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QUASI-HYPERIDEALS IN MULTIPLICATIVE HYPERRINGS

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จงกล ทำสวน : ควอซี-ไฮเพอร์ไอเดียลในไฮเพอร์ริงการคูณ (QUASI-HYPERIDEALS IN MULTIPLI - CATIVE HYPERRINGS) อ. ที่ปรึกษา : ผศ. ดร. อมร วาสนาวิจิตร์, อ. ที่ปรึกษาร่วม : รศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์ 50 หน้า. ISBN 974-17-1948-5

เราจะกล่าวว่าริงย่อย Q ของริง A เป็น *ควอซี-ไอดีล* ของ A ถ้า $AQ \cap QA \subseteq Q$ โดยที่ AQ [QA] หมายถึง เซตที่ประกอบด้วยสมาชิกที่อยู่ในรูปผลบวกจำกัด $\sum a_i q_i$ [$\sum q_i a_i$] เมื่อ $a_i \in A$ และ $q_i \in Q$ ควอซี-ไอดีลเป็นนัยทั่วไปของไอดีลซ้ายและไอดีลขวา ได้มีการศึกษาควอซี-ไอดีลในริงกันมาเป็นเวลานานแล้ว อีกทั้งได้มีทฤษฎีบทที่น่าสนใจซึ่งเกี่ยวข้องกับควอซี-ไอดีลในริงเกิดขึ้นจำนวนมากเช่นกัน

การดำเนินการไฮเพอร์ บนเซตไม่ว่าง H คือฟังก์ชัน $\circ : H \times H \to P^*(H)$ โดยที่ $P(H)$ หมายถึงเซตกำลังของ H และ $P^*(H)$ หมายถึง $P(H)\backslash\{\varnothing\}$ ในกรณีนี้จะเรียก $(H,\circ)$ ว่า *ไฮเพอร์กรุปพอยด์* และสำหรับเซตย่อยไม่ว่าง X และ Y ของ H ให้ $X\circ Y$ แทนส่วนรวมของเซตในรูปแบบ $x\circ y$ ทั้งหมด โดย $x \in X$ และ $y \in Y$ *กึ่งไฮเพอร์กรุป* คือ ไฮเพอร์กรุปพอยด์ $(H,\circ)$ ซึ่ง $(x\circ y)\circ z = x\circ(y\circ z)$ สำหรับ $x, y, z \in H$ ใดๆ

ไฮเพอร์ริงการคูณ หมายถึง ระบบ $(A, +, \circ)$ ซึ่ง

(i) $(A, +)$ เป็นอาบีเลียนกรุป (กรุปสลับที่)

(ii) $(A,\circ)$ เป็นกึ่งไฮเพอร์กรุป

(iii) $x\circ(y+z) \subseteq x\circ y + x\circ z$ และ $(y+z)\circ x \subseteq y\circ x + z\circ x$ สำหรับ $x, y, z \in A$ ใดๆ

(iv) $(-x)\circ y = x\circ(-y) = -(x\circ y)$ สำหรับ $x, y \in A$ ใดๆ

ถ้าการเป็นสับเซตทั้งสองใน (iii) เป็นการเท่ากัน เราจะกล่าวว่าไฮเพอร์ริงการคูณ $(A, +, \circ)$ *แจกแจงอย่างเข้ม* ไฮเพอร์ริงย่อย ไฮเพอร์ไอดีลซ้าย [ขวา] ไฮเพอร์ไอดีล และควอซี-ไฮเพอร์ไอดีลของไฮเพอร์ริงการคูณนั้นมีบทนิยามในทำนองเดียวกันกับริงย่อย ไอดีลซ้าย [ขวา] ไอดีล และควอซี-ไอดีลของริง ตามลำดับ เราได้ด้วยว่า ควอซี-ไฮเพอร์ไอดีลเป็นนัยทั่วไปของไฮเพอร์ไอดีลซ้ายและไฮเพอร์ไอดีลขวา โดยเฉพาะอย่างยิ่ง ควอซี-ไฮเพอร์ไอดีลในไฮเพอร์ริงการคูณและควอซี-ไฮเพอร์ไอดีลในไฮเพอร์ริงการคูณที่แจกแจงอย่างเข้มเป็นนัยทั่วไปของควอซี-ไอดีลในริง

ในการวิจัยนี้ เราให้ทฤษฎีบทเกี่ยวกับควอซี-ไฮเพอร์ไอดีลในไฮเพอร์ริงการคูณหรือไฮเพอร์ริงการคูณที่แจกแจงอย่างเข้มซึ่งเป็นนัยทั่วไปของทฤษฎีบทเกี่ยวกับควอซี-ไอดีลในริงที่รู้จักกันดีหลายทฤษฎีบท ดังนั้นความจริงในริงที่รู้จักกันดีเหล่านี้จะกลายเป็นกรณีเฉพาะของทฤษฎีบทของเรา

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต......................................
สาขาวิชา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา
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A subring Q of a ring A is called a *quasi-ideal* of A if $AQ \cap QA \subseteq Q$ where AQ [QA] denotes the set of all finite sums of the form $\sum a_i q_i$ [$\sum q_i a_i$] where $a_i \in A$ and $q_i \in Q$. Quasi-ideals are a generalization of left ideals and right ideals. Quasi-ideals in rings have long been studied and a lot of interesting theorems relating to quasi-ideals in rings have been provided.

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to P^*(H)$ where $P(H)$ is the power set of H and $P^*(H) = P(H) \setminus \{\varnothing\}$. In this case, $(H, \circ)$ is called a *hypergroupoid* and for nonempty subsets X and Y of H, let $X \circ Y$ denote the union of all sets $x \circ y$ where x and y run over X and Y, respectively. A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

A *multiplicative hyperring* is a system $(A, +, \circ)$ such that

(i) $(A, +)$ is an abelian group,

(ii) $(A, \circ)$ is a semihypergroup,

(iii) $x \circ (y + z) \subseteq x \circ y + x \circ z$ and $(y + z) \circ x \subseteq y \circ x + z \circ x$ for all $x, y, z \in A$,

(iv) $(-x) \circ y = x \circ (-y) = -(x \circ y)$ for all $x, y \in A$.

If both containments in (iii) are equalities we say that $(A, +, \circ)$ is *strongly distributive*. Subhyperrings, left [right] hyperideals, hyperideals and quasi-hyperideals of multiplicative hyperrings are similar in definitions to subrings, left [right] ideals, ideals and quasi-ideals of rings, respectively. We also have that quasi-hyperideals generalize left hyperideals and right hyperideals. Especially, quasi-hyperideals in multiplicative hyperrings and quasi-hyperideals in strongly distributive multiplicative hyperrings generalize quasi-ideals in rings.

In this research, many well-known theorems on quasi-ideals in rings are generalized to theorems on quasi-hyperideals in multiplicative hyperrings or strongly distributive multiplicative hyperrings. Then those well-known facts in rings become our special cases.

Department **Mathematics**	Student's signature......................................
Field of study **Mathematics**	Advisor's signature......................................
Academic year **2002**	Co-advisor's signature ....

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CONTENTS

page

CHAPTER I

INTRODUCTION AND PRELIMINARIES

Let $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ denote respectively the set of natural numbers (positive integers), the set of integers and the set of real numbers.

For a nonempty set X of an abelian group $(A, +)$, let $\mathbb{Z}X$ denote the set of all finite sums of the form $\sum k_i x_i$ where $k_i \in \mathbb{Z}$ and $x_i \in X$. For $x \in A$, let $\mathbb{Z}x$ denote $\mathbb{Z}\{x\}$. Note that $\mathbb{Z}X$ is the subgroup of $(A, +)$ generated by X.

For nonempty subsets X and Y of a ring $A = (A, +, \cdot)$, let XY denote the set of all finite sums of the form $\sum x_i y_i$ where $x_i \in X$ and $y_i \in Y$. If X consists of a single element x, we write xY for XY. Similarly, if $Y = \{y\}$, we write Xy for XY. For a nonempty subset X of A and $x \in A$, the notations $\mathbb{Z}X$ and $\mathbb{Z}x$ are defined as those in the abelian group $(A, +)$. A *quasi-ideal* of a ring A is a subring Q of A such that $AQ \cap QA \subseteq Q$. Every one-sided ideal of a ring A is clearly a quasi-ideal . The notion of quasi-ideal in rings was first introduced by O. Steinfeld [10] in 1953. Note that if A is commutative, then the quasi-ideals and the ideals of A coincide.

Example 1.1. Let R be a division ring, $n \in \mathbb{N}$ and $M_n(R)$ the ring of all $n \times n$ matrices over R under the usual addition and multiplication of matrices. For $A \in M_n(R)$, let A_{ij} denote the entry of A in the i^{th} row and the j^{th} column. For $k, l \in \{1, 2, \ldots, n\}$, let $Q_n^{kl}(R)$ be the subset of $M_n(R)$ consisting of all matrices $A \in M_n(R)$ such that

$$A_{ij} = 0 \quad \text{if} \quad i \neq k \quad \text{or} \quad j \neq l$$

Then for $k, l \in \{1, 2, \ldots, n\}$, $Q_n^{kl}(R)$ is a subring of $M_n(R)$,

$$M_n(R)Q_n^{kl}(R) = \left\{ \begin{array}{c} \begin{array}{ccccccc} & & & l & & & \\ & & & \downarrow & & & \end{array} \\ \begin{bmatrix} 0 & \ldots & 0 & x_1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & x_2 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & x_n & 0 & \ldots & 0 \end{bmatrix} \end{array} \;\middle|\; x_1, x_2, \ldots, x_n \in R \right\}$$

and

$$Q_n^{kl}(R)M_n(R) = \left\{ \begin{array}{cc} & \\ k \rightarrow & \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ x_1 & x_2 & \ldots & x_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \end{array} \;\middle|\; x_1, x_2, \ldots, x_n \in R \right\}$$

which implies that $M_n(R)Q_n^{kl}(R) \cap Q_n^{kl}(R)M_n(R) = Q_n^{kl}(R)$, so $Q_n^{kl}(R)$ is a quasi-ideal of $M_n(R)$. Moreover, if $n > 1$, then for all $k, l \in \{1, 2, \ldots, n\}$, $Q_n^{kl}(R)$ is neither a left ideal nor a right ideal of $M_n(R)$.

Example 1.1 shows that quasi-ideals of rings are a generalization of one-sided ideals.

It is well-known in ring theory that if a ring A is not a zero ring, then A is a division ring if and only if A and $\{0\}$ are the only left [right] ideals of A. This is also true if the word “left [right] ideals” is replaced by “quasi-ideals”.

Proposition 1.2. ([11], page 6). *Let A be a ring such that $A^2 \neq \{0\}$. Then A is a division ring if and only if A and $\{0\}$ are the only quasi-ideals of A.*

We also have

Proposition 1.3. ([11], page 10 and 12). *Let A be a ring. Then the intersection of a set of quasi-ideals of A is a quasi-ideal of A.*

For a subset X of a ring A, let $(X)_q$ denote the intersection of all quasi-ideals of A containing X. Then for $X \subseteq A, (X)_q$ is the smallest quasi-ideal of A containing X.

Proposition 1.4. (H.J. Wilnert [13]). *For a nonempty subset X of a ring A,*

$$(X)_q = \mathbb{Z}X + (AX \cap XA).$$

A ring A is said to be a (Von Neumann) *regular ring* if for every $x \in A, x = xyx$ for some $y \in A$. These two facts are known.

Proposition 1.5. ([11], page 69). *Let A be a ring . Then A is regular if and only if $QAQ = Q$ for every quasi-ideal Q of A.*

Proposition 1.6. ([11], page 69). *Let A be a ring . Then A is regular if and only if $RL = R \cap L$ for all right ideal R and left ideal L of A.*

It is clearly seen that the intersection of a left ideal and a right ideal of a ring A is a quasi-ideal. However, a quasi-ideal of A may not be obtained in this way. See [11], page 8 , [6], [1] and [4] for examples. A quasi-ideal Q of A is said to have *the intersection property* if $Q = L \cap R$ for some left ideal L and right ideal R of A, and we say that A *has the intersection property of quasi-ideals* if every quasi-ideal of A has the intersection property.

It is known that every ring with a one-sided identity and every regular ring has the intersection property of quasi-ideals. This is a special case of the following proposition.

Proposition 1.7. ([11], page 9). *Let Q be a quasi-ideal of a ring A. If $Q \subseteq QA$ or $Q \subseteq AQ$, then*

$$Q = (Q + AQ) \cap (Q + QA).$$

In this case, Q has the intersection property (since $Q + AQ$ and $Q + QA$ are a left ideal and right ideal of A, respectively.)

H.J. Wilnert [13] and Z. Moucheng and etc. [6] characterized quasi-ideals of rings having the intersection property as follows:

Proposition 1.8. (H.J. Wilnert [13]). *Let Q be a quasi-ideal of a ring A. Then the following statements are equivalent.*

(i) *Q has the intersection property.*

(ii) $(Q + AQ) \cap (Q + QA) = Q$.

(iii) $AQ \cap (Q + QA) \subseteq Q$.

(iv) $QA \cap (Q + AQ) \subseteq Q$.

Proposition 1.9. (Z. Moucheng and etc. [6]). *Let X be a nonempty subset of a ring A. Then the following statements are equivalent.*

(i) *$(X)_q$ has the intersection property.*

(ii) $(\mathbb{Z}X + AX) \cap (\mathbb{Z}X + XA) = (X)_q$.

(iii) $AX \cap (\mathbb{Z}X + XA) \subseteq (X)_q$.

(iv) $XA \cap (\mathbb{Z}X + AX) \subseteq (X)_q$.

Z. Moucheng and etc. [6] also characterized rings having the intersection property of quasi-ideals.

Proposition 1.10. (Z. Moucheng and etc. [6]). *The following statements for a ring A are equivalent.*

(i) *A has the intersection property of quasi-ideals.*

(ii) *For any finite nonempty subset X of A, we have*

$$AX \cap (\mathbb{Z}X + XA) \subseteq \mathbb{Z}X + (AX \cap XA)\,(= (X)_q)\,.$$

(iii) *For any finite subset $X = \{x_1, x_2, \ldots, x_n\}$ of A and $a_1, a_2, \ldots, a_n \in A$, if*

$$\sum_{i=1}^{n}(a_i x_i + k_i x_i + x_i a_i') = 0,$$

for some $a_i' \in A$ and $k_i \in \mathbb{Z}$, then $\sum\limits_{i=1}^{n} a_i x_i \in (X)_q$.

We call a nonzero quasi-ideal Q of a ring A a *minimal quasi-ideal* of A if Q does not properly contain a nonzero quasi-ideal of A. The following fact is clearly true.

Proposition 1.11. *A nonzero quasi-ideal Q of a ring A is a minimal quasi-ideal of A if and only if $(x)_q = Q$ for all $x \in Q \backslash \{0\}$.*

Recall that for an element x of a ring A, the principal left [right] ideal of A generated by x is $\mathbb{Z}x + Ax$ [$\mathbb{Z}x + xA$] which is denoted by $(x)_l$ [$(x)_r$]. P. N. Stewart [12] gave a necessary and sufficient condition for a quasi-ideal of a ring to be minimal in terms of principal left ideals and principal right ideals as follows:

Proposition 1.12. (P. N. Stewart [12]). *A quasi-ideal Q of a ring A is a minimal quasi-ideal of A if and only if for any two nonzero elements x and y of Q,*

$$(x)_l = (y)_l \quad \textit{and} \quad (x)_r = (y)_r.$$

A *minimal left* [*right*] *ideal* of a ring A is a nonzero left [right] ideal of A which does not properly contain a nonzero left [right] ideal of A. There is a relationship among minimal quasi-ideals, minimal left ideals and minimal right ideals as follows:

Proposition 1.13. ([11], page 34). *If L and R are a minimal left ideal and a minimal right ideal of a ring A, then either $L \cap R = \{0\}$ or $L \cap R$ is a minimal quasi-ideal of A.*

A sufficient condition for a quasi-ideal of a ring A to be minimal are as follows:

Proposition 1.14. ([11], pages 37). *Let Q be a quasi-ideal of a ring A. If Q is a division subring of A, then Q is a minimal quasi-ideal of A.*

Next, we shall give the definitions of a multiplicative hyperring and their subhyperrings, left [right] hyperideals, hyperideals quasi-hyperideals accordingly as those in rings.

For a set X, let $P(X)$ denote the power set of X and let $P^*(X) = P(X) \backslash \{\varnothing\}$.

A *hyperoperation* on a nonempty set H is a mapping of $H \times H$ into $P^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets X and Y of H, let

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} (x \circ y)$$

and let $X \circ x = X \circ \{x\}$ and let $x \circ X = \{x\} \circ X$ for all $x \in H$. An element e of H is called an *identity* of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$. An element e of H is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$ for all $x \in H$. Then $(H, \circ)$ has at most one scalar identity.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. A *subsemihypergroup* of a semihypergroup $(H, \circ)$ is a nonempty subset H_1 of H which forms a semihypergroup under the hyperoperation $\circ$ of H restricted to H_1. It is clear that the intersection of a set of subsemihypergroups of $(H, \circ)$ is a subsemihypergroup of $(H, \circ)$ if it is nonempty.

The following example of a semihypergroup is known.

Proposition 1.15. ([2], page 11). *Let P be a nonempty subset of a semigroup S. Define a hyperoperation $\circ$ on S by*

$$x \circ y = xPy \qquad \textit{for all } x, y \in S.$$

Then $(S, \circ)$ is a semihypergroup.

R. Rota [8] first introduced the notion of multiplicative hyperrings in 1982 as follows:

A *multiplicative hyperring* is a triple $(A, +, \circ)$ such that

(i) $(A, +)$ is an abelian group,

(ii) $(A, \circ)$ is a semihypergroup,

(iii) $x \circ (y + z) \subseteq x \circ y + x \circ z$ and $(y + z) \circ x \subseteq y \circ x + z \circ x$ for all $x, y, z \in A$,

(iv) $(-x) \circ y = x \circ (-y) = -(x \circ y)$ for all $x, y \in A$.

The operation $+$ and the hyperoperation $\circ$ of a multiplicative hyperring $(A, +, \circ)$ are called the *addition* and the *multiplication* of $(A, +, \circ)$, respectively. If both containments in (iii) are equalities, we say that $(A, +, \circ)$ is *strongly distributive*. That is, a strongly distributive multiplicative hyperring is a triple $(A, +, \circ)$ such that

(i) $(A, +)$ is an abelian group,

(ii) $(A, \circ)$ is a semihypergroup,

(iii) $x \circ (y + z) = x \circ y + x \circ z$ and $(y + z) \circ x = y \circ x + z \circ x$ for all $x, y, z \in A$,

(iv) $(-x) \circ y = x \circ (-y) = -(x \circ y)$ for all $x, y \in A$.

A multiplicative hyperring $(A, +, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$ for all $x, y \in A$. A *unitary multiplicative hyperring* is a multiplicative hyperring $(A, +, \circ)$ such that $(A, \circ)$ has a scalar identity which is called the *unitary element* of $(A, +, \circ)$ and it is usually denoted by u. A *unitary multiplicative hyperring* is a multiplicative hyperring with a unitary element. Multiplicative hyperrings are clearly a generalization of rings. By the definitions, strongly distributive multiplicative hyperrings give a closer generalization of rings.

For a nonempty subset X of a multiplicative hypering $(A, +, \circ)$ and $x \in A$, we use the notations $\mathbb{Z}X$ and $\mathbb{Z}x$ in the same meaning as those in a ring, that is,

$$\begin{aligned} \mathbb{Z}X &= \text{the set of all finite sums of the form } \sum k_i x_i \\ &\quad \text{where } k_i \in \mathbb{Z} \text{ and } x_i \in X, \\ \mathbb{Z}x &= \mathbb{Z}\{x\}, \end{aligned}$$

so $\mathbb{Z}x = \{nx \mid n \in \mathbb{Z}\}$. If $(A, +, \circ)$ is a multiplicative hyperring, then for $a, b, c, d \in A$, $a \circ b$ and $c \circ d$ are nonempty subsets of A and $a \circ b \circ c \circ d = (a \circ b) \circ (c \circ d)$ which is the set $\bigcup\limits_{\substack{s \in a\circ b \\ t \in c \circ d}} (s \circ t)$, not the union of all finite sums of the form $\sum s_i \circ t_i$ where $s_i \in a \circ b$ and $t_i \in c \circ d$. Because of this fact, we cannot use the notation $X \circ Y$ in the similar meaning as that in rings where X and Y are nonempty subsets of a multiplicative hyperring. To distinguish between the following two subsets of a multiplicative hyperring $(A, +, \circ)$: $\bigcup\limits_{\substack{x \in X \\ y \in Y}} x \circ y$ and the union of all finite sums of the form $\sum x_i \circ y_i$ where $x_i \in X$ and $y_i \in Y$ for nonempty subsets X, Y of A , we shall let $X \circ Y$ and $< X \circ Y >$ denote the first set and the second one, respectively, that is,

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} (x \circ y), \quad < X \circ Y > \ = \bigcup_{\substack{x_i \in X, y_i \in Y \\ n \in \mathbb{N}}} (x_1 \circ y_1 + x_2 \circ y_2 + \cdots + x_n \circ y_n).$$

Because of the associative law of the hyperoperation $\circ$ of $(A, +, \circ)$, it follows that if $X_1, X_2, \ldots, X_n$ are nonempty subsets of $(A, +, \circ)$, then

$$X_1 \circ X_2 \circ \cdots \circ X_n = \bigcup_{x_1 \in X_1, \ldots, x_n \in X_n} (x_1 \circ x_2 \circ \cdots \circ x_n),$$

$$< X_1 \circ X_2 \circ \cdots \circ X_n > \;\; = \text{the union of all finite sums of the form}$$
$$\sum x_i^{(1)} \circ x_i^{(2)} \circ \cdots \circ x_i^{(n)} \quad \text{where } x_i^{(1)} \in X_1,$$
$$x_i^{(2)} \in X_2, \ldots, x_i^{(n)} \in X_n.$$

For $a \in A$ and $\varnothing \neq X \subseteq A$, let $X \circ a$ and $a \circ X$ denote $X \circ \{a\}$ and $\{a\} \circ X$, respectively. For elements a and b of A and $n \in N$, we shall let $n(a \circ b)$ and $(-n)(a \circ b)$ denote

$$a \circ b + a \circ b + \cdots + a \circ b \quad (n \;\text{ copies}) \qquad \text{and}$$

$$(-(a \circ b)) + (-(a \circ b)) + \cdots + (-(a \circ b)) \quad (n \;\text{ copies}),$$

respectively. The following facts are obvious.

Proposition 1.16. *If $(A, +, \circ)$ is a strongly distributive multiplicative hyperring, then for $a, b \in A$,*

$$< A \circ a > = \bigcup_{x \in A} (x \circ a) = A \circ a, \quad < a \circ A > = \bigcup_{x \in A} (a \circ x) = a \circ A,$$
$$< a \circ A \circ b > = \bigcup_{x \in A} (a \circ x \circ b) = a \circ A \circ b.$$

The following three propositions are known.

Proposition 1.17. (R. Rota [8]). *In a multiplicative hyperring $(A, +, \circ)$,*

$$(a + b) \circ (c + d) \subseteq a \circ c + a \circ d + b \circ c + b \circ d \quad \textit{for all } a, b, c, d \in A.$$

In particular, if $(A, +, \circ)$ is strongly distributive, then

$$(a + b) \circ (c + d) = a \circ c + a \circ d + b \circ c + b \circ d \quad \textit{for all } a, b, c, d \in A.$$

The following proposition will be useful and we omit its easy proof.

Proposition 1.18. *Let $(A,+,\circ)$ is a multiplicative hyperring. Then the following statements hold.*

1. *For $x,y\in A$ and $n\in\mathbb{Z}$, $x\circ(ny)\subseteq n(x\circ y)$ and $(nx)\circ y\subseteq n(x\circ y)$.*
2. *For nonempty subsets X,Y and Z of A,*

$$<< X >> =< X >=< X > + < X >,$$

$$< X\circ Y > =<< X > \circ < Y >> \quad \text{(from Proposition 1.17)},$$

$$< X\circ < Y\circ Z >> =< X\circ Y\circ Z >=<< X\circ Y > \circ Z > .$$

3. *If A_1 is a subgroup of $(A,+)$ then $\mathbb{Z}A_1 = A_1$.*
4. *For nonempty subsets X,Y,Z and V of A, $< X+Y > \circ < Z+V >\subseteq< X\circ Z > + < X\circ V > + < Y\circ Z > + < Y\circ V >$* (from Proposition 1.17).
5. *For nonempty subsets X and Y of A, $< X\circ(\mathbb{Z}Y) >\subseteq \mathbb{Z}(X\circ Y)$ and $< (\mathbb{Z}X)\circ Y >\subseteq \mathbb{Z}(X\circ Y)$* (from (1)).
6. *For nonempty subsets X and Y of A, $\mathbb{Z}(A\circ X) =< A\circ X >$, $\mathbb{Z}(X\circ A) = < X\circ A >$ and $\mathbb{Z}(X\circ A\circ Y) =< X\circ A\circ Y >$.*

If $(A,+,\circ)$ is strongly distributive, then " $\subseteq$" in 1. and 5. can be replaced by " $=$" as follows:

7. *For $x,y\in A$ and $n\in\mathbb{N}$, $x\circ(ny) = n(x\circ y) = (nx)\circ y$.*
8. *For nonempty subsets X and Y of A, $< X\circ(\mathbb{Z}Y) >= \mathbb{Z}(X\circ Y) =< (\mathbb{Z}X)\circ Y >$.*

Proposition 1.19. (R. Rota [8]). *In a strongly distributive multiplicative hyperring $(A,+,\circ)$, $0\in a\circ 0$ and $0\in 0\circ a$ for every $a\in A$.*

Some examples of multiplicative hyperrings are as follows:

Example 1.20. (R. Rota [8]). Let I be an ideal of a ring $(R,+,\cdot)$. Define a hyperoperation $\circ$ on R by

$$x\circ y = xy + I \quad \text{for all } x,y\in R.$$

Then $(R, +, \circ)$ is a strongly distributive multiplicative hyperring. If $|I| > 1$, then it is not unitary.

