\times U(1)$ case and given in (4.1.9). The scalar potential is different from the previous two cases and it has the following form

$$\begin{aligned} V &= -\frac{1}{8}g^2\Big[192+30\alpha^2-32(-4+\alpha^2)\cosh(\sqrt{2}b_1)+2\alpha^2\cosh(2\sqrt{2}b_1) \\ &\quad +32\cosh[\sqrt{2}(b_1-b_2)]+16\alpha^2\cosh[\sqrt{2}(b_1-b_2)]-\alpha^2\cosh[2\sqrt{2}(b_1-b_2)] \\ &\quad +128\cosh(\sqrt{2}b_2)-32\alpha^2\cosh(\sqrt{2}b_2)+2\alpha^2\cosh(2\sqrt{2}b_2) \\ &\quad +32\cosh[\sqrt{2}(b_1+b_2)]+16\alpha^2\cosh[\sqrt{2}(b_1+b_2)] \\ &\quad -\alpha^2\cosh[2\sqrt{2}(b_1+b_2)]\Big]\,. \end{aligned} \tag{4.1.14}$$

Three critical points are found. We again set $b_1=b_2=b$ at the critical points as we did in the previous cases. The critical points in this cases used primed roman number to distinguish from previous cases.

	b	V_0	unbroken SUSY
I′	0	$-64g^2$	$(4,2)$
II′	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha+2}{\alpha}\right),\ \alpha>0$	$-\frac{16g^2(1+2\alpha)^2}{\alpha^2}$	$(2,2)$
III′	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha-2}{\alpha}\right),\ \alpha<0$	$-\frac{16g^2(1-2\alpha)^2}{\alpha^2}$	$(2,0)$

Table III: Critical points of $SO(4)\times SO(2)\times SU(2)\times U(1)$ gauging for the $k=2$ case.

	b	V_0	unbroken SUSY
I′	0	$-64g^2$	$(3,3)$
II′	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha+2}{\alpha}\right),\ \alpha>0$	$-\frac{16g^2(1+2\alpha)^2}{\alpha^2}$	$(1,2)$
III′	$\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\alpha-2}{\alpha}\right),\ \alpha<0$	$-\frac{16g^2(1-2\alpha)^2}{\alpha^2}$	$(2,1)$

Table IV: Critical points of $SO(3)\times SO(3)\times SU(2)\times U(1)$ gauging for the $k=2$ case.

The unbroken gauge symmetries at each critical point are as follows

- Critical point I′ : $SO(4)\times SO(2)\times SU(2)\times U(1)$
- Critical point II′ : $U(1)_{\text{diag}}\times U(1)\times U(1)$
- Critical point III′ : $U(1)_{\text{diag}}\times U(1)\times U(1)$

The scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h,\bar{h})$	$SO(4)\times SU(2)\sim SU(2)\times SU(2)\times SU(2)$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$(\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{1},\mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(\mathbf{1},\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{1},\mathbf{2})$

$SO(3)\times SO(3)\times SU(2)\times U(1)$ **gauging**

In this case, we use the coset representative as in $SO(4)\times SO(2)\times SU(2)\times U(1)$ and $SO(6)\times SO(2)\times SU(2)\times U(1)$ gauging. The scalar potential is the same as in the two cases. The primed roman number indicates the similarity of critical point compared to previous one. and distinctiveness compared to the first two cases. The critical points are listed in the table IV.

The unbroken gauge symmetries at each critical point are as follows

- Critical point I′ : $SO(3)\times SO(3)\times SU(2)\times U(1)$
- Critical point II′ : $U(1)_{\text{diag}}\times U(1)$
- Critical point III′ : $U(1)_{\text{diag}}\times U(1)$

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2L^2	$(h,\bar{h})$	$SO(3)\times SO(3)\times SU(2)$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$(\mathbf{2},\mathbf{2},\mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(\mathbf{2},\mathbf{2},\mathbf{2})$

Note that there are some remarks worth mentioning here. It involves $SU(k)$ coupling associated with the relative coupling constant α. For $\alpha=0$, no $SU(k)$ coupling, $SO(6)\times SU(2)\times U(1)$ and $SO(5)\times SU(2)\times U(1)$ gaugings contain only one non-trivial critical point. In addition, for the others, $SO(3)\times SO(3)\times SU(2)\times U(1)$ and $SO(4)\times SO(2)\times SU(2)\times U(1)$ gaugings, there is no non-trivial critical point at all. This pattern repeats itself for $k=3$ and $k=4$ cases as we will present in the next sections. This glitch might be rooted from our specific form of coset representative. Even though many additional attempts are made under the restriction on computational complexity , the statement about non-trivial critical point above is not altered.

4.1.3 The $k=3$ case

The number of scalar of $N=6$ theory is $8k$, so in this case we are dealing with 24 scalar. The submanifold we picked is invariant under $U(1)_{\text{diag}}$ as in the previous cases. There are two coset representatives corresponding to various gauge groups. The criteria is the same as the previous cases; therefore, we have one for $SO(5)\times SU(3)\times U(1)$ and the other for $SO(6)\times SU(3)\times U(1)$, $SO(4)\times SO(2)\times SU(3)\times U(1)$, and $SO3)\times SO(3)\times SU(3)\times U(1)$. As a result, we have coset representative $\frac{SU(2,3)}{SU(3)\times SU(2)\times U(1)}$ parametrized by 12 scalars.

For $SO(5)\times SU(3)\times U(1)$ case, $U(1)_{\text{diag}}$ generator is given by $X_{12}+X_{34}$. As a result, the coset representative is given by

$$L=e^{a_1\Lambda_3}e^{a_2c_{34}}e^{a_3\Lambda_3}e^{a_4c_{45}}e^{a_5\Lambda_8}e^{a_6\Lambda_3}e^{a_7c_{34}}e^{a_8\Lambda_3}e^{a_9\kappa}e^{a_{10}c_{14}}e^{\frac{i}{\sqrt{2}}b_1c_{20}}e^{\frac{i}{\sqrt{2}}b_1c_{31}} \tag{4.1.15}$$

where

$$\Lambda_3=\frac{2}{\sqrt{10}}c_{24}-\frac{3}{\sqrt{15}}c_{35},\qquad \Lambda_8=\frac{2}{\sqrt{30}}c_{24}+\frac{2}{3\sqrt{5}}c_{35}-\frac{7}{3\sqrt{7}}c_{48}, \tag{4.1.16}$$

and κ is given by (4.1.12).

For the other three cases, the submanifold is invariant under $U(1)_{\text{diag}}$ which the generator is given by $X_{12}+X_{56}$. Consequently, the coset representative is the following

$$L=e^{a_1\Lambda_3}e^{a_2c_{34}}e^{a_3\Lambda_3}e^{a_4c_{45}}e^{a_5\Lambda_8}e^{a_6\Lambda_3}e^{a_7c_{34}}e^{a_8\Lambda_3}e^{a_9M_3}e^{a_{10}M_2}e^{\frac{i}{\sqrt{2}}b_1c_{18}}e^{\frac{i}{\sqrt{2}}b_1c_{42}} \tag{4.1.17}$$

where M_2 and M_3 are given in (4.1.10).

The scalar potential and values at critical points is similar to $k = 2$ case, so are the residual supersymmetries. The only difference, of course, is the residual gauge symmetry. Note that the trivial critical point always preserve full compact gauge symmetry. The non-trivial points II, III, II′ and III′ have the following residual symmetries.

- $SO(6) \times SU(3) \times U(1)$ gauge group: $SU(2) \times SU(2) \times U(1) \times U(1)$
- $SO(5) \times SU(3) \times U(1)$ gauge group: $SU(2) \times SU(2) \times U(1)$
- $SO(4) \times SO(3) \times SU(2) \times U(1)$ gauge group: $U(1) \times U(1) \times U(1) \times U(1)$
- $SO(3) \times SO(3) \times SU(3) \times U(1)$ gauge group: $U(1) \times U(1) \times U(1)$

The scalar mass spectra are given below.

- For $SO(6) \times SU(3) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	(h, h)	$SO(6) \times SU(3)$
$-\frac{3}{4}$	$(\frac{3}{4}, \frac{3}{4})$	$(\mathbf{4}, \bar{\mathbf{3}})$
$-\frac{3}{4}$	$(\frac{1}{4}, \frac{1}{4})$	$(\bar{\mathbf{4}}, \mathbf{3})$

- For $SO(5) \times SU(3) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(5) \times SU(3)$
$-\frac{3}{4}$	$(\frac{3}{4}, \frac{3}{4})$	$(\mathbf{4}, \bar{\mathbf{3}})$
$-\frac{3}{4}$	$(\frac{1}{4}, \frac{1}{4})$	$(\mathbf{4}, \mathbf{3})$

- For $SO(4) \times SO(2) \times SU(3) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(4) \times SU(3) \sim SU(2) \times SU(2) \times SU(3)$
$-\frac{3}{4}$	$(\frac{3}{4}, \frac{3}{4})$	$(\mathbf{2}, \mathbf{1}, \bar{\mathbf{3}}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}})$
$-\frac{3}{4}$	$(\frac{1}{4}, \frac{1}{4})$	$(\mathbf{1}, \mathbf{2}, \mathbf{3}) + (\mathbf{2}, \mathbf{1}, \mathbf{3})$

- For $SO(3) \times SO(3) \times SU(3) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(3) \times SO(3) \times SU(3)$
$-\frac{3}{4}$	$(\frac{3}{4}, \frac{3}{4})$	$(\mathbf{2}, \mathbf{2}, \bar{\mathbf{3}})$
$-\frac{3}{4}$	$(\frac{1}{4}, \frac{1}{4})$	$(\mathbf{2}, \mathbf{2}, \mathbf{3})$

4.1.4 The $k = 4$ case

The $k = 4$ full scalar manifold contains 32 scalar. To tackle problem with 32 variables is impossible. Reducing to $U(1)_{\text{diag}}$ invariant submanifold is still not easy to deal with. One way to subdue the difficulty is reducing the size of submanifold. Consequently, we pick the submanifold that is invariant under $U(1)_{\text{diag}} \times SU(2)$. The $U(1)_{\text{diag}}$ sector is the same as in the previous cases. The additional $SU(2)$ sector is a subgroup of $SU(k) = SU(4)$. The $\alpha \to 0$ limit is meaningless here, α is always included, since the residual gauge symmetry contains subgroup of $SU(2) \subset SU(4)$. The manifold reduces to eight scalars. We use the same coset representative as in $k = 2$ case. For $p = 3, 4, 6$, the coset representative is (4.1.9) and for $p = 5$, the coset representative is (4.1.11).

The scalar potential and structure of the critical points are similar to the $k = 2$ case except for residual gauge symmetries of the non-trivial critical points. The non-trivial points II, III, II′ and III′ have the following unbroken gauge symmetries.

- $SO(6) \times SU(4) \times U(1)$ gauge group: $SU(2) \times SU(2) \times SU(2) \times U(1) \times U(1)$
- $SO(5) \times SU(4) \times U(1)$ gauge group: $SU(2) \times SU(2) \times SU(2)$
- $SO(4) \times SO(2) \times SU(4) \times U(1)$ gauge group: $SU(2) \times U(1) \times U(1) \times U(1) \times U(1)$
- $SO(3) \times SO(3) \times SU(4) \times U(1)$ gauge group: $SU(2) \times U(1) \times U(1) \times U(1)$

Other properties are the same as those given in table in $k = 2$ case. The scalar mass spectra are given below.

- For $SO(6) \times SU(4) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(6) \times SU(4)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$(\mathbf{4}, \bar{\mathbf{4}})$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$(\bar{\mathbf{4}}, \mathbf{4})$

- For $SO(5) \times SU(4) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(5) \times SU(4)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$(\mathbf{4}, \bar{\mathbf{4}})$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$(\mathbf{4}, \mathbf{4})$

- For $SO(4) \times SO(2) \times SU(4) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(4) \times SU(4) \sim SU(2) \times SU(2) \times SU(4)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$(\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$(\mathbf{1}, \mathbf{2}, \mathbf{4}) + (\mathbf{2}, \mathbf{1}, \mathbf{4})$

- For $SO(3) \times SO(3) \times SU(4) \times U(1)$ gauge group, the scalar mass spectrum at the trivial critical point is given below.

m^2L^2	$(h, \bar{h})$	$SO(3) \times SO(3) \times SU(4)$
$-\frac{3}{4}$	$\left(\frac{3}{4}, \frac{3}{4}\right)$	$(\mathbf{2}, \mathbf{2}, \bar{\mathbf{4}})$
$-\frac{3}{4}$	$\left(\frac{1}{4}, \frac{1}{4}\right)$	$(\mathbf{2}, \mathbf{2}, \mathbf{4})$

4.2 Non-compact gauge groups and their vacua

We have discussed the compact gauging of the form $SO(p) \times SO(6-p) \times SU(k) \times U(1)$ with limit to $k \leq 4$. We now take turn the non-compact gauge groups. All the compact gauging cases are already proved to be admissible [37], so to check their consistency is redundant. However, the story is quite different for the non-compact cases, since they are not checked The T-tensors constraint must be satisfied to justify gauging.

For non-compact gauging, at the trivial critical point, the gauge group cannot remain in full gauge symmetry but broken into maximal compact subgroup. At non-trivial point, the residual symmetry gauge group is even smaller. In non-compact gaugings, the scalar sector that overlapped to the gauge group will drop out from the scalar potential, so the non-compact generators that used to paremetrize the manifold are all non-compact generators living outside the gauge group.

The reporting format is quite similar to the compact cases except for the beginning part. In the non-compact cases, we list all possible gaugings via the embedding tensors before we proceed. The format of the rest is similar to the compact cases. After we categorize the embedding tensors, we then parametrize the manifold or submanifold thereof via Euler angle parametrization or unitary gauge. Then the scalar potential are computed and their critical points are searched afterwards. Supersymmetries and residual gauge symmetry are be identified at each critical point. The mass spectrum at each trivial critical point are given together with their representation under residual gauge symmetry. Unlike the compact gauging cases, none of which will be studied in RG flow section.

At the trivial critical point where all scalars are zero, the gauge group is broken down to its maximal compact subgroup which constitutes the residual symmetry of the associated critical point. Furthermore, this point preserves full supersymmetry namely $N = 6$ in three dimensions. It is convenient to express the number of supersymmetries in the case of AdS critical points in the two dimensional language of the corresponding dual CFT's in the form (n_-, n_+) as in the compact gaugings.

4.2.1 The $k = 1$ case

This is the smallest manifold to deal with. The global symmetry group G is $SU(4, 1)$. The non-compact subgroups thereof that pass the T-tensors constraint are the following:

$$
\begin{aligned}
SU(3,1)\times U(1) \quad &: \qquad \Theta = \Theta_{SU(3,1)} - \frac{3}{5}\Theta_{U(1)} \\
SU(2,1)\times SU(2)\times U(1) \quad &: \qquad \Theta = \Theta_{SU(2,1)} - \Theta_{SU(2)} - \frac{1}{5}\Theta_{U(1)} \\
SU(1,1)\times SU(3)\times U(1) \quad &: \qquad \Theta = \Theta_{SU(1,1)} - \Theta_{SU(3)} + \frac{1}{5}\Theta_{U(1)}\,.
\end{aligned} \tag{4.2.1}
$$

$SU(3,1)\times U(1)$ **gauging**

Recall that the number of non-compact generator of $SU(n,m)$ is $2nm$, so in this case there are six non-compact generators in the gauge group. We are left with only two from eight non-compact generators that can parametrize coset manifold. This is good for calculation process since we are dealing with two variables while parametrizing full manifold. The left-over non-compact generators are in $SU(1,1)\subset SU(3,1)$. We choose the parametrization

$$
L = e^{aX}e^{\frac{i}{\sqrt{2}}bc_{16}}e^{-aX}, \qquad X = -\frac{1}{\sqrt{2}}\left[c_{16},c_{17}\right]. \tag{4.2.2}
$$

The scalar potential is given by

$$
V = 8g^2(3\cosh(\sqrt{2}b)-5). \tag{4.2.3}
$$

Obviously there is no non-trivial critical point. At trivial critical point, $b=0$, the value of the potential is $V_0=-16g^2$. The eigenvalue of A_1 tensor at this point implies that the $N=(0,6)$ supersymmetry is preserved. The scalar mass spectrum is given below

m^2L^2	$(h,\bar{h})$	$SU(3)$
3	$(\frac{3}{2},\frac{3}{2})$	$2\times\mathbf{1}$

There are two massive particles with mass $m^2L^2=3$. They are singlet under $SU(3)$. The other six particles are massless and drop out from the potential. They correspond to Goldstone bosons. According to Higgs mechanism, there are six massive vector bosons corresponding to symmetry breaking $SU(3,1)\to SU(3)$. The right moving sector supercharges $N=(0,6)$ transforms as $\mathbf{3}+\bar{\mathbf{3}}$ under R-symmetry $U(3)\sim SU(3)\times U(1)\subset SU(4)$. Its superconformal algebra is $SU(1,1|3)$ where $SU(1,1)\sim SO(2,1)\sim SL(2,\mathbb{R})$ is a part of AdS_3 isometry group $SO(2,2)\sim SO(2,1)\times SO(2,1)$. The twelve supercharges of superconformal group transform as $(\mathbf{2},\mathbf{3})+(\mathbf{2},\bar{\mathbf{3}})$ under full non-compact bosonic subalgebra $SU(1,1)\times SU(3)\times U(1)$. Consequently the $(0,6)$ supersymmetric AdS_3 vacuum has a background isometry group $SU(1,1)\times SU(1,1|3)$ where $SU(1,1)$ is non-supersymmetric left moving part. Note that if the full $SU(4)\sim SO(6)$ R-symmetry is preserved, the associated superconformal algebra now becomes $OSp(6|2,\mathbb{R})$.

$SU(2,1) \times SU(2) \times U(1)$ gauging

The coset manifold is parametrized by four scalar associated with non-compact generators $SU(2,1) \subset SU(4,1)$ which is not the subgroup of the gauge group. The coset representative we define to be

$$L = e^{a_1 q_1} e^{a_2 q_2} e^{a_3 q_3} e^{-\frac{i}{\sqrt{2}} b c_{16}} e^{-a_3 q_3} e^{-a_2 q_2} e^{-a_1 q_1} \tag{4.2.4}$$

where

$$q_i = \frac{1}{2} c_i \,. \tag{4.2.5}$$

The scalar potential is a function of one variable, it reads

$$V = 8g^2 \left[-1 + \cosh(\sqrt{2} b) \right] . \tag{4.2.6}$$

There is only one critical point and it is trivial. The value at trivial critical point is $V_0 = 0$ when $b = 0$, so we have Minkowskian background. It preserves $N = 6$ supersymmetry. The scalar mass at this point is given by the table below

m^2	$SU(2) \times SU(2)$
$16g^2$	$2 \times (\mathbf{1}, \mathbf{2})$

$SU(1,1) \times SU(3) \times U(1)$ gauging

The coset representative is defined by six scalars corresponding to non-compact generators of $SU(3,1) \subset SU(4,1)$. Under $U(3)$ Euler angle parametrization, the coset representative is given by

$$L = e^{a_1 c_3} e^{a_2 c_2} e^{a_3 c_3} e^{a_4 c_5} e^{a_5 c_8} e^{\frac{i}{\sqrt{2}} b c_{17}} \,. \tag{4.2.7}$$

The scalar potential is found to be

$$V = -8g^2 \left(1 + \cosh(\sqrt{2} b) \right) . \tag{4.2.8}$$

No non-trivial critical point is found. At trivial critical point $b = 0$ the value of the potential is $V_0 = -16g^2$. The scalar mass spectrum is given in the table below.

$m^2 L^2$	$(h, \bar{h})$	$SU(3)$
-1	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\mathbf{3} + \bar{\mathbf{3}}$

The trivial critical point is $N = (0, 6)$ supersymmetric. The residual gauge symmetry is $SU(3) \times U(1) \times U(1)$ with the first two factors being a subgroup of the $SU(4)$ R-symmetry. The corresponding superconformal group is $SU(1,1|3)$.

4.2.2 The $k = 2$ case

For $k = 2$ the global symmetry group of the theory is $SU(4,2)$. The admissible gauge groups are among subgroup thereof. There are six gauge groups that satisfy T-tensor constraint, their embedding tensors are the following:

$$\begin{aligned}
SU(3,2)\times U(1) &: \quad \Theta = \Theta_{SU(3,2)} - \frac{2}{3}\Theta_{U(1)} \\
SU(2,2)\times SU(2)\times U(1) &: \quad \Theta = \Theta_{SU(2,2)} - \Theta_{SU(2)} - \frac{1}{3}\Theta_{U(1)} \\
SU(1,2)\times SU(3) &: \quad \Theta = \Theta_{SU(1,2)} - \Theta_{SU(3)} \\
SU(3,1)\times SU(1,1)\times U(1) &: \quad \Theta = \Theta_{SU(3,1)} - \Theta_{SU(1,1)} - \frac{1}{3}\Theta_{U(1)} \\
SU(2,1)\times SU(2,1) &: \quad \Theta = \Theta_{SU(2,1)} - \Theta_{SU(2,1)} \\
SU(4,1)\times U(1) &: \quad \Theta = \Theta_{SU(4,1)} - \frac{2}{3}\Theta_{U(1)}\,.
\end{aligned} \tag{4.2.9}$$

We now proceed to the study of the critical points and the mass spectrum.

$SU(3,2)\times U(1)$ gauging

This manifold $\frac{SU(2,1)}{SU(2)\times U(1)}$ is parametrized by four scalars and its coset representative is given by

$$L = e^{a_1 Q_1} e^{a_2 Q_2} e^{a_3 Q_3} e^{\frac{i}{\sqrt{2}} b c_{16}} \tag{4.2.10}$$

where

$$Q_1 = \frac{1}{2} c_{33}, \qquad Q_2 = \frac{1}{2} c_{34}, \qquad Q_3 = \frac{1}{2}\left(\frac{2}{\sqrt{10}} c_{24} - \frac{3}{\sqrt{15}} c_{35}\right). \tag{4.2.11}$$

The computation gives us the scalar potential

$$V = 8g^2\left(-5 + 3\cosh(\sqrt{2}b)\right) \tag{4.2.12}$$

Like the previous cases, there is only one critical point and it is trivial. At $b = 0$ the value of the critical point is $V_0 = -16g^2$ with $(0,6)$ supersymmetry. The superconformal group for this vacuum is $SU(1,1|3)$ with the R-symmetry group $SU(3)\times U(1)$ being a subgroup of the $SU(4)\sim SO(6)$ R-symmetry. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)\times SU(2)$
3	$(\frac{3}{2},\frac{3}{2})$	$2\times(\mathbf{1},\mathbf{2})$

$SU(2,2)\times SU(2)\times U(1)$ gauging

This coset manifold $\frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)}$ is parametrized by eight non-compact generators outside gauge groups. The coset representative is given by

$$L = e^{a_1 P_1} e^{a_2 P_2} e^{a_3 P_3} e^{a_4 Q_1} e^{a_5 Q_2} e^{a_6 Q_3} e^{\frac{i}{\sqrt{2}} b_1 c_{16}} e^{\frac{i}{\sqrt{2}} b_2 c_{27}}, \tag{4.2.13}$$

where $P_i = \frac{1}{2}c_i$ and Q_i's are given in (4.2.11). After the computation, we get the scalar potential with two variables

$$V = -8g^2\left[3 + \cosh(\sqrt{2}b_1)(-2 + \cosh(\sqrt{2}b_2)) - 2\cosh(\sqrt{2}b_2)\right] \tag{4.2.14}$$

Two critical points are found:

- Trivial critical point at $b_1 = b_2 = 0$ with $V_0 = 0$ and $N = 6$ supersymmetry
- at $b_1 = b_2 = \frac{1}{\sqrt{2}}\cosh^{-1} 2$ with $V_0 = 8g^2$. The residual symmetry is $SU(2)_{\text{diag}}\times SU(2)\times U(1)$ symmetry. The $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)$ factor in the full gauge group and one of the $SU(2)$'s in $SU(2,2)$.

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2	$SU(2)\times SU(2)\times SU(2)$
$16g^2$	$2\times(\mathbf{1},\mathbf{2},\mathbf{2})$

$SU(1,2)\times SU(3)$ **gauging**

From $8k = 16$ scalars, there are four non-compact generators in gauge groups, so the coset representative can be parametrized by twelve scalars. They are non-compact generators $SU(3,2)\subset SU(4,2)$. It reads

$$L = e^{a_1c_3}e^{a_2c_2}e^{a_3c_3}e^{a_4c_5}e^{\frac{1}{\sqrt{3}}a_5c_8}e^{a_6c_3}e^{a_7c_2}e^{a_8c_3}e^{a_9Q_3}e^{a_{10}Q_2}e^{\frac{i}{\sqrt{2}}b_1c_{16}}e^{\frac{i}{\sqrt{2}}b_2c_{27}}, \tag{4.2.15}$$

where Q_i's are given in (4.2.11). The scalar potential is found to be

$$V = -8g^2\left[1 + \cosh(\sqrt{2}b_1)\cosh(\sqrt{2}b_2)\right]. \tag{4.2.16}$$

In this case, we have solely a trivial critical point given by $b_1 = b_2 = 0$ with $V_0 = -16g^2$ The residual supersymmetry is $(0,6)$ and the conformal group is $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)\times SU(2)$
-1	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(\mathbf{3},\mathbf{2})+(\bar{\mathbf{3}},\mathbf{2})$

$SU(3,1)\times SU(1,1)\times U(1)$ **gauging**

The gauge group has eight non-compact generators, six from $SU(3,1)$ and two from $SU(1,1)$, so there are eight scalars parametrized by coset $\frac{SU(1,1)}{U(1)}\times\frac{SU(3,1)}{SU(3)\times U(1)}$ The coset representative is parametrized by

$$L = e^{-\frac{1}{2\sqrt{2}}a_1[c_{16},c_{17}]}e^{\frac{i}{\sqrt{2}}b_1c_{16}}e^{a_2w_3}e^{a_3w_2}e^{a_4w_3}e^{a_5w_5}e^{\frac{1}{\sqrt{3}}a_6w_8}e^{\frac{i}{\sqrt{2}}b_2c_{28}}, \tag{4.2.17}$$

where

$$
\begin{aligned}
w_2 &= \frac{1}{2}c_7, \qquad w_3 = -\frac{1}{4}(c_3 - \sqrt{3}c_8), \\
w_5 &= \frac{1}{2}c_{12}, \qquad w_8 = \frac{1}{4}(\sqrt{3}c_3 + c_8 - 4\sqrt{2}c_{15}).
\end{aligned}
\tag{4.2.18}
$$

The scalar potential is given by

$$
V = 8g^2\left[-3 - 2\cosh(\sqrt{2}b_2) + \cosh(\sqrt{2}b_1)(2 + \cosh(\sqrt{2}b_2))\right]. \tag{4.2.19}
$$

Only one critical point is found at $b_1 = b_2 = 0$ with $V_0 = -16g^2$. It is $(0,6)$ supersymmetric as determined by eigenvalue of A_1 tensor. The superconformal algebra is given by $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)$
-1	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\mathbf{3}+\bar{\mathbf{3}}$
3	$\left(\frac{3}{2},\frac{3}{2}\right)$	$2\times\mathbf{1}$

$SU(2,1)\times SU(2,1)$ **gauging**

The coset manifold is $\frac{SU(2,1)}{SU(2)\times U(1)} \times \frac{SU(2,1)}{SU(2)\times U(1)}$ parametrized by eight scalars. Note that the two $SU(2,1) \subset SU(4,2)$ is not the same as a subgroup in gauge group. The coset representative can be written as

$$
L = e^{a_1 q_1}e^{a_2 q_2}e^{a_3 q_3}e^{\frac{i}{\sqrt{2}}b_1 c_{16}}e^{a_4\tilde{w}_1}e^{a_5\tilde{w}_2}e^{a_6\tilde{w}_3}e^{\frac{i}{\sqrt{2}}b_2 c_{29}}, \tag{4.2.20}
$$

where q_i's are given in (4.2.5) and

$$
\tilde{w}_1 = \frac{1}{2}c_{13}, \qquad \tilde{w}_2 = \frac{1}{2}c_{14}, \qquad \tilde{w}_3 = \frac{1}{2}\left(-\frac{1}{\sqrt{3}}c_8 + \frac{2}{\sqrt{6}}c_{15}\right). \tag{4.2.21}
$$

The scalar potential is given by

$$
\begin{aligned}
V &= 2g^2\Big[\cosh[\sqrt{2}(b_1+b_2)] - \sinh[\sqrt{2}(b_1+b_2)](1 + \cosh(2\sqrt{2}b_1) + \cosh(2\sqrt{2}b_2) \\
&\quad -4\cosh(\sqrt{2}(b_1+b_2)) + \cosh(2\sqrt{2}(b_1+b_2)) + \sinh(2\sqrt{2}b_1) + \sinh(2\sqrt{2}b_2) \\
&\quad -4\sinh(\sqrt{2}(b_1+b_2)) + \sinh(2\sqrt{2}(b_1+b_2))\Big].
\end{aligned}
\tag{4.2.22}
$$

There is only one critical point and it is trivial. It is located at $b_1 = b_2 = 0$ with $V_0 = 0$ and preserves $N = 6$ supersymmetry. The scalar mass spectrum is given in the table below.

m^2	$SU(2)\times SU(2)$
$16g^2$	$2\times(\mathbf{2},\mathbf{1}) + 2\times(\mathbf{1},\mathbf{2})$

$SU(4,1) \times U(1)$ **gauging**

There are eight non-compact generators of gauge group $SU(4,1) \times U(1)$. We are left with another eight non-compact generators to parametrize coset manifold $\frac{SU(4,1)}{SU(4)\times U(1)}$. The corresponding coset representative is

$$L = e^{a_1 c_3} e^{a_2 c_2} e^{a_3 c_3} e^{a_4 c_5} e^{\frac{1}{\sqrt{3}} a_5 c_8} e^{a_6 c_{10}} e^{\frac{1}{\sqrt{6}} a_7 c_{15}} e^{\frac{i}{\sqrt{2}} b c_{26}}. \tag{4.2.23}$$

As a result, the scalar potential is a function of one variable:

$$V = -8g^2 \left(5 + 3\cosh(\sqrt{2}b)\right) \tag{4.2.24}$$

Obviously it admits only trivial critical point. The potential has value $V_0 = -64g^2$ at $b = 0$. This critical point, like the other cases, is $(0,6)$ supersymmetric. This vacuum preserves residual gauge symmetry $SU(4) \subset SU(4,1)$ which is also full R-symmetry group. In this case, the superconformal algebra is $OSp(6|2,\mathbb{R})$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(4)$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$\mathbf{4}$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$\bar{\mathbf{4}}$

4.2.3 The $k = 3$ case

For $k = 3$, the gauge groups are subgroup of $SU(4,3)$ which satisfy constraint on T-tensors, they are as follows

$$\begin{aligned}
SU(3,3)\times U(1) &: \quad \Theta = \Theta_{SU(3,3)} - \frac{5}{7}\Theta_{U(1)} \\
SU(2,3)\times SU(2)\times U(1) &: \quad \Theta = \Theta_{SU(2,3)} - \Theta_{SU(2)} - \frac{3}{7}\Theta_{U(1)} \\
SU(1,3)\times SU(3)\times U(1) &: \quad \Theta = \Theta_{SU(1,3)} - \Theta_{SU(3)} - \frac{1}{7}\Theta_{U(1)} \\
SU(3,2)\times SU(1,1)\times U(1) &: \quad \Theta = \Theta_{SU(3,2)} - \Theta_{SU(1,1)} - \frac{3}{7}\Theta_{U(1)} \\
SU(2,2)\times SU(2,1)\times U(1) &: \quad \Theta = \Theta_{SU(2,2)} - \Theta_{SU(2,1)} - \frac{1}{7}\Theta_{U(1)} \\
SU(1,2)\times SU(3,1)\times U(1) &: \quad \Theta = \Theta_{SU(1,2)} - \Theta_{SU(3,1)} + \frac{1}{7}\Theta_{U(1)} \\
SU(4,1)\times SU(2)\times U(1) &: \quad \Theta = \Theta_{SU(4,1)} - \Theta_{SU(2)} - \frac{3}{7}\Theta_{U(1)} \\
SU(4,2)\times U(1) &: \quad \Theta = \Theta_{SU(4,2)} - \frac{5}{7}\Theta_{U(1)}\,.
\end{aligned} \tag{4.2.25}$$

We are now in the position to study the critical points and their properties for each case.

$SU(3,3) \times U(1)$ **gauging**

For full scalar manifold, there are $3k = 24$ scalars. Since there are 18 non-compact generators from gauge group $SU(3,3) \times U(1)$, so they are left with six scalars parametrizing the coset $\frac{SU(3,1)}{SU(3)\times U(1)}$. The coset representative is then given by

$$L = e^{a_1 L_3} e^{a_2 L_2} e^{a_3 L_3} e^{a_4 L_5} e^{\frac{1}{\sqrt{3}} a_5 L_8} e^{\frac{i}{\sqrt{2}} b c_{17}}, \tag{4.2.26}$$

where

$$\begin{aligned} L_2 &= \frac{1}{2} c_{34}, \qquad L_3 = \frac{1}{10}\left(\sqrt{10} c_{24} - \sqrt{15} c_{35}\right), \\ L_5 &= \frac{1}{2} c_{45}, \qquad L_8 = -\frac{1}{40}\left(2\sqrt{15} c_{24} + 2\sqrt{10} c_{35} - 5\sqrt{14} c_{48}\right). \end{aligned} \tag{4.2.27}$$

The scalar potential now takes the form

$$V = 8g^2(3\cosh(\sqrt{2}b) - 5). \tag{4.2.28}$$

There is only one critical point found. It is trivial critical point at $b = 0$ with $V_0 = -16g^2$ and $(0,6)$ supersymmetric. The superconformal algebra is given by $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h, \bar{h})$	$SU(3) \times SU(3)$
3	$(\frac{3}{2}, \frac{3}{2})$	$(\mathbf{3}, \mathbf{1}) + (\bar{\mathbf{3}}, \mathbf{1})$

$SU(2,3) \times SU(2) \times U(1)$ **gauging**

The coset $\frac{SU(2,3)}{SU(2)\times SU(3)\times U(1)}$ is parametrized by twelve scalars associated with non-compact generators that do not overlap with the gauge group in question. The coset representative reads

$$L = e^{a_1 L_3} e^{a_2 L_2} e^{a_3 L_3} e^{a_4 L_5} e^{\frac{1}{\sqrt{3}} a_5 L_8} e^{a_6 L_3} e^{a_7 L_2} e^{a_8 L_3} e^{a_9 c_3} e^{a_{10} c_2} e^{\frac{i}{2} b_1 c_{16}} e^{\frac{i}{2} b_2 c_{27}}, \tag{4.2.29}$$

where L_i's are given in (4.2.11). The scalar potential is given by

$$V = -8g^2[3 + \cosh(\sqrt{2}b_1)(-2 + \cosh(\sqrt{2}b_2)) - 2\cosh(\sqrt{2}b_2)]\,. \tag{4.2.30}$$

In this case, two critical points are found:

- $b_1 = b_2 = 0$ with $V_0 = 0$ and $(0,6)$ supersymmetry
- $b_1 = b_2 = \frac{1}{\sqrt{2}} \cosh^{-1} 2$ with $V_0 = 8g^2$ and residual gauge symmetry $SU(2)_{\text{diag}} \times SU(2) \times U(1) \times U(1)$. The $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)$ and the $SU(2)$ in the $SU(3) \subset SU(2,3)$.

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2	$SU(3) \times SU(2) \times SU(2)$
$16g^2$	$(\mathbf{3}, \mathbf{1}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})$

$SU(1,3) \times SU(3)$ **gauging**

The coset $\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$ is parametrized by eighteen scalars associated with non-compact generators that do not overlap with the gauge group $SU(1,3) \times SU(3)$. The coset representative in this case is given by

$$\begin{aligned} L &= e^{a_1 q_3} e^{a_2 q_2} e^{a_3 q_3} e^{a_4 q_5} e^{\frac{1}{\sqrt{3}} a_5 q_8} e^{a_6 q_3} e^{a_7 q_2} e^{a_8 q_3} e^{a_9 L_3} e^{a_{10} L_2} e^{a_{11} L_3} e^{a_{12} L_5} e^{\frac{1}{\sqrt{3}} a_{13} L_8} e^{a_{14} L_3} \times \\ & \quad e^{a_{15} L_2} e^{\frac{i}{\sqrt{2}} b_1 c_{16}} e^{\frac{i}{\sqrt{2}} b_2 c_{27}} e^{\frac{i}{\sqrt{2}} b_3 c_{40}}, \end{aligned} \tag{4.2.31}$$

where q_i's and L_i's are the same as those given in (4.2.5) and (4.2.27), respectively. The scalar potential is given by

$$\begin{aligned} V &= -4g^2 \left[\frac{3}{2} - \frac{1}{2}\cosh(2\sqrt{2}b_1) - \frac{1}{2}\cosh(2\sqrt{2}b_2) + \frac{1}{4}\Big(-2 + 2\cosh(\sqrt{2}b_1) \right. \\ & \left. \quad +2\cosh(\sqrt{2}b_2) + 2\cosh(\sqrt{2}b_3) \Big)^2 - \frac{1}{2}\cosh(2\sqrt{2}b_3) \right]. \end{aligned} \tag{4.2.32}$$

Even though the potential is more complicated than many cases, there exists only trivial critical point. It is $(0,6)$ supersymmetric and located at $b_1 = b_2 = b_3 = 0$ with $V_0 = -16g^2$. The superconformal group for this trivial critical point is $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3) \times SU(3)$
-1	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(\mathbf{3},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{3})$

$SU(3,2) \times SU(1,1) \times U(1)$ **gauging**

Since there is twelve and two non-compact gauge group generators from $SU(3,2)$ and $SU(1,1)$, respectively, we are left with another ten non-compact generators to parametrize coset manifold. It is in the form of $\frac{SU(2,1)}{SU(2)\times U(1)} \times \frac{SU(3,1)}{SU(3)\times U(1)}$ whose the coset representative is given by

$$L = e^{a_1 c_{33}} e^{a_2 c_{34}} e^{\frac{1}{\sqrt{2}} a_3 [c_{33}, c_{34}]} e^{\frac{i}{\sqrt{2}} b_1 c_{16}} e^{a_4 w_3} e^{a_5 w_2} e^{a_6 w_3} e^{a_7 w_5} e^{\frac{1}{\sqrt{3}} \frac{a_8}{\sqrt{3}} w_8} e^{\frac{i}{\sqrt{2}} b_2 c_{39}}, \tag{4.2.33}$$

where w_i's are given in (4.2.18). The scalar potential is given by

$$\begin{aligned} V &= 2g^2 \Big[\cosh(\sqrt{2}(b_1+b_2)) - \sinh(\sqrt{2}(b_1+b_2)))(1 - 4\cosh(\sqrt{2}b_1) + \cosh(2\sqrt{2}b_1) \\ & \quad +4\cosh(\sqrt{2}b_2) + \cosh(2\sqrt{2}b_2) - 12\cosh(\sqrt{2}(b_1+b_2)) + \cosh(2\sqrt{2}(b_1+b_2)) \\ & \quad +4\cosh(\sqrt{2}(2b_1+b_2)) - 4\cosh(\sqrt{2}(b_1+2b_2)) - 4\sinh(\sqrt{2}b_1) + \sinh(2\sqrt{2}b_1) \\ & \quad +4\sinh(\sqrt{2}b_2) + \sinh(2\sqrt{2}b_2) - 12\sinh(\sqrt{2}(b_1+b_2)) + \sinh(2\sqrt{2}(b_1+b_2)) \\ & \quad +4\sinh(\sqrt{2}(2b_1+b_2)) - 4\sinh(\sqrt{2}(b_1+2b_2)) \Big]. \end{aligned} \tag{4.2.34}$$

The potential above has only one critical point, and obviously it is trivial one. Like many cases previously found, it is $(0,6)$ supersymmetric critical point. At $b_1 = b_2 = 0$ its value is $V_0 = -16g^2$. The superconformal algebra at this point is given by $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)\times SU(2)$
3	$(\frac{3}{2},\frac{3}{2})$	$2\times(\mathbf{1},\mathbf{2})$
-1	$(\frac{1}{2},\frac{1}{2})$	$(\mathbf{3},\mathbf{1})+(\bar{\mathbf{3}},\mathbf{1})$

$SU(2,2)\times SU(2,1)\times U(1)$ gauging

In this case, the coset $\frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)}\times\frac{SU(2,1)}{SU(2)\times U(1)}$ is described by twelve scalars. The corresponding coset representative is

$$L=e^{a_1q_1}e^{a_2q_2}e^{a_3q_3}e^{a_4Q_1}e^{a_5Q_2}e^{a_6Q_3}e^{\frac{i}{\sqrt{2}}b_1c_{16}}e^{\frac{i}{\sqrt{2}}b_2c_{27}}e^{a_7Z}e^{a_8c_{14}}e^{a_9Z}e^{\frac{i}{\sqrt{2}}b_3c_{40}}, \quad (4.2.35)$$

where $q_i=\frac{1}{2}c_i$,

$$Z=\frac{1}{\sqrt{2}}[c_{13},c_{14}], \quad (4.2.36)$$

and Q_i's are given in (4.2.11). The scalar potential is given by

$$\begin{aligned} V &= -g^2\Big[6-2\cosh(2\sqrt{2}b_1)-2\cosh(2\sqrt{2}b_2)-2\cosh(2\sqrt{2}b_3) \\ &\quad +4\left(-1+\cosh(\sqrt{2}b_1)+\cosh(\sqrt{2}b_2)-\cosh(\sqrt{2}b_3)\right)^2\Big]. \end{aligned} \quad (4.2.37)$$

There exists two critical point corresponding the scalar potential above:

- Trivial critical point at $b_1=b_2=b_3=0$ with $V_0=0$ and $(0,6)$ supersymmetry
- at $b_1=b_2=\frac{1}{\sqrt{2}}\cosh^{-1}2$, $b_3=0$ with $V_0=8g^2$ and residual gauge symmetry $SU(2)_{\text{diag}}\times SU(2)\times U(1)\times U(1)$ where $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)\subset SU(2,1)$ and one of the $SU(2)\subset SU(2,2)$.

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2	$SU(2)\times SU(2)\times SU(2)$
$16g^2$	$2\times(\mathbf{1},\mathbf{2},\mathbf{1})+2\times(\mathbf{2},\mathbf{1},\mathbf{2})$

$SU(1,2)\times SU(3,1)\times U(1)$ gauging

In this case, fourteen scalars parametrizes the coset $\frac{SU(1,1)}{U(1)}\times\frac{SU(3,2)}{SU(3)\times SU(2)\times U(1)}$ whose coset representative is

$$\begin{aligned} L &= e^{a_1c_3}e^{a_2c_2}e^{a_3c_3}e^{a_4c_5}e^{\frac{1}{\sqrt{3}}a_5c_8}e^{a_6c_3}e^{a_7c_2}e^{a_8c_3}e^{a_9Q_3}e^{a_{10}Q_2}e^{\frac{i}{\sqrt{2}}b_1c_{16}}e^{\frac{i}{\sqrt{2}}b_2c_{27}}\times \\ &\quad e^{\frac{1}{\sqrt{2}}a_{11}[c_{23},c_{24}]}e^{\frac{i}{\sqrt{2}}b_3c_{42}}, \end{aligned} \quad (4.2.38)$$

where Q_i's are given in (4.2.11). The scalar potential is given by

$$\begin{aligned} V &= -g^2\Big[6-2\cosh(2\sqrt{2}b_1)-2\cosh(2\sqrt{2}b_2)-2\cosh(2\sqrt{2}b_3) \\ &\quad +4\left(1+\cosh(\sqrt{2}b_1)+\cosh(\sqrt{2}b_2)-\cosh(\sqrt{2}b_3)\right)^2\Big]. \end{aligned} \quad (4.2.39)$$

Only trivial critical point is found and located at $b_1 = b_2 = b_3 = 0$ with $V_0 = -16g^2$. It is $(0,6)$ supersymmetric critical point and its superconformal algebra is found to be $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)\times SU(2)$
-1	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(\mathbf{3},\mathbf{2})+(\bar{\mathbf{3}},\mathbf{2})+2\times(\mathbf{1},\mathbf{1})$

$SU(4,1)\times SU(2)\times U(1)$ **gauging**

There are eight non-compact generators associated with gauge group $SU(4,1)\times SU(2)\times U(1)$, so the sixteen scalars corresponding to the other non-compact generators parametrize coset $\frac{SU(4,2)}{SU(4)\times SU(2)\times U(1)}$ whose coset representative is given by

$$\begin{aligned} L &= e^{a_1q_3}e^{a_2q_2}e^{a_3q_3}e^{a_4q_5}e^{\frac{1}{\sqrt{3}}a_5q_8}e^{a_6q_{10}}e^{a_7q_3}e^{a_8q_2}e^{a_9q_3}e^{a_{10}q_5}e^{\frac{1}{\sqrt{3}}a_{11}q_8}e^{a_{12}\tilde{q}_3}e^{a_{13}\tilde{q}_2}\times \\ &\quad e^{a_{14}\tilde{q}_3}e^{\frac{i}{\sqrt{2}}b_1c_{25}}e^{\frac{i}{\sqrt{2}}b_2c_{38}} \end{aligned} \quad (4.2.40)$$

where $q_i = \frac{1}{2}c_i$ and

$$\tilde{q}_2 = \frac{1}{2}c_{47}, \qquad \tilde{q}_3 = -\frac{1}{12}\left(\sqrt{15}c_{35}-\sqrt{21}c_{48}\right). \quad (4.2.41)$$

The scalar potential is found to be

$$V = -8g^2\left(3+2\cosh(\sqrt{2}b_2)+\cosh(\sqrt{2}b_1)(2+\cosh(\sqrt{2}b_2))\right). \quad (4.2.42)$$

In this case, non-trivial critical point does not exist. The trivial critical point at $b_1 = b_2 = 0$ with $V_0 = -64g^2$ is $(0,6)$ supersymmetric. The superconformal symmetry at this critical point is given by $OSp(6|2,\mathbb{R})$ since the full $SO(6)\sim SU(4)$ R-symmetry is preserved. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(4)\times SU(2)$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(\mathbf{4},\mathbf{2})$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$(\bar{\mathbf{4}},\mathbf{2})$

$SU(4,2)\times U(1)$ **gauging**

There are eight scalars parametrizing coset $\frac{SU(4,1)}{SU(4)\times U(1)}$ with the coset representative is given by

$$L = e^{a_1q_3}e^{a_2q_2}e^{a_3q_3}e^{a_4q_5}e^{\frac{1}{\sqrt{3}}a_5q_8}e^{a_6q_{10}}e^{\frac{1}{\sqrt{6}}a_7q_{15}}e^{\frac{i}{\sqrt{2}}bc_{37}}, \quad (4.2.43)$$

where q_i's are given in (4.2.5). The scalar potential is obtained to be

$$V = -8g^2\left(5+3\cosh(\sqrt{2}b)\right). \quad (4.2.44)$$

The potential is so simple and obviously admits no non-trivial critical point. The trivial one at $b=0$ with $V_0 = -64g^2$ is $(0,6)$ supersymmetric and $OSp(6|2,\mathbb{R})$ is its superconformal algebra. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(4)\times SU(2)$
$-\frac{3}{4}$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(\mathbf{4},\mathbf{1})$
$-\frac{3}{4}$	$\left(\frac{3}{4},\frac{3}{4}\right)$	$(\bar{\mathbf{4}},\mathbf{1})$

4.2.4 The $k=4$ case

In this case, we consider $k=4$ cases whose number of full scalars is 32. The actual coset representative is smaller since some non-compact generators may overlap gauge generators. We rule out some of gauge groups that cannot pass T-tensor constraint test, so the admissible gauge groups are as follows:

$$\begin{aligned}
SU(3,4)\times U(1) \quad &: \quad \Theta=\Theta_{SU(3,4)}-\frac{3}{4}\Theta_{U(1)} \qquad (4.2.45)\\
SU(2,4)\times SU(2)\times U(1) \quad &: \quad \Theta=\Theta_{SU(2,4)}-\Theta_{SU(2)}-\frac{1}{2}\Theta_{U(1)}\\
SU(1,4)\times SU(3)\times U(1) \quad &: \quad \Theta=\Theta_{SU(1,4)}-\Theta_{SU(3)}-\frac{1}{4}\Theta_{U(1)}\\
SU(3,3)\times SU(1,1)\times U(1) \quad &: \quad \Theta=\Theta_{SU(3,3)}-\Theta_{SU(1,1)}-\frac{1}{2}\Theta_{U(1)}\\
SU(2,3)\times SU(2,1)\times U(1) \quad &: \quad \Theta=\Theta_{SU(2,3)}-\Theta_{SU(2,1)}-\frac{1}{4}\Theta_{U(1)}\\
SU(1,3)\times SU(3,1) \quad &: \quad \Theta=\Theta_{SU(1,3)}-\Theta_{SU(3,1)}\\
SU(2,2)\times SU(2,2) \quad &: \quad \Theta=\Theta_{SU(2,2)}-\Theta_{SU(2,2)}\,.
\end{aligned}$$

We are ready to study the critical points and some of their properties for each case.

$SU(3,4)\times U(1)$ **gauging**

In this case, there are 24 non-compact generators corresponding to $SU(3,4)\times U(1)$ gauge group. The 32 non-compact generators of full global symmetry are deducted by the number of non-compact generators of gauge group, so there are eight generators left to construct the coset $\frac{SU(4,1)}{SU(4)\times U(1)}$ whose coset representative is defined as follows

$$L=e^{a_1 j_3}e^{a_2 j_2}e^{a_3 j_3}e^{a_4 j_5}e^{\frac{1}{\sqrt{3}}a_5 j_8}e^{a_6 j_{10}}e^{\frac{1}{\sqrt{6}}a_7 j_{15}}e^{\frac{i}{\sqrt{2}}bc_{17}}, \qquad (4.2.46)$$

where

$$\begin{aligned}
j_2 &= \frac{1}{2}c_{34}, \qquad j_3=\frac{1}{2\sqrt{2}}\left[c_{33},c_{34}\right], \qquad j_5=\frac{1}{2}c_{45},\\
j_8 &= \frac{1}{15}(3\sqrt{10}c_{24}+2\sqrt{15}c_{35}-5\sqrt{21}c_{48}), \qquad j_{10}=\frac{1}{2}c_{58},\\
j_{15} &= \frac{1}{105}(21\sqrt{10}c_{24}+14\sqrt{15}c_{35}+10\sqrt{21}c_{48}-90\sqrt{7}c_{63}). \qquad (4.2.47)
\end{aligned}$$

After running the code, we find the scalar potential

$$V = 8g^2 \left(-5 + 3\cosh(\sqrt{2}b)\right). \tag{4.2.48}$$

There is a critical point found which is trivial and located at the origin $b = 0$ whose value at that point is $V_0 = -16g^2$. Furthermore, it is a $(0, 6)$ supersymmetric critical point with $OSp(6|2, \mathbb{R})$ superconformal algebra. The scalar mass spectrum is given in the table below.

m^2L^2	$(h, \bar{h})$	$SU(4) \times SU(3)$
3	$(\frac{3}{2}, \frac{3}{2})$	$(\mathbf{4}, \mathbf{1}) + (\bar{\mathbf{4}}, \mathbf{1})$

$SU(2, 4) \times SU(2) \times U(1)$ gauging

In this case, we dealing with coset $\frac{SU(4,2)}{SU(4)\times SU(2)\times U(1)}$ whose parametrized by sixteen scalars associated with leftover non-compact generators. The coset representative takes the form

$$\begin{aligned} L &= e^{a_1 j_3} e^{a_2 j_2} e^{a_3 j_3} e^{a_4 j_5} e^{\frac{1}{\sqrt{3}} a_5 j_8} e^{a_6 j_3} e^{a_7 j_3} e^{a_8 j_2} e^{a_9 j_3} e^{a_{10} j_5} e^{\frac{1}{\sqrt{3}} a_{11} j_8} e^{a_{12} c_3} e^{a_{13} c_2} \times \\ & \quad e^{a_{14} c_3} e^{\frac{i}{\sqrt{2}} b_1 c_{16}} e^{\frac{i}{\sqrt{2}} b_2 c_{27}}, \end{aligned} \tag{4.2.49}$$

where j_i's are given in (4.2.47). The scalar potential is given by

$$V = -8g^2 \left[3 + \cosh(\sqrt{2}b_1)(-2 + \cosh(\sqrt{2}b_2)) - 2\cosh(\sqrt{2}b_2)\right]. \tag{4.2.50}$$

Two critical points are found, they are as follows:

- Trivial critical point at $b_1 = b_2 = 0$ with $V_0 = 0$ and $(0, 6)$ supersymmetry
- at $b_1 = b_2 = \frac{1}{\sqrt{2}} \cosh^{-1} 2$ with $V_0 = 8g^2$ and residual gauge symmetry $SU(2)_{\text{diag}} \times SU(2) \times SU(2) \times U(1) \times U(1)$. The $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)$ and an $SU(2)$ in the $SU(4) \subset SU(2, 4)$.

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2	$SU(4) \times SU(2) \times SU(2)$
$16g^2$	$(\mathbf{4}, \mathbf{1}, \mathbf{2}) + (\bar{\mathbf{2}}, \mathbf{1}, \mathbf{2})$

$SU(1, 4) \times SU(3) \times U(1)$ gauging

The coset $\frac{SU(4,3)}{SU(4)\times SU(3)\times U(1)}$ is parametrized by twenty four scalars whose coset representative takes the form

$$\begin{aligned} L &= e^{a_1 c_3} e^{a_2 c_2} e^{a_3 c_3} e^{a_4 c_5} e^{\frac{1}{\sqrt{3}} a_5 c_8} e^{a_6 c_3} e^{a_7 c_3} e^{a_8 j_3} e^{a_9 j_2} e^{a_{10} j_3} e^{a_{11} j_5} e^{\frac{1}{\sqrt{3}} a_{12} j_8} e^{a_{13} j_{10}} e^{a_{14} j_3} \times \\ & \quad e^{a_{14} j_3} e^{a_{15} j_2} e^{a_{16} j_3} e^{a_{17} j_5} e^{\frac{1}{\sqrt{3}} a_{18} j_8} e^{a_{19} j_3} e^{a_{20} j_2} e^{a_{21} j_3} e^{\frac{i}{\sqrt{2}} b_1 c_{16}} e^{\frac{i}{\sqrt{2}} b_2 c_{27}} e^{\frac{i}{\sqrt{2}} b_3 c_{40}}, \end{aligned} \tag{4.2.51}$$

where j_i's are given in (4.2.47). The scalar potential is given by

$$V = -8g^2\Big[2+\cosh(\sqrt{2}b_2)(-1+\cosh(\sqrt{2}b_3))-\cosh(\sqrt{2}b_3)+\cosh(\sqrt{2}b_1)\times (-1+\cosh(\sqrt{2}b_2)+\cosh(\sqrt{2}b_3))\Big]. \quad (4.2.52)$$

Even though we have quite complicated potential with three variables, there exists only trivial critical point which is at $b_1 = b_2 = b_3 = 0$ with $V_0 = -16g^2$ and $(0,6)$ supersymmetric. The superconformal algebra for this critical point is $OSp(6|2,\mathbb{R})$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(4)\times SU(3)$
-1	$(\frac{1}{2},\frac{1}{2})$	$(\mathbf{4},\bar{\mathbf{3}})+(\bar{\mathbf{4}},\mathbf{3})$

$SU(3,3)\times SU(1,1)\times U(1)$ **gauging**

In this case, we parametrize the product of coset manifold $\frac{SU(3,1)}{SU(3)\times U(1)}\times\frac{SU(3,1)}{SU(3)\times U(1)}$ with twelve scalars associated with non-overlapped non-compact generators. The coset representative takes the following form

$$L = e^{a_1w_3}e^{a_2w_2}e^{a_3w_3}e^{a_4w_5}e^{\frac{1}{\sqrt{3}}a_5w_8}e^{\frac{i}{\sqrt{2}}b_1c_{52}}e^{a_6L_3}e^{a_7L_2}e^{a_8L_3}e^{a_9L_5}e^{\frac{1}{\sqrt{3}}a_{10}L_8}e^{\frac{i}{\sqrt{2}}b_2c_{17}}, \quad (4.2.53)$$

where w_i's and L_i's are given in (4.2.18) and (4.2.27), respectively. The scalar potential is found to be

$$V = 8g^2[-3+\cosh(\sqrt{2}b_1)(-2+\cosh(\sqrt{2}b_2))+2\cosh(\sqrt{2}b_2)]. \quad (4.2.54)$$

There is only one critical point for this potential. It is trivial critical point with $(0,6)$ supersymmetry. Furthermore, it is specified by $b_1 = b_2 = 0$ with $V_0 = -16g^2$ whereas its superconformal algebra is $SU(1,1|3)$. The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)\times SU(3)$
3	$(\frac{3}{2},\frac{3}{2})$	$(\mathbf{1},\mathbf{3})+(\mathbf{1},\bar{\mathbf{3}})$
-1	$(\frac{1}{2},\frac{1}{2})$	$(\mathbf{3},\mathbf{1})+(\bar{\mathbf{3}},\mathbf{1})$

$SU(2,3)\times SU(2,1)\times U(1)$ **gauging**

The sixteen scalars described by the coset $\frac{SU(2,1)}{SU(2)\times U(1)}\times\frac{SU(3,2)}{SU(3)\times SU(2)\times U(1)}$ are parametrized by the coset representative

$$L = e^{a_1L_3}e^{a_2L_2}e^{a_3L_3}e^{a_4L_5}e^{\frac{1}{\sqrt{3}}a_5L_8}e^{a_6L_3}e^{a_7L_2}e^{a_8L_3}e^{a_9q_3}e^{a_{10}q_2}e^{\frac{i}{\sqrt{2}}b_2c_{16}}e^{\frac{i}{\sqrt{2}}b_3c_{27}}\times e^{a_{11}z_1}e^{a_{12}z_2}e^{a_{13}z_3}e^{\frac{i}{\sqrt{2}}b_1c_{44}}, \quad (4.2.55)$$

where

$$z_1 = \frac{1}{2}c_{13}, \qquad z_2 = \frac{1}{2}c_{14}, \qquad z_3 = \frac{1}{2}\left(\frac{-1}{\sqrt{3}}c_8+\frac{2}{\sqrt{6}}c_{15}\right), \quad (4.2.56)$$

and q_i's and L_i's are given in (4.2.5) and (4.2.27), respectively. We find the potential

$$V = 8g^2\Big[-2+\cosh(\sqrt{2}b_3)+\cosh(\sqrt{2}b_1)(-1+\cosh(\sqrt{2}b_2)+\cosh(\sqrt{2}b_3)) -2\cosh(\sqrt{2}b_2)\sinh^2\left(\frac{b_3}{\sqrt{2}}\right)\Big]. \quad (4.2.57)$$

There are two critical points:

- at $b_1=b_2=b_3=0$ with $V_0=0$ and $N=6$ supersymmetry
- at $b_1=0$, $b_2=b_3=\frac{1}{\sqrt{2}}\cosh^{-1}2$ with $V_0=8g^2$ and residual symmetry $SU(2)_{\text{diag}}\times SU(2)\times U(1)\times U(1)\times U(1)$. The $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)\subset SU(2,1)$ and the $SU(2)$ subgroup of the $SU(3)\subset SU(2,3)$.

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2	$SU(3)\times SU(2)\times SU(2)$
$16g^2$	$2\times(\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{3})+(\mathbf{2},\mathbf{1},\bar{\mathbf{3}})$

$SU(1,3)\times SU(3,1)$ **gauging**

In this case, we are dealing with the coset $\frac{SU(1,1)}{U(1)}\times\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$ which is parametrized by twenty scalars. The coset representative is given by

$$L = e^{\frac{1}{\sqrt{2}}a_1[c_{31},c_{32}]}e^{\frac{i}{\sqrt{2}}b_1c_{56}}e^{a_2c_3}e^{a_4c_3}e^{a_5c_5}e^{\frac{1}{\sqrt{3}}a_6c_8}e^{a_7c_3}e^{a_8c_2}e^{a_9c_3}e^{a_{10}L_3}e^{a_{11}L_2}e^{a_{12}L_3}e^{a_{13}L_5}\times e^{\frac{1}{\sqrt{3}}a_{14}L_8}e^{a_{15}L_3}e^{a_{16}L_2}e^{\frac{i}{\sqrt{2}}b_2c_{16}}e^{\frac{i}{\sqrt{2}}b_3c_{27}}e^{\frac{i}{\sqrt{2}}b_4c_{40}}, \quad (4.2.58)$$

where L_i's are given in (4.2.27). The scalar potential is found to be

$$V = -g^2\Big[6-2\cosh(2\sqrt{2}b_2)-2\cosh(2\sqrt{2}b_3)+4\Big(-\cosh(\sqrt{2}b_1)+\cosh(\sqrt{2}b_2) +\cosh(\sqrt{2}b_3)+\cosh(\sqrt{2}b_4)\Big)^2-2\cosh(2\sqrt{2}b_4)-4\sinh^2(\sqrt{2}b_1)\Big]. \quad (4.2.59)$$

From the potential above, we found only one critical point which is at $b_1=b_2=b_3=0$ with $V_0=-16g^2$. It is $(0,6)$ supersymmetric with superconformal algebra $SU(1,1|3)$ The scalar mass spectrum is given in the table below.

m^2L^2	$(h,\bar{h})$	$SU(3)\times SU(3)$
3	$\left(\frac{3}{2},\frac{3}{2}\right)$	$2\times(\mathbf{1},\mathbf{1})$
-1	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(\mathbf{3},\bar{\mathbf{3}})+(\bar{\mathbf{3}},\mathbf{3})$

$SU(2,2)\times SU(2,2)$ **gauging**

This is the last case for non-compact gauging. In this case, the sixteen scalars characterized by the coset $\frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)}\times\frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)}$. Note that $SU(2,2)\subset SU(4,4)$ is not the same as those appear in the gauge group. The coset representative for this gauging is

$$\begin{aligned} L &= e^{a_1q_1}e^{a_2q_2}e^{a_3q_3}e^{a_4Q_1}e^{a_5Q_2}e^{a_6Q_3}e^{\frac{i}{\sqrt{2}}b_1c_{16}}e^{\frac{i}{\sqrt{2}}b_2c_{27}}e^{a_7z_1}e^{a_8z_2}e^{a_9z_3}e^{a_{10}\tilde{z}_1}e^{a_{11}\tilde{z}_2}\times \\ & e^{\frac{1}{\sqrt{2}}a_{12}[\tilde{z}_1,\tilde{z}_2]}e^{\frac{i}{\sqrt{2}}b_3c_{40}}e^{\frac{i}{\sqrt{2}}b_4c_{55}}, \end{aligned} \tag{4.2.60}$$

where

$$\tilde{z}_{13}=\frac{1}{2}c_{61},\qquad \tilde{z}_{14}=\frac{1}{2}c_{62}, \tag{4.2.61}$$

and q_i's, z_i's and Q_i's are given in (4.2.5), (4.2.56) and (4.2.11), respectively. The scalar potential is given by

$$\begin{aligned} V &= -g^2\Big[8-2\cosh(2\sqrt{2}b_1)-2\cosh(2\sqrt{2}b_2)-2\cosh(2\sqrt{2}b_3)-2\cosh(2\sqrt{2}b_4) \\ &\quad +4\Big(\cosh(\sqrt{2}b_1)+\cosh(\sqrt{2}b_2)-\cosh(\sqrt{2}b_3)-\cosh(\sqrt{2}b_4)\Big)^2\Big]. \end{aligned} \tag{4.2.62}$$

In this cases, two critical points are found and they are characterized as follows:

- at Trivial critical point $b_1=b_2=b_3=b_4=0$ with $V_0=0$ and $(0,6)$ supersymmetry
- at $b_1=b_2=\frac{1}{\sqrt{2}}\cosh^{-1}2$, $b_3=b_4=0$ with $V_0=8g^2$ which is equivalent to the critical point at $b_3=b_4=\frac{1}{\sqrt{2}}\cosh^{-1}2$, $b_1=b_2=0$ with the same V_0. In both cases, the residual gauge symmetry is $SU(2)_{\text{diag}}\times SU(2)\times SU(2)\times U(1)$. The $SU(2)_{\text{diag}}$ is a diagonal subgroup of the $SU(2)$ factors from each $SU(2,2)$.

The scalar mass spectrum at the trivial critical point is given in the table below.

m^2	$SU(2)\times SU(2)\times SU(2)\times SU(2)$
$16g^2$	$2\times(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})+2\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$

4.3 Non-semisimple $N=6$, $SO(6)\ltimes\mathbf{T}^{15}$ gauged supergravity

In this case, we consider non-semisimple case for $N=6$ with gauge group $SO(6)\ltimes\mathbf{T}^{15}$. Originally, this section is not a part in [40], but it is later added to [39] as an important supplement since the non-semisimple gauging has a direct relation to higher dimensional theories on compactified on some manifolds or orbifolds. Recall that the full coset manifold for this gauge group is $\frac{SU(4,4)}{S(U(4)\times U(4))}$. Non-semisimple group $SO(6)\ltimes\mathbf{T}^{15}$ is a subgroup of global symmetry group $SU(4,4)$. The $SO(6)$

subgroup (or $SO(6)_{\rm diag}$) is defined by the diagonal subgroup of $SO(6)\times SO(6)\sim SU(4)\times SU(4)\subset SU(4,4)$. The 32 scalar transform as

$$(\mathbf{4},\mathbf{4})+(\mathbf{4},\mathbf{4}) \tag{4.3.1}$$

under $SU(4)\times SU(4)$ and

$$(\mathbf{4}\times\mathbf{4})+(\mathbf{4}\times\mathbf{4})=\mathbf{1}+\mathbf{15}+\mathbf{1}+\mathbf{15}. \tag{4.3.2}$$

under $SO(6)_{\rm diag}$. There are two singlets and two $\mathbf{15}$. The $\mathbf{15}$ is adjoint representation of $SO(6)$ and they are used to construct $\mathbf{T}^{15}$.

The $SO(6)\ltimes T^{15}$ generators are given by

$$\begin{aligned} SO(6)\ &: \qquad J_{\rm a}^{ij}=\bar{T}^{ij}+\tilde{\bar{T}}^{ij}, \qquad i,j=1,\ldots,6\\ \mathbf{T}^{15}\ &: \qquad J_{\rm b}^{ij}=\bar{T}^{ij}-\tilde{\bar{T}}^{ij}+\tilde{\bar{Y}}^{ij} \end{aligned} \tag{4.3.3}$$

where

$$\begin{aligned}
\tilde{\bar{T}}^{12} &= i\left(\frac{1}{\sqrt{10}}\lambda_{24}-\sqrt{\frac{3}{20}}\lambda_{35}-\sqrt{\frac{3}{28}}\lambda_{48}+\frac{1}{\sqrt{7}}\lambda_{63}\right),\\
\tilde{\bar{T}}^{34} &= i\left(\frac{1}{\sqrt{10}}\lambda_{24}-\sqrt{\frac{3}{20}}\lambda_{35}+\sqrt{\frac{3}{28}}\lambda_{48}-\frac{1}{\sqrt{7}}\lambda_{63}\right),\\
\tilde{\bar{T}}^{56} &= i\left(\frac{1}{\sqrt{10}}\lambda_{24}+\frac{1}{\sqrt{15}}\lambda_{35}-\frac{2}{\sqrt{21}}\lambda_{48}-\frac{1}{\sqrt{7}}\lambda_{63}\right),\\
\tilde{\bar{T}}^{13} &= \frac{i}{2}\left(\lambda_{34}+\lambda_{62}\right), \qquad \tilde{\bar{T}}^{23}=-\frac{i}{2}\left(\lambda_{33}-\lambda_{61}\right), \qquad \tilde{\bar{T}}^{14}=-\frac{i}{2}\left(\lambda_{33}+\lambda_{61}\right),\\
\tilde{\bar{T}}^{24} &= \frac{i}{2}\left(\lambda_{62}-\lambda_{34}\right), \qquad \tilde{\bar{T}}^{45}=\frac{i}{2}\left(\lambda_{58}-\lambda_{47}\right), \qquad \tilde{\bar{T}}^{15}=\frac{i}{2}\left(\lambda_{59}-\lambda_{44}\right),\\
\tilde{\bar{T}}^{25} &= -\frac{i}{2}\left(\lambda_{45}+\lambda_{60}\right), \qquad \tilde{\bar{T}}^{35}=\frac{i}{2}\left(\lambda_{46}+\lambda_{57}\right), \qquad \tilde{\bar{T}}^{16}=\frac{i}{2}\left(\lambda_{60}-\lambda_{45}\right),\\
\tilde{\bar{T}}^{26} &= \frac{i}{2}\left(\lambda_{44}+\lambda_{59}\right), \qquad \tilde{\bar{T}}^{36}=\frac{i}{2}\left(\lambda_{47}+\lambda_{58}\right), \qquad \tilde{\bar{T}}^{46}=\frac{i}{2}\left(\lambda_{46}-\lambda_{57}\right)
\end{aligned} \tag{4.3.4}$$

and

$$\begin{aligned}
\tilde{\bar{Y}}^{12} &= -\frac{1}{2}\left(\lambda_{27}-\lambda_{16}+\lambda_{40}-\lambda_{55}\right), \qquad \tilde{\bar{Y}}^{34}=-\frac{1}{2}\left(\lambda_{55}-\lambda_{16}+\lambda_{27}-\lambda_{40}\right),\\
\tilde{\bar{Y}}^{56} &= -\frac{1}{2}\left(\lambda_{55}-\lambda_{16}-\lambda_{27}+\lambda_{40}\right), \qquad \tilde{\bar{Y}}^{13}=-\frac{1}{2}\left(\lambda_{54}-\lambda_{19}+\lambda_{26}-\lambda_{43}\right),\\
\tilde{\bar{Y}}^{23} &= -\frac{1}{2}\left(\lambda_{53}-\lambda_{18}-\lambda_{25}+\lambda_{42}\right), \qquad \tilde{\bar{Y}}^{14}=\frac{1}{2}\left(\lambda_{18}+\lambda_{25}+\lambda_{42}+\lambda_{53}\right),\\
\tilde{\bar{Y}}^{24} &= -\frac{1}{2}\left(\lambda_{19}-\lambda_{26}-\lambda_{43}+\lambda_{54}\right), \qquad \tilde{\bar{Y}}^{45}=-\frac{1}{2}\left(\lambda_{50}-\lambda_{23}+\lambda_{30}-\lambda_{39}\right),\\
\tilde{\bar{Y}}^{15} &= -\frac{1}{2}\left(\lambda_{31}-\lambda_{20}-\lambda_{36}+\lambda_{51}\right), \qquad \tilde{\bar{Y}}^{25}=-\frac{1}{2}\left(\lambda_{21}+\lambda_{32}-\lambda_{37}-\lambda_{52}\right),\\
\tilde{\bar{Y}}^{35} &= -\frac{1}{2}\left(\lambda_{22}+\lambda_{29}+\lambda_{38}+\lambda_{49}\right), \qquad \tilde{\bar{Y}}^{16}=-\frac{1}{2}\left(\lambda_{21}-\lambda_{32}-\lambda_{37}+\lambda_{52}\right),\\
\tilde{\bar{Y}}^{26} &= -\frac{1}{2}\left(\lambda_{20}+\lambda_{31}+\lambda_{36}+\lambda_{51}\right), \qquad \tilde{\bar{Y}}^{36}=-\frac{1}{2}\left(\lambda_{50}-\lambda_{23}-\lambda_{30}+\lambda_{39}\right),\\
\tilde{\bar{Y}}^{46} &= -\frac{1}{2}\left(\lambda_{29}-\lambda_{22}+\lambda_{38}-\lambda_{49}\right).
\end{aligned} \tag{4.3.5}$$

The embedding tensor is in the same form as in $N = 5$ theory discussed in the previous chapter. However, the constraint on the embedding tensor $\mathbb{P}_{R_0}\Theta = 0$ requires $g_2 = 0$. This structure is similar to $N = 16, 10, 8$ theories [50, 57, 58]. Consequently, the embedding tensor reduces to $\Theta = g_1\Theta_{\text{ab}}$. The two singlets under $SO(6)_{\text{diag}}$ discussed above correspond to the non-compact generators

$$\begin{aligned} Y_{s1} &= \frac{1}{2}(Y^1 + Y^{11} + Y^{21} + Y^{31}), \\ Y_{s2} &= \frac{1}{2}(Y^2 + Y^{12} + Y^{22} + Y^{32}). \end{aligned} \tag{4.3.6}$$

In this case, the coset representative is given by

$$L = e^{\sqrt{2}b_1 Y_{s1}} e^{\sqrt{2}b_2 Y_{s2}} \tag{4.3.7}$$

where $\sqrt{2}$ is the normalization constant that is required so that the scalar potential contains no multiplicative constant. The scalar potential is found to be

$$V = -224g^2 \left(\cosh b_1 \cosh b_2 - \sinh b_2\right)^2 . \tag{4.3.8}$$

Note that we have only one coupling constant, so we redefine $g = g_1$. There is no critical point for this scalar potential, so the vacuum must be half-supersymmetric domain wall. The next and the last task is to find the domain wall solution corresponding to this theory.

The process in searching for domain wall solution is quite similar those used to obtain flows in chapter 5. The BPS equations for this problem are given by the vanishing of supersymmetry transformation of $\delta\psi^I_\mu$ and $\delta\chi^{iI}$ together with the traditional domain wall ansatz in chapter 5. We obtain a set of differential equations as follows

$$b_1' = 8g\text{sech}b_2 \sinh b_1, \tag{4.3.9}$$

$$b_2' = -8g\left(\cosh b_2 - \cosh b_1 \sinh b_2\right), \tag{4.3.10}$$

$$A' = -16g\left(\cosh b_1 \cosh b_2 - \sinh b_2\right) \tag{4.3.11}$$

where $'$ denotes $\frac{d}{dr}$. These equations are not easy to solve in the exact form, so simply consider subclass of general solution. We simply set $b_1 = 0$. The equation (4.3.10) now becomes

$$b_2' = -8ge^{-b_2} . \tag{4.3.12}$$

After integration, the solution is simply given by

$$b_2 = \ln\left(-8gr + c_1\right) \tag{4.3.13}$$

where c_1 is an integration constant. With $b_1 = 0$ and b_2 given by (4.3.13), equation (4.3.11) becomes

$$A' = \frac{-16g}{c_1 - 8gr} \tag{4.3.14}$$

whose solution is easily found to be

$$A = 2\ln\left(-8gr + c_1\right) + c_2 \tag{4.3.15}$$

with another integration constant c_2. As in the RG flow cases, the integration constants are not significant, since we can redefine the coordinates. We can write the metric in the form of a warped AdS_3 as

$$ds^2 = \frac{1}{(8g)^4\rho^2}\left(\frac{dx_{1,1}^2 + d\rho^2}{\rho^2}\right) \tag{4.3.16}$$

where $\rho = -\frac{1}{(8g)^2 r}$.

CHAPTER V

RG Flow Solutions for N=5 and N=6 Theory

In this section, we solve RG flow solutions interpolating between a pair of AdS critical points obtained from previous sections. We study only supersymmetric flows, so the critical points we picked must be supersymmetric. One of the critical points is trivial and preserves maximal supersymmetry, and another critical point is non-trivial and preserves some supersymmetry. Recall that gauged supergravity requires the existence of a scalar potential whose critical points may give rise to AdS background solution. According to AdS/CFT, the bulk theory with AdS space characterized by radius L is dual to conformal theory at boundary characterized by central charges c. If the conformal field theory is perturbed by an operator with a particular scale dimension, the theory will make a transition (in the space of coupling) to another fixed points called IR CFT, so we name the unperturbed theory UV CFT. Another scenario that can initiate the flow is that the operator acquires non-vanishing expectation value and from all flows we study, this case is not found. In dual picture, this process is illustrated by the domain wall solution of the bulk theory. The near boundary limit of the bulk field corresponds to UV CFT and the deep interior limit associates with IR CFT. We study four flows for both $N = 5$ and $N = 6$ theory

We solve not only analytical flow solutions but also identify the scale dimension of operator Δ that perturbed theory by expanding the scalar field around the critical point and comparing to asymptotic solution discussed in chapter 2. Moreover, we calculate the ratio of central charges c_{UV}/c_{IR} for every case and they are all larger than one as predicted by c-theorem. Our study of holographic RG flows are quite similar to those in other gauged supergravities in three dimensions [53, 94, 56, 55].

5.1 RG flow solutions for N=5 theory

We consider only RG flows in $k = 2$ case because the structure of critical points of $k = 4$ cases are quite similar. We do not study RG flow solutions for all possible cases since the methodology is the same. In this work, we explore RG flows in compact gauge group $SO(5) \times USp(2)$ and non-compact one with $USp(2, 2) \times USp(2)$. In order to investigate the flow, AdS_3 critical points from chapter 3 are required. Since we are interested in supersymmetric RG flows, each critical point must preserve some supersymmetry. Nonsupersymmtric critical points are automatically excluded from consideration. We study flow from only one active scalar denoted $b(r)$. This analysis is legitimate since BPS equations are still valid when some scalars are turned off.

5.1.1 An RG flow between $(5,0)$ and $(4,0)$ CFT's in $SO(5)\times USp(2)$ gauging

Recall that in this case the two critical points are $b=0$ and $b=\cosh^{-1}(\frac{-2g_1+g_2}{2g_1+g_2})$ corresponding to supersymmetry $(5,0)$ and $(4,0)$ respectively. Our study involves only one scalar and the potential depends on b associated with generator Y^7, so the coset representative can be given by

$$L = e^{b(r)Y^7}\,. \tag{5.1.1}$$

The vanishing of variation of fermions , i.e. $\delta\chi^{iI}=0$, give us a flow equation:

$$\frac{db}{dr} = [2g_1 - g_2 + (2g_1+g_2)\cosh b]\sinh b. \tag{5.1.2}$$

Because on the r.h.s. of the equation is a function of $b(r)$ only, so we can integrate for r. The solution becomes

$$\begin{aligned} r &= \frac{1}{8g_1g_2}\Big[4g_1\ln\cosh\frac{b}{2} - (2g_1+g_2)\ln[2g_1-g_2+(2g_1+g_2)\cosh b] \\ &\quad +2g_2\ln\sinh\frac{b}{2}\Big]\,. \end{aligned} \tag{5.1.3}$$

The integration constant is ignored; this does not affect the analysis here, because we can redefine a new r that differ from the old one by the constant.

By Taylor expanding around $b=0$, we obtain

$$b(r) \sim e^{4g_1 r} = e^{-\frac{r}{2L_{UV}}}\,,\ \ L_{UV} = \frac{1}{8|g_1|}. \tag{5.1.4}$$

For $g_1<0$, the solution is valid for $r\to\infty$, i.e. near boundary; therefore we can identify this point with UV point. From asymptotic analysis, we can extract the scale dimension of operator that drives the flow. We refer reader to chapter 2 for detailed discussion. Now we simply give you an important equation:

$$b(r) \sim c_1 e^{\frac{(\Delta-d)r}{L}} + c_2 e^{-\frac{\Delta r}{L}} \tag{5.1.5}$$

where

$$\Delta = \frac{1}{2}\left(d+\sqrt{d^2+4m^2L^2}\right). \tag{5.1.6}$$

c_2 is zero because the solution must be finite at large r. After we substitute $m^2L^2=-\frac{3}{4}$, we have $\Delta_{UV}=\frac{3}{2}$, so the operator is relevant. In the dual theory, the interpretation is that the flow is driven by relevant operator of dimension $\Delta_{UV} = \frac{3}{2}$. The other critical point can be identify with IR point, by Taylor expansion around that point we get

$$b(r) \sim e^{-\frac{8g_1 g_2 r}{2g_1+g_2}} = e^{\frac{g_2 r}{(g_1+g_2)L_{IR}}}, \quad L_{IR} = -\frac{2g_1+g_2}{8g_1(g_1+g_2)} > 0. \tag{5.1.7}$$

In order to make this solution valid for deep interior region, $r \to -\infty$, now we need additional constraint, i.e. $g_2 > -2g_1$. At IR point, c_2 in equation (5.1.5) is always 0. Consequently, Δ_{IR} is $\frac{3g_2+2g_1}{g_1+g_2}$ which is always greater than two; so it is irrelevant operator. The ratio of the central charges is

$$\frac{c_{UV}}{c_{IR}} = \frac{L_{UV}}{L_{IR}} = \sqrt{\frac{V_{0IR}}{V_{0UV}}} = \frac{g_1+g_2}{2g_1+g_2} > 1 \tag{5.1.8}$$

for $g_1 < 0$ and $g_2 > -2g_1$. The equation above satisfies the holographic c-theorem: $c_{UV} > c_{IR}$.

The vanishing of variation of gravitini , i.e. $\delta\psi^I_\mu = 0$, give us another flow equation solving for metric part $A(r)$:

$$\begin{aligned}\frac{dA}{dr} &= \frac{1}{4}[4g_2\cosh b - 22g_1 - 3g_2 - 8g_1\cosh b \\ &\quad -2g_1\cosh(2b) - g_2\cosh(2b)]\end{aligned} \tag{5.1.9}$$

Applying chain rule, it becomes

$$\frac{dA}{db} = -\frac{[22g_1 + 3g_2 + (8g_1 - 4g_2)\cosh b + (2g_1+g_2)\cosh(2b)]\,\mathrm{csch} b}{8g_1 - 4g_2 + 4(2g_1+g_2)\cosh b}. \tag{5.1.10}$$

This equation is easily solved by merely integrating; as mentioned before we neglect integration constant. It reads

$$\begin{aligned}A &= \frac{1}{g_2}\Bigg[(g_1+g_2)\ln[2g_1 - g_2 + (2g_1+g_2)\cosh b] - (2g_1+g_2)\ln\cosh\frac{b}{2} \\ &\quad -2g_2\ln\sinh\frac{b}{2}\Bigg].\end{aligned} \tag{5.1.11}$$

5.1.2 An RG flow between $(5,0)$ and $(1,0)$ CFT's in $SO(5)\times USp(2)$ gauging

In this case, we study the other RG flow in $SO(5)\times USp(2)$ gauging which interpolating between critical points $b = 0$ and $b = \cosh^{-1}(\frac{6g_1+g_2}{2g_1+g_2})$ corresponding to residual supersymmetry $(5,0)$ and $(1,0)$. With the analysis previously discussed, we obtain an RG flow equation:

$$\frac{db}{dr} = [6g_1 + g_2 - (2g_1+g_2)\cosh b]\sinh b \tag{5.1.12}$$

Following the same strategy as in the previous section, we obtain the solution

$$\begin{aligned} r &= -\frac{1}{8g_1(4g_1+g_2)}\left[4g_1 \ln\cosh\frac{b}{2} + (2g_1+g_2)\ln\left[(2g_1+g_2)\cosh b\right.\right. \\ &\quad \left.\left. -6g_1 - g_2\right] - 2(4g_1+g_2)\ln\sinh\frac{b}{2}\right]. \end{aligned} \tag{5.1.13}$$

The fluctuation around $b = 0$, UV point, it reads

$$b(r) \sim e^{4g_1 r} = e^{-\frac{r}{2L_{UV}}}, L_{UV} = \frac{1}{8|g_1|} \tag{5.1.14}$$

which is equal to $(5,0)$ to $(4,0)$ case. It is a UV point for $g_1 < 0$ when r approaching the boundary. The flow is driven by a relevant operator with scale dimension $\Delta_{UV} = \frac{3}{2}$. Near the other critical point which is identified as an IR, $b(r)$ is approximately

$$b(r) \sim e^{-\frac{8g_1(4g_1+g_2)r}{2g_1+g_2}} = e^{\frac{(4g_1+g_2)r}{(3g_1+g_2)L_{IR}}}, \;\; L_{IR} = -\frac{2g_1+g_2}{8g_1(3g_1+g_2)}. \tag{5.1.15}$$

Note that $g_1 < 0$ and $g_2 < -2g_1$. The constraint on g_2 is not the same as in the previous case. The operator has scale dimension $\Delta_{IR} = \frac{10g_1+3g_2}{3g_1+g_2}$. The ratio of the central charges is

$$\frac{c_{UV}}{c_{IR}} = \frac{3g_1+g_2}{2g_1+g_2} > 1, \qquad \text{for} \qquad g_1 < 0 \text{ and } g_2 < -2g_1\,. \tag{5.1.16}$$

The equation above satisfies the holographic c-theorem: $c_{UV} > c_{IR}$.

The differential equation for $A(r)$ in this case is

$$\frac{dA}{dr} = \frac{1}{4}\left[3g_2 - 10g_1 - 4(6g_1+g_2)\cosh b + (2g_1+g_2)\cosh(2b)\right]. \tag{5.1.17}$$

The technique applied to the case at hand is analogous to the previous one, now the equation becomes

$$\frac{dA}{db} = \frac{\left[10g_1 - 3g_2 + 4(6g_1+g_2)\cosh b - (2g_1+g_2)\cosh(2b)\right]\operatorname{csch} b}{4(2g_1+g_2)\cosh b - 4(6g_1+g_2)} \tag{5.1.18}$$

It is easily solved by *Mathematica.* The solution is

$$\begin{aligned} A &= \frac{1}{4g_1+g_2}\left[(3g_1+g_2)\ln\left((2g_1+g_2)\cosh b - 6g_1 - g_2\right)\right. \\ &\quad \left. -(2g_1+g_2)\ln\cosh\frac{b}{2} - 2(4g_1+g_2)\ln\sinh\frac{b}{2}\right]. \end{aligned} \tag{5.1.19}$$

5.1.3 An RG flow between $(4,1)$ and $(4,0)$ CFT's in $USp(2)\times USp(2,2)$ gauging

In the last section, we present two examples of the RG flow in compact gauging case. Now we turn our attention to the non-compact case; we choose $USp(2)\times USp(2,2)$ gauging case. Recall that in this case we have found four critical points. For breivity, we identify each with their residual supersymmetry: (4,1),(4,0),(1,0) and (0,0). Note that (4,1) is the trivial one and (0,0) is non-supersymmetric. Because we focus on supersymmetric flows, the nonsupersymmetric one is ruled out. The flow between (4,0) and (1,0) is also ruled out because the killing spinor generating both cases cannot be the same. Therefore, we consider two cases of flows between (4,1) and (4,0) and between (4,1) and (1,0).

The scalar potential in this case has two scalar fields, but all critical points we have found are of the form $(b_1, b_2 = 0)$. So, in the analysis of the flows, we need only one scalar. b_2 is set to zero and we replace b_1 to b for the sake of convenience. The coset representative is given by

$$L = e^{b(r)Y_7} \tag{5.1.20}$$

for both cases. The strategies are analogous to the previous two cases, so from now on we merely present the results and discuss about some constraints that may occur.

In this section, we start with studying flow interpolating between $b = 0$ to $b = \cosh^{-1}(\frac{-g_1+g_2}{g_1+g_2})$. From $\delta\chi^{iI} = 0$, we get a differential equation

$$\frac{db}{dr} = (g_1 - g_2 + (g_1+g_2)\cosh b)\sinh b \tag{5.1.21}$$

where the solution is

$$\begin{aligned} r &= \frac{1}{4g_1g_2}\Bigg[2g_2\ln\sinh\frac{b}{2} + 2g_1\ln\cosh\frac{b}{2} \\ &\qquad - (g_1+g_2)\ln\left[g_1 - g_2 + (g_1+g_2)\cosh b\right]\Bigg]. \end{aligned} \tag{5.1.22}$$

The fluctuation around $b = 0$ in the limit $r \to \infty$ is

$$b(r) \sim e^{2g_1 r} = e^{\frac{g_1 r}{(g_1+g_2)L_{UV}}}, \quad L_{UV} = \frac{1}{2(g_1+g_2)} \tag{5.1.23}$$

where $g_1 < 0$ and $g_2 > -g_1$. We obtain the relevant operator with dimension $\frac{3g_1+2g_2}{g_1+g_2} < 2$. The other critical point can be identify with IR point, by Taylor expansion around that point we get

$$b(r) \sim e^{-\frac{4g_1g_2r}{g_1+g_2}} = e^{\frac{2g_2r}{|2g_1+g2|L_{IR}}}, \quad L_{IR} = \frac{g_1+g_2}{2|g_1(2g_1+g_2)|}. \tag{5.1.24}$$

In IR the operator is irrelevant with dimension $\Delta = \frac{2g_2}{|2g_1+g2|} + 2$. The ratio of the central charge is given by

$$\frac{c_{UV}}{c_{IR}} = \frac{|g_1(2g_1+g_2)|}{(g_1+g_2)^2}. \tag{5.1.25}$$

The equation from $\delta\psi^I_\mu = 0$ reads

$$\frac{dA}{dr} = -2\left[g_2 + g_1\cosh^4\frac{b}{2} + g_2\sinh^4\frac{b}{2}\right]. \tag{5.1.26}$$

From chain rule, A as a function of b is

$$\frac{dA}{db} = -\frac{2\,\mathrm{csch}b\left(g_2 + g_1\cosh^4\frac{b}{2} + g_2\sinh^4\frac{b}{2}\right)}{g_1 - g_2 + (g_1+g_2)\cosh b}. \tag{5.1.27}$$

The solution is easily found to be

$$\begin{aligned} A &= \frac{1}{2g_1}\left[(2g_1+g_2)\ln\left[g_1 - g_2 + (g_1+g_2)\cosh b\right] - 4g_1\ln\cosh\frac{b}{2}\right. \\ &\quad \left. -2(g_1+g_2)\ln\sinh\frac{b}{2}\right]. \end{aligned} \tag{5.1.28}$$

5.1.4 An RG flow between $(4,1)$ and $(1,0)$ CFT's in $USp(2)\times USp(2,2)$ gauging

In this last case, we investigate flow interpolating between $b = 0$ with (4,1) supersymmetry to $b = \cosh^{-1}(\frac{-g_1-3g_2}{g_1+g_2})$ with $(1,0)$ supersymmetry critical point. The equation $\delta\chi^{iI} = 0$ leads to a differential equation:

$$\frac{db}{dr} = -\left[g_1 + 3g_2 + (g_1+g_2)\cosh b\right]\sinh b, \tag{5.1.29}$$

The corresponding solution takes the following form

$$\begin{aligned} r &= -\frac{1}{4g_2(g_1+2g_2)}\left[(g_1+g_2)\ln\left[g_1 + 3g_2 + (g_1+g_2)\cosh b\right]\right. \\ &\quad \left. +2g_2\ln\sinh\frac{b}{2} - 2(g_1+2g_2)\ln\cosh\frac{b}{2}\right], \end{aligned} \tag{5.1.30}$$

The fluctuation around $b = 0$ at $r \to \infty$ is

$$b(r) \sim e^{-2(g_1+2g_2)r} = e^{\frac{(g_1+2g_2)r}{(g_1+g_2)L_{UV}}}, \qquad L_{UV} = -\frac{1}{2(g_1+g_2)}, \tag{5.1.31}$$

where $g_1 < 0$ and $-\frac{g_1}{2} < g_2 < -g_1$ or $g_1 + g_2 < 0$ which are different from the previous case. We obtain the relevant operator with dimension $\frac{g_1}{g_1+g_2}$. The flow is driven by the relevant operator with scale dimension $\Delta = \frac{3g_1+4g_2}{g_1+g_2}$. The other critical point can be identified with IR point and by Taylor expansion around that point the solution becomes

$$b(r) \sim e^{-\frac{4g_2(g_1+2g_2)r}{g_1+g_2}} = e^{\frac{2g_2(g_1+2g_2)r}{|g_1(2g_1+3g2)|L_{IR}}}, \qquad L_{IR} = -\frac{(g_1+g_2)}{2|g_1(2g_1+3g_2)|}. \qquad (5.1.32)$$

In the IR, the operator becomes irrelevant with scale dimension $\Delta = \frac{2g_2}{|2g_1+g_2|} + 2$. The ratio of the central charge is given by

$$\frac{c_{UV}}{c_{IR}} = \frac{|g_1(2g_1+3g_2)|}{(g_1+g_2)^2}. \qquad (5.1.33)$$

The vanishing of the gravitini leads to a flow equation

$$\frac{dA}{dr} = \frac{1}{4}\left[3g_1 - 5g_2 + 4(g_1+3g_2)\cosh b + (g_1+g_2)\cosh(2b)\right]. \qquad (5.1.34)$$

Using chain rule, the equation now reads

$$\frac{dA}{db} = -\frac{[3g_1 - 5g_2 + 4(g_1+3g_2)\cosh b + (g_1+g_2)\cosh(2b)]\,\text{csch} b}{-4(6g_1+g_2) + 4(2g_1+g_2)\cosh b}. \qquad (5.1.35)$$

The solution is the following

$$\begin{aligned} A &= \frac{1}{2(g_1+2g_2)}\Big[(2g_1+3g_2)\ln\left[g_1+3g_2+(g_1+g_2)\cosh b\right] \\ &\quad -4(g_1+2g_2)\ln\cosh\frac{b}{2} - 2(g_1+g_2)\ln\sinh\frac{b}{2}\Big]. \end{aligned} \qquad (5.1.36)$$

5.2 RG flow solutions for $N = 6$ theory

In this section, we study RG flows for $N = 6$ theories. The methodology is similar to $N = 5$ case, so we may neglect some steps and simply present the final results. We study only the flow of $k = 4$ in compact gauge group; they are $SO(6) \times SU(4) \times U(1)$ and $SO(4) \times SO(2) \times SU(4) \times U(1)$. In each gauge group, we consider two flows from different supersymmetric critical points obtained from chapter 4. As a results, we study four cases in total. We give the full list below:

$SO(6) \times SU(4) \times U(1)$ case

- RG flow between (6,0) and (4,0) points
- RG flow between (6,0) and (2,0) points

$SO(4) \times SO(2) \times SU(4) \times U(1)$ case

- RG flow between (4,2) and (2,2) points

- RG flow between (4,2) and (2,0) points

Note that the flows studied here are constructed from two identical scalars which are different from $N = 5$ cases. We are looking for supersymmetric RG flows that interpolate between two critical points with $(b_1, b_2) = (0, 0)$ and $(b_1, b_2) = (b, b)$ in which the two scalar fields b_1 and b_2 take the same value. According the argument on the ansatz in the beginning of this chapter, the scalar $b(r)$ depends solely on radial coordinate. The coset representative of the form

$$L = e^{b(r)Y_3}e^{b(r)Y_{15}} = e^{b(r)(Y_3+Y_{15})} \tag{5.2.1}$$

where all the Y^A's generators are non-compact generators for $N = 6$ case. Note that the expression is derived by using $[Y_3, Y_{15}] = 0$.

In the $SO(6) \times SU(4) \times U(1)$ case, there are two singlet which are $Y_3 + Y_{15}$ itself and $Y_6 + Y_{16}$ under residual symmetry $SO(4) \times SU(2) \times U(1) \times U(1)$. The coset representative above with Y_6+Y_{16} sector truncated is verified as a consistent truncation. For $SO(4)\times SO(2)\times SU(4)\times U(1)$ gauge group, there are four singlet under residual symmetry $SU(2) \times U(1)^4$ which are

$$Y_3 + Y_{15}, \qquad Y_3 - Y_{15}, \qquad Y_4 + Y_{16}, \qquad Y_4 - Y_{16}\,.$$

As in the previous case, the consistent truncation is checked.

5.2.1 RG Flows in $SO(6) \times SU(4) \times U(1)$ gauging

We explore the RG flows for the theory with $SO(6)\times SU(4)\times U(1)$ gauge group. Its critical points and other properties are discussed in the previous section. In order to complete the analysis on RG flows, the constraint on the relative coupling α have to be specified. It is divided into two classes either $\alpha < -1$ or $\alpha > -1$. The trivial critical point with supersymmetry $(6, 0)$ and $SO(6)\times SU(4)\times U(1)$ residual gauge symmetry is identified with UV point. The critical points with supersymmetry $(4, 0)$ and $(2, 0)$ points with $SU(2)^3\times U(1)^2$ residual gauge symmetry are identified with IR points.

An RG flow between $(6, 0)$ and $(4, 0)$ critical points

The vanishing of the fermions $\delta\chi^{iI} = 0$ gives rise to a differential equation as follows

$$\frac{db(r)}{dr} = \sqrt{2}g\left[1 - \alpha + (1+\alpha)\cosh(\sqrt{2}b(r))\right]\sinh(\sqrt{2}b(r)). \tag{5.2.2}$$

Note that in this case $\alpha < -1$. The differential equation above can easily be solved by integration. The result is given by

$$\begin{aligned} r &= \frac{1}{4g\alpha}\ln\left[\cosh\frac{b}{\sqrt{2}}\right] - \frac{1+\alpha}{8g\alpha}\ln\left[1-\alpha+(1+\alpha)\cosh(\sqrt{2}b)\right] \\ &\quad + \frac{1}{4g}\ln\left[\sinh\frac{b}{\sqrt{2}}\right]. \end{aligned} \tag{5.2.3}$$

As mentioned in the $N = 5$ case, the additive constant from integration can be neglected without any problem since we can always redefine a new radial coordinate which is different from the original r by a constant.

The solution near $b = 0$ is

$$b(r) \sim e^{4gr} \sim e^{-\frac{r}{2L_{\rm UV}}}, \qquad L_{\rm UV} = \frac{1}{8|g|}\,. \tag{5.2.4}$$

We identify this critical point with the UV point in which $r \to \infty$, so there must be $g < 0$. The scale dimension of the operator that drives the flow can be determined by asymptotic analysis. From the same analysis we applied for $N = 5$ case, we conclude that this flow is driven by relevant operator with dimension $\Delta = \frac{3}{2}$.

We identify another point $\frac{1}{\sqrt{2}} \cosh^{-1} \frac{\alpha-1}{\alpha+1}$ with IR. By expanding around this point, we get an asymptotic equation

$$b(r) \sim e^{-\frac{8g\alpha r}{1+\alpha}} = e^{\frac{2\alpha}{(1+2\alpha)}\frac{r}{L_{\rm IR}}}, \qquad L_{\rm IR} = \frac{1+\alpha}{4|g|(1+2\alpha)}\,. \tag{5.2.5}$$

In this IR, the operator is irrelevant with scale dimension $\Delta = \frac{2(1+3\alpha)}{1+2\alpha} > 2$ for $\alpha < -1$. The ratio of the central charge is given by

$$\frac{c_{\rm UV}}{c_{\rm IR}} = \frac{L_{\rm UV}}{L_{\rm IR}} = \sqrt{\frac{V_{0\rm IR}}{V_{0\rm UV}}} = \frac{1+2\alpha}{2(1+\alpha)} > 1, \qquad \text{for } \alpha < -1\,. \tag{5.2.6}$$

The next task is determining the metric of the domain wall ansatz characterized by $A(r)$. The vanishing of the gravitini, $\delta\psi^I_\mu = 0$, can provide us the solution and it reads

$$\begin{aligned}\frac{dA(r)}{dr} \;=\; & -g\Big[5+\alpha-(\alpha-1)\cosh(\sqrt{2}b(r)) \\ & +\cosh(\sqrt{2}b(r))\Big\{1-\alpha+(1+\alpha)\cosh(\sqrt{2}b(r))\Big\}\Big]\,. \end{aligned} \tag{5.2.7}$$

Using chain rule, we can derive above equation as follows

$$\frac{dA(b)}{db} = -\frac{5+\alpha-(\alpha-1)\cosh(\sqrt{2}b)+\cosh(\sqrt{2}b)(1-\alpha+(1+\alpha)\cosh(\sqrt{2}b))}{\sqrt{2}\left[1-\alpha+(1+\alpha)\cosh(\sqrt{2}b)\right]\sinh(\sqrt{2}b)}\,. \tag{5.2.8}$$

According to the reason we have given earlier, the additive constant are neglected.

If the scalar is canonically normalized, we must get the correct dimension of the operator by simply reading mass squared from the scalar potential at quadratic order. In our case, to check whether $\sqrt{2}b(r)$ is canonically normalized we simply read off the value of the mass squared from the the scalar potential at quadratic order. At UV point, the potential (4.1.13) expanded up to quadratic order is given by

$$V = -64g^2 - 48g^2b^2. \tag{5.2.9}$$

This potential gives $m^2L_{\text{UV}}^2 = -\frac{3}{4}$ and according to the relation between mass and scale, it provides the correct scale dimension at UV $\Delta = \frac{3}{2}$ obtained from asymptotic analysis of $b(r)$. In the same way as UV case, we obtain the correct scale dimension $\Delta = \frac{2(1+3\alpha)}{1+2\alpha}$ and mass $m^2L_{\text{IR}}^2 = \frac{4\alpha(1+3\alpha)}{(1+2\alpha)^2}$ from from quadratic order of the scalar potential

$$V = -\frac{16g^2(1+2\alpha)^2}{(1+\alpha)^2} + \frac{64g^2\alpha(1+3\alpha)}{(1+\alpha)^2}(b-b_0)^2 \tag{5.2.10}$$

where $b_0 = \frac{1}{\sqrt{2}}\cosh^{-1}\frac{\alpha-1}{\alpha+1}$.

An RG flow between $(6,0)$ and $(2,0)$ critical points

In this case, we study flow between the same UV point as the previous case to another supersymmetri critical point (2,0). The strategy is analogous to precedent case. Note that this case is valid only if $\alpha > -1$ which is distinct from the $(6,0)$ and $(4,0)$ flow analysis.

The scalar field $b(r)$ can be obtained via the BPS equation $\delta\chi^{iI} = 0$ as follows

$$\frac{db(r)}{dr} = -\sqrt{2}g\left[-3-\alpha+(1+\alpha)\cosh(\sqrt{2}b(r))\right]\sinh(\sqrt{2}b(r)). \tag{5.2.11}$$

We can solve the equation above and get the undesired form r as a function of b:

$$\frac{db(r)}{dr} = -\sqrt{2}g\left[-3-\alpha+(1+\alpha)\cosh(\sqrt{2}b(r))\right]\sinh(\sqrt{2}b(r)). \tag{5.2.12}$$

Asymptotic expansion near the UV point $b = 0$, the scalar field becomes

$$b(r) \sim e^{4gr} = e^{-\frac{r}{2L_{\text{UV}}}}, \qquad L_{\text{UV}} = \frac{1}{8|g|}. \tag{5.2.13}$$

The equation above is justified with a requirement $g < 0$ when $r \to \infty$. After matching parameters with the general solution, we conclude that it is a relevant operator with dimension $\Delta = \frac{3}{2}$ that drives the flow.

In the vicinity of IR point $b = \frac{1}{\sqrt{2}}\cosh^{-1}\frac{\alpha+3}{\alpha+1}$, the solution is approximately

$$b(r) \sim e^{-\frac{8g(2+\alpha)r}{1+\alpha}} = e^{\frac{2(2+\alpha)r}{(3+2\alpha)L_{\text{IR}}}}, \qquad L_{\text{IR}} = \frac{1+\alpha}{4|g|(3+2\alpha)}. \tag{5.2.14}$$

This leads to the conclusion that the operator has dimension $\Delta = \frac{2(5+3\alpha)}{3+2\alpha} > 2$, for $\alpha > -1$.

The geometrical quantity $A(r)$ can be derived from equation $\delta\psi_\mu^I = 0$ where its solution is given by

$$\begin{aligned}\frac{dA(r)}{dr} &= g\Big[\alpha-3-(3+\alpha)\cosh(\sqrt{2}b(r)) \\ &\qquad +\cosh(\sqrt{2}b(r))\{-3-\alpha+(1+\alpha)\cosh(\sqrt{2}b(r))\}\Big]\end{aligned} \tag{5.2.15}$$

Having utilized chain rule, we can rearrange the more desired differential equation as follows

$$\frac{dA}{db} = -\frac{\alpha - 3 - (3+\alpha)\cosh(\sqrt{2}b) + \cosh(\sqrt{2}b)\left[-\alpha - 3 + (1+\alpha)\cosh(\sqrt{2}b)\right]}{\sqrt{2}\left[-3-\alpha+(1+\alpha)\cosh(\sqrt{2}b)\right]\sinh(\sqrt{2}b)}\,. \tag{5.2.16}$$

After integration and discarding integration constant, $A(r)$ now becomes

$$\begin{aligned} A &= -\frac{1+\alpha}{2+\alpha}\ln\left[2\cosh\frac{b}{\sqrt{2}}\right] + \frac{3+2\alpha}{2(2+\alpha)}\ln\left[-3-\alpha+(1+\alpha)\cosh(\sqrt{2}b)\right] \\ &\quad -2\ln\left[2\sinh\frac{b}{\sqrt{2}}\right]. \end{aligned} \tag{5.2.17}$$

The holographic c-theorem can be checked by considering the ratio

$$\frac{c_{\rm UV}}{c_{\rm IR}} = \frac{3+2\alpha}{2(1+\alpha)} > 1, \qquad \text{for} \qquad \alpha > -1\,, \tag{5.2.18}$$

so it perfectly concurs with the c-theorem.

5.2.2 RG Flows in $SO(4)\times SO(2)\times SU(4)\times U(1)$ gauging

We dedicate the rest of this chapter to RG Flows in $SO(4)\times SO(2)\times SU(4)\times U(1)$ gauging. Two RG flows of this gauging are studied. They are the flow between trivial critical point to the other two non-trivial ones with supersymmetry (2,2) and (2,0). Recall that the trivial critical point is located at $b_1 = b_2 = 0$ has $(4,2)$ supersymmetry. The two non-trivial critical points are given by $b_1 = b_2 = \frac{1}{\sqrt{2}}\cosh^{-1}\frac{2+\alpha}{\alpha}$ with $(2,2)$ supersymmetry and $b_1 = b_2 = \frac{1}{\sqrt{2}}\cosh^{-1}\frac{\alpha-2}{\alpha}$ with $(2,0)$ supersymmetry. The methodology is the same as in the previous cases, so some detailed steps are omitted and only key results are presented.

An RG flow between $(4,2)$ and $(2,2)$ critical points

In $(4,2)$ critical points the left-handed preserved supercharges is outnumbered the right, but for $(2,2)$ case they are equal. We are dealing with the flow between a chiral supersymmetric theory to a non-chiral one. A constraint for this flow is $\alpha > 0$. The equation $\delta\chi^{iI} = 0$ gives us the flow equation

$$\frac{db(r)}{dr} = -\sqrt{2}g\left[2+\alpha-\alpha\cosh(\sqrt{2}b(r))\right]\sinh(\sqrt{2}b(r)). \tag{5.2.19}$$

Solving for r by integration, we found

$$\begin{aligned} r &= \frac{1}{4g(1+\alpha)}\ln\left[\cosh\frac{b}{\sqrt{2}}\right] + \frac{\alpha}{8g(1+\alpha)}\ln\left[\alpha\cosh(\sqrt{2}b) - \alpha - 2\right] \\ &\quad -\frac{1}{4g}\ln\left[\sinh\frac{b}{\sqrt{2}}\right]. \end{aligned} \tag{5.2.20}$$

At the UV point, for validity of the solution, we pick $g > 0$. The asymptotic solution for scalars now becomes

$$b(r) \sim e^{-4gr} = e^{-\frac{r}{2L_{\rm UV}}}, \qquad L_{\rm UV} = \frac{1}{8g}. \tag{5.2.21}$$

As commonly found in many cases, the flow is driven by the relevant operator of dimension $\frac{3}{2}$.

At the IR point, the solution is given by

$$b(r) \sim e^{\frac{8g(1+\alpha)r}{\alpha}} = e^{\frac{2(1+\alpha)r}{(1+2\alpha)L_{\rm IR}}}, \qquad L_{\rm IR} = \frac{\alpha}{4g(1+2\alpha)}. \tag{5.2.22}$$

The dimension of this operator is given by $\Delta = \frac{2(2+3\alpha)}{1+2\alpha}$ which is larger than two when $\alpha > 0$.

The flow solution from $\delta\psi^I_\mu = 0$ equation is given by

$$\frac{dA}{dr} = -\frac{1}{2}g\left[8 - 3\alpha + 4(2+\alpha)\cosh(\sqrt{2}b(r)) - \alpha\cosh(2\sqrt{2}b(r))\right] \tag{5.2.23}$$

written in term of b as

$$\frac{dA(b)}{db} = -\frac{8 - 3\alpha + 4(2+\alpha)\cosh(\sqrt{2}b(r)) - \alpha\cosh(2\sqrt{2}b(r))}{2\sqrt{2}\left[2+\alpha-\alpha\cosh(\sqrt{2}b)\right]\sinh(\sqrt{2}b)}. \tag{5.2.24}$$

Its solution is given by

$$\begin{aligned} A &= \frac{\alpha}{1+\alpha}\ln\left(2\cosh\frac{b}{\sqrt{2}}\right) - \frac{1+2\alpha}{2(1+\alpha)}\ln\left[\alpha\cosh(\sqrt{2}b) - \alpha - 2\right] \\ &\quad +2\ln\left(2\sinh\frac{b}{\sqrt{2}}\right). \end{aligned} \tag{5.2.25}$$

The ratio of the central charges is given by

$$\frac{c_{\rm UV}}{c_{\rm IR}} = \frac{1+2\alpha}{2\alpha} > 1, \qquad \text{for} \qquad \alpha > 0. \tag{5.2.26}$$

An RG flow between $(4,2)$ and $(2,0)$ critical points

We are now at the end of our journey on RG flow. This last example is the flow between chiral supersymmetric theories. The equation $\delta\chi^{iI} = 0$ gives us the flow equation

$$\frac{db(r)}{dr} = -\sqrt{2}g\left[\alpha - 2 - \alpha\cosh(\sqrt{2}b(r))\right]\sinh(\sqrt{2}b(r)) \tag{5.2.27}$$

After integration the solution is found to be

$$\begin{aligned} r &= \frac{1}{4g(\alpha-1)}\ln\left(\cosh\frac{b}{\sqrt{2}}\right) - \frac{\alpha}{8g(\alpha-1)}\ln\left[2 - \alpha + \alpha\cosh(\sqrt{2}b)\right] \\ &\quad +\frac{1}{4g}\ln\left(\sinh\frac{b}{\sqrt{2}}\right). \end{aligned} \tag{5.2.28}$$

At the UV point, we have found the asymptotic solution

$$b(r) \sim e^{4gr} = e^{-\frac{r}{2L_{\rm UV}}}, \qquad L_{\rm UV} = \frac{1}{8|g|} \tag{5.2.29}$$

The constraint $g < 0$ must satisfy in order to identify this point with UV point corresponding to $r \to \infty$. The flow is again driven by a relevant operator of dimension $\frac{3}{2}$.

At the IR point, $b = \frac{1}{\sqrt{2}} \cosh \frac{\alpha-2}{\alpha}$ corresponding to $r \to -\infty$, we have found

$$b(r) \sim e^{-\frac{8g(\alpha-1)r}{\alpha}} = e^{\frac{(2\alpha-1)r}{(2\alpha-1)L_{\rm IR}}}, \qquad L_{\rm IR} = \frac{\alpha}{4|g|(2\alpha-1)}\,. \tag{5.2.30}$$

The operator becomes irrelevant in the IR with dimension $\Delta = \frac{2(2-3\alpha)}{1-2\alpha} > 2$ for $\alpha < 0$.

The $A(r)$ function obtained from $\delta\psi^I_\mu = 0$ equation is as follows

$$\frac{dA(r)}{dr} = -\frac{1}{2} g \left[8 + 3\alpha - 4(\alpha-2)\cosh(\sqrt{2}b(r)) + \alpha(2\sqrt{2}b(r))\right] \tag{5.2.31}$$

Applying chain rule, it becomes

$$\frac{dA(b)}{db} = \frac{8 + 3\alpha - 4(\alpha-2)\cosh(\sqrt{2}b(r)) + \alpha(2\sqrt{2}b(r))}{2\sqrt{2}\left[\alpha - 2 - \alpha\cosh(\sqrt{2}b)\right]\sinh(\sqrt{2}b)}\,. \tag{5.2.32}$$

The solution is found to be

$$\begin{aligned} A \;=\; & \frac{\alpha}{\alpha-1} \ln\left[2\cosh\frac{b}{\sqrt{2}}\right] + \frac{1-2\alpha}{2(\alpha-1)} \ln\left[2 - \alpha + \alpha\cosh(\sqrt{2}b)\right] \\ & -2\ln\left(2\sinh\frac{b}{\sqrt{2}}\right). \end{aligned} \tag{5.2.33}$$

The ratio of the central charges is given by

$$\frac{c_{\rm UV}}{c_{\rm IR}} = \frac{2\alpha-1}{2\alpha} > 1, \qquad \text{for} \qquad \alpha < 0\,. \tag{5.2.34}$$

It perfectly satisfies c-theorem.

CHAPTER VI

Conclusion

In this dissertation, we have studied $N = 5$ and $N = 6$ gauged supergravity in three dimensions and the applications thereof. The scalar manifolds are in the form of $\frac{USp(4,k)}{USp(4)\times USp(k)}$ and $\frac{SU(4,k)}{S(U(4)\times U(k))}$ where k is an integer for $N = 6$ and an even integer for $N = 5$. We limit ourselves to study only small number in k; for $N = 5$ we study $k = 2, 4$ and for $N = 6$ we study $k = 1, 2, 3, 4$. The larger number in k, the larger the manifold and the more complex the problem. Moreover, to the best of our knowledge, studying the large k may not lead to any interesting features.

The gauge groups we studied are classified as the followings: compact, non-compact, and non-semisimple gaugings. For compact gauge groups, the embedding tensors have been identified in [37]. On the other hand, for non-compact gauge groups, we identified some of which via the criteria on constraint on T-tensors. Non-semisimple gaugings deserve a special attention since they are directly connected to higher dimensional theory via dimensional reduction on an orbifold. For non-semisimple gauging, we study the gauge groups in the form of $G_0 \ltimes \mathbf{T}^{\dim G_0}$ where G_0 is a semisimple group and $\mathbf{T}^{\dim G_0}$ is translational group with dimension of $\dim G_0$. For $N = 5$ Chern-Simons gauging, the gauge group is $SO(5) \ltimes \mathbf{T}^{10}$ and it is equivalent to $SO(5)$ Yang-Mills gauged supergravity. The theory with this gauge group also can be obtained by dimensional reduction of $N = 5$ gauged supergravity in four dimensions on the orbifold $S^1/\mathbb{Z}_2$. With identifying $g_2 = -g_1$, we obtain maximally supersymmetric critical point at origin. In the dual theory, it corresponds to a two dimensional superconformal theory with supergroup $Osp(5|2, \mathbb{R}) \times Sp(2, \mathbb{R})$. For $N = 6$ gauged supergravity theory, it is a theory with non-semisimple gauge group $SO(6) \ltimes \mathbf{T}^{15}$. It is equivalent to $SO(6)$ Yang-Mills gauged supergravity and can be obtained from dimensional reduction of four dimensional $N = 6$ gauged supergravity theory on the orbifold $S^1/\mathbb{Z}_2$. There is no critical points for the resulting scalar potential, so maximally supersymmetric AdS_3 background does not exist. However, the half-maximal domain wall solution is admitted, so it is useful in the context of DW/QFT.

We focus on the study of the critical points of the scalar potential. In order to obtain the scalar potential we first need to parametrize the coset manifold. There are two ways to parametrization used in this dissertation. The first one is the traditional unitary gauge; it is easy for setup but it put a heavy load on computer while running. In Euler angle parametrization, on the other hand, the setup is much more complicated but it consumes less computer resources. Another reason why the Euler angle parametrization is preferred to the others is that the resulting scalar potential is in a very compact form due to gauge invariant and this parametrization has some sectors that overlap with the gauge group.

We have found many supersymmetric AdS vacua which are indispensable for study RG flows. Having identified critical points, we explore further some of their properties. They include the residual supersymmetry and residual gauge symmetry at each critical point. The mass spectra at both trivial and non-trivial critical points are also determined for $N = 5$ theory. For $N = 6$ theory, we present the mass spectra only at trivial critical points.

In application to AdS/CFT, holographic RG flows are studied. In order to study the RG flow, one of the basic requirement is that the gauging in question must have at least two supersymmetric critical points, so the gaugings that have only trivial critical points are not considered here. For $N = 5$ theory, we pick two cases from compact gaugings and the other two from non-compact cases. For $N = 6$ theory, all four cases are in compact gaugings since non-compact gaugings do not admit non-trivial critical points. RG flow solutions are analytically obtained via solving BPS equations. All flows we study for both $N = 5$ and $N = 6$ theories, at UV point, are driven by relevant operator with dimension $\frac{3}{2}$ while at IR the operators become irrelevant with dimensions varied among cases. We obtain $c_{UV}/c_{IR} > 1$ for every case, so it is in agreement with the c-theorem. In 2011 the new method for obtaining critical points has been suggested in the study of $N = 8$ gauged supergravity in four dimensions [96]. By using the variation of embedding tensor instead of extremization on the scalar potential, some new critical points are found. We hope that this technique can be useful in searching for more critical points from many previous works, since it might overcome the complication of computation within the framework of extremization of multivariables function.

Although the construction gauged supergravity was firmly established in 2003 and the applications thereof have been sprung since then; nowadays, its construction still is in favor of the theoretical physicists. Recently, a sector of $N = 4$ gauged supergravity is constructed via dimensional reduction from $N = (1, 0)$ in six dimensional supergravity coupled to a chiral tensor multiplet on $AdS_3 \times S^3$ [97]. It relates to the previous work done by our colleague [98] on $SU(2)$ manifold reduction of $N = 1$ supergravity in six dimensions. The authors also point out that it is possible to embed $N = 8$ gauged supergravity in three dimension to $N = (1, 1)$ and $N = (2, 0)$ theories in six dimensions.

The research on solution of supergravity in three dimensions have made some progress as well. In [99], all timelike supersymmetric solutions of $N = 8$ ungauged supergravity in three dimensions are classified. This paper elaborated the work in [100] published by the same authors. They also pointed out that the success on this work can extend to $N = 16$ case. We hopefully extend the series of this work to the other Ns as well as the gauged version thereof.

Recently, only three weeks before the time of writing, a major breakthrough in AdS/CFT correspondence was made. A persistent unsolved problem in AdS/ CFT is unraveled. The problem of whether the truncation of type IIB supergravity to maximal supergravity in five dimensions is consistent was simply a conjecture without a solid proof, even though many evidences supported the claim. All the evidences were explored only some sectors by setting some fields to vanish or postulating some symmetries on them. In [101], they show that, with the help of exceptional field theory, the full non-linear reduction ansatz for the $AdS_5 \times S^5$ and $AdS_5 \times H^{p,6-p}$ in type IIB supergravity can reduce to the maximal supergravity in five dimensions with gauge groups $SO(6)$ and $SO(p, q)$, respectively. And more importantly the resulting theory is a consistent truncation. Note that the corresponding exceptional group is $E_{6(6)}$, since it is global symmetry group for maximal supergravity in five dimensions. The key idea of exceptional field theory is that spactime is extended to exceptional groups and various fields are covariant with respect to those groups. For exceptional group $E_{6(6)}$ [102], those fields are of M-theory and Type IIB supergravity. The exceptional group $E_{7(7)}$ case which can be useful in studying the reduction to four dimensional supergravity can be found in [103]. Analogous to the previous cases, $E_{8(8)}$ [104] hopefully can be applied to supergravity in three dimensions. The consistent Kaluza-Klien truncations of those exceptional field theories are outlined in [105]. This research might open a new door to abundant of new research on consistent truncation including those theories in involving in supergravity in three dimensions.

References

[1] G. 't Hooft, *A Planar Diagram Theory For Strong Interactions*, Nucl. Phys. B72 (1974) 461.

[2] J. M. Maldacena, *The large N limit of superconformal Field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231-252, arXiv: hepth/ 9711200.

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253, arXiv: hep-th/9802150.

[4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. 323 (2000) 183–386, arXiv: hep-th/9905111.

[5] Eric D'Hoker and Daniel Z. Freedman, *Supersymmetric Gauge Theories and the AdS/CFT Correspondence*, TASI 2001 Lecture Notes, arXiv: hep-th/ 0201253.

[6] P. Karndumri, *On Hologrphic RG Flows*, Ph. D. Thesis, Theoretical Particle Physics, Scuola Internazionale Superiore di Studi Avanzati (SISSA), 2011.

[7] Daniel Z. Freedman and Antoine Van Proeyen, *Supergravity*, New York: Cambridge University Press, 2012.

[8] Horatiu Nastase, *Introduction to AdS/CFT*, arXiv: hep-th/0712.0689

[9] Clifford V. Johnson, *D-Branes*, Cambridge: Cambridge University Press, 2003.

[10] Alberto Zaffaroni, *Introduction to the AdS/CFT Correspondence*, 2009.

[11] Johanna Erdmenger, *Introduction to Gauge Gravity Duality*, 2010.

[12] M. Günaydin, L. J. Romans and N. P. Warner, *Gauged $N = 5$ Supergravity in Five Dimensions*, Phys. Lett. 154B (1985) 268.

[13] M. Günaydin, L. J. Romans and N. P. Warner, *Compact and Non-Compact Gauged Supergravity Theories in Five Dimensions*, Nucl. Phys. B272 (1986) 598.

[14] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged $N = 8$ $D = 5$ Supergravity*, Nucl. Phys. B259 (1985) 460.

[15] B. de Wit, H. Nicolai, *The Consistency of the S^7 Truncation in $d = 11$ Supergravity*, Nucl. Phys. B281 (1987) 211.

[16] Nicholas P. Warner, *Some New Extrema of the Scalar Potential of Gauged $N = 8$ Supergravity*, Phys. Lett. 128B (1983) 169.

[17] Alexei Khavaev, Krzysztof Pilch and Nicholas P. Warner, *New Vacua of Gauged $N = 8$ Supergravity in Five Dimensions*, Physics Letters B 487 (2000) 14-24 , arXiv: hep-th/9812035.

[18] L.Girardello, M.Petrini, M.Porrati and A. ZaKaroni, *Novel Local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, JHEP 12 (1998) 022, arXiv: hep-th/9810126.

[19] D.Z. Freedman, S. Gubser, N. Warner and K. Pilch, *Renormalization Group Flows from Holography-Supersymmetry and a c-Theorem*, Adv. Theor. Math. Phys. 3 (1999), arXiv: hep-th/9904017.

[20] J. Distler and F. Zamora ,*Non-supersymmetric Conformal Field Theories from Stable Anti-de Sitter Spaces*, Adv. Theor. Math. Phys. 2 1405 (1999), arXiv: hep-th/9810206.

[21] Nicholas P. Warner, *Renormalization Group Flows from Five-Dimensional Supergravity*, Class. Quantum Grav. 17 1287 (2000), arXiv: hep-th/9911240.

[22] Alexei Khavaev and Nicholas P. Warner, *A Class of $N = 1$ Supersymmetric RG Flows from Five-dimensional $N = 8$ Supergravity*, Phys. Lett. 495 (2000) 215-222, arXiv: hep-th/0009159.

[23] A. Staruszkiewicz, *Gravitation theory in three-dimensional space*, Acta Physica Polonica 6 (1963) 735-740.

[24] S. Deser, R. Jackiw and G. 't Hooft, *Three-dimensional Einstein gravity: Dynamics of flat space*, Ann. Phys. 152 (1984) 220

[25] S. Deser and R. Jackiw, *Three-Dimensional Cosmological Gravity: Dynamics Of Constant Curvature*, Annals Phys. 153 (1984) 405.

[26] G. 't Hooft, *Cosmology in 2+1 Dimensions*, Nucl. Phys. B30 (Proc. Suppl.) (1993) 200-203.

[27] E. Witten, *2 + 1 Dimensional Gravity as an Exactly Soluble System*, Nucl. Phys. B311 (1988) 46-78.

[28] S. Carlip. *Quantum Gravity in 2+1 Dimensions*, Cambridge: Cambridge Unversity Press, 1998.

[29] J. D. Brown and Marc Henneaux, *Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity*, Commun. Math. Phys. 104 (1986) 207.

[30] M. Banados, K. Bautier, O. Coussaert, M. Henneaux and M. Ortiz, *Anti-de Sitter/CFT correspondence in three-dimensional supergravity*, Phys. Rev. D58 (1998) 085020, arXiv: hep-th/9805165.

[31] M. Henneaux, L. Maoz and A. Schwimmer, *Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity*, Annals Phys. 282 (2000) 31, arXiv: hep-th/9910013.

[32] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on* AdS_3, Adv. Theor. Math. Phys. 2 (1998) 733, arXiv: hep-th/9806194

[33] D. Kutasov and N. Seiberg, *More comments on string theory on* AdS_3, JHEP 9904 (1999) 008, arXiv: hep-th/9903219.

[34] H. Nicolai and H. Samtleben, *Kaluza-Klein supergravity on* $AdS_3 \times S^3$, JHEP 0309 (2003) 036, arXiv: hep-th/0306202.

[35] N. S. Deger, *Renormalization group flows from D = 3, N = 2 matter coupled gauged supergravities*, JHEP 0211 (2002) 025, arXiv: hep-th/0209188.

[36] B. de Wit, A. K. Tollstén, and H. Nicolai, *Locally supersymmetric* $D = 3$ *nonlinear sigma models*, Nucl. Phys. B392 (1993) 3-38, arXiv: hep-th/9208074.

[37] B. de Wit, I. Herger and H. Samtleben, *Gauged Locally Supersymmetric* $D = 3$ *Nonlinear Sigma Models*, Nucl. Phys. B671 (2003) 175-216, arXiv: hep-th/0307006.

[38] B. de Wit, B. and H. Nicolai and H. Samtleben, *Gauged supergravities in three-dimensions: A Panoramic overview*, PoS. jhw(2003) 016, arXiv:hep-th/0403014.

[39] A. Chatrabhuti, P. Karndumri and B. Ngamwatthanakul, *New N=5,6, 3D gauged supergravities and holography*, JHEP 1401 (2014) 159, arXiv:hep-th/1312.4275.

[40] A. Chatrabhuti, P. Karndumri and B. Ngamwatthanakul, *3D N=6 Gauged Supergravity: Admissible Gauge Groups,Vacua and RG Flows*, JHEP 07 (2012) 057, arXiv: hep-th/1202.1043.

[41] P. Di Vecchia and S. Ferrara, *Classical Solutions in Two-dimensional Supersymmetric Field Theory*, Nucl. Phys. B130 (1977) 93-104.

[42] B. Zumino, *Supersymmetric and Kähler Manifolds*, Physics Letters , 5 November 1979, Volume 87B, number 3 ,203-206.

[43] Luis Alvarez-Gaume and Daniel Z. Freedman, *Gemetrical Structure and Ultraviolet Finiteness in Supersymmetric* σ*-model*, Commun. Math. Phys. 80 (1981) ,443-451.

[44] Jonathan Bagger and Edward Witten, *The Gauge Invariant Supersymmetric Nonlinear Sigma model*, Physics Letter, 2 December 1982, Voume 118B, number 1,2,3, 103-106.

[45] N. Marcus and J. H. Schwarz, *Three-dimensional supergravity theories* , Nucl. Phys. B228 (1983) 145-162.

[46] A. Achúcarro and P.K. Townsend, *A Chern-Simons action for three-dimensional anti-de sitter supergravity theories*, Phys. Lett. 180B (1986) 89-92.

[47] S. Deser and J. H. Kay, *Topologically massive supergravity*, Phys. Lett. 120B (1983) 97-100.

[48] H. Nicolai and H. Samtleben, *Maximal gauged supergravity in three dimensions*, Phys. Rev. Lett. 86 (2001) 1686-1689, arXiv: hep-th/0010076.

[49] H. Nicolai and H. Samtleben, *Compact and noncompact gauged maximal supergravities in three dimensions*, JHEP 04 (2001) 022, arXiv: hep-th/ 0103032.

[50] T. Fischbacher, H. Nicolai and H. Samtleben, *Non-semisimple and Complex Gaugings of $N = 16$ Supergravity*, Commun.Math.Phys. 249 (2004) 475-496, arXiv: hep-th/0306276.

[51] H. Nicolai and H. Samtleben, *$N = 8$ matter coupled AdS_3 supergravities*, Phys. Lett. B514 (2001) 165-172, arXiv: hep-th/0106153.

[52] T. Fischbacher, H. Nicolai and H. Samtleben, *Vacua of Maximal Gauged $D = 3$ Supergravities*, Class.Quant.Grav. 19 (2002) 5297-5334, arXiv: hep-th//0207206.

[53] M. Berg and H. Samtleben, *An exact holographic RG Flow between 2d Conformal Field Theories*, JHEP 05 (2002) 006, arXiv: hep-th/0112154.

[54] A. Chatrabhuti and P. Karndumri, *Vacua of $N = 10$ three dimensional gauged supergravity*, Class.Quant.Grav.28:125027,2011, arXiv: 1011.5355.

[55] A. Chatrabhuti and P. Karndumri, Vacua and *RG flows in $N = 9$ three dimensional gauged supergravity*, JHEP 10 (2010) 098, arXiv: 1007.5438.

[56] E. Gava, P. Karndumri and K. S. Narain, *AdS_3 Vacua and RG Flows in Three Dimensional Gauged Supergravities*, JHEP 04 (2010) 117, arXiv: 1002.3760.

[57] P. Karndumri, *$\frac{1}{2}$-BPS Domain wall from $N = 10$ three dimensional gauged supergravity*, JHEP 11 (2013) 023, arXiv: 1307.6641.

[58] P. Karndumri, *Domain walls in three dimensional gauged supergravity*, JHEP 10 (2012) 001, arXiv: 1207.1027.

[59] G. W. Gibbons, C. M. Hull and N. P. Warner, *The Stability of Gauged Supergraavity* , Nucl. Phys. B218 (1983) 173-190.

[60] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc., 234 (1953), 428-443

[61] S. Okubo. *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Cambridge:Cambridge University Press, 1995.

[62] S. Helgason. *Differential Geometry and Symmetric Spaces*, Massachusetts: Academic Press, 1962.

[63] A. L. Besse. *Einstein Manifold*, New York: Springer, 2008.

[64] J. Simons, *On the Transitivity of Holonomy Systems*, The Annals of Mathematics 76 (1962) 213-234.

[65] M. Berger, *Sur les groupes d'holonomie homogè des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France, 83 (1955), 279-330.

[66] H. Nicolai and H. Samtleben, *Chern-Simons vs Yang-Mills gaugings in three dimensions*, Nucl. Phys. B638 (2002) 207-219 , arXiv: hep-th/0303213.

[67] B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and Gaugings of Maximal Supergravities*, , arXiv: hep-th/0212239.

[68] B. de Wit, *The Maximal D=5 Supergravities*, , arXiv: hep-th/0412173.

[69] S. Chern and J. Simons, *Characteristic Forms and Geometric Invariants*, Annals of Mathematics, Second Series, Vol. 99. No1 (Jan 1974), pp. 48-69.

[70] M. Günaydin, G. Sierra, P. K. Townsend, *The Unitary Supermultiplets of d=3 Anti-de Sitter and d=2 Conformal Superalgebras*, Nucl. Phys. B274 (1986) 429-447.

[71] N. P. Warner, *Some New Extrema of the Scalar Potential of Gauged $N = 8$ Supergravity*, Phys. Lett. B128 (1983) 169.

[72] N. P. Warner, *Some Properties of the Scalar Potential in Gauged Supergravity Theories*, Nucl. Phys. B231 (1984) 250-268.

[73] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, JHEP 9807 (1998) 023, arXiv: hep-th/9806087.

[74] S. Deser, M. J. Duff and C. J. Isham, *Non-local Comformal Anomalies*, Nucl. Phys. B111, 45 (1976)

[75] M. J. Duff, *Twenty years of the Weyl anomaly*, Class. Quant. Grav. 11, 1387 (1994), arXiv: hep-th/9308075.

[76] A. Polyakov, *Quantum Geometry of Bosonic Strings*, Phys. Lett. 103B, 207 (1981)

[77] A. Polyakov, *Quantum Geometry of Fermionic Strings*, Phys. Lett. 103B, 211 (1981)

[78] A. Cappelli, D. Friedan and J. I. Latorre, *C Theorem And Spectral Representation*, Nucl. Phys. B352, 616 (1991).

[79] C. Fefferman and C. Robin Graham, *Conformal Invariants*, Elie Cartan et les Mathématiques d'aujourd'hui (Astérisque, 1985) 95.

[80] D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, Nucl. Phys. B526, 543 (1998), arXiv: hep-th/9708042.

[81] A. Cappelli, R. Guida and N. Magnoli, *Exact consequences of the trace anomaly in four dimensions*, Nucl. Phys. B618, 371 (2001), arXiv: hep-th/0103237.

[82] J. L. Cardy, *Is There A C Theorem In Four-Dimensions?*, Phys. Lett. B 215, 749 (1988).

[83] Z. Komargodski and A. Schwimmer, *On Renormalization Group Flows in Four Dimensions*, JHEP 1112, 099 (2011), arXiv: 1107.3987.

[84] S. Forte and J. I. Latorre, *A proof of the irreversibility of renormalization group flows in four dimensions*, Nucl. Phys. B535, 709 (1998), arXiv: hep-th/9805015.

[85] D. Anselmi, J. Erlich, D. Z. Freedman and A. A. Johansen, *Positivity constraints on anomalies in supersymmetric gauge theories*, Phys. Rev. D 57, 7570 (1998), arXiv: hep-th/9711035.

[86] I. I. Kogan, M. A. Shifman and A. I. Vainshtein, *Matching conditions and duality in N=1 SUSY gauge theories in the conformal window*, Phys. Rev. D53, 4526 (1996), arXiv: hep-th/9507170.

[87] M. Cvetic, H. Lu and C. N. Pope, *Brane-world Kaluza-Klein reductions and branes on the brane*, J. Math. Phys. 42, 3048 (2001), arXiv: hep-th/0009183.

[88] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, *Diffeomorphisms and Holographic Anomalies*, Class. Quant. Grav. 17 (2000) 1129-1138, arXiv: hep-th/9910267.

[89] A. Karch, D. Lust and A. Miemiec, *New $N = 1$ superconformal field theories and their supergravity description*, Phys. Lett. B 454, 265 (1999), arXiv: hep-th/9901041.

[90] A. B. Zamolodchikov, *'Irreversibility' Of The Flux Of The Renormalization Group In A 2D Field Theory*, JETP Lett. 43, 730 (1986)

[91] R. S. Halbersma. *Geometry of Strings and Branes*, Ph.D. thesis, University of Groningen, 2002.

[92] P. Breitenlohner and D. Z. Freedman, *Stability In Gauged Extended Supergravity*, Annals Phys. 144 (1982) 249.

[93] M. Pernici, K. Pilch, P. van Nieuwenhuizen and N. Warner, *Nonconpact Gaugings and Critical Points of Maximal Supergravity in Seven-dimensions*, Nucl. Phys. B249, 381 (1985)

[94] P. Karndumri, *Gaugings of $N = 4$ three dimensional gauged supergravity with exceptional coset manifolds*, JHEP 08 (2012) 007, arXiv: 1206.2150.

[95] O. Hohm and H. Samtleben, *Effective actions for massive Kaluza-Klein states on $AdS_3 \times S^3 \times S^3$*, JHEP 05 (2005) 027, arXiv: hep-th/0503088.

[96] G. Dall' Agata and G. Inverso, *On the vacua of $N = 8$ gauged supergravity in 4 dimensions*, arXiv: 1112.3345.

[97] N. S. Deger, H. Samtleben, Özgür Sarıoğlu and D. Van den Bleeken, *A Supersymmetric reduction on three sphere*, arXiv: 1410.7168v2.

[98] Edi Gava, Parinya Karndumri and K. S. Narain, *3D gauged supergravity from SU(2) reduction of $N = 1$ 6D supergravity*, JHEP 09 (2010) 028, arXiv: 1006.4997.

[99] N. S. Deger, G. Moutsopoulos, H. Samtleben, and Özgür Sarıoğlu, *All timelike supersymmetric solutions of three-dimensional half-maximal supergravity*, arXiv: 1503.09146v2.

[100] N. S. Deger, H. Samtleben, and O. Sarıoğlu, *On the supersymmetric solutions of half-maximal supergravities*, Nucl. Phys. B840 (2010) 29–53, arXiv: 1003.3119.

[101] A. Baguet, O. Hohm and H. Samtleben, *Consistent Type IIB Reductions to Maximal 5D Supergravity*, MIT-CTP/4670, arXiv: 1506.01385.

[102] O. Hohm and H. Samtleben, *Exceptional field theory I: $E_{6(6)}$ covariant form of M-theory and type IIB*, Phys.Rev. D89 (2014) 066016, arXiv:1312.0614.

[103] O. Hohm and H. Samtleben, *Exceptional field theory II: $E_{7(7)}$*, Phys.Rev. D89 (2014) 066017, arXiv: 1312.4542.

[104] O. Hohm and H. Samtleben, *Exceptional field theory III: $E_{8(8)}$*, Phys.Rev. D90 (2014) 066002, arXiv: 1406.3348.

[105] O. Hohm and H. Samtleben, *Consistent Kaluza-Klein truncations via exceptional field theory*, JHEP 1501 (2015) 131, arXiv: 1410.8145.

[106] J. Wess and J. Bagger. *Supersymmetry and Supergravity*, second edition, New Jersey: Princeton University Press, 1992.

[107] S. Weinberg. *The Quantum Theory of Fields*, Volume 3, Cambridge: Cambridge University Press, 2005.

[108] L. Castellani, R. D'Auria and P. Fré. *Supergravity and Superstrings*, Volume 1,2, Singapore: World Scientific, 1991.

[109] D. Bailin and A. Love *Supersymmetric Gauge Field Theory an String Theory*, Bristol: Institute of Physics Publishing, 1994.

[110] Prem P. Srivastava, *Supersymmetry, Superfields and Supergravity: an Intriduction*, Bristol: IOP Publishing, 1986.

[111] Harald J. W. Müller-Kirsten and Armin Wiedemann, *Introduction to Supersymmetry*, second edition, Singapore: World Scientific, 2010.

[112] H. Samtleben, *Introduction to Supergravity*, 2007.

[113] H. Samtleben, *Lectures on Gauged Supergravity and Flux Conpactification*, 2008.

[114] Y. Tanii, *Introduction to Supergravities in Diverse Dimensions*, arXiv: hep-th/9802138.

[115] B. de Wit, *Supergravity*, arXiv: hep-th/0212245.

[116] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rept. 79 (1981) 1.

[117] S. Bertini, S. L. Cacciatori, B. L. Cerchiai, *On the Euler angles for $SU(N)$*, J. Math. Phys. 47 (2006) 043510, arXiv: math-ph/0510075.

[118] Sergio L. Cacciatori and B. L. Cerchiai, *Exceptional groups, symmetric spaces and applications*, arXiv: 0906.0121.

APPENDICES

APPENDIX A

Supersymmetry and Supergravity

In this appendix, we briefly review Supersymmetry and Supergravity in four dimensions. It is intended to be sketchy and conceptual, so many technical details are omitted. The excellent reviews are as follows [7, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115].

It is well-known that Supersymmetry is the only non-trivial way to combine space-time symmetry with internal symmetries of S-matrix. It unites particles with various spins into sets called supermultiplets. Bosons and fermions are mingled in such a way that every particle has its superpartner, a particle with the same mass but opposite spin statistic. Supersymmetry necessitates the additional generators to the conventional Lorentz group. The spacetime algebra is now extended and we call the new generators supercharges. These supercharges make senses only when they are fermionic and the algebra for spacetime symmetry is now graded Lie algebra rather than Lie algebra. Besides the mathematical affectionate reason mentioned above, there are some physical reasons that cherish the existence of supersymmetry. It is well known that supersymmetry can tame the divergence coming from loop diagrams in quantum field theory. It is naturally done by cancelation of fermionic loops to bosonic loops and vice versa. If supersymmetry exists and is unbroken, there must be superpartners of those known particles with the same mass but different spin. However, a sensible reason of why we do not observe those particles in nature because supersymmetry might be broken and renders the superpartners very massive. To detect those massive particles, it requires a powerful particle collider. As the time of the writing, no direct evidences support the claim of the existence of supersymmetry in nature even in the promising LHC.

We follow the mainstream in the discussion of Poincaré supersymmetry by starting with the algebra of the theory. The algebra is established by two classes of generators: bosonic (we label B) and fermionic (F). The supercharges Qs are fermionic while the others are bosonic. Schematically, the structure of the algebra are as follows $[B, B] \sim F$, $\{F, F\} \sim B$ and $[F, B] \sim F$. The irreducible representation of Q depends on spacetime in which they are constructed. For example, in four dimensions, the irreducible representation can be either Majorana or Weyl. The $N = 1$ supersymmetry algebra involving Q given below will be presented in

Majorana representation:

$$\{Q_\alpha, \bar{Q}^\beta\} = -\frac{1}{2}(\gamma_\mu)_\alpha{}^\beta P^\mu , \tag{A.0.1}$$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta , \tag{A.0.2}$$

$$[P_\mu, Q_\alpha] = 0 . \tag{A.0.3}$$

A one-particle representation is represented by $|p, \lambda\rangle$ where p is momentum and λ is helicity. A supercharge acting on a state leads to the helicity reduced by 1/2, i.e. $Q_\alpha|p, \lambda\rangle = |p, \lambda - 1/2\rangle$, so Q_α changes boson to fermion or vice versa. The operator P^2 has m^2 as an eigenvalue, since $[P_\mu, Q_\alpha] = 0$ bosons and fermions in the same multiplet must have the same mass.

The algebra alone is not enough to construct a theory. A supersymmetric theory is established by a field theory that realizes supersymmetry. Under supersymmetry, a generic field $\Phi(x)$ transforms as follows

$$\delta\Phi(x) = [\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)] , \tag{A.0.4}$$

where ϵ is a fermionic parameter of supersymmetry transformation which does not depend on spacetime, so sometimes we may called global supersymmetry as opposed to local supersymmetry in which the parameter becomes spacetime dependent $\epsilon(x)$. The local supersymmetry is synonymous to supergravity as we will explained more later. Supersymmetry can be realized by considering commutation relation of variations of a generic field. The variations δ_1, δ_2 correspond to parameters ϵ_1, ϵ_2, respectively. The commutation reads

$$\begin{aligned} [\delta_1, \delta_2]\,\Phi(x) &= \delta_1\,\delta_2\,\Phi(x) - \delta_2\,\delta_1\,\Phi(x) \\ &= \bar{\epsilon}_1^\alpha\,[\{Q_\alpha, \bar{Q}^\beta\}, \Phi(x)]\,\epsilon_{2\beta} \\ &= -\frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\,\partial_\mu\Phi(x) , \end{aligned} \tag{A.0.5}$$

where we replace momentum generator P_μ with a partial derivative ∂_μ and $(\bar{\epsilon}_1\gamma^\mu\epsilon_2) = a^\mu$ is a constant parameter of translation. We can say that the transformation above realizes supersymmetry. However, in general, this is not the case. If the supersymmetry of a particular theory is realized without requiring equations of motion, we say that the algebra of the theory is closed off-shell. If equations of motion are required, we say that the algebra of the theory is closed on-shell.

Supersymmetry can have more than one set of supercharge; we denote the addition degrees of freedom by indices $i, j = 1, ..., N$, so the supercharges now become $Q_{i\alpha}$. Combining with ordinary Poincaré algebra and defining relations among supercharged, we call them extended superalgebra. An operation that rotates among supercharges and commutes with Poincaré generators is called R-symmetry. Mathematically speaking, R-symmetry is an automorphism of supersymmetry algebra. We denote them by generators T_As and those generators can form a group . The commutation relation with supercharges are as follows

$$[T_A, Q_{\alpha i}] = (U_A)_i{}^j\, Q_{\alpha j} , \tag{A.0.6}$$

where $(U_A)_i{}^j$ is an element of matrix U which is a particular representation of T_A as can be shown by Jacobi identity. Note that R-symmetry is an optional symmetry of the theory which may not be realized as symmetry of the action. However in AdS superalgebra, we can add R-symmetry group with other generators to form a single supergroup. This scenario cannot happen in other algebra such as dS.

Recall that supersymmetry parameters is spacetime independent, if we promote those parameters to become spacetime dependent

$$(\bar{\epsilon}_1\gamma^\mu\epsilon_2)(x) = a^\mu(x)\,, \tag{A.0.7}$$

so we get a spactime dependent of translation parameter or infinitesimal transformation of general coordinate transformation on spacetime (diffeomorphism) . This new transformation embraces diffeomorphism as a new invariance, so does Einstein's general relativity. As a result, we have supersymmetry with theory of gravity and we call it supergravity. Sometimes it is also referred to as local supersymmetry due to spacetime dependent of supersymmetry parameters.

There are many ways to construct a supergravity theory. We give a brief review on the four widely used methods. The first method is called the Noether method which is the most wildly used. This method starts from linearised theory of terms in the action and transformation rules and then modifies them systematically until reach the final form of the theory. The second is superspace formalism which is a generalization of superspace in supersymmetry. It makes use of the supersymmetric version of vielbein called supervielbein along with spin connection defined in superspace. Unfortunately, this formalism is valid for a small number of theories. The third method is the conformal approach which is constructed from various superconformal multiplets and then the redundant fields are removed by gauge fixing to get the final form of supergravity with correct multiplets. The forth method is achieved by dimensional reduction. This method is very powerful in construction of supergravity from higher dimensional supergravity, since supergravity in higher dimensions are much more simple than those theories in the lower ones. The resulting theories may vary depending on the shape of compactified manifold. The trivial cases are the compactification on n-dimensional torus which gives rise to ungauged theories in the lower dimensional theories. For non-trivial case, if we do the dimensional reduction from 11-dimensional supergravity on 7-sphere, we get $N = 8$ $D = 4$ gauged supergravity with gauge group $SO(8)$.

Apart from those particles contained in global supersymmetric theory, supergravity introduces two new types of particles into the theory: particle spin-2 called graviton ($g_{\mu\nu}$ or $e_\mu{}^a$) which is responsible for gravitational interaction and spin-3/2 particle called gravitino (ψ_μ). In the language of gauge theory, supergravity is considered as a gauged global supersymmetry. Analogous to typical gauge theory, there must be gauge fields; in this case, it is gravitino as implying by its vector index. Moreover, since parameters of supersymmetry transformation are spinors as a result gravitino must be spinor as well.

In the discussion on supergravity below, we will employ the vielbein formalism instead of the traditional one since we are dealing with spinors which are spinorial representation associating with local Lorentz group which is obviously used in vielbein formalism. As an example, we will discuss the simplest form of supergravity, i.e. $D = 4$ $N = 1$ pure supergavity. Its Lagragian is composed of Einstein-Hilbert action (Palatini's action, to be exact), kinetic term of spin-3/2 field also known as Rarita-Schwinger term and the other modification that keeps the theory supersymmetric. We discuss only some important steps and some important features. Note that we will use the so-called the 1.5$^{\text{th}}$ order formalism. Recall that the first order formalism treats spin connection $\omega_{\mu ab}$ as an independent variable whereas the second order formalism is set up by treating spin connection as a function of vielbein and torsion tensors and then puts it back to the action, therefore, the action is very complicated in terms of vielbein and gravitino. As the name implies, the 1.5$^{\text{th}}$ order formalism is somewhere between those two. First, let's consider the action in the first order formalism:

$$\begin{aligned} S &= S_2 + S_{3/2} = \int d^4x\, \mathcal{L}[e, \omega, \psi_\mu]\,, \\ S_2 &= \frac{1}{2\kappa^2}\int d^4x\, e e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega)\,, \\ S_{3/2} &= -\frac{1}{2\kappa^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\sigma D_\nu \psi_\rho\,. \end{aligned} \tag{A.0.8}$$

Then we search for equation of motion of spin connection via $\delta S[e, \omega, \psi_\mu]/\delta\omega = 0$. The solution is given by

$$\begin{aligned} \hat{\omega}_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab}\,, \\ \omega_{\mu ab}(e) &= 2e^\nu_{[a}\partial_{[\mu} e_{\nu]b]} - e^\nu_{[a} e_{b]}{}^\sigma e_{\mu c}\partial_\nu e_\sigma{}^c\,, \\ K_{\mu\nu\rho} &= -\frac{1}{4}\left(\bar{\psi}_\mu\gamma_\rho\psi_\nu - \bar{\psi}_\nu\gamma_\mu\psi_\rho + \bar{\psi}_\rho\gamma_\nu\psi_\mu\right)\,. \end{aligned} \tag{A.0.9}$$

The variation of the action with evaluating at $\omega = \hat{\omega}$ can be written as

$$\delta S = \frac{\delta S}{\delta e}\bigg|_{\omega=\hat{\omega}} + \frac{\delta S}{\delta \psi}\bigg|_{\omega=\hat{\omega}} + \underbrace{\frac{\delta S}{\delta \omega}\bigg|_{\omega=\hat{\omega}}}_{=0}\left(\frac{\delta\omega}{\delta e}\delta e + \frac{\delta\omega}{\delta\psi}\delta\psi\right)\,. \tag{A.0.10}$$

In 1.5$^{\text{th}}$ order formalism, we can neglect the contribution from variation of spin connection, since it is identically zero by construction. Consequently, it reduces complications involving spin connection. The supersymmetry transformations are:

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\ \delta\psi_\mu &= D_\mu(\hat{\omega})\,\epsilon(x) \;=\; \partial_\mu\epsilon(x) + \frac{1}{4}\hat{\omega}_{\mu ab}\gamma^{ab}\epsilon(x)\,. \end{aligned} \tag{A.0.11}$$

Note that in the first order formalism, we need supersymmetric transformation of spin connection $\delta\omega$ as well. The commutation relation of two variations on vielbein is found to be

$$[\delta_1, \delta_2]\, e^a_\mu = \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) e^a_\mu\,, \tag{A.0.12}$$

where $\xi^a = 1/2\,\bar{\epsilon}_2\gamma^a\epsilon_1$ and its variation corresponds to infinitesimal of diffeomorphism which is as follows

$$\delta_\xi e^a_\mu = \xi^\rho\partial_\rho e^a_\mu + \partial_\mu\xi^\rho e^a_\rho\,. \tag{A.0.13}$$

The second term $\delta_{\hat{\lambda}} e^a_\mu = \xi^\rho\hat{\omega}_\rho{}^a{}_b e^b_\mu$ is infinitesimal of local Lorentz transformation with parameter $\hat{\lambda}_{ab} = \xi^\rho\omega_{\rho ab}$. The last term $\delta_{\hat{\epsilon}} e^a_\mu = \xi^\rho\bar{\psi}_\rho\gamma^a\psi_\mu$ is local supersymmetry transformation with parameter $\hat{\epsilon} = \xi^\rho\psi_\rho$. As a result, the commutation in (A.0.12) implies closure of local symmetries of the theory, so this theory, at this stage, is justified as a supergravity theory. However, the commutation on gravitino $[\delta_1,\delta_2]\psi_\mu$ requires equation of motion of ψ_μ itself to furnish the closure of algebra, so this theory is closed only on-shell. The off-shell theory that equations of motion are unnecessary for closure of the algebra requires more fields called auxiliary fields into the multiplet. The discussion on off-shell theory can be found in various literatures on supergravity.

In addition to graviton multiplet previously discussed, there are also other multiplets such as chiral multiplet (ϕ^i,χ^i) and vector multiplets (χ^A, A^A_μ) and both of which sometimes are referred as matter multiplets. The chiral multiplet contains two particles which are a spin-1/2 fermion and a complex scalar whereas the vector multiplet is composed of a vector and another spin-1/2 fermion. To construct more realistic supergravity theory, one can couple matter multiplets to graviton multiplet. In this discussion, we simply restrict ourselves to some important terms in bosonic sector. The kinetic term of scalar sector is in the form of non-linear sigma model with scalar manifold metric $G_{jk}(\phi^i)$. Another important term of this sector is the scalar potential which plays important role in studying RG flows this dissertation. The Lagrangian is given by

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi^i\,\partial_\nu\phi^k\,G_{jk}(\phi^i) - \sqrt{g}V(\phi^i)\,. \tag{A.0.14}$$

In vector sector, the important terms are the kinetic term and a topological term which does not depend on spacetime metric $g_{\mu\nu}$. Its Lagrangian is given by

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F^A_{\mu\nu}F^B_{\rho\sigma}\,\mathcal{I}_{AB}(\phi^i) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}F^B_{\rho\sigma}\,\mathcal{R}_{AB}(\phi^i)\,. \tag{A.0.15}$$

The vector fields at this stage are abelian, and scalars and fermions are not charged under these abelian groups. However, these vector fields can be extended to non-abelian gauge group with leaving supersymmetry unspoiled and scalars and fermions are now charged under this group. This theory is called gauged supergravity which is the central aspect of this dissertation. Note that the scalar dependent quantities $\mathcal{I}_{AB}(\phi^i)$, $\mathcal{R}_{AB}(\phi^i)$ and scalar potential $V(\phi^i)$ are restricted by supersymmetry as well as scalar manifold metric $G_{jk}(\phi^i)$, so supersymmetry restricts the shape of manifold. In this case, the scalar manifold is a Kähler manifold.

We have discussed theories with $N = 1$ supersymmetry so far and now we will briefly discuss theories with $N > 1$ which are referred as extended supergravity. Supersymmetry is very restrict; however, there are some arbitrariness that gives rise to variety of theories such as central charges and number of supersymmetry N. The more number of supersymmetry N, the more number of particles in the multiple. Of course, the simplest case is $N = 1$. For $D = 4$ dimensional spacetime, the maximal number of N is 8. The number of supersymmetry is bound to this number because we require that particle spin-2 or graviton is the highest spin. The situation is quite different for $D = 3$ case since it relies on purely mathematical argument. For $N = 2$ supergravity, there are three multiplets: supergravity, vector and hyper multiplets. The scalar manifold for hyper multiplets is quaternion Kähler manifold. The $N = 2$ supergravity is thoroughly discussed in [7] via the conformal approach. For $N = 4$ supergravity, there are two multiplets: supergravity and vector multiplet. The matter coupled theory can be constructed by a gravity multiplet and n vector multiplets. As a result, the scalar manifold is in the form of coset space:

$$\frac{SO(6,n)}{SO(6)\times SO(n)}\times\frac{SL(2)}{SO(2)}\,. \tag{A.0.16}$$

There are other theories with other N such as 5 and 6 but they are not widely discussed. Last but not least, it is the maximal (or $N = 8$) supergravity. It is a unique theory because there is only a gravity multiplet which contains a graviton, 8 gravitini, 28 vectors, 56 spin-1/2 fermions and 70 scalars. The scalar manifold is in the form of coset space:

$$\frac{E_{7(7)}}{SU(8)}\,. \tag{A.0.17}$$

This theory also can be constructed from dimensional reduction of 11-dimensional supergravity on 7-dimensional torus T^7. At this stage, the theory is an ungauged supergravity since all vectors are abelian and none of matter fields are charged. It was early 1980s that there was an attempt to promote $D = 4$ $N = 8$ supergravity into the theory of everything but later it was realized that maximal possible gauge group is $SO(8)$ which is impossible to embed gauge group of standard model.

There are many topics that we do not discuss here such as ungauged and gauged supergravity in higher dimensions, dimensional reduction and various applications thereof. We encourage enthusiastic readers to study from references given in the beginning of this appendix.

APPENDIX B

Branching of T-tensor for $N = 5, 6$

We devote this section to discuss the representation of T-tensor branching under maximal compact subgroup $H = SO(N) \times H'$. The representation of the embedding tensor coincides with the T-tensor at $\Theta = T_{\mathcal{V}=\mathrm{I}}$, so this section is crucial in specifying the embedding tensor. According to [37], they are given in Dynkin label; however, since we deal explicitly with $k = 2, 4$ for $N = 5$ theory and $k = 1, 2, 3, 4$ for $N = 6$ theory, we give them in conventional form with the help from [116].

B.1 $N = 5$ with $USp(4, k) \supset USp(4) \times USp(k)$

The adjoint representation of $G = USp(4, k)$, under $USp(4) \times USp(k)$, can be decomposed into

$$X^{IJ}: \ (\mathbf{10}\,,\ (0,0,0,\dots))\ , \quad X^{\alpha}: \ (\mathbf{1}\,,\ (2,0,0,\dots))\ , \quad Y^{A}: \ (\mathbf{4}\,,\ (1,0,0,\dots))\ .$$

The T-tensor representations are given by

$$\begin{aligned}
T^{IJ,KL} &= (\mathbf{1}+\mathbf{5}+\mathbf{14}\,,\ (0,0,0,\dots))\ ,\\
T^{\alpha\beta} &= (\mathbf{1}\,,\ (0,0,0,\dots)+(0,1,0,\dots)+(0,2,0,\dots))\ ,\\
T^{IJ\alpha} &= (\mathbf{10}\,,\ (2,0,0,\dots))\ ,\\
T^{AB} &= (\mathbf{1}+\mathbf{5}\,,\ (0,0,0,\dots)+(0,1,0,\dots))+(\mathbf{10}\,,\ (2,0,0,\dots))\ ,\\
T^{IJ,A} &= (\mathbf{4}+\mathbf{16}\,,\ (1,0,0,\dots))\ ,\\
T^{\alpha A} &= (\mathbf{4}\,,\ (1,0,0,\dots)+(1,1,0,\dots))\ .
\end{aligned} \tag{B.1.1}$$

Note that the two representations $(\mathbf{10},(2,0,0,\dots))$ in $T^{IJ\alpha}$ and T^{AB} coincide. Next we give them for each k starting with $k = 2$. In conventional notaion, for $k = 2$, the adjoint representation of $G = USp(4, 2)$, under $USp(4) \times USp(2) \simeq SO(5) \times SU(2)$, can be decomposed into $\mathbf{21} = (\mathbf{10},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{4},\mathbf{2})$. For $k = 4$, the adjoint representation of $G = USp(4, 4)$, under $USp(4) \times USp(4) \simeq SO(5) \times SU(2)$, can be decomposed into $\mathbf{21} = (\mathbf{10},\mathbf{1}) + (\mathbf{1},\mathbf{10}) + (\mathbf{4},\mathbf{4})$.

k	$\mathbf{SU(4,k)}$	$\mathbf{SU(4)\times SU(k)\times U(1)}$
1	$SU(4,1)$	$SU(4)\times U(1)$
	$\mathbf{24}$	$\mathbf{15}^0+\mathbf{1}^0+\mathbf{4}^{-5}+\overline{\mathbf{4}}^5$
2	$SU(4,2)$	$SU(4)\times SU(2)\times U(1)$
	$\mathbf{35}$	$(\mathbf{15},\mathbf{1})^0+(\mathbf{1},\mathbf{1})^0+(\mathbf{1},\mathbf{3})^0+(\mathbf{4},\mathbf{2})^{-3}+(\overline{\mathbf{4}},\mathbf{2})^3$
3	$SU(4,3)$	$SU(4)\times SU(3)\times U(1)$
	$\mathbf{48}$	$(\mathbf{15},\mathbf{1})^0+(\mathbf{1},\mathbf{1})^0+(\mathbf{1},\mathbf{8})^0+(\mathbf{4},\mathbf{3})^{-a}+(\overline{\mathbf{4}},\mathbf{3})^a$
4	$SU(4,4)$	$SU(4)\times SU(4)\times U(1)$
	$\mathbf{63}$	$(\mathbf{15},\mathbf{1})^0+(\mathbf{1},\mathbf{1})^0+(\mathbf{1},\mathbf{15})^0+(\mathbf{4},\mathbf{4})^{-b}+(\overline{\mathbf{4}},\mathbf{4})^b$

Table I: The decomposition of adjoint representation of $SU(4,k)$ under its maximal subgroup $SU(4)\times SU(k)$ for $k=1,2,3,4$ where the elements of decompositions correspond to an element of X^{IJ}, two elements of X^α and two elements of Y^A, respectively Note that for $k=1$ the maximal compact subgroup is $SU(4)\times U(1)$. a and b are some $U(1)$ charges which are not relevant in our analysis.

B.2 $N=6$ with $SU(4,k)\supset SU(4)\times SU(k)$

The adjoint representation of $G=SU(4,k)$, under $SU(4)\times SU(k)$, can be decomposed into

$$X^{IJ}: \quad (\mathbf{15}\,,\ (0,0,\ldots,0,0))\ , \qquad X^\alpha: \quad (\mathbf{1}\,,\ (0,0,\ldots,0,0)+(1,0,\ldots,0,1))\ ,$$
$$Y^A: \quad \big(\overline{\mathbf{4}}\,,\ (1,0,\ldots,0,0)\big)+(\mathbf{4}\,,\ (0,0,\ldots,0,1))\ .$$

The T-tensor representations are given by

$$\begin{aligned}
T^{IJ,KL} &= (\mathbf{1}+\mathbf{15}+\mathbf{20'}\,,\ (0,0,\ldots,0,0))\ ,\\
T^{\alpha\beta} &= (\mathbf{1}\,,\ 2\cdot(0,0,\ldots,0,0)+2\cdot(1,0,\ldots,0,1)+(0,1,\ldots,1,0))\ ,\\
T^{IJ\alpha} &= (\mathbf{15}\,,\ (0,0,\ldots,0,0)+(1,0,\ldots,0,1))\ ,\\
T^{AB} &= (\mathbf{1}+\mathbf{15}\,,\ (0,0,\ldots,0,0)+(1,0,\ldots,0,1))\\
&\quad +(\mathbf{6}\,,\ (0,1,\ldots,0,0)+(0,0,\ldots,1,0))\ ,\\
T^{IJ,A} &= \big(\overline{\mathbf{4}}+\mathbf{20}\,,\ (1,0,\ldots,0,0)\big)+\big(\mathbf{4}+\overline{\mathbf{20}}\,,\ (0,0,\ldots,0,1)\big)\ ,\\
T^{\alpha A} &= (\mathbf{4}\,,\ (1,0,\ldots,1,0)+2\cdot(0,0,\ldots,0,1))\ ,\\
&\quad +\big(\overline{\mathbf{4}}\,,\ (0,1,\ldots,0,1)+2\cdot(1,0,\ldots,0,0)\big)\ ,
\end{aligned} \tag{B.2.1}$$

Note that the two representations $(\mathbf{15}\,,\ (1,0,\ldots,0,1))$ in $T^{IJ\alpha}$ and T^{AB} coincide. For $k=1,2,3,4$ the adjoint representation $SU(4,k)$ are given in the table I.

APPENDIX C

On the Euler angle parametrization

In this section, the topic of discussion is on the Euler angle parametrization which is used throughout this dissertation. This is a generalized concept of the Euler angles used in the study rotation of rigid body in classical mechanics. In many cases, this parametrization is preferred to the traditional unitary gauge due to the reason we mentioned earlier. We review it in an elementary and sketchy fashion. The completed discussion can be found in [118].

Let G be a Lie group and H be its subgroup. The elements of group are given by $g \in G$ and $h \in H$, and the algebra of G and H are given by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We choose H to be maximal compact subgroup. We denote the non-compact generators $\tau_i \in \mathfrak{p}$, then $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. We first parametrize $g \in G$ and the coset space G/H is simply the result after modding out the H factor. In general the parametrization of $g \in G$ is in the form of

$$g = e^a e^b\,, \quad a \in \mathfrak{p},\, b \in \mathfrak{h}. \tag{C.0.1}$$

If we parametrize $a = a_1\tau_1 + a_2\tau_2 + \cdots +$, we get the unitary gauge. Now we will parametrize e^a by Euler angle. The first task is to construct a subspace $V \subset \mathfrak{p}$ that all elements in $\mathfrak{p}$ are generated by commutation relation $[V, h] \subset \mathfrak{p}$. In the language of Mathematics, it is that the whole space $\mathfrak{p}$ is generated from V via the adjoint action of H, i.e. $\mathfrak{p} = Ad_H(V) = hVh^{-1}$. Now g can be written as

$$g = e^{\tilde{b}} e^v e^b, \quad b, \tilde{b} \in \mathfrak{h},\, v \in V \tag{C.0.2}$$

Note that $e^{\tilde{b}}$ contains redundancy. We can get rid of the redundancy generators by first seeking for the set generators that generates the automorphisms of V, we denote them H_0. The automorphism of space V can be written as

$$Ad_{H_0} : V \to V\,. \tag{C.0.3}$$

The final form of G now is

$$G = Be^V H\,, \tag{C.0.4}$$

where $B = H/H_0$.

For example, we parametrize full coset manifold $\frac{USp(4,2)}{USp(4)\times USp(2)}$ for $k = 2$ in $N = 5$ theory. We pick Y^7 which is the only one member in minimum set of non-compact generators that generates the other generators by adjoint action. The redundancy H_0 is given by the generators in $USp(4)\times USp(2)$ that commute with Y^7. They are as follows: $j_4, j_5, j_6, j_1+j_{11}, j_2+j_{12}, j_3+j_{13}$. As a result, $B=\frac{H}{H_0}$ is composed of $j_1-j_{11}, j_2-j_{12}, j_3-j_{13}, j_7, j_8, j_9, j_{10}$. With a proper normalization factor, we get the coset representative

$$L = e^{a_1 X_1} e^{a_2 X_2} e^{a_3 X_3} e^{a_4 J_7} e^{a_5 J_8} e^{a_6 J_9} e^{a_7 J_{10}} e^{bY^7} \tag{C.0.5}$$

where X_i's are defined by

$$X_1 = \frac{1}{\sqrt{2}}(J_1 - J_{11}), \qquad X_2 = \frac{1}{\sqrt{2}}(J_2 - J_{12}), \qquad X_3 = \frac{1}{\sqrt{2}}(J_3 - J_{13}). \tag{C.0.6}$$

The next example is taken from $N=6$ theory. We parametrize 24 scalars by the coset $G/H = \frac{SU(4,3)}{SU(4)\times SU(3)\times U(1)}$ for $k=3$ with gauge group $SU(1,4)\times SU(3)\times U(1)$. The subspace V is chosen to be Y_1, Y_{11}, Y_{21}. The elements of H that commute with Y_1, Y_{11}, Y_{21} are $U(1)\times U(1)\times U(1)$, so we have $B=\frac{SU(4)\times SU(3)\times U(1)}{U(1)\times U(1)\times U(1)}$. We can identify one of the $U(1)$ in $U(1)\times U(1)\times U(1)$ with the $U(1)$ factor in H. Moreover, we also choose to remove the remaining $U(1)\times U(1)$ in H_0 by modding out one $U(1)$ factor from $SU(4)$ and the other one from $SU(3)$. Finally, we are now left with $B=\frac{SU(4)}{U(1)}\times\frac{SU(3)}{U(1)}$. Note that there are other possible ways to getting rid of the redundancy, but they are equivalent after redefinition of the scalars. Finally, the coset is given by

$$\begin{aligned} L &= e^{a_1 c_3} e^{a_2 c_2} e^{a_3 c_3} e^{a_4 c_5} e^{\frac{1}{\sqrt{3}} a_5 c_8} e^{a_6 c_3} e^{a_7 c_3} e^{a_8 j_3} e^{a_9 j_2} e^{a_{10} j_3} e^{a_{11} j_5} e^{\frac{1}{\sqrt{3}} a_{12} j_8} e^{a_{13} j_{10}} \times \\ & \quad e^{a_{14} j_3} e^{a_{14} j_3} e^{a_{15} j_2} e^{a_{16} j_3} e^{a_{17} j_5} e^{\frac{1}{\sqrt{3}} a_{18} j_8} e^{a_{19} j_3} e^{a_{20} j_2} e^{a_{21} j_3} e^{b_1 Y_1} e^{b_2 Y_{11}} e^{b_3 Y_{21}}, \end{aligned} \tag{C.0.7}$$

where the $SU(4)$ generators j_i's are defined in (4.2.47). The $SU(3)$ generators is labelled by c_i. The explicit parametrization of both $SU(4)$ and $SU(3)$ can be found in [117].

Vitae

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Publications

Author names are in Alphabetical order. For the up-to-date publication list and citation numbers, search `NGAMWATTHANAKUL or BOONPITHAK` at the SPIRES homepage `http://inspirehep.net/`

1. A. Chatrabhuti, P. Karndumri and B. Ngamwatthanakul, *New N=5,6, 3D gauged supergravities and holography*, JHEP 1401 (2014) 159, arXiv:hep-th/1312.4275.
2. A. Chatrabhuti, P. Karndumri and B. Ngamwatthanakul, *3D N=6 Gauged Supergravity: Admissible Gauge Groups,Vacua and RG Flows*, JHEP 07 (2012) 057, arXiv:hep-th/1202.1043.