ขอบเขตไม่เอกรูปในการประมาณด้วยการแจกแจงปกติโดยวิธีของสไตน์และคู่ค่าที่เอนเอียงขนาดทางเดียวแบบมีขอบเขต

นางสาวกมลรัตน์ กำจรกิตติคุณ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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NON-UNIFORM BOUNDS ON NORMAL APPROXIMATION BY STEIN'S METHOD AND BOUNDED MONOTONE SIZE BIASED COUPLINGS

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วิทยานิพนธ์ฉบับนี้ประกอบด้วยสองส่วนหลัก โดยส่วนที่หนึ่งเป็นการให้ขอบเขตไม่เอกรูปแบบเลขชี้กำลังในการประมาณด้วยการแจกแจงปกติโดยวิธีของสไตน์และคู่ค่าที่เอนเอียงขนาดทางเดียวแบบมีขอบเขต ส่วนที่สองเป็นการประยุกต์ใช้ทฤษฎีบทหลักเพื่อให้ได้ขอบเขตไม่เอกรูปในการประมาณด้วยการแจกแจงปกติสำหรับผลรวมของตัวแปรสุ่มที่เป็นอิสระต่อกัน จำนวนหลอดไฟที่สว่าง ณ เวลาสุดท้ายในกระบวนการไลท์บับ และจำนวนของ m รัน

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This dissertation contains two main parts. First, we give a non-uniform exponential bound on normal approximation by using the Stein's method and bounded monotone size biased couplings. Second, applications of the main theorem to give the bound on normal approximation for sum of independent random variables, the number of bulbs on at the terminal time in the lightbulb process, and the number of m runs are provided.

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CONTENTS

CHAPTER I
INTRODUCTION

Stein's method introduced by Stein [44] in 1972 is a powerful tool to give uniform and non-uniform bounds on normal approximation. There are many approaches of the Stein's method, one of them is the size biased couplings approach.

For a non-negative random variable Y with finite and positive mean μ_Y, we say Y^s has the Y**-size biased distribution** if

$$E(Yf(Y)) = \mu_Y Ef(Y^s)$$

for all functions f for which these expectations exist.

The size biased technique is a useful tool to give bounds in normal approximation which can be seen in [9] and [21].

In 2011, the bounded monotone size biased couplings, that is,

$$Y \leq Y^s \leq Y + B$$

with probability 1, for some non-negative constant B, are mentioned in [26] by Goldstein and Zhang. They also gave a uniform Berry-Esseen bound for bounded monotone size biased coupling via Stein's method.

Let $X_1, X_2, \dots, X_n$ be non-negative random variables with finