CHAPTER IV

SIMPLE Γ-SEMINEARRINGS

The final chapter is concerned with simple Γ-seminearrings and 0-simple Γ-seminearrings.

4.1 Simple Γ-seminearrings

First of all, recall that a semigroup R is called left (right) simple if and only if the only left (right) ideal of R is itself and R is called simple if and only if the only ideal of R is itself. Then, we define a left and a right simple Γ-seminearring.

Definition 4.1.1. A Γ-seminearring R is called *left (right) simple* if the only left (right) ideal of R is itself. Furthermore, R is called *simple* if the only ideal of R is itself.

Theorem 4.1.1. *Let R be a Γ-seminearring. If R is left (right) zero, then R is left (right) simple.*

Proof. It is enough to assume that R is left zero. To show that R is left simple, let A be a left ideal of R and $x \in R$. Fix $a \in A$ and $\alpha \in \Gamma$. Then $x = x\alpha a \in A$ since x is a left zero and A is a left ideal of R. This implies that $R \subseteq A$ so that $R = A$. As a result, R is left simple. □

The following theorems give conditions for being one-sided simple or simple.

Theorem 4.1.2. *Let R be a Γ-seminearring.*

(i) If $R\Gamma x = R$ for all $x \in R$, then R is left simple.

(ii) $x\Gamma R = R$ for all $x \in R$ if and only if R is right simple.

(iii) If $(R\Gamma x)\Gamma R = R$ for all $x \in R$, then R is simple.

Proof. (i) Assume that $R\Gamma x = R$ for all $x \in R$. Let L be a left ideal of R and $a \in L$. Then $R = R\Gamma a \subseteq R\Gamma L \subseteq L$ so that $R = L$. Hence R is left simple.

(ii) First, assume that $x\Gamma R = R$ for all $x \in R$. Let A be a right ideal of R and $a \in A$. Then $R = a\Gamma R \subseteq A\Gamma R \subseteq A$ so that $R = A$. Therefore, R is right simple.

Conversely, assume that R is right simple. For each $x \in R$, we know from Corollary 2.2.6 that $x\Gamma R$ is a right ideal of R so that $x\Gamma R = R$.

(iii) Assume that $(R\Gamma x)\Gamma R = R$ for all $x \in R$. Let I be an ideal of R and $a \in I$ Then $R = (R\Gamma a)\Gamma R \subseteq I\Gamma R \subseteq I$ so that $R = I$. Hence R is simple.

□

If a Γ-seminearring is distributively generated, then the converses of (i) and (iii) in Theorem 4.1.2 hold.

Theorem 4.1.3. *Let R be a distributively generated Γ-seminearring.*

(i) $R\Gamma x = R$ for all $x \in R$ if and only if R is left simple.

(ii) $(R\Gamma x)\Gamma R = R$ for all $x \in R$ if and only if R is simple.

Proof. (i) It suffices to show the converse direction only. Assume that R is left simple. Since R is distributively generated, $R\Gamma x$ is a left ideal of R for all $x \in R$ by Corrollary 2.2.9. Hence $R\Gamma x = R$ for all $x \in R$.

(ii) The proof of this is obtained similarly to that of (i). Note that $(R\Gamma x)\Gamma R$ is an ideal of R for all $x \in R$ because R is d.g., and then $(R\Gamma x)\Gamma R = R$ for all $x \in R$ since R is simple.

□

4.2 0-simple Γ-seminearrings

We institute 0-simple Γ-seminearrings and find some properties.

Proposition 2.2.4 suggests if R is a zero-symmetric Γ-seminearring, then R is neither left simple nor right simple. This inspires us to study so called 0-simple.

R. Chinram gave the definitions of one-sided 0-simple semigroups in [3].

A semigroup R having more than one element with the zero 0 is called left (right) 0-simple if and only if $R^2 \neq \{0\}$ and R has no left (right) ideals other than $\{0\}$ and itself. Moreover, R is called 0-simple if and only if $R^2 \neq \{0\}$ and R has no ideals other than $\{0\}$ and itself.

