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ALTERNATIVE JENSEN TYPE FUNCTIONAL EQUATION

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ให้ X และ Y เป็นปริภูมิเชิงเส้นเหนือฟีลด์ $\mathbb{F}$ โดยที่ $\mathbb{F}=\mathbb{Q},\mathbb{R}$ หรือ $\mathbb{C}$ และให้ $f:X\to Y$ เป็นฟังก์ชันใดๆ กำหนดให้ค่าคงที่ $p\in\mathbb{R}$ ซึ่ง $p\neq 0,1$ เราพิสูจน์ว่าสมการเชิงฟังก์ชันแบบเจนเซนทางเลือก

$$pf(x)+(1-p)f(y)=\pm f(px+(1-p)y)$$

สมมูลกับสมการเชิงฟังก์ชันแบบเจนเซน

$$pf(x)+(1-p)f(y)=f(px+(1-p)y)$$

นอกจากนี้เราพิสูจน์ว่าผลเฉลยทั่วไปเมื่อ $p\in\mathbb{Q}$ คือ $f(x)=A(x)+c$ โดยที่ $A:X\to Y$ เป็นฟังก์ชันการบวก และ $c\in Y$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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Let X and Y be linear spaces over a field $\mathbb{F}$ where $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ and let $f : X \to Y$ be arbitrary function. Given a constant $p \in \mathbb{R}$ such that $p \neq 0, 1$, we prove that the alternative Jensen type functional equation

$$pf(x) + (1-p)f(y) = \pm f(px + (1-p)y)$$

is equivalent to the Jensen type functional equation

$$pf(x) + (1-p)f(y) = f(px + (1-p)y).$$

Moreover, we prove that the general solution when $p \in \mathbb{Q}$ is $f(x) = A(x) + c$ where $A : X \to Y$ is an additive function and $c \in Y$.

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CONTENTS

CHAPTER I
INTRODUCTION

In this chapter, first we present a brief introduction to a functional equation and give some examples of functional equations. Then we treat an alternative functional equation and its literature. Furthermore we refer to the motivation of our proposed problem.

1.1 Functional Equations

A ***functional equation*** is an equation which unknown variables are function. M. Kuczma [8] said that *"On several occasions in investigations of functional equations it has been observed that the family of the solutions of the equation in question depends quite essentially on the domain in which the validity of the equation is postulated."* The following example demonstrates how to solve functional equation problems.

Example 1.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(x+y) + f(x-y) + f(y) = 2x + y \tag{1.1}$$

for all $x, y \in \mathbb{R}$.

<u>**Solution**</u>: Assume that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1).

Setting $y = 0$ into (1.1) yields

$$\begin{aligned} 2f(x) &= 2x - f(0); \\ f(x) &= x - \frac{f(0)}{2} \qquad \text{for all } x \in \mathbb{R}. \end{aligned} \tag{1.2}$$

Replacing $(x, y) = (0, 0)$ into (1.1) yields

$$f(0) = 0.$$

Substituting $f(0) = 0$ into (1.2), we obtain

$$f(x) = x \qquad \text{for all } x \in \mathbb{R}.$$

On the other hand, if a function is defined by $f(x) = x$ for all $x \in \mathbb{R}$, then we have

$$f(x+y) + f(x-y) + f(y) = 2x + y.$$

Therefore, all solutions of the equation (1.1) are of the form $f(x) = x$. □

In general, a functional equation may not necessarily have a solution. The following example shows that a functional equation has no solution.

Example 1.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(1-x) + f(x) = x + 1 \tag{1.3}$$

for all $x \in \mathbb{R}$.

Solution: Assume that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.3).
Plugging $x = 0$ into (1.3) yields

$$f(1) + f(0) = 1. \tag{1.4}$$

Putting $x = 1$ into (1.3), we get

$$f(0) + f(1) = 2. \tag{1.5}$$

Then, from (1.4) and (1.5) we obtain $1 = 2$ which is a contradiction. Therefore, there is no function $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation (1.3). □

One of the most well-known functional equations is the ***Cauchy functional equation***

$$f(x+y) = f(x) + f(y). \tag{1.6}$$

In 1821, A.L. Cauchy [2] proved that the general continuous solution $f : \mathbb{R} \to \mathbb{R}$ of the equation (1.6) is of the form $f(x) = cx$ where c is an arbitrary constant. The general solution of the equation (1.6) is called an ***additive function***. An important properties of an additive function $A : X \to Y$, where X and Y are linear spaces over a field $\mathbb{C}$, is that $A(rx) = rA(x)$ for all $r \in \mathbb{Q},\ x \in X$ (please refer to [3] for more information). In 1905, G. Hamel [6] succeeded in constructing a discontinuous solution of the equation (1.6) by using the axiom of choice. In addition, Hamel proved that, for a discontinuous additive function $A : \mathbb{R} \to \mathbb{R}$, the graph $G(A) = \{(x, A(x)) : x \in \mathbb{R}\}$ is dense in the plane $\mathbb{R}^2$.

Another well-known functional equation particularly associated with the Cauchy functional equation is the ***Jensen's functional equation***

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}. \tag{1.7}$$

For a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.7), by introducing $\tilde{f}(x) = f(x) - f(0)$, it can be shown that $\tilde{f}$ satisfies (1.6). This means that the general solution of (1.7) is of the form $f(x) = A(x)+c$ for some additive function $A : \mathbb{R} \to \mathbb{R}$ and $c = f(0)$. The continuous solution of the equation (1.7) is given by $f(x) = ax + b$ where a and b are arbitrary constants (please refer to [1] for details).

1.2 Alternative Functional Equations

To introduce an alternative functional equation, we start with the following functional equation. We consider a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) + f(y) = \pm f(x + y) \tag{1.8}$$

or equivalently

$$f(x) + f(y) = f(x + y) \quad \text{or} \quad f(x) + f(y) = -f(x + y).$$

for all $x, y \in \mathbb{R}$. In other words, for each $x, y \in \mathbb{R}$, f either satisfies $f(x) + f(y) = f(x + y)$ or $f(x) + f(y) = -f(x + y)$. The functional equation (1.8) is called an ***alternative Cauchy functional equation***. It has been widely studied or explored and was solved by several authors [4, 7, 9, 11, 12], under various hypotheses of the functions, domain and co-domain. In 1981, G.L. Forti and L. Paganoni [5] have been studied another version of an alternative Cauchy functional equation

$$f(x) + f(y) = f(x + y) \quad \text{or} \quad f(x) + f(y) = f(x + y) + a \tag{1.9}$$

where X, Y are arbitrary abelian groups, $a \in Y$ and $f : X \to Y$. They proved that the functional equation (1.9) is also equivalent to the Cauchy functional equation (1.6).

One approach of alternative functional equation problems is finding an alternative functional equation which is equivalent to the original functional equation. Next example shows that the alternative Cauchy functional equation (1.8) is equivalent to the Cauchy functional equation (1.6).

Example 1.3. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x) + f(y) = \pm f(x + y) \qquad \text{for all} \ \ x, y \in \mathbb{R} \tag{1.10}$$

if and only if f satisfies

$$f(x) + f(y) = f(x + y) \qquad \text{for all} \ \ x, y \in \mathbb{R}. \tag{1.11}$$

Proof. Since (1.11) certainly implies (1.10), it is sufficient to prove that (1.10) implies (1.11). Assume that a functions $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.10).
Putting $(x, y) = (0, 0)$ into (1.10), we obtain

$$2f(0) = f(0) \quad \text{or} \quad 2f(0) = -f(0).$$

Then

$$f(0) = 0.$$

Replacing $(x, y) = (x, -x)$ into (1.10) gives

$$f(x) + f(-x) = f(0) \quad \text{or} \quad f(x) + f(-x) = -f(0)$$

and so, by $f(0) = 0$, we have

$$f(-x) = -f(x). \tag{1.12}$$

Next, we want to show that f satisfies (1.11) for all $x, y \in \mathbb{R}$. Suppose not, then there exist $a, b \in \mathbb{R}$ such that