Example 1.21. (R. Rota [8]). Let V be a vector space over a field F. Define a hyperoperation $\circ$ on V by

$$u \circ v = \text{the subspace of } V \text{ generated by } u \text{ and } v \qquad \text{for all } u, v \in V,$$

that is,

$$u \circ v = Fu + Fv \qquad \text{for all } u, v \in V.$$

Then $(V, +, \circ)$ is a commutative multiplicative hyperring where $+$ is the addition on V, $0 \in 0 \circ u$ for every $u \in V$ but it is not strongly distributive if dim $V \geq 1$. Moreover, it is not unitary if dim $V > 0$.

Example 1.22. ([7], page 79). Let $(R, +, \cdot)$ be a ring. Define a hyperoperation $\circ$ on R by

$$a \circ b = \{ab, 2ab, 3ab, \ldots\} \qquad \text{for all } a, b \in R.$$

Then $(R, +, \circ)$ is a multiplicative hyperring but it need not be strongly distributive and it need not be unitary. Moreover, $a \circ 0 = \{0\} = 0 \circ a$ for all $a \in R$.

Next, let $(A, +, \circ)$ be a multiplicative hyperring. For a nonempty subset B of A, one says that B is a *subhyperring* of $(A, +, \circ)$ if B is itself a multiplicative hyperring under the operation $+$ and the hyperoperation $\circ$ on A restricted to B. A subhyperring B of $(A, +, \circ)$ is called a *left* [*right*] *hyperideal* of $(A, +, \circ)$ if $A \circ B \subseteq B$ $[B \circ A \subseteq B]$. If B is both a left and a right hyperideal of $(A, +, \circ)$, then it is called a (*two-sided*)*hyperideal* of $(A, +, \circ)$. A subhyperring Q of $(A, +, \circ)$ is called a *quasi-hyperideal* of $(A, +, \circ)$ if $< A \circ Q > \cap < Q \circ A > \subseteq Q$. Then quasi-hyperideals are also a generalization of left hyperideals and right hyperideals. Especially, quasi-hyperideals in a multiplicative hyperrings generalize quasi-ideals in rings. Note that if $(A,+,\circ)$ is commutative, then quasi-hyperideals, left [right] hyperideals and hyperideals of $(A,+,\circ)$ coincide.

For a multiplicative hyperring $(A,+,\circ)$ and a hyperideal I of $(A,+,\circ)$, R. Rota [8] constructed a multiplicative hypering from $(A,+,\circ)$ by I which may be called the *quotient multiplicative hyperring of* $(A,+,\circ)$ *by* I as follows:

Example 1.23. (R. Rota [8]). Let $(A,+,\circ)$ be a multiplicative hyperring and I a hyperideal of $(A,+,\circ)$. Define a hyperoperation $*$ on A/I by

$$(a+I)*(b+I) = \{\ c+I \mid c \in a \circ b\ \} \text{ for all } a,b \in A.$$

Then $(A/I,\oplus,*)$ is a multiplicative hyperring where $(A/I,\oplus)$ is the quotient group of $(A,+)$ relative to I. Moreover $(A/I,\oplus,*)$ is strongly distributive if $(A,+,\circ)$ is. If $|I|>1$, then it is not unitary.

An element a of a multiplicative hyperring $(A,+,\circ)$ is said to be *regular* if $a \in a\circ x\circ a$ for some $x \in A$ and we call $(A,+,\circ)$ is a *regular multiplicative hyperring* if every element of A is regular in $(A,+,\circ)$.

The intersection of a left hyperideal and a right hyperideal of a multiplicative hyperring $(A,+,\circ)$ is clearly a quasi-hyperideal. However, a quasi-hyperideal may not be obtained in this way. A quasi-hyperideal of a multiplicative hyperring $(A,+,\circ)$ is said to have *the intersection property* if it is the intersection of a left hyperideal and a right hyperideal of $(A,+,\circ)$ and we say that $(A,+,\circ)$ *has the intersection property of quasi-hyperideals* if every quasi-hyperideal of A has the intersection property.

A nonzero left hyperideal L of a multiplicative hyperring $(A,+,\circ)$ is said to be *minimal* if L does not properly contain a nonzero left hyperideal of $(A,+,\circ)$. A minimal right [two-sided, quasi-] hyperideal of $(A,+,\circ)$ is defined similarly.

In Chapter II, Proposition 1.2 to Proposition 1.6 are generalized to quasi-hyperideals in multiplicative hyperrings. In this chapter, we construct a multiplicative hyperring from a ring by using Proposition 1.15 and some of its quasi-hyperideals are provided.

We generalize Proposition 1.7 – Proposition 1.10 to quasi-hyperideals in multtiplicative hyperrings in Chapter III.

Finally, minimal quasi-hyperideals of multiplicative hyperrings are studied to generalize Proposition 1.11 – Proposition 1.14 in Chapter IV. The study of minimal quasi-ideals of rings of strictly upper triangular $n\times n$ matrices by Y. Kemprasit and P. Juntarakhajorn in [5] motivates us to investigate minimal quasi-hyperideals of the multiplicative hyperrings of upper triangular $n \times n$ matrices with a hyperoperation defined as in Proposition 1.15. Many nice results relating to minimal quasi-hyperideals of these multiplicative hyperrings are given in this chapter.

CHAPTER II

GENERAL PROPERTIES AND EXAMPLES

In this chapter, we first generalize Proposition 1.2 – Proposition 1.6 of Chapter I. Then a multiplicative hyperring defined from a ring is provided and we show that every quasi-ideal of the given ring is a quasi-hyperideal of the constructed multiplicative hyperring.

To generalize Proposition 1.2, the following three lemmas are required.

Lemma 2.1. *Let $(A, +, \circ)$ be a multiplicative hyperring. Then the following statements hold.*

(i) *The intersection of a collection of subhyperrings of $(A, +, \circ)$ is a subhyperring of $(A, +, \circ)$.*

(ii) *The intersection of a collection of left [right] hyperideals of $(A, +, \circ)$ is a left [right] hyperideal of $(A, +, \circ)$.*

Proof. Let $\{C_\alpha \mid \alpha \in \Lambda\}$ be a set of subhyperrings of $(A, +, \circ)$. Then $\bigcap_{\alpha\in\Lambda} C_\alpha$ is a subgroup of $(A, +)$. Hence $\bigcap_{\alpha\in\Lambda} C_\alpha$ is a subhyperring (see page 6). If each C_α is a left hyperideal of $(A, +, \circ)$, we have that for every $\beta \in \Lambda$,

$$A \circ (\bigcap_{\alpha\in\Lambda} C_\alpha) \subseteq A \circ C_\beta \subseteq C_\beta$$

which implies that, $A \circ (\bigcap_{\alpha\in\Lambda} C_\alpha) \subseteq \bigcap_{\alpha\in\Lambda} C_\alpha$. Hence $\bigcap_{\alpha\in\Lambda} C_\alpha$ is a left hyperideal of $(A, +, \circ)$. Dually, if each C_α is a right hyperideal of $(A, +, \circ)$, then $\bigcap_{\alpha\in\Lambda} C_\alpha$ is a right hyperideal of $(A, +, \circ)$. Hence the lemma is proved. $\square$

Lemma 2.2. *Let $(A, +, \circ)$ be a multiplicative hyperring and $\varnothing \neq X \subseteq A$. Then the following statements hold.*

(i) $0 \in < A \circ X >$ *and* $0 \in < X \circ A >$.

(ii) $< A \circ X >$ *and* $< X \circ A >$ *are a left hyperideal and a right hyperideal of* $(A, +, \circ)$, *respectively.*

(iii) $\mathbb{Z}X+ < A \circ X >$ *and* $\mathbb{Z}X+ < X \circ A >$ *are respectively a left hyperideal and a right hyperideal of* $(A, +, \circ)$ *containing* X.

Proof. (i) Let $a \in A$ and $x \in X$. If $y \in a \circ x$, then $-y \in -(a \circ x) = (-a) \circ x \subseteq A \circ X$ which implies that

$$0 = y - y \in a \circ x + (-a) \circ x \subseteq < A \circ X > .$$

We can show similarly that $0 \in < X \circ A >$.

(ii) By Proposition 1.18(2), $< A \circ X > + < A \circ X > = < A \circ X >$. By (i), $0 \in < A \circ X >$. Let $a \in < A \circ X >$. Then $a \in \sum\limits_{i=1}^{n} r_i \circ x_i$ for some $r_i \in A, x_i \in X$ and $n \in \mathbb{N}$. Since $-a \in -(\sum\limits_{i=1}^{n} r_i \circ x_i) = \sum\limits_{i=1}^{n}(-(r_i \circ x_i)) = \sum\limits_{i=1}^{n}(-r_i) \circ x_i$, we have that $-a \in < A \circ X >$. Then $< A \circ X >$ is a subgroup of $(A, +)$. By Proposition 1.18(2), $A \circ < A \circ X > \subseteq < A \circ < A \circ X >> = < A \circ A \circ X > \subseteq < A \circ X >$. Hence $< A \circ X >$ is a left hyperideal of $(A, +, \circ)$.

Similarly, $< X \circ A >$ is a right hyperideal of $(A, +, \circ)$.

(iii) Since $0 \in < A \circ X >$ from (i) and $X \subseteq \mathbb{Z}X$, it follows that $X \subseteq \mathbb{Z}X+ < A \circ X >$. We know that $\mathbb{Z}X$ is a subgroup of $(A, +)$ and by (ii), $< A \circ X >$ is a subgroup of $(A, +)$. It then follows that $\mathbb{Z}X+ < A \circ X >$ is a subgroup of $(A, +)$. We also have

$$\begin{aligned} A \circ (\mathbb{Z}X+ < A \circ X >) &\subseteq A \circ (\mathbb{Z}X) + A \circ < A \circ X > \\ &\subseteq < A \circ (\mathbb{Z}X) > + < A \circ < A \circ X >> \end{aligned}$$

$\subseteq < \mathbb{Z}(A \circ X) > + < A \circ A \circ X >$

from Proposition 1.18(2) and (5)

$\subseteq << A \circ X >> + < A \circ X >$ from Proposition 1.18(6)

$\subseteq < A \circ X >$ from Proposition 1.18(2)

$\subseteq \mathbb{Z}X + < A \circ X >$ since $0 \in \mathbb{Z}X$.