Definition 4.2.1. A zero-symmetric Γ-seminearring R with more than one element is called *left (right)* 0*-simple* if $R\Gamma R \neq \{0\}$ and R has no left (right) ideals other than $\{0\}$ and itself. Furthermore, R is called 0*-simple* if $R\Gamma R \neq \{0\}$ and R has no ideals other than $\{0\}$ and itself.

Note that, in the above definition, the zero-symmetric property is compulsory otherwise $\{0\}$ may not be an ideal of R according to Proposition 2.2.4.

Theorem 4.2.1. *Let R be a zero-symmetric Γ-seminearring having more than one element.*

(i) If $R\Gamma x = R$ for all $x \in R\setminus\{0\}$, then R is left 0-simple.

(ii) $x\Gamma R = R$ *for all* $x \in R \setminus \{0\}$ *if and only if* R *is right* 0*-simple.*

(iii) If $(R\Gamma x)\Gamma R = R$ *for all* $x \in R \setminus \{0\}$*, then* R *is* 0*-simple.*

Proof. (i) Assume that $R\Gamma x = R$ for all $x \in R \setminus \{0\}$. Let $x \in R \setminus \{0\}$. Then $\{0\} \neq R = R\Gamma x \subseteq R\Gamma R$, i.e., $R\Gamma R \neq \{0\}$. Let L be a nonzero left ideal of R and $a \in L \setminus \{0\}$. Then $R = R\Gamma a \subseteq R\Gamma L \subseteq L$ so that $R = L$. Therefore, R is left 0-simple.

(ii) For the proof that R is right 0-simple provided that $x\Gamma R = R$ for all $x \in R \setminus \{0\}$ can be obtained in the same way as the proof of (i).

Conversely, assume that R is right 0-simple. Let

$L = \{x \in R \mid x\alpha r = 0$ for all $r \in R$ and $\alpha \in \Gamma\}$. Since R is zero-symmetric, $0 \in L$ so that $L \neq \varnothing$. We show that L is a right ideal of R. Let $x, y \in L$. Then, for each $r \in R$ and $\alpha \in \Gamma$, we see that $x\alpha r = 0 = y\alpha r$ so that $(x+y)\alpha r = x\alpha r + y\alpha r = 0 + 0 = 0$ since 0 is the additive identity of R. Thus $x + y \in L$. Next, $(x\beta s)\alpha r = 0\alpha r = 0$ for all $r, s \in R$ and $\alpha, \beta \in \Gamma$ and then $x\beta s \in L$. This implies that L is a right ideal of R. If $L = R$, then $R\Gamma R = L\Gamma R = \{0\}$ which is a contradiction because R is right 0-simple. Thus $L = \{0\}$. Finally, let $x \in R \setminus \{0\}$. Then $x \notin L$. So $x\alpha r \neq 0$ for some $r \in R$ and $\alpha \in \Gamma$. This implies that $x\Gamma R \neq \{0\}$. Recall that $x\Gamma R$ is a right ideal of R. Hence $x\Gamma R = R$ because R is right 0-simple.

(iii) The proof follows directly from the proof of (i).

□

The following theorem shows that the converses of (i) and (iii) in Theorem 4.2.1 hold if the distributively generated property is given.

Theorem 4.2.2. *Let a zero-symmetric* Γ*-seminearring* R *be distributively generated and have more than one element.*

(i) $R\Gamma x = R$ *for all* $x \in R\smallsetminus\{0\}$ *if and only if* R *is left* 0*-simple.*

(ii) $(R\Gamma x)\,\Gamma R = R$ *for all* $x \in R\smallsetminus\{0\}$ *if and only if* R *is* 0*-simple.*

Proof. (i) Assume that R is left 0-simple. Let

$L = \{x \in R \,|\, r\alpha x = 0 \text{ for all } r \in R \text{ and } \alpha \in \Gamma\}$. If we can show that $L = \{0\}$, then we are done as follows. Let $x \in R\smallsetminus\{0\}$. Then $x \notin L$ so that there exists $r \in R$ and $\alpha \in \Gamma$ such that $r\alpha x \neq 0$. This implies that $R\Gamma x \neq \{0\}$. Since R is left 0-simple and $R\Gamma x$ is a left ideal of R, we can conclude that $R\Gamma x = R$.