$$f(a) + f(b) \neq f(a + b). \tag{1.13}$$

Since $f(a)+f(b)=\pm f(a+b)$, we are left with

$$f(a)+f(b)=-f(a+b). \tag{1.14}$$

Setting $(x,y)=(-a,a+b)$ into (1.10) yields

$$f(-a)+f(a+b)=f(b) \quad \text{or} \quad f(-a)+f(a+b)=-f(b). \tag{1.15}$$

Using (1.12) in (1.15), we have

$$-f(a)+f(a+b)=f(b) \quad \text{or} \quad -f(a)+f(a+b)=-f(b).$$

Since $f(a)+f(b)\neq f(a+b)$, we get

$$f(a+b)=f(a)-f(b). \tag{1.16}$$

Comparing (1.16) and (1.14), we obtain

$$f(a)=0. \tag{1.17}$$

Similarly, plugging $(x,y)=(a+b,-b)$ in (1.10) and using (1.12), we have

$$f(a+b)-f(b)=f(a) \quad \text{or} \quad f(a+b)-f(b)=-f(a).$$

Since $f(a)+f(b)\neq f(a+b)$, we have $f(a+b)=-f(a)+f(b)$ and then solving it with (1.14) implies

$$f(b)=0. \tag{1.18}$$

Substituting $f(a)$ and $f(b)$ from (1.17) and (1.18), respectively, into (1.14), we have

$$f(a+b)=0 \tag{1.19}$$

It follows from (1.17), (1.18) and (1.19) that $f(a+b)=f(a)+f(b)$, which is a contradiction to (1.13). Therefore $f(x)+f(y)=f(x+y)$ for all $x,y\in\mathbb{R}$. □

Another example of an alternative functional equation is the ***alternative Jensen's functional equation***

$$\frac{f(x)+f(y)}{2}=\pm f\left(\frac{x+y}{2}\right). \tag{1.20}$$

However, the alternative Jensen's functional equation (1.20) has not been widely investigated. In 2012, P. Nakmahachalasint [10] proved that the alternative Jensen's functional equation (1.20) where f is a function from a 2-divisible semigroup to a divisible abelian group is equivalent to the Jensen's functional equation (1.7).

1.3 Proposed Problem

Throughout this thesis, let X and Y be linear spaces over a field $\mathbb{F}$ where $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ and $f : X \to Y$ be an arbitrary function.

In this thesis, we are interested in the functional equation

$$pf(x) + (1-p)f(y) = f(px + (1-p)y) \tag{1.21}$$

where $p \in \mathbb{R} \setminus \{0, 1\}$. Note that, when $p = \frac{1}{2}$, it is the Jensen's functional equation. The equation (1.21) is then called a Jensen type functional equation.

The aim of this work is to prove that the alternative Jensen type functional equation

$$pf(x) + (1-p)f(y) = \pm f(px + (1-p)y) \quad \text{for all} \quad x, y \in X \tag{1.22}$$

is equivalent to the Jensen type functional equation (1.21).

CHAPTER II

ALTERNATIVE JENSEN'S FUNCTIONAL EQUATION

In this chapter, we investigate the alternative Jensen's functional equation

$$\frac{f(x)+f(y)}{2} = \pm f\left(\frac{x+y}{2}\right).$$

By using different approach to P. Nakmahachalasint's, we prove that the alternative Jensen's functional equation is equivalent to the Jensen's functional equation.

Theorem 2.1. *A function $f : X \to Y$ satisfies*

$$\frac{f(x)+f(y)}{2} = \pm f\left(\frac{x+y}{2}\right) \qquad \textit{for all} \;\; x, y \in X \tag{2.1}$$

if and only if f satisfies

$$\frac{f(x)+f(y)}{2} = f\left(\frac{x+y}{2}\right) \qquad \textit{for all} \;\; x, y \in X. \tag{2.2}$$

Proof. Since (2.2) readily implies (2.1), it is sufficient to prove that (2.1) implies (2.2). Assume that a functions $f : X \to Y$ satisfies (2.1). Now suppose that there exist $a, b \in X$ such that

$$\frac{f(a)+f(b)}{2} \neq f\left(\frac{a+b}{2}\right). \tag{2.3}$$

Since $\frac{f(a)+f(b)}{2} = \pm f\left(\frac{a+b}{2}\right)$, we are left with

$$\frac{f(a)+f(b)}{2} = -f\left(\frac{a+b}{2}\right). \tag{2.4}$$

If $f\left(\frac{a+b}{2}\right) = 0$, then (2.4) gives $f(a)+f(b) = 0$, which in turn implies that $\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right)$, a contradiction to (2.3). Therefore,

$$f\left(\frac{a+b}{2}\right) \neq 0. \tag{2.5}$$

Putting $(x, y) = (a, \frac{a+b}{2})$ into (2.1), we obtain

$$f(a) + f\left(\frac{a+b}{2}\right) = \pm 2f\left(\frac{3a+b}{4}\right). \tag{2.6}$$