Hence $\mathbb{Z}X + < A \circ X >$ is a left hyperideal of $(A, +, \circ)$ containing X. We can obtain similarly that $\mathbb{Z}X + < X \circ A >$ is a right hyperideal of $(A, +, \circ)$ containing X. $\square$

Let $(A, +, \circ)$ be a multiplicative hyperring. Then by Lemma 2.1, the intersection of a collection of left [right] hyperideals of $(A, +, \circ)$ is also a left [right] hyperideal of $(A, +, \circ)$. For $\varnothing \neq X \subseteq A$, let $(X)_l[(X)_r]$ denote the intersection of all left [right] hyperideals of $(A, +, \circ)$ containing X. Therefore, $(X)_l[(X)_r]$ is the smallest left [right] hyperideal of $(A, +, \circ)$ containing X and it is called the *left [right] hyperideal of $(A, +, \circ)$ generated by X*. For $a \in A$, let $(a)_l$ $[(a)_r]$ denote $(\{a\})_l$ $[(\{a\})_r]$ which is called the *principal left [right] hyperideal of $(A, +, \circ)$ generated by a.*

Lemma 2.3. *In a multiplicative hyperring $(A, +, \circ)$,*

$$(X)_l = \mathbb{Z}X + < A \circ X > \quad \textit{and} \quad (X)_r = \mathbb{Z}X + < X \circ A >$$

for every nonempty subset X of A. In particular, for $a \in A$,

$$(a)_l = \mathbb{Z}a + < A \circ a > \quad \textit{and} \quad (a)_r = \mathbb{Z}a + < a \circ A > .$$

Proof. From Lemma 2.2 (iii), $(X)_l \subseteq \mathbb{Z}X + < A \circ X >$. Since $(X)_l$ is a subgroup of $(A, +)$ containing X, we have that $\mathbb{Z}X \subseteq (X)_l$ and $< A \circ X > \subseteq < A \circ (X)_l > \subseteq < (X)_l > = (X)_l$. Hence $\mathbb{Z}X + < A \circ X > \subseteq (X)_l$. Therefore we deduce that $(X)_l = \mathbb{Z}X + < A \circ X >$. We can show similarly that $(X)_r = \mathbb{Z}X + < X \circ A >$. $\square$

Theorem 2.4. *Let $(A, +, \circ)$ be a multiplicative hyperring such that $A \circ A \neq \{0\}$. Then $< A \circ x >= A =< x \circ A >$ for all $x \in A \backslash \{0\}$ if and only if $(A, +, \circ)$ has no proper nonzero quasi-hyperideals.*

Proof. Assume that for every $x \in A \backslash \{0\}, < A \circ x >= A =< x \circ A >$. Let Q be a nonzero quasi-hyperideal of $(A, +, \circ)$. Then there exists $k \in Q \backslash \{0\}$. By assumption, $A =< A \circ k >=< k \circ A >$, so $A =< k \circ A > \cap < A \circ k > \subseteq < Q \circ A > \cap < A \circ Q > \subseteq Q$. Thus $A = Q$. This prove that $(A, +, \circ)$ has no proper nonzero quasi-hyperideals.

Conversely, assume that $(A, +, \circ)$ has no proper nonzero quasi-hyperideals. Let $a \in A \backslash \{0\}$. Since $(a)_l$ is the principal left hyperideal of $(A, +, \circ)$ generated by a, $(a)_l$ is a nonzero quasi-hyperideal of $(A, +, \circ)$. By assumption, $(a)_l = A$. Hence

$$
\begin{aligned}
< A \circ a > &\subseteq < A \circ (a)_l > \\
&\subseteq < A \circ (\mathbb{Z}a + < A \circ a >) > && \text{by Lemma 2.3} \\
&\subseteq < A \circ (\mathbb{Z}a) + A \circ < A \circ a >> \\
&\subseteq < \mathbb{Z}(A \circ a) + < A \circ A \circ a >> && \text{by Proposition 1.18(2) and (5)} \\
&\subseteq << A \circ a > + < A \circ a >> && \text{by Proposition 1.18(6)} \\
&=< A \circ a > && \text{by Proposition 1.18(2)}
\end{aligned}
$$

which implies that $< A \circ a >=< A \circ (a)_l >=< A \circ A > \neq \{0\}$. But $< A \circ a >$ is a left hyperideal of $(A, +, \circ)$ by Lemma 2.2(ii), so $< A \circ a >$ is a nonzero quasi-hyperideal of $(A, +, \circ)$. It then follows from the assumption that $< A \circ a >= A$. Similarly, we obtain that $< a \circ A >= A$. Therefore $< A \circ a >= A =< a \circ A >$. □

We know that for a ring A with $|A| > 1$, A is a division ring if and only if $Ax = A = xA$ for all $x \in A \backslash \{0\}$. Then Proposition 1.2 is a corollary of the above theorem.

Corollary 2.5. *Let A be a ring such that $A^2 \neq \{0\}$. Then A is a division ring if and only if A and $\{0\}$ are the only quasi-ideals of A.*

We also have

Theorem 2.6. *Let $(A,+,\circ)$ be a multiplicative hyperring. Then the intersection of a set of quasi-hyperideals of $(A,+,\circ)$ is a quasi-hyperideal of $(A,+,\circ)$.*

Proof. Let Q_α $(\alpha \in \Lambda)$ be a set of quasi-hyperideals of $(A,+,\circ)$. Then $\bigcap\limits_{\alpha\in\Lambda} Q_\alpha$ is a subgroup of $(A,+)$ since each Q_α is a subgroup of $(A,+)$. We have that for every $\beta \in \Lambda$,

$$< A \circ \big(\bigcap_{\alpha\in\Lambda} Q_\alpha\big) > \cap < \big(\bigcap_{\alpha\in\Lambda} Q_\alpha\big) \circ A > \subseteq < A \circ Q_\beta > \cap < Q_\beta \circ A > \subseteq Q_\beta.$$

Consequently, $< A \circ \big(\bigcap\limits_{\alpha\in\Lambda} Q_\alpha\big) > \cap < \big(\bigcap\limits_{\alpha\in\Lambda} Q_\alpha\big) \circ A > \subseteq \bigcap\limits_{\alpha\in\Lambda} Q_\alpha$. Hence $\bigcap\limits_{\alpha\in\Lambda} Q_\alpha$ is a quasi-hyperideal of $(A,+,\circ)$. □

Proposition 1.3 is an immediate consequence of the above theorem.

Corollary 2.7. *Let A be a ring. Then the intersection of a set of quasi-ideals of A is a quasi-ideal of A.*

Let $(A,+,\circ)$ be a multiplicative hyperring. For $X \subseteq A$, the *quasi-hyperideal of* $(A,+,\circ)$ *generated by* X is the intersection of all quasi-hyperideals of $(A,+,\circ)$ containing X which is denoted by $(X)_q$. Then for $X \subseteq A$, $(X)_q$ is the smallest quasi-hyperideal of $(A,+,\circ)$ containing X. For $a \in A$, let $(a)_q$ denote $(\{a\})_q$ and it is called the *principal quasi-hyperideal of* $(A,+,\circ)$ *generated by* a .

Theorem 2.8. *For a nonempty subset X of a multiplicative hyperring $(A,+,\circ)$,*

$$(X)_q = \mathbb{Z}X + (< A \circ X > \cap < X \circ A >).$$

In particular, for $a \in A$*,*

$$(a)_q = \mathbb{Z}a + (< A \circ a > \cap < a \circ A >).$$

Proof. First, we show that $\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$ is a quasi-hyperideal of $(A, +, \circ)$ containing X. Since $X \subseteq \mathbb{Z}X$ and $0 \in < A \circ X > \cap < X \circ A >$ (from Lemma 2.2(i)), $X \subseteq \mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$. By Lemma 2.2(ii), $< A \circ X >$ and $< X \circ A >$ are subgroups of $(A, +)$. Then $(< A \circ X > \cap < X \circ A >)$ is a subgroup of $(A, +)$. Since $\mathbb{Z}X$ is a subgroup of $(A, +)$, we have that $\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$ is a subgroup of $(A, +)$. Since

$$
\begin{aligned}
&< A \circ (\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)) > \\
&\qquad \subseteq < A \circ (\mathbb{Z}X + < A \circ X >) > \\
&\qquad \subseteq < A \circ (\mathbb{Z}X) + A \circ < A \circ X >> \\
&\qquad \subseteq < \mathbb{Z}(A \circ X) + < A \circ A \circ X >> \\
&\qquad\qquad \text{by Proposition 1.18(2) and (5)} \\
&\qquad \subseteq << A \circ X > + < A \circ X >> \\
&\qquad\qquad \text{by Proposition 1.18(6)} \\
&\qquad \subseteq << A \circ X >> \subseteq < A \circ X > \\
&\qquad\qquad \text{from Proposition 1.18(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&< (\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)) \circ A > \\
&\qquad \subseteq < (\mathbb{Z}X + < X \circ A >) \circ A > \\
&\qquad \subseteq < (\mathbb{Z}X) \circ A + < X \circ A > \circ A > \\
&\qquad \subseteq < \mathbb{Z}(X \circ A) + < X \circ A \circ A >> \\
&\qquad\qquad \text{by Proposition 1.18(2) and (5)}
\end{aligned}
$$

$$\subseteq << X \circ A > + < X \circ A >>$$

by Proposition 1.18(6)

$$\subseteq << X \circ A >> \subseteq < X \circ A >$$

by Proposition 1.18(2)

it follows that

$$< A \circ (\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)) > \cap < (\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)) \circ A >$$

$$\subseteq < A \circ X > \cap < X \circ A >$$

$$\subseteq \mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$$

from Lemma 2.2(i).

Hence $\mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$ is a quasi-hyperideal of $(A, +, \circ)$ containing X. Then we deduce that $(X)_q \subseteq \mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$.

Since $((X)_q, +)$ is a subgroup of $(A, +)$ containing X, $\mathbb{Z}X \subseteq (X)_q$. We also have $< A \circ X > \cap < X \circ A > \subseteq < A \circ (X)_q > \cap < (X)_q \circ A > \subseteq (X)_q$ which implies that $\mathbb{Z}X + \ (< A \circ X > \cap < X \circ A >) \subseteq (X)_q$. Therefore $(X)_q = (< A \circ X > \cap < X \circ A >)$. □

Proposition 1.4 becomes a consequence of Theorem 2.8.

Corollary 2.9. *For a nonempty subset X of a ring A,*

$$(X)_q = \mathbb{Z}X + (AX \cap XA).$$

Theorem 2.10. *Let $(A, +, \circ)$ be a multiplicative hyperring.*

(i) *If $(A, +, \circ)$ is regular, then $Q = Q \circ A \circ Q$ for every quasi-hyperideal Q of $(A, +, \circ)$.*

(ii) *The converse of* (i) *holds if $(A, +, \circ)$ is strongly distributive.*

Proof. (i) Assume that $(A,+,\circ)$ is regular and let Q be a quasi-hyperideal of $(A,+,\circ)$. Then for $x \in Q$, $x \in x \circ A \circ x \subseteq Q \circ A \circ Q$. That means $Q \subseteq Q \circ A \circ Q$. But $Q \circ A \circ Q = (Q \circ A) \circ Q \subseteq A \circ Q$ and $Q \circ A \circ Q = Q \circ (A \circ Q) \subseteq Q \circ A$, so we have

$$Q \circ A \circ Q \subseteq (A \circ Q) \cap (Q \circ A) \subseteq < A \circ Q > \cap < Q \circ A > \subseteq Q.$$

Hence $Q = Q \circ A \circ Q$.

(ii) Assume that$(A,+,\circ)$ is strongly distributive and $Q \circ A \circ Q = Q$ for every quasi-hyperideal Q of $(A,+,\circ)$. To show that $(A,+,\circ)$ is regular, let $a \in A$. Then $(a)_l \cap (a)_r$ is a quasi-hyperideal of $(A,+,\circ)$. It thus follows that

$$\begin{aligned}
a &\in (a)_l \cap (a)_r \\
&= ((a)_l \cap (a)_r) \circ A \circ ((a)_l \cap (a)_r) \qquad \text{by assumption} \\
&\subseteq (a)_r \circ A \circ (a)_l \\
&= (\mathbb{Z}a + a \circ A) \circ A \circ (\mathbb{Z}a + A \circ a) \\
&\qquad\qquad \text{from Proposition 1.16 and Lemma 2.3} \\
&\subseteq (\mathbb{Z}a) \circ A \circ (\mathbb{Z}a) + (\mathbb{Z}a) \circ A \circ (A \circ a) \\
&\quad + (a \circ A) \circ A \circ (\mathbb{Z}a) + (a \circ A) \circ A \circ (A \circ a) \\
&\qquad\qquad \text{from Proposition 1.18(4)} \\
&\subseteq \mathbb{Z}(a \circ A \circ a) + \mathbb{Z}(a \circ A \circ A \circ a) + \mathbb{Z}(a \circ A \circ A \circ a) \\
&\quad + (a \circ A \circ a) \\
&\qquad\qquad \text{from Proposition 1.18(5)} \\
&\subseteq \mathbb{Z}(a \circ A \circ a) + \mathbb{Z}(a \circ A \circ a) + \mathbb{Z}(a \circ A \circ a) \\
&\quad + \mathbb{Z}(a \circ A \circ a) \\
&= a \circ A \circ a + a \circ A \circ a + a \circ A \circ a + a \circ A \circ a \\
&\qquad\qquad \text{from Proposition 1.16 and Proposition 1.18(6)}
\end{aligned}$$

$$= a \circ A \circ a \quad \text{from Proposition 1.16 and Proposition 1.18(2).}$$

This implies that a is a regular element of $(A, +, \circ)$. Hence (ii) is proved. $\square$

From Theorem 2.10, we have the following corollary which is Proposition 1.5.