We claim that $L = \{0\}$. Since R is zero-symmetric, $0 \in L$. So $L \neq \varnothing$. First, we show that L is a left ideal of R. Let $r, s \in R$, $x, y \in L$ and $\alpha, \beta \in \Gamma$. Then $r = \sum d_i$ where all d_i's are distributive elements of R for all i. So $r\alpha(x+y) = \big(\sum d_i\big)\alpha(x+y) = \sum \big(d_i\alpha x + d_i\alpha y\big) = 0$ and $r\alpha(s\beta x) = r\alpha 0 = 0$. This implies that $x+y, s\beta x \in L$. Thus L is a left ideal of R. Next, since R is left 0-simple, $L = R$ or $L = \{0\}$. If $L = R$, then $R\Gamma R = R\Gamma L = \{0\}$, which is a contradiction. Thus $L = \{0\}$ as claimed.

(ii) Assume that R is 0-simple. Frist, we show that $(R\Gamma R)\,\Gamma R = R$. Since $R\Gamma R$ is an ideal of R, this implies that $R\Gamma R \neq \{0\}$ and then $R\Gamma R = R$. Thus $(R\Gamma R)\,\Gamma R = R\Gamma R = R$.

Next, let $I = \{x \in R \mid r\alpha x\beta s = 0 \text{ for all } r, s \in R \text{ and } \alpha, \beta \in \Gamma\}$. Since R is zero-symmetric, $0 \in I$. We claim that I is an ideal of R. Let $r, s, t \in R$, $x, y \in I$ and $\alpha, \beta, \gamma \in \Gamma$. Then $r = \sum d_i$ where all d_i's are distributive elements of R for all i. Then

$$r\alpha(x+y)\beta s = \left(\sum d_i\right)\alpha(x+y)\beta s = \sum(d_i\alpha(x+y)\beta s)$$
$$= \sum(d_i\alpha x\beta s + d_i\alpha y\beta s) = 0.$$

Thus $x+y \in I$. We then verify that $x\gamma t$, $t\gamma x \in I$. Since $x, y \in I$ and 0 is the zero of R, we obtain that $r\alpha(x\gamma t)\beta s = (r\alpha x\gamma t)\beta s = 0\beta s = 0$ and $r\alpha(t\gamma x)\beta s = r\alpha(t\gamma x\beta s) = r\alpha 0 = 0$. Thus $x\gamma t, t\gamma x \in I$. Hence I is an ideal of R. Since R is 0-simple, $I = \{0\}$ or $I = R$. If $I = R$, then $R = (R\Gamma R)\Gamma R = (R\Gamma I)\Gamma R = \{0\}$ which is absurd. Thus $I = \{0\}$. Finally, let $x \in R \setminus \{0\}$. Then $x \notin I$. This implies that $r\alpha x\beta s \neq 0$ for some $r, s \in R$, $\alpha, \beta \in \Gamma$. Then $(R\Gamma x)\Gamma R \neq \{0\}$. As a result, $(R\Gamma x)\Gamma R = R$ since $(R\Gamma x)\Gamma R$ is an ideal of R.

□

Finally, 0-simple Γ-seminearrings are in fact prime Γ-seminearrings.

Theorem 4.2.3. *If R is a 0-simple Γ-seminearring, then R is a prime Γ-seminearring.*

Proof. Assume that R is a 0-simple Γ-seminearring. Suppose R is not prime. Then there exists ideals A and B of R such that $A\Gamma B = \{0\}$ but $A \neq \{0\}$ and $B \neq \{0\}$. This implies that A and B are nonzero ideals of R. Since R is 0-simple, $A = R = B$. So $R\Gamma R = A\Gamma B = \{0\}$, a contradiction. Hence R is prime. □