Substituting $(x, y) = (\frac{a+b}{2}, b)$ into (2.1), we have

$$f\left(\frac{a+b}{2}\right) + f(b) = \pm 2f\left(\frac{a+3b}{4}\right). \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+3b}{4}\right) = \pm\left(f(a) + f\left(\frac{a+b}{2}\right)\right) \pm \left(f\left(\frac{a+b}{2}\right) + f(b)\right). \tag{2.8}$$

Replacing $(x, y) = (\frac{3a+b}{4}, \frac{a+3b}{4})$ into (2.1), we have

$$f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) = \pm 2f\left(\frac{a+b}{2}\right). \tag{2.9}$$

Eliminating $f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)$ from (2.8) and (2.9), we can see that

$$\left(f(a) + f\left(\frac{a+b}{2}\right)\right) \pm \left(f\left(\frac{a+b}{2}\right) + f(b)\right) = \pm 4f\left(\frac{a+b}{2}\right). \tag{2.10}$$

First, suppose that the left-hand side of (2.10) takes the plus sign; that is

$$f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) = \pm 4f\left(\frac{a+b}{2}\right).$$

Solving the above equation with (2.4) gives $f\left(\frac{a+b}{2}\right) = 0$, a contradiction to (2.5). Back to (2.10), we are now left with

$$f(a) - f(b) = \pm 4f\left(\frac{a+b}{2}\right).$$

Case 1 First, consider the possibility where

$$f(a) - f(b) = 4f\left(\frac{a+b}{2}\right). \tag{2.11}$$

Substituting $f(a) = -2f\left(\frac{a+b}{2}\right) - f(b)$ from (2.4) into (2.11) yields

$$f(b) = -3f\left(\frac{a+b}{2}\right). \tag{2.12}$$

Replacment of $f(b)$ from (2.12) into (2.11) shows that

$$f(a) = f\left(\frac{a+b}{2}\right). \tag{2.13}$$

Putting $(x, y) = (\frac{a+b}{2}, \frac{-a+3b}{2})$ into (2.1), we obtain

$$f\left(\frac{a+b}{2}\right) + f\left(\frac{-a+3b}{2}\right) = \pm 2f(b). \tag{2.14}$$

Next, substituting $f(b)$ from (2.12) into the right-hand side of (2.14) yields

$$f\left(\frac{-a+3b}{2}\right) = 5f\left(\frac{a+b}{2}\right) \text{ or } f\left(\frac{-a+3b}{2}\right) = -7f\left(\frac{a+b}{2}\right). \tag{2.15}$$

Setting $(x, y) = (a, \frac{-a+3b}{2})$ into (2.1), we get

$$f(a) + f\left(\frac{-a+3b}{2}\right) = \pm 2f\left(\frac{a+3b}{4}\right). \tag{2.16}$$

Substituting $f(a)$ from (2.13) and $f\left(\frac{-a+3b}{2}\right)$ from (2.15) into (2.16) yield

$$f\left(\frac{a+3b}{4}\right) = \pm 3f\left(\frac{a+b}{2}\right). \tag{2.17}$$

Replacement of $f(b)$ from (2.12) into (2.7) shows that

$$f\left(\frac{a+3b}{4}\right) = \pm f\left(\frac{a+b}{2}\right). \tag{2.18}$$

Comparing (2.17) and (2.18), we have

$$3f\left(\frac{a+b}{2}\right) = \pm f\left(\frac{a+b}{2}\right),$$

which always leads to $f\left(\frac{a+b}{2}\right) = 0$, a contradiction to (2.5).

Case 2 Now consider the possibility where

$$f(a) - f(b) = -4f\left(\frac{a+b}{2}\right). \tag{2.19}$$

Substituting $f(a) = -2f\left(\frac{a+b}{2}\right) - f(b)$ from (2.4) into (2.19) yields

$$f(b) = f\left(\frac{a+b}{2}\right). \tag{2.20}$$

Replacing $f(b)$ from (2.20) into (2.19) shows that

$$f(a) = -3f\left(\frac{a+b}{2}\right). \tag{2.21}$$

Putting $(x, y) = (\frac{3a-b}{2}, \frac{a+b}{2})$ into (2.1), we get

$$f\left(\frac{3a-b}{2}\right) + f\left(\frac{a+b}{2}\right) = \pm 2f(a). \tag{2.22}$$