Corollary 2.11. *A ring A is regular if and only if $QAQ = Q$ for every quasi-ideal Q of A.*

Theorem 2.12. *Let $(A, +, \circ)$ be a multiplicative hyperring.*

(i) *If $(A, +, \circ)$ is regular, then $R \circ L = R \cap L$ for every right hyperideal R and for every left hyperideal L of $(A, +, \circ)$.*

(ii) *The converse of* (i) *holds if $(A, +, \circ)$ is strongly distributive.*

Proof. (i) Assume that $(A, +, \circ)$ is regular. Let R and L be a right hyperideal and a left hyperideal of $(A, +, \circ)$, respectively. Then $R \circ L \subseteq R \cap L$. Let $a \in R \cap L$. Since $(A, +, \circ)$ is regular, $a \in a \circ y \circ a$ for some $y \in A$. Since $a \in R$ and $y \circ a \subseteq L$, $a \in a \circ y \circ a \subseteq R \circ L$. Hence $R \cap L \subseteq R \circ L$, and consequently, $R \circ L = R \cap L$.

(ii) Assume that $(A, +, \circ)$ is strongly distributive and $R \circ L = R \cap L$ for every right hyperideal R and left hyperideal L of $(A, +, \circ)$. To show that $(A, +, \circ)$ is regular, let $a \in A$. By Proposition 1.16 and Lemma 2.2(ii), $a \circ A$ and $A \circ a$ are a right hyperideal and a left hyperideal of $(A, +, \circ)$, respectively. Therefore

$$\begin{aligned}
a &\in (a)_r \cap (a)_l \\
&= (a)_r \circ (a)_l && \text{by assumption} \\
&\subseteq (a)_r \circ A \\
&= (\mathbb{Z}a + a \circ A) \circ A && \text{by Proposition 1.16 and Lemma 2.3} \\
&\subseteq (\mathbb{Z}a) \circ A + (a \circ A) \circ A \\
&\subseteq \mathbb{Z}(a \circ A) + a \circ A && \text{by Proposition 1.18(5)}
\end{aligned}$$

$= a \circ A + a \circ A$ by Proposition 1.16 and Proposition 1.18(6)

$= a \circ A$ by Proposition 1.16 and Proposition 1.18(2)

and similarly, $a \in A \circ a$. Hence

$a \in a \circ A \cap A \circ a$

$= (a \circ A) \circ (A \circ a)$ by assumption, Proposition 1.16 and Lemma 2.2(ii)

$\subseteq a \circ A \circ a,$

so a is a regular element of $(A, +, \circ)$. Therefore (ii) is proved. $\square$

As a consequence of Theorem 2.12, we have

Corollary 2.13. *Let A be a ring. Then A is regular if and only if $RL = R \cap L$ for every right ideal R and for every left ideal L of A.*

Theorem 2.14. *Let $(R, +, \cdot)$be a ring and $\varnothing \neq P \subseteq R$. Define a hyperoperation $\circ$ on R by*

$$x \circ y = \{\ xty \mid t \in P\ \} \qquad \textit{for all } x, y \in R.$$

Then $(R, +, \circ)$ is a multiplicative hyperring.

Proof. From Proposition 1.15, $(R, \circ)$ is a semihypergroup. Next, let $x, y, z \in R$. Then

$$\begin{aligned} x \circ (y+z) &= \{\ xt(y+z) \mid t \in P\ \} \\ &= \{\ xty + xtz \mid t \in P\ \} \\ &\subseteq \{\ xty \mid t \in P\ \} + \{\ xtz \mid t \in P\ \} \\ &= x \circ y + x \circ z \end{aligned}$$

and we obtain similarly that $(y+z) \circ x \subseteq y \circ x + z \circ x$. Moreover,

$$(-x) \circ y = \{\ (-x)ty \mid t \in P\ \}$$

$$= \{ \, xt(-y) \mid t \in P \, \} = x \circ (-y)$$

$$= \{ -(xty) \mid t \in P \, \}$$

$$= -\{ \, xty \mid t \in P \, \} = -(x \circ y).$$

This proves that $(R, +, \circ)$ is a multiplicative hyperring, as required. □

Lemma 2.15. *Let $(R, +, \cdot)$ be a ring, P a nonempty subset of R and $(R, +, \circ)$ the multiplicative hyperring defined from $(R, +, \cdot)$ and P where*

$$x \circ y = \{ \, xty \mid t \in P \, \} \qquad \text{for all } x, y \in R.$$

If K_1 and K_2 are nonempty subsets of R, then

$$< K_1 \circ K_2 >= K_1 P K_2,$$

that is, the set $< K_1 \circ K_2 >$ in $(R, +, \circ)$ is equal to $K_1 P K_2$ in $(R, +, \cdot)$ which is the set of all finite sums of the form $\sum a_i p_i b_i$ where $a_i \in K_1, p_i \in P$ and $b_i \in K_2$. In particular,

$$< a \circ b >= aPb \qquad \text{for all } a, b \in R.$$

Proof. Let $x \in < K_1 \circ K_2 >$. Then $x \in a_1 \circ b_1 + a_2 \circ b_2 + \cdots + a_m \circ b_m$ for some $a_1, a_2, \ldots, a_m \in K_1$ and $b_1, b_2, \ldots, b_m \in K_2$. It therefore follows that

$$x = a_1 p_1 b_1 + a_2 p_2 b_2 + \cdots + a_m p_m b_m \qquad \text{for some } p_1, p_2, \ldots, p_m \in P.$$

Consequently, $x \in K_1 P K_2$.

Conversely, if $x \in K_1 P K_2$, then $x = a_1 p_1 b_1 + a_2 p_2 b_2 + \cdots + a_m p_m b_m$ for some $a_1, a_2, \ldots, a_m \in K_1$, $p_1, p_2, \ldots, p_m \in P$ and $b_1, b_2, \ldots, b_m \in K_2$. This implies that

$$x \in a_1 \circ b_1 + a_2 \circ b_2 + \cdots + a_m \circ b_m \subseteq < K_1 \circ K_2 > .$$

Hence the lemma is proved. □

Theorem 2.16. *Let $(R, +, \cdot)$ be a ring, P a nonempty subset of R and $(R, +, \circ)$ the multiplicative hyperring where $\circ$ is the hyperoperation defined from P by*

$$x \circ y = \{ \ xty \mid t \in P \ \} \qquad \text{for all } x, y \in R.$$

If Q is a quasi-ideal of $(R, +, \cdot)$, then Q is a quasi-hyperideal of the multiplicative hyperring $(R, +, \circ)$.

Proof. Recall that $(R, +, \circ)$ is really a multiplicative hyperring by Theorem 2.14.

Let Q be a quasi-ideal of $(R, +, \cdot)$. Then Q is a subgroup of $(A, +)$ and $RQ \cap QR \subseteq Q$. It remains to show that $< R \circ Q > \cap < Q \circ R > \subseteq Q$. But

$$\begin{aligned} < R \circ Q > &= RPQ \qquad \text{by Lemma 2.15} \\ &= (RP)Q \\ &\subseteq RQ \end{aligned}$$

and similarly $< Q \circ R > \subseteq QR$. It then follows that

$$< R \circ Q > \cap < Q \circ R > \subseteq RQ \cap QR \subseteq Q.$$

This proves that Q is a quasi-hyperideal of $(R, +, \circ)$, as required. $\square$

The converse of Theorem 2.16 is not true in general. If Q is a subgroup of $(R, +)$ which is not a quasi-ideal of $(R, +, \cdot)$ (that is, $RQ \cap QR \nsubseteq Q$) and $P = \{0\}$, then Q is clearly a quasi-hyperideal of $(R, +, \circ)$. The following example is not a trivial one.

Example 2.17. Let R be a ring with identity $1 \neq 0$ and $M_4(R)$ the ring of all 4×4 matrices over R. Let

$$U_4(R) = \{A \in M_4(R) \mid A \text{ is upper triangular}\}$$

and

$$P = \left\{ \begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, x, y, z \in R \right\}.$$

Define a hyperoperation $\circ$ on $M_4(R)$ by

$$A \circ B = \{ACB \mid C \in P\} \qquad \text{for all } A, B \in M_4(R).$$

Then $U_4(R)$ is a subring of $(M_4(R), +, \cdot)$ and P is clearly an ideal of $(U_4(R), +, \cdot)$. Since $U_4(R)$ contains the identity matrix of $M_4(R)$, we have

$$M_4(R)U_4(R) \cap U_4(R)M_4(R) = M_4(R) \nsubseteq U_4(R).$$

Thus $U_4(R)$ is not a quasi-ideal of $(M_4(R), +, \cdot)$. By Lemma 2.15,

$$\begin{aligned} &< M_4(R) \circ U_4(R) > \cap < U_4(R) \circ M_4(R) > \\ &\qquad = M_4(R)PU_4(R) \cap U_4(R)PM_4(R) \\ &\qquad = M_4(R)(PU_4(R)) \cap (U_4(R)P)M_4(R) \\ &\qquad \subseteq M_4(R)P \cap PM_4(R) \qquad\qquad \text{since } P \text{ is an ideal of } U_4(R). \end{aligned}$$

But an element of $M_4(R)P$ is of the form

$$\begin{bmatrix} 0 & 0 & x_1 & y_1 \\ 0 & 0 & x_2 & y_2 \\ 0 & 0 & x_3 & y_3 \\ 0 & 0 & x_4 & y_4 \end{bmatrix}$$

and an element of $PM_4(R)$ is of the form

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so each element of $M_4(R)P \cap PM_4(R)$ is of the form

$$\begin{bmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is upper triangular. Hence

$$< M_4(R) \circ U_4(R) > \cap < U_4(R) \circ M_4(R) > \subseteq U_4(R)$$

which implies that $U_4(R)$ is a quasi-hyperideal of $(M_4(R), +, \circ)$.

Note that if "4"is replaced by "n"which is greater than 4, the above result is still true.

CHAPTER III

MULTIPLICATIVE HYPERRINGS HAVING THE INTERSECTION PROPERTY OF QUASI-HYPERIDEALS

The main purpose of this chapter is to generalize Proposition 1.7 – Proposition 1.10 by characterizing when quasi-hyperideals of multiplicative hyperrings have the intersection property and when multiplicative hyperrings have the intersection property of quasi-hyperideals.

The first theorem of this chapter is a generalization of Proposition 1.7. We first give a lemma which follows directly from Proposition 1.18(3) and Lemma 2.2(iii)

Lemma 3.1. *If S is a subhyperring of a multiplicative hyperring $(A, +, \circ)$, then $S + < A \circ S >$ and $S + < S \circ A >$ are respectively a left hyperideal and a right hyperideal of $(A, +, \circ)$ containing S.*

Theorem 3.2. *If Q is a quasi-hyperideal of a multiplicative hyperring $(A, +, \circ)$, and either $Q \subseteq < Q \circ A >$ or $Q \subseteq < A \circ Q >$, then*

$$Q = (Q + < A \circ Q >) \cap (Q + < Q \circ A >).$$

In this case, Q has the intersection property.

Proof. Suppose that Q is a quasi-hyperideal of $(A, +, \circ)$. Let $D = (Q + < Q \circ A >)$ $\cap$ $(Q + < A \circ Q >)$. By Lemma 2.2(i), $Q \subseteq D$. Now assume that $Q \subseteq < A \circ Q >$. Then $Q + < A \circ Q > = < A \circ Q >$ by Lemma 2.2(ii), and so $D = (Q +$

$< Q \circ A >) \cap < A \circ Q >$. Next, we will show that $D \subseteq Q$. Let $d \in D$. Then $d \in Q + < Q \circ A >$ and $d \in < A \circ Q >$. Thus $d = k + c$ for some $k \in Q$ and $c \in < Q \circ A >$. Then

$$\begin{aligned} c &= d + (-k) \\ &\in < A \circ Q > + Q \\ &= < A \circ Q > \end{aligned}$$

which implies that $c \in < A \circ Q > \cap < Q \circ A > \subseteq Q$, and therefore $d = k + c \in Q + Q \subseteq Q$. Hence $D \subseteq Q$. Similarly, $Q \subseteq < Q \circ A >$ implies that $D \subseteq Q$. We then conclude that $D = Q$. By Lemma 3.1, Q has the intersection property. $\square$

As a consequence of Theorem 3.2, we have

Corollary 3.3. *Let Q be a quasi-ideal of a ring A. If $Q \subseteq QA$ or $Q \subseteq AQ$, then*

$$Q = (Q + AQ) \cap (Q + QA).$$

In this case, Q has the intersection property (since $Q + AQ$ and $Q + QA$ are a left ideal and a right ideal of A, respectively.)

We note that if $(A, +, \circ)$ is a unitary multiplicative hyperring, then for every quasi-hyperideal Q of $(A, +, \circ)$, $Q = Q \circ u \subseteq Q \circ A \subseteq < Q \circ A >$ where u is the unitary element of $(A, +, \circ)$, so by Theorem 3.2, $Q = (Q + < A \circ Q >) \cap (Q + < Q \circ A >)$. We then deduce that every unitary multiplicative hyperring has the intersection property of quasi-hyperideals. Moreover, if $(A, +, \circ)$ is a regular multiplicative hyperring, then for every quasi-hyperideal Q of $(A, +, \circ)$, $Q \subseteq Q \circ A \circ Q \subseteq Q \circ A \subseteq < Q \circ A >$, so $Q = (Q + < A \circ Q >) \cap (Q + < Q \circ A >)$ by Theorem 3.2. Hence a regular multiplicative hyperring has the intersection property of quasi-hyperideals.