Next, substituting $f(a)$ from (2.21) into the right-hand side of (2.22) yields

$$f\left(\frac{3a-b}{2}\right) = 5f\left(\frac{a+b}{2}\right) \text{ or } f\left(\frac{3a-b}{2}\right) = -7f\left(\frac{a+b}{2}\right). \tag{2.23}$$

Setting $(x, y) = (\frac{3a-b}{2}, b)$ into (2.1), we obtain

$$f\left(\frac{3a-b}{2}\right) + f(b) = \pm 2f\left(\frac{3a+b}{4}\right). \tag{2.24}$$

Substituting $f(b)$ from (2.20) and $f\left(\frac{3a-b}{2}\right)$ from (2.23) into the left-hand side of (2.24) yield

$$f\left(\frac{3a+b}{4}\right) = \pm 3f\left(\frac{a+b}{2}\right). \tag{2.25}$$

Replacement of $f(a)$ from (2.21) into (2.6) shows that

$$f\left(\frac{3a+b}{4}\right) = \pm f\left(\frac{a+b}{2}\right). \tag{2.26}$$

Comparing (2.25) and (2.26), we have

$$3f\left(\frac{a+b}{2}\right) = \pm f\left(\frac{a+b}{2}\right),$$

which implies that $f\left(\frac{a+b}{2}\right) = 0$, a contradiction to (2.5). From all of the consideration above, the assumption (2.3) always leads to a contradiction. Therefore,

$$\frac{f(x)+f(y)}{2} = f\left(\frac{x+y}{2}\right) \text{ for all } x, y \in X.$$

□

CHAPTER III
ALTERNATIVE JENSEN TYPE FUNCTIONAL EQUATION

This chapter consist of two sections. In the first section, we determine the general solution of the Jensen type functional equation in case of $p \in \mathbb{Q}$ and in the last section, we prove the equivalence of the alternative Jensen type functional equation and the Jensen type functional equation.

3.1 Jensen Type Functional Equation

In this section, we extend the Jensen's functional equation to the Jensen type functional equation. The following theorem gives the general solution of the Jensen type functional equation when $p \in \mathbb{Q}$.

Theorem 3.1. *Given a rational number $p \neq 0, 1$. A function $f : X \to Y$ satisfies*

$$f(px + (1-p)y) = pf(x) + (1-p)f(y) \tag{3.1}$$

for all $x, y \in X$ if and only if there exist an additive function $A : X \to Y$ and a constant $c \in Y$ such that

$$f(x) = A(x) + c$$

for all $x \in X$.

Proof. Define function $\tilde{f} : X \to Y$ by

$$\tilde{f}(x) = f(x) - f(0) \qquad \text{for all} \quad x \in X. \tag{3.2}$$

Replacing $x = 0$ into (3.2), we have

$$\tilde{f}(0) = 0.$$

Since the sum of the coefficients of $f(x)$ and $f(y)$ in the right-hand side of (3.1) is 1, $\tilde{f}$ still satisfies (3.1). That is

$$\tilde{f}(px + (1-p)y) = p\tilde{f}(x) + (1-p)\tilde{f}(y). \tag{3.3}$$

Putting $(x, y) = \left(\frac{x}{p}, 0\right)$ into (3.3), and noting that $\tilde{f}(0) = 0$, we have

$$\tilde{f}(x) = p\tilde{f}\left(\frac{x}{p}\right). \tag{3.4}$$

Plugging $(x, y) = \left(0, \frac{y}{1-p}\right)$ into (3.3), and using $\tilde{f}(0) = 0$, we see that

$$\tilde{f}(y) = (1-p)\tilde{f}\left(\frac{y}{1-p}\right). \tag{3.5}$$

Setting $(x, y) = \left(\frac{x}{p}, \frac{y}{1-p}\right)$ into (3.3) gives

$$\tilde{f}(x+y) = p\tilde{f}\left(\frac{x}{p}\right) + (1-p)\tilde{f}\left(\frac{y}{1-p}\right). \tag{3.6}$$

Using (3.4) and (3.5) in (3.6), it follows that

$$\tilde{f}(x+y) = \tilde{f}(x) + \tilde{f}(y).$$

Hence, $\tilde{f}$ is an additive function. That is there exists an additive function $A : X \to Y$ such that $\tilde{f}(x) = A(x)$. Since $\tilde{f}(x) = f(x) - f(0)$, we conclude that $f(x) = A(x) + c \quad$ where $c = f(0)$.

Conversely, suppose that there exist an additive function $A : X \to Y$ and a constant $c \in Y$ such that

$$f(x) = A(x) + c \qquad \text{for all} \quad x \in X,$$

so that for each $x, y \in X$,

$$f(px + (1-p)y) = A(px + (1-p)y) + c \tag{3.7}$$

By the properties of the additive function A,

$$\begin{aligned} A(px + (1-p)y) + c &= A(px) + A((1-p)y) + c \\ &= pA(x) + (1-p)A(y) + c. \end{aligned}$$

Note that $pA(x) + (1-p)A(y) + c = p(A(x) + c) + (1-p)(A(y) + c)$. Therefore $f(px + (1-p)y) = pf(x) + (1-p)f(y)$. Hence, f satisfies (3.1) for all $x, y \in X$. This completes the proof. $\square$

3.2 Alternative Jensen Type Functional Equation

In this section, we apply the technique used in Theorem 2.1 to show that the alternative Jensen type functional equation is equivalent to the Jensen type functional equation.

Theorem 3.2. *Given a constant* $p \in \mathbb{R} \setminus \{0,1\}$. *A function* $f : X \to Y$ *satisfies*

$$pf(x) + (1-p)f(y) = \pm f(px + (1-p)y) \qquad \text{for all } \ x, y \in X \tag{3.8}$$

if and only if f *satisfies*

$$pf(x) + (1-p)f(y) = f(px + (1-p)y) \qquad \text{for all } \ x, y \in X. \tag{3.9}$$

Proof. Since (3.9) evidently implies (3.8), it is sufficient to prove that (3.8) implies (3.9). Assume that a functions $f : X \to Y$ satisfies (3.8). Theorem 2.1 takes care the case $p = \dfrac{1}{2}$. Now assume that $p \neq \dfrac{1}{2}$. Suppose that there exist $a, b \in X$ such that

$$pf(a) + (1-p)f(b) \neq f(c) \tag{3.10}$$

where $c = pa + (1-p)b$. Since $pf(a) + (1-p)f(b) = \pm f(c)$, we are left with

$$pf(a) + (1-p)f(b) = -f(c). \tag{3.11}$$

If $pf(a) + (1-p)f(b) = 0$, then $f(c) = 0$, which in turn implies that $pf(a) + (1-p)f(b) = f(c)$, a contradiction to (3.10). Therefore,

$$pf(a) + (1-p)f(b) \neq 0. \tag{3.12}$$

Let

$$\begin{aligned} x_1 &= pa + (1-p)c, \\ x_2 &= pc + (1-p)a, \\ y_1 &= pb + (1-p)c, \\ y_2 &= pc + (1-p)b. \end{aligned}$$

In order to understand more, the following graph illustrates the position of $a, x_1, x_2, c, y_2, y_1, b$ in case of $p = \frac{2}{3}$.

Setting $(x, y) = (a, c), (c, a), (b, c)$ and (c, b), respectively, in (3.8), we obtain

$$pf(a) + (1-p)f(c) = \pm f(x_1), \tag{3.13}$$

$$pf(c) + (1-p)f(a) = \pm f(x_2), \tag{3.14}$$

$$pf(b) + (1-p)f(c) = \pm f(y_1), \tag{3.15}$$

$$pf(c) + (1-p)f(b) = \pm f(y_2). \tag{3.16}$$

We can verify that $px_1 + (1-p)y_1 = c$ and $px_2 + (1-p)y_2 = c$.
Replacing $(x, y) = (x_1, y_1), (x_2, y_2)$, respectively, in (3.8), we have

$$pf(x_1) + (1-p)f(y_1) = \pm f(c), \tag{3.17}$$

$$pf(x_2) + (1-p)f(y_2) = \pm f(c), \tag{3.18}$$

Substituting $f(x_1)$ from (3.13) and $f(y_1)$ from (3.15) into (3.17) yield

$$p(pf(a) + (1-p)f(c)) \pm (1-p)(pf(b) + (1-p)f(c)) = \pm f(c). \tag{3.19}$$

There are four possible cases in (3.19) which will be considered in detail as follows:

Case 1 $\quad p^2 f(a) + p(1-p)f(b) + (1-p)f(c) = f(c)$

Substituting $f(c)$ from (3.11) into the above equation, we have

$$\begin{aligned} p^2 f(a) + p(1-p)f(b) - p(-pf(a) - (1-p)f(b)) &= 0; \\ 2p^2 f(a) + 2p(1-p)f(b) &= 0. \end{aligned}$$

Since $p \neq 0$, we have $pf(a) + (1-p)f(b) = 0$, a contradiction to (3.12).