The following two theorems give some equivalent conditions for a quasi-hyperideal of a multiplicative hyperring to have the intersection property.

Theorem 3.4. *Let Q be a quasi-hyperideal of a multiplicative hyperring $(A, +, \circ)$. Then the following statements are equivalent.*

(i) *Q has the intersection property.*

(ii) $(Q+ < A \circ Q >) \cap\ (Q+ < Q \circ A >) = Q$.

(iii) $< A \circ Q > \cap\ (Q+ < Q \circ A >) \subseteq Q$.

(iv) $< Q \circ A > \cap\ (Q+ < A \circ Q >) \subseteq Q$.

Proof. (i)⇒(ii) Assume that Q has the intersection property. Then there exist a left-hyperideal L and a right-hyperideal R of $(A, +, \circ)$ such that $Q = L \cap R$. Then $Q \subseteq L$ and $Q \subseteq R$ and so $< A \circ Q > \subseteq < A \circ L > \subseteq < L > = L$ and $< Q \circ A > \subseteq < R \circ A > \subseteq < R > = R$. Then $Q+ < A \circ Q > \subseteq L$ and $Q+ < Q \circ A > \subseteq R$. Consequently, $(Q+ < A \circ Q >) \cap (Q+ < Q \circ A >) \subseteq L \cap R = Q$. But $Q \subseteq (Q+ < A \circ Q >) \cap (Q+ < Q \circ A >)$ by Lemma 2.2(i), so (ii) holds.

(ii)⇒(i) This follows from Lemma 3.1.

(ii)⇒(iii) This is obvious since $< A \circ Q > \subseteq Q+ < A \circ Q >$.

(iii)⇒(ii) Assume that $< A \circ Q > \cap\ (Q+ < Q \circ A >) \subseteq Q$. By Lemma 2.2(i), $Q \subseteq (Q+ < A \circ Q >) \cap (Q+ < Q \circ A >)$. To prove the reverse inclusion, let $x \in (Q+ < A \circ Q >) \cap (Q+ < Q \circ A >)$. Thus $x = t_1 + c$ and $x = t_2 + d$ for some $t_1, t_2 \in Q, c \in < A \circ Q >$ and $d \in < Q \circ A >$. Since Q is a subgroup of $(A, +)$, $c = -t_1 + x = -t_1 + t_2 + d = (-t_1 + t_2) + d \in Q+ < Q \circ A >$. Now, we have that $c \in < A \circ Q > \cap\ (Q+ < Q \circ A >)$. By (iii), $c \in Q$. This implies that $x = t_1 + c \in Q$. Then (ii) holds.

Similarly, we can prove that (ii) ⇔ (iv). □

The following theorem strengthens Theorem 3.4.

Theorem 3.5. *Let X be a nonempty subset of a multiplicative hyperring $(A, +, \circ)$. Then the following statements are equivalent.*

(i) $(X)_q$ *has the intersection property.*

(ii) $(\mathbb{Z}X + < A \circ X >) \cap (\mathbb{Z}X + < X \circ A >) = (X)_q$.

(iii) $< A \circ X > \cap (\mathbb{Z}X + < X \circ A >) \subseteq (X)_q$.

(iv) $< X \circ A > \cap (\mathbb{Z}X + < A \circ X >) \subseteq (X)_q$.

Proof. (i)⇒(ii) Since $(X)_q$ has the intersection property, there exist a left-hyperideal L and a right-hyperideal R of $(A, +, \circ)$ such that $(X)_q = L \cap R$. Then $X \subseteq L$ and so $\mathbb{Z}X \subseteq L$ and $< A \circ X > \subseteq < A \circ L > \subseteq L$. Thus $\mathbb{Z}X + < A \circ X > \subseteq L$. Similarly, we have $\mathbb{Z}X + < X \circ A > \subseteq R$. Therefore, $(\mathbb{Z}X + < A \circ X >) \cap (\mathbb{Z}X + < X \circ A >) \subseteq L \cap R = (X)_q$. By Theorem 2.8, $(X)_q = \mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$, so $(X)_q = \mathbb{Z}X + (< A \circ X > \cap < X \circ A >) \subseteq (\mathbb{Z}X + < A \circ X >) \cap (\mathbb{Z}X + < X \circ A >)$. Then (ii) holds.

(ii)⇒(i) It is true because of Lemma 2.2(iii).

(ii)⇒(iii) This implication is clear.

(iii)⇒(ii) Assume that $< A \circ X > \cap (\mathbb{Z}X + < X \circ A >) \subseteq (X)_q$. By Theorem 2.8, $(X)_q = \mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$. Then $(X)_q \subseteq (\mathbb{Z}X + < A \circ X >) \cap (\mathbb{Z}X + < X \circ A >)$. By Theorem 2.8, we remain to show that $\mathbb{Z}X + (< A \circ X > \cap < X \circ A >) \supseteq (\mathbb{Z}X + < A \circ X >) \cap (\mathbb{Z}X + < X \circ A >)$. Let $t \in (\mathbb{Z}X + < A \circ X >) \cap (\mathbb{Z}X + < X \circ A >)$. Then $t = t_1 + q_1$ and $t = t_2 + q_2$ for some $t_1, t_2 \in \mathbb{Z}X, q_1 \in < A \circ X >$ and $q_2 \in < X \circ A >$. Then $q_1 = -t_1 + t = -t_1 + (t_2 + q_2) = (-t_1 + t_2) + q_2 \in \mathbb{Z}X + < X \circ A >$. Hence $q_1 \in < A \circ X > \cap (\mathbb{Z}X + < X \circ A >) \subseteq (X)_q$. This implies that $t = t_1 + q_1 \in \mathbb{Z}X + (X)_q = (X)_q$. Therefore (ii) holds.

Similarly, we can prove that (ii) ⇔ (iv). □

Proposition 1.8 and Proposition 1.9 are special cases of Theorem 3.4 and Theorem 3.5, respectively .

Corollary 3.6. *Let Q be a quasi-hyperideal of a ring A. Then the following statements are equivalent.*

(i) *Q has the intersection property.*

(ii) $(AQ+Q)\cap\ (QA+Q)=Q$.

(iii) $AQ\cap\ (Q+QA)\subseteq Q$.

(iv) $QA\cap\ (Q+AQ)\subseteq Q$.

Corollary 3.7. *Let X be a nonempty subset of a ring A. Then the following statements are equivalent.*

(i) *$(X)_q$ has the intersection property.*

(ii) $(\mathbb{Z}X+AX)\cap\ (\mathbb{Z}X+XA)=(X)_q$.

(iii) $AX\cap\ (\mathbb{Z}X+XA)\subseteq (X)_q$.

(iv) $XA\cap\ (\mathbb{Z}X+AX)\subseteq (X)_q$.

The following theorem gives some equivalent conditions for a multiplicative hyperring to have the intersection property of quasi-hyperideals.

Theorem 3.8. *Let $(A,+,\circ)$ be a multiplicative hyperring and let* (i),(ii) *and* (iii) *be the statements given as follows.*

(i) *$(A,+,\circ)$ has the intersection property of quasi-hyperideals.*

(ii) *For any finite nonempty subset X of A,*

$$< A\circ X > \cap(\mathbb{Z}X+ < X\circ A >)\subseteq \mathbb{Z}X+(< A\circ X > \cap < X\circ A >)\ (=(X)_q).$$

(iii) *For a finite subset $X=\{x_1,x_2,\ldots,x_n\}$ of A and $a_1,a_2,\ldots a_n\in A$, if*

$$y\in(\sum_{i=1}^{n}a_i\circ x_i)\cap(\sum_{i=1}^{n}k_ix_i+\sum_{i=1}^{n}x_i\circ a_i')$$

for some $a_i'\in A$ and $k_i\in\mathbb{Z}$, then $y\in(X)_q$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), *and if $(A,+,\circ)$ is strongly distributive, then* (iii) $\Rightarrow$(ii).

Proof. (i)$\Rightarrow$(ii) Suppose that $(A,+,\circ)$ has the intersection property of quasi-hyperideals and let X be a finite nonempty subset of A. Then $(X)_q$ has the intersection property. Therefore (ii) holds by Theorem 2.8 and Theorem 3.5.

(ii)$\Rightarrow$(i) Assume that (ii) true. Let Q be any quasi-hyperideal of $(A,+,\circ)$. We need to show that $< A\circ Q > \cap\, (Q+ < Q\circ A >) \subseteq Q$. Let $y \in < A\circ Q > \cap\, (Q+ < Q\circ A >)$. Then $y \in \sum_{i=1}^{n} a_i \circ q_i$ and $y \in q + \sum_{j=1}^{m} q'_j \circ a'_j$ for some $a_i, a'_j \in A, q, q_i, q'_j \in Q$ and $m, n \in \mathbb{N}$. Consider $X = \{q_1, q_2, \ldots, q_n, q, q'_1, \ldots, q'_m\}$. Then $X \subseteq Q$ and $|X| < \infty$. By assumption, $< A\circ X > \cap\, (\mathbb{Z}X+ < X\circ A >) \subseteq \mathbb{Z}X + (< A\circ X > \cap < X \circ A >)$, so we have

$$\begin{aligned} y \in < A\circ X > \cap\, (\mathbb{Z}X+ < X\circ A >) &\subseteq \mathbb{Z}X + (< A\circ X > \cap < X\circ A >) \\ &\subseteq Q + (< A\circ Q > \cap < Q\circ A >) \\ &\subseteq Q + Q \subseteq Q. \end{aligned}$$

This shows that $< A\circ Q > \cap\, (Q+ < Q\circ A >) \subseteq Q$. By Theorem 3.4, Q has the intersection property. Hence (i) is proved.

(ii)$\Rightarrow$(iii) Assume (ii) holds. Let $X = \{x_1, x_2, \ldots, x_n\} \subseteq A$ and $a_1, a_2, \ldots, a_n \in A$ and let $y \in (\sum_{i=1}^{n} a_i \circ x_i) \cap (\sum_{i=1}^{n} k_i x_i + \sum_{i=1}^{n} x_i \circ a'_i)$ for some $a'_i \in A, k_i \in \mathbb{Z}$. But $(\sum_{i=1}^{n} a_i \circ x_i) \cap (\sum_{i=1}^{n} k_i x_i + \sum_{i=1}^{n} x_i \circ a'_i) \subseteq < A\circ X > \cap\, (\mathbb{Z}X+ < X\circ A >) \subseteq (X)_q$, so $y \in (X)_q$

(iii)$\Rightarrow$(ii) Assume that $(A,+,\circ)$ is strongly distributive and (iii) holds. Let X be a finite nonempty subset of A, say $X = \{x_1, x_2, \ldots, x_n\}$, and $y \in < A\circ X > \cap\, (\mathbb{Z}X+ < X\circ A >)$. Then $y \in < A\circ X >$ and $y \in \mathbb{Z}X+ < X\circ A >$. Since $(A,+,\circ)$ is strongly distributive, by Proposition 1.19, $0 \in 0\circ x_i$ and $0 \in x_i \circ 0$ for all $i \in \{\ 1, 2, \ldots, n\ \}$. It then follows that

$$y \in \sum_{i=1}^{n} a_i \circ x_i \quad \text{and} \quad y \in \sum_{i=1}^{n} l_i x_i + \sum_{i=1}^{n} x_i \circ a'_i$$

for some $a_i, a_i' \in A$ and $l_i \in \mathbb{Z}$ for $i \in \{\ 1, 2, \ldots, n\ \}$. Therefore

$$y \in (\sum_{i=1}^{n} a_i \circ x_i) \cap \ (\sum_{i=1}^{n} l_i x_i + \sum_{i=1}^{n} x_i \circ a_i')$$

which implies that by (iii) that $y \in (X)_q$. This proves that $< A \circ X > \cap\, (\mathbb{Z}X + < X \circ A >) \subseteq \mathbb{Z}X + (< A \circ X > \cap < X \circ A >)$. Hence (ii) holds. $\square$

A direct consequence of Theorem 3.8 is Proposition 1.10.

Corollary 3.9. *The following statements for a ring A are equivalent.*

(i) *Every quasi-ideal of A has the intersection property.*

(ii) *For any finite nonempty subset X of A,*

$$AX \cap (\mathbb{Z}X + XA) \subseteq \mathbb{Z}X + (AX \cap XA)\ (= (X)_q).$$

(iii) *For any finite subset $X = \{x_1, x_2, \ldots, x_n\}$ of A and $a_1, a_2, \ldots, a_n \in A$, if*

$$\sum_{i=1}^{n} (a_i x_i + k_i x_i + x_i a_i') = 0,$$

for some $a_i' \in A$ and $k_i \in \mathbb{Z}$, then $\sum_{i=1}^{n} a_i x_i \in (X)_q$.