Case 2 $p^2 f(a) + p(1-p)f(b) + (1-p)f(c) = -f(c)$

Substituting $f(c)$ from (3.11) into the above equation, we obtain

$$\begin{aligned} p^2 f(a) + p(1-p)f(b) + (2-p)(-pf(a) - (1-p)f(b)) &= 0; \\ 2p(p-1)f(a) + 2(1-p)(p-1)f(b) &= 0. \end{aligned}$$

Since $p \neq 1$, we get $pf(a) + (1-p)f(b) = 0$, a contradiction to (3.12).

Case 3 $p^2 f(a) - p(1-p)f(b) + (-2p^2 + 3p - 1)f(c) = f(c)$

Substituting $f(c)$ from (3.11) into the above equation, we see that

$$p^2 f(a) - p(1-p)f(b) + (-2p^2 + 3p - 2)(-pf(a) - (1-p)f(b)) = 0;$$

$$p(p^2 - p + 1)f(a) + (1-p)^3 f(b) = 0. \tag{3.20}$$

Case 4 $p^2 f(a) - p(1-p)f(b) + (-2p^2 + 3p - 1)f(c) = -f(c)$

Substituting $f(c)$ from (3.11) into the above equation, it follows that

$$\begin{aligned} p^2 f(a) - p(1-p)f(b) + (-2p^2 + 3p)(-pf(a) - (1-p)f(b)) &= 0; \\ -2p^2(1-p)f(a) - 2p(2-p)(1-p)f(b) &= 0. \end{aligned}$$

Since $p \neq 1$ and $p \neq 0$, we have

$$pf(a) + (2-p)f(b) = 0. \tag{3.21}$$

From all four cases, we conclude that

$$p(p^2 - p + 1)f(a) + (1-p)^3 f(b) = 0 \quad \text{or} \quad pf(a) + (2-p)f(b) = 0. \tag{3.22}$$

Similarly, substituting $f(x_2)$ from (3.14) and $f(y_2)$ from (3.16) into (3.18) yield

$$p(pf(c) + (1-p)f(a)) \pm (1-p)(pf(c) + (1-p)f(b)) = \pm f(c). \tag{3.23}$$

There are four possible cases in (3.23) which will be considered in detail as follows:

Case 1 $p(1-p)f(a) + (1-p)^2 f(b) + pf(c) = f(c)$

Substituting $f(c)$ from (3.11) into the above equation, it follows that

$$\begin{aligned} p(1-p)f(a) + (1-p)^2 f(b) - (1-p)(-pf(a) - (1-p)f(b)) &= 0; \\ 2p(1-p)f(a) + 2(1-p)^2 f(b) &= 0. \end{aligned}$$

Since $p \neq 1$, we obtain $pf(a) + (1-p)f(b) = 0$, a contradiction to (3.12).

Case 2 $p(1-p)f(a) + (1-p)^2 f(b) + pf(c) = -f(c)$

Substituting $f(c)$ from (3.11) into the above equation, we have

$$\begin{aligned} p(1-p)f(a) + (1-p)^2 f(b) + (p+1)(-pf(a) - (1-p)f(b)) &= 0; \\ -2p^2 f(a) - 2p(1-p)f(b) &= 0. \end{aligned}$$

Since $p \neq 0$, it follows that $pf(a)+(1-p)f(b) = 0$, a contradiction to (3.12).