CHAPTER IV

MINIMAL QUASI-HYPERIDEALS OF MULTIPLICATIVE HYPERRINGS

In the last chapter, minimal quasi-hyperideals of multiplicative hyperrings are studied. The first aim is to generalize Proposition 1.11 to Proposition 1.14 by considering minimal quasi-hyperideals of multiplicative hyperrings. The second purpose is to investigate minimal quasi-ideals of the multiplicative hyperring defined from the ring of all upper triangular $n \times n$ matrices over a division ring R as in Theorem 2.14 by using P to be the set of all strictly upper triangular $n \times n$ matrices over a division ring R.

Theorem 4.1. *A nonzero quasi-hyperideal Q of a multiplicative hyperring $(A, +, \circ)$ is a minimal quasi-hyperideal if and only if $(x)_q = Q$ for all $x \in Q\backslash\{0\}$.*

Proof. Suppose that Q is a minimal quasi-hyperideal of $(A, +, \circ)$ and let $x \in Q\backslash\{0\}$. Since $(x)_q$ is a nonzero quasi-hyperideal of $(A, +, \circ)$ contained in Q, by the minimality of Q, $(x)_q = Q$.

Conversely, assume that $(x)_q = Q$ for all $x \in Q\backslash\{0\}$. Let Q' be a nonzero quasi-hyperideal of $(A, +, \circ)$ contained in Q. Then there exists a nonzero element in Q', say y, so $(y)_q = Q$. Thus $Q = (y)_q \subseteq Q'$. Hence $Q = Q'$. Therefore Q is a minimal quasi-hyperideal of $(A, +, \circ)$. $\square$

We obtain that Proposition 1.11 is an immediate consequence of the above theorem.

Corollary 4.2. *A nonzero quasi-ideal Q of a ring A is a minimal quasi-ideal of A if and only if $(x)_q = Q$ for all $x \in Q\backslash\{0\}$.*

Next, a necessary and sufficient condition for a quasi-hyperideal of a multiplicative hyperring to be minimal in terms of principal left hyperideals and principal right hyperideals are givens as follows:

Theorem 4.3. *A quasi-hyperideal Q of a multiplicative hypering $(A, +, \circ)$ is minimal if and only if for any elements $x, y \in Q\backslash\{0\}$,*

$$(x)_l = (y)_l \quad \textit{and} \quad (x)_r = (y)_r.$$

Proof. Assume that Q is a minimal quasi-hyperideal of $(A, +, \circ)$. Let $x, y \in Q\backslash\{0\}$. Then by Lemma 2.6, $(x)_l \cap Q$ is a nonzero quasi-hyperideal of $(A, +, \circ)$ contained in Q. By the minimality of Q, we have that $Q = (x)_l \cap Q$. Then $Q \subseteq (x)_l$, this implies that $y \in (x)_l$. Therefore $(y)_l \subseteq (x)_l$. By a similar argument, we obtain $(x)_l \subseteq (y)_l$. Hence $(x)_l = (y)_l$. Dually, we can show that $(x)_r = (y)_r$.

Conversely, assume that for all $x, y \in Q\backslash\{0\}, (x)_l = (y)_l$ and $(x)_r = (y)_r$. Let Q' be a nonzero quasi-hyperideal of $(A, +, \circ)$ such that $Q' \subseteq Q$. Let $x \in Q\backslash\{0\}$.

Case 1: $< A \circ Q' > \cap\, Q \neq \{0\}$ and $< Q' \circ A > \cap\, Q \neq \{0\}$. Let $q \in (< A \circ Q' > \cap\, Q)\backslash\{0\}$ and $s \in (< Q' \circ A > \cap\, Q)\backslash\{0\}$. By assumption, $(x)_l = (q)_l$ and $(x)_r = (s)_r$, so $x \in (q)_l$ and $x \in (s)_r$. Thus

$$\begin{aligned}
x \in (q)_l &= \mathbb{Z}q + < A \circ q > && \text{by Lemma 2.3}\\
&\subseteq \mathbb{Z} < A \circ Q' > + < A \circ < A \circ Q' >> && \text{since } q \in < A \circ Q' >\\
&= \mathbb{Z} < A \circ Q' > + < A \circ A \circ Q' > && \text{by Proposition 1.18(2)}\\
&\subseteq \mathbb{Z} < A \circ Q' > + < A \circ Q' > &&\\
&\subseteq < A \circ Q' > + < A \circ Q' > &&
\end{aligned}$$

by Lemma 2.2(ii) and Proposition 1.18(3)

$$=< A \circ Q' >$$

and

$$x \in (s)_r = \mathbb{Z}s + < s \circ A > \qquad \text{by Lemma 2.3}$$

$$\subseteq \mathbb{Z} < Q' \circ A > + << Q' \circ A > \circ A > \qquad \text{since } s \in < Q' \circ A >$$

$$= \mathbb{Z} < Q' \circ A > + < Q' \circ A \circ A > \qquad \text{by Proposition 1.18(2)}$$

$$\subseteq \mathbb{Z} < Q' \circ A > + < Q' \circ A >$$

$$\subseteq < Q' \circ A > + < Q' \circ A >$$

by Lemma 2.2(ii) and Proposition 1.18(3)

$$=< Q' \circ A >,$$

so $x \in < A \circ Q' > \cap < Q' \circ A > \subseteq Q'$.

Case 2: $< A \circ Q' > \cap Q = \{0\}$. Let $y \in Q' \backslash \{0\}$. Then $y \in Q \backslash \{0\}$ and $(x)_l = (y)_l$, so $x \in (y)_l = \mathbb{Z}y + < A \circ y >$. Thus $x = ny + t$ for some $t \in < A \circ y >$, so $x - ny = t \in < A \circ Q' > \cap Q = \{0\}$. Hence $x - ny = 0$. Thus $x = ny \in Q'$.

Case 3: $< Q' \circ A > \cap Q = \{0\}$. Dually to Case 2, one can prove that $x \in Q'$.

From the above three cases, we obtain $Q = Q'$. Therefore Q is a minimal quasi-hyperideal of $(A, +, \circ)$. $\square$

A consequence of Theorem 4.3 is Proposition 1.12.

Corollary 4.4. *A quasi-ideal Q of a ring A is a minimal quasi-ideal of A if and only if any two nonzero elements x and y of Q,*

$$(x)_l = (y)_l \quad \text{and} \quad (x)_r = (y)_r.$$

There are some relationships among minimal quasi-hyperideals, minimal left hyperideals and minimal right hyperideals as follows:

Theorem 4.5. *The intersection of a minimal left hyperideal L and a minimal right hyperideal R of a multiplicative hyperring $(A, +, \circ)$ is either $\{0\}$ or a minimal quasi-hyperideal of $(A, +, \circ)$.*

Proof. We have that $Q = R \cap L$ is a quasi-hyperideal of $(A, +, \circ)$. Assume that $Q \neq \{0\}$. We shall show that Q is minimal. Suppose that there exists a quasi-hyperideal Q' of $(A, +, \circ)$ such that $\{0\} \neq Q' \subsetneq Q$. Then $Q' \subsetneq L$, so $< A \circ Q' >$ is a left hyperideal of $(A, +, \circ)$ contained in L. But L is a minimal left hyperideal of $(A, +, \circ)$, so it follows that $< A \circ Q' >= \{0\}$ or $< A \circ Q' >= L$. If $< A \circ Q' >= \{0\}$, then Q' is a left hyperideal of $(A, +, \circ)$ such that $\{0\} \neq Q' \subsetneq L$ which contradicts the minimality of L. Then $< A \circ Q' >= L$. Similarly, one can show that $< Q' \circ A >= R$. Hence $Q = R \cap L =< Q' \circ A > \cap < A \circ Q' >\subseteq Q'$, which contradicts that $Q' \subsetneq Q$. We then deduce that Q is a minimal quasi-hyperideal of $(A, +, \circ)$. □

Proposition 1.13 is directly obtained from Theorem 4.5.

Corollary 4.6. *If L and R are a minimal left ideal and a minimal right ideal of a ring A, then either $L \cap R = \{0\}$ or $L \cap R$ is a minimal quasi-ideal of A.*

Theorem 4.7. *If a nonzero quasi-hyperideal Q of a multiplicative hyperring $(A, +, \circ)$ such that $< x \circ Q >= Q =< Q \circ x >$ for all $x \in Q \backslash \{0\}$, then Q is a minimal quasi-hyperideal of $(A, +, \circ)$.*

Proof. Let Q' be a nonzero quasi-hyperideal of $(A, +, \circ)$ such that $Q' \subseteq Q$. This implies that Q' is a quasi-hyperideal of Q. Since $< Q \circ x >= Q =< x \circ Q >$ for all $x \in Q \backslash \{0\}$, we have $< Q \circ Q' >= Q$ and $< Q' \circ Q >= Q$. Hence $Q =< Q' \circ Q >$

$\cap < Q \circ Q' > \subseteq < Q' \circ A > \cap < A \circ Q' > \subseteq Q'$. Then $Q = Q'$. Therefore Q is a minimal quasi-hyperideal of $(A, +, \circ)$. $\square$

We have mentioned in Chapter II that for a ring A with $|A| > 1$, A is a division ring if and only if $Ax = xA = A$ for all $x \in A \backslash \{0\}$. Then the following result can be considered as a consequence of Theorem 4.7.

Corollary 4.8. *Let Q be a quasi-ideal of a ring A. If Q is a division subring of A, then Q is a minimal quasi-ideal of A.*

In the remainder, let R be a division ring, $n \in \mathbb{N}$ and $U_n(R) = (U_n(R), +, \cdot)$ the ring of all upper triangular $n \times n$ matrices over R where $+$ and $\cdot$ are the usual addition and the multiplication of matrices, respectively. Let $SU_n(R)$ be the set of all strictly upper triangular $n \times n$ matrices over R. Define a hyperoperation $\circ$ on $U_n(R)$ by

$$A \circ B = \{ACB \mid C \in SU_n(R)\} \qquad \text{for all } A, B \in U_n(R) \ .$$

Since $SU_n(R)$ is a subring of $(U_n(R), +, \cdot)$, we have

$$A \circ B = ASU_n(R)B \qquad \text{for all } A, B \in U_n(R).$$

By Theorem 2.14, $(U_n(R), +, \circ)$ is a multiplicative hyperring which may be denoted by $(U_n(R), +, SU_n(R))$. Note that $SU_n(R)$ is an ideal of $(U_n(R), +, \cdot)$.

The main purpose of this part is to prove the following results.

1) If char $R = 0$, then $(U_n(R), +, SU_n(R))$ has no minimal quasi-hyperieal.

2) Let char $R = p > 0$. Then the following statements hold.

   2.1) For $A \in U_n(R)$, if $\text{rank}(A) = 1$, then $(A)_q$ is a minimal quasi-hyperideal of $(U_n(R), +, SU_n(R))$.

2.2) The converse of 2.1) holds if and only if $n \leq 2$.

We note that if char $R = 0$, then $m1_R \neq 0$ for all $m \in \mathbb{Z}\backslash\{0\}$ where 1_R is the identity of R. Also, if char $R = p > 0$, then

$$\mathbb{Z}1_R = \{\ 0,\ 1_R,\ 2(1_R), \ldots,\ (p-1)(1_R)\ \}$$

so for $x \in R$,

$$\mathbb{Z}x = \{\ 0,\ x,\ 2x, \ldots,\ (p-1)x\ \}$$

and $|\mathbb{Z}x| = p$ if $x \neq 0$.

Theorem 4.9. *If char $R = 0$, then $(U_n(R), +, SU_n(R))$ has no minimal quasi-hyperideal.*

Proof. Let char $R = 0$. To prove that $(U_n(R), +, SU_n(R))$ has no minimal quasi-hyperideal, by Theorem 4.1, it suffices to prove that for every $A \in U_n(R)\backslash\{0\}$, there exists $B \in U_n(R)\backslash\{0\}$ such that $(B)_q \subsetneq (A)_q$ in $(U_n(R), +, SU_n(R))$.