Case 3 $p(1-p)f(a) - (1-p)^2 f(b) + (2p^2 - p)f(c) = f(c)$

Substituting $f(c)$ from (3.11) into the above equation, we get

$$\begin{aligned} p(1-p)f(a) - (1-p)^2 f(b) + (2p^2 - p - 1)(-pf(a) - (1-p)f(b)) &= 0; \\ 2p(1-p)(1+p)f(a) + 2p(1-p)^2 f(b) &= 0. \end{aligned}$$

Since $p \neq 1$ and $p \neq 0$, we have

$$(1+p)f(a) + (1-p)f(b) = 0. \tag{3.24}$$

Case 4 $p(1-p)f(a) - (1-p)^2 f(b) + (2p^2 - p)f(c) = -f(c)$

Substituting $f(c)$ from (3.11) into the above equation, we obtain

$$\begin{aligned} p(1-p)f(a) - (1-p)^2 f(b) + (2p^2 - p + 1)(-pf(a) - (1-p)f(b)) &= 0; \\ -2p^3 f(a) + 2(1-p)(-p^2 + p - 1)f(b) &= 0. \end{aligned}$$

Then

$$p^3 f(a) + (1-p)(p^2 - p + 1)f(b) = 0. \tag{3.25}$$

From all four cases, we conclude that

$$(1+p)f(a)+(1-p)f(b)=0 \quad \text{or} \quad p^3 f(a)+(1-p)(p^2-p+1)f(b)=0. \tag{3.26}$$

Comparing (3.22) and (3.26), we have four cases to consider as follows:

Case 1 $p(p^2-p+1)f(a)+(1-p)^3 f(b)=0$ and $(1+p)f(a)+(1-p)f(b)=0$

Therefore, we obtain

$$p(p^2-p+1)f(a)+(1-p)^3 f(b)=0 \text{ and } (1+p)(1-p)^2 f(a)+(1-p)^3 f(b)=0. \tag{3.27}$$

Then

$$\begin{aligned} p(p^2-p+1)f(a) &= (1+p)(1-p)^2 f(a); \\ (1-2p)f(a) &= 0. \end{aligned}$$

Since $p \neq \frac{1}{2}$, we have $f(a)=0$ and substituting the resulting in (3.27) shows that $f(b)=0$.

Case 2 $p(p^2-p+1)f(a)+(1-p)^3 f(b)=0$ and $p^3 f(a)+(1-p)(p^2-p+1)f(b)=0$

Thus, we see that

$$p^3(p^2-p+1)f(a)+p^2(1-p)^3 f(b)=0 \text{ and } p^3(p^2-p+1)f(a)+(1-p)(p^2-p+1)^2 f(b)=0. \tag{3.28}$$

Then

$$\begin{aligned} p^2(1-p)^3 f(b) &= (1-p)(p^2-p+1)^2 f(b); \\ p^2(1-p)^2 f(b) &= (p^2-p+1)^2 f(b); \\ (1-2p+2p^2)f(b) &= 0. \end{aligned}$$

Since $p \in \mathbb{R}$, we have $1-2p+2p^2 \neq 0$, so we obtain $f(b)=0$. Replacing $f(b)=0$ into (3.28), it follows that $f(a)=0$.

Case 3 $pf(a)+(2-p)f(b)=0$ and $(1+p)f(a)+(1-p)f(b)=0$

This means that

$$p(1+p)f(a)+(1+p)(2-p)f(b)=0 \text{ and } p(1+p)f(a)+p(1-p)f(b)=0. \tag{3.29}$$

Then

$$\begin{aligned} (1+p)(2-p)f(b) &= p(1-p)f(b); \\ 2f(b) &= 0. \end{aligned}$$

Thus, we have $f(b) = 0$. Substituting $f(b) = 0$ into (3.29), we get $f(a) = 0$.

Case 4 $\quad pf(a) + (2-p)f(b) = 0$ and $p^3 f(a) + (1-p)(p^2 - p + 1)f(b) = 0$

That is

$$p^3 f(a) + p^2(2-p)f(b) = 0 \text{ and } p^3 f(a) + (1-p)(p^2 - p + 1)f(b) = 0. \quad (3.30)$$

Then

$$\begin{aligned} p^2(2-p)f(b) &= (1-p)(p^2-p+1)f(b); \\ (1-2p)f(b) &= 0. \end{aligned}$$

Since $p \neq \frac{1}{2}$, we have $f(b) = 0$. Replacement of $f(b) = 0$ into (3.30) shows that $f(a) = 0$.

Hence from all four cases, we conclude that $f(a) = 0$ and $f(b) = 0$. Then $pf(a) + (1-p)f(b) = 0$, a contradiction to (3.12). From all of the consideration above, the assumption (3.10) always leads to a contradiction. Therefore,

$$pf(x) + (1-p)f(y) = f(px + (1-p)y) \qquad \text{for all } \; x, y \in X.$$

$\square$

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