Let $A \in U_n(R)$ and $A \neq 0$. Then $2A \in U_n(R)$. Since char $R = 0$ and $A \neq 0$, we have $2A \neq 0$. Since $(A)_q$ is a subgroup of $(U_n(R), +)$, $2A \in (A)_q$ which implies that $(2A)_q \subseteq (A)_q$. Suppose that $(2A)_q = (A)_q$. By Theorem 2.8 and Lemma 2.15,

$$\begin{aligned} (A)_q &= \mathbb{Z}A + (< A \circ U_n(R) > \cap < U_n(R) \circ A >) \\ &= \mathbb{Z}A + (ASU_n(R)U_n(R) \cap U_n(R)SU_n(R)A)\,, \end{aligned} \tag{1}$$

so

$$\begin{aligned} (2A)_q &= \mathbb{Z}(2A) + ((2A)SU_n(R)U_n(R) \cap U_n(R)SU_n(R)(2A)) \\ &= 2(\mathbb{Z}A + (ASU_n(R)U_n(R) \cap U_n(R)SU_n(R)A)) \\ &= 2(A)_q. \end{aligned}$$

Since $A \in (A)_q = (2A)_q = 2(A)_q$ and $SU_n(R)$ is an ideal of $U_n(R)$, we have form (1) that there exist $m \in \mathbb{Z}, B \in U_n(F)$ and $C \in SU_n(F)$ such that

$$A = 2(mA + CA) = 2mA + 2CA$$

which implies that

$$(1 - 2m)A = 2CA. \tag{2}$$

Since A is upper triangular, $A_{ij} = 0$ for all $i, j \in \{ 1, 2, \ldots, n \}$ and $i > j$. Since $A \neq 0, A_{ij} \neq 0$ for some $i \in \{ 1, 2, \ldots, n \}$ and $j \in \{ i, i+1, \ldots, n \}$. Let

$$k = \max\{ i \in \{ 1, 2, \ldots, n \} \mid A_{ij} \neq 0 \text{ for some } j \in \{ i, i+1, \ldots, n \}\}, \tag{3}$$

and let $l \in \{ k, k+1, \ldots, n \}$ be such that $A_{kl} \neq 0$. From (2), we have

$$((1 - 2m)A)_{kl} = (2CA)_{kl}.$$

It then follows that

$$\begin{aligned} (1 - 2m)A_{kl} &= 2\sum_{j=1}^{n} C_{kj}A_{jl}. \\ &= 2\sum_{j=1}^{k} C_{kj}A_{jl} + 2\sum_{j=k+1}^{n} C_{kj}A_{jl} \\ &= 2\sum_{j=1}^{k} C_{kj}A_{jl} \qquad \text{from(3)}. \end{aligned} \tag{4}$$

Since C is strictly upper triangular, $C_{k1} = C_{k2} = \ldots = C_{kk} = 0$. Thus we have from (4) that

$$(1 - 2m)A_{kl} = 0.$$

But char $R = 0$ and $A_{kl} \neq 0$, so $1-2m = 0$. Thus $2m = 1$ which is a contradiction. Hence $(2A)_q \subsetneq (A)_q$.

This proves that $(U_n(R), +, SU_n(R))$ has no minimal quasi-hyperideal, as required. $\square$

Lemma 4.10. *For $A \in U_n(R)$, if rank$(A) = 1$, then*

$$U_n(R)SU_n(R)A \ \cap \ ASU_n(R)U_n(R) = \{0\}.$$

Proof. Let $A \in U_n(R)$ be such that rank$(A) = 1$. Then $A \neq 0$ and $A_{ij} = 0$ for all $i, j \in \{ 1,\ 2, \ldots, n \}$ with $i > j$, so $A_{ij} \neq 0$ for some $i \in \{ 1,\ 2, \ldots,\ n \}$ and $j \in \{ i,\ i+1, \ldots,\ n \}$. Let

$$k = \max\{ i \in \{ 1,\ 2, \ldots,\ n \} \mid A_{ij} \neq 0 \ \text{ for some } j \in \{ i,\ i+1, \ldots,\ n \}\},$$

and

$$l = \min\{ j \in \{ k,\ k+1, \ldots,\ n \} \mid A_{kj} \neq 0\}.$$

It thus follows from the properties of $k,\ l$ and the fact that rank$(A) = 1$ that

$$A = \begin{bmatrix} 0 & \ldots & 0 & x_1 A_{kl} & x_1 A_{k,l+1} & \ldots & x_1 A_{kn} \\ 0 & \ldots & 0 & x_2 A_{kl} & x_2 A_{k,l+1} & \ldots & x_2 A_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & x_{k-1} A_{kl} & x_{k-1} A_{k,l+1} & \ldots & x_{k-1} A_{kn} \\ 0 & \ldots & 0 & x_k A_{kl} & x_k A_{k,l+1} & \ldots & x_k A_{kn} \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

for some $x_1,\ x_2, \ldots,\ x_{k-1} \in R$ where $x_k = 1$. Let $B \in < U_n(R) \circ A > \cap < A \circ U_n(R) >$. Since $SU_n(R)$ is an ideal of $(U_n(R), +, \cdot)$, by Lemma 2.15, we have $B = CA = AD$ for some $C,\ D \in SU_n(R)$. Then

$$CA = \begin{bmatrix} 0 & \dots & 0 & \sum_{j=2}^{k}(C_{1j}x_j)A_{kl} & \sum_{j=2}^{k}(C_{1j}x_j)A_{k,l+1} & \dots & \sum_{j=2}^{k}(C_{1j}x_j)A_{kn} \\ 0 & \dots & 0 & \sum_{j=3}^{k}(C_{2j}x_j)A_{kl} & \sum_{j=3}^{k}(C_{2j}x_j)A_{k,l+1} & \dots & \sum_{j=3}^{k}(C_{2j}x_j)A_{kn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & C_{k-1,k}x_kA_{kl} & C_{k-1,k}x_kA_{k,l+1} & \dots & C_{k-1,k}x_kA_{kn} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (1)$$

Since $A \in U_n(R)$ and $D \in SU_n(R)$, $A_{i1} = A_{i2} = \ldots = A_{i,l-1} = 0$ for all $i \in \{ 1,\ 2,\ \ldots,\ n \}$, $D_{ll} = D_{l+1,l} = \ldots = D_{nl} = 0$. Then $(AD)_{il} = \sum_{t=1}^{l-1}(A_{it}D_{tl}) + \sum_{t=l}^{n}(A_{it}D_{tl}) = 0 + 0 = 0$ for all $i \in \{ 1,\ 2, \ldots,\ n \}$. But $CA = AD$, so $(CA)_{il} = (AD)_{il} = 0$ for all $i \in \{ 1,\ 2, \ldots,\ n \}$. Since $A_{kl} \neq 0$ and $(CA)_{il} = 0$ for all $i \in \{1, 2, \ldots, n\}$, we have from (1) that

$$\sum_{j=2}^{k}(C_{1j}x_j) = 0$$

$$\sum_{j=3}^{k}(C_{2j}x_j) = 0$$

$$\vdots$$

$$\sum_{j=k-1}^{k}(C_{k-2,j}x_j) = 0$$

$$C_{k-1,k}x_k = 0.$$

which implies by (1) that $CA = 0$. But $B = CA$, so $B = 0$. This proves that $U_n(R)SU_n(R)A \cap ASU_n(R)U_n(R) = \{0\}$, as desired. □

Corollary 4.11. *For $A \in U_n(R)$, if rank$(A) = 1$, then*

$$(A)_q = \mathbb{Z}A \quad \text{in } (U_n(R), +, SU_n(R)).$$

Proof. Let $A \in U_n(R)$ be such that $\text{rank}(A) = 1$. By Lemma 4.10, $U_n(R)SU_n(R)A \cap ASU_n(R)U_n(R) = \{0\}$. Then

$$\begin{aligned}
(A)_q &= \mathbb{Z}A + (< A \circ U_n(R) > \cap < U_n(R) \circ A >) && \text{by Theorem 2.8} \\
&= \mathbb{Z}A + (U_n(R)SU_n(R)A \cap ASU_n(R)U_n(R)) && \text{by Lemma 2.15} \\
&= \mathbb{Z}A. && \text{by Lemma 4.10}
\end{aligned}$$
□

Lemma 4.12. *Let char $R = p > 0$. Then for $A \in U_n(R)\backslash\{0\}$, $(A)_q = \mathbb{Z}A$ in $(U_n(R), +, SU_n(R))$, then $(A)_q = \{\ 0,\ A,\ 2A,\ \ldots,\ (p-1)A\ \}$, $|(A)_q| = p$ and $(A)_q$ is a minimal quasi-hyperideal of $(U_n(R), +, SU_n(R))$.*

Proof. Since char $R = p$,

$$\mathbb{Z}1_R = \{\ 0,\ 1_R,\ 2(1_R), \ldots,\ (p-1)(1_R)\},$$

and so

$$\mathbb{Z}A = (\mathbb{Z}1_R)A = \{\ 0,\ A,\ 2A, \ldots,\ (p-1)A\ \}$$

where 1_R is the identity of R. Then $(A)_q = \{\ 0,\ A,\ 2A, \ldots,\ (p-1)A\ \}$. Because $A \neq 0$, $A_{ij} \neq 0$ for some $i, j \in \{1, 2, \ldots, n\}$. Since char $R = p$ and $A_{ij} \neq 0$, we have that $0,\ A_{ij},\ 2A_{ij}, \ldots,\ (p-1)A_{ij}$ are all distinct. Hence $|(A)_q| = p$. Let $B \in (A)_q$ and $B \neq 0$. Then $(B)_q$ is an additive subgroup of $(A)_q$ and $(B)_q \neq \{0\}$. But $|(B)_q| \Big| |(A)_q|$, so $|(B)_q| = p$. Hence $(B)_q = (A)_q$. Therefore $(A)_q$ is a minimal quasi-hyperideal of $(U_n(R), +, SU_n(R))$. □

The following theorem is obtained directly from Corollary 4.11 and Lemma 4.12.

Theorem 4.13. *Let char $R = p > 0$ and $A \in U_n(R)$. If rank$(A) = 1$, then in $(U_n(R), +, SU_n(R))$, $(A)_q$ is a minimal quasi-hyperideal.*

Theorem 4.14. *Let char $R = p > 0$. Then the following statements are equivalent.*

(i) *For $A \in U_n(R)$, if $(A)_q$ is a minimal quasi-hypereideal of $(U_n(R), +, SU_n(R))$, then rank$(A) = 1$.*

(ii) $n \leq 2$.

Proof. To prove (i) implies (ii) by contrapositive, assume that $n > 2$. Let

$$A = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $A \in U_n(R)$ and rank$(A) = 2$. To show that $U_n(R)SU_n(R)A \cap ASU_n(R)U_n(R) = \{0\}$, let $B \in U_n(R)SU_n(R)A \cap ASU_n(R)U_n(R)$. Since $SU_n(R)$ is an ideal of $(U_n(R), +, \cdot)$, $B = CA = AD$ for some $C,\ D \in SU_n(R)$. But

$$CA = \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & C_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & C_{12} & 0 \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AD = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & D_{12} & D_{13} & \dots & D_{1n} \\ 0 & 0 & D_{23} & \dots & D_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & D_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & D_{n-1,n} \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $B = CA = AD$, so we have that $B = 0$. This proves that $U_n(R)SU_n(R)A \cap ASU_n(R)U_n(R) = \{0\}$. By Theorem 2.8, Lemma 2.15 and Lemma 4.12, $(A)_q$ is a minimal quasi-hyperideal of $(U_n(R), +, SU_n(R))$.

Conversely, to prove that (ii) implies (i), assume that $n \leq 2$ and let $A \in U_n(R)$ be such that $(A)_q$ is a minimal quasi-hyperideal of $(U_n(R), +, SU_n(R))$. Then $A \neq 0$. If $n = 1$, then $\text{rank}(A) = 1$.

Next, assume that $n = 2$. To prove that $\text{rank}(A) = 1$, suppose not. Since

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix},$$

we have that $a_{11} \neq 0$ and $a_{22} \neq 0$. Let

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Because

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_{22}^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 0 & a_{11}^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

we have that $B \in U_2(R)SU_2(R)A \cap ASU_2(R)U_2(R)$. But $(A)_q = \mathbb{Z}A + (U_2(R)SU_2(R)A \cap ASU_2(R)U_2(R))$ by Theorem 2.8 and Lemma 2.15, so $B \in (A)_q$ and hence $(B)_q \subseteq (A)_q$. Since $\operatorname{rank}(B) = 1$, by Corollary 4.11, $(B)_q = \mathbb{Z}B$ in $(U_2(R), +, SU_2(R))$, so

$$(B)_q = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \;\middle|\; m \in \mathbb{Z} \right\}.$$

Then $A \notin (B)_q$ since $a_{11} \neq 0$ and $a_{22} \neq 0$. Therefore $\{0\} \subsetneq (B)_q \subsetneq (A)_q$. It is a contradiction since $(A)_q$ is a minimal quasi-hyperideal of $(U_2(R), +, SU_2(R))$. This proves that $\operatorname{rank}(A) = 1$, as required. $\square$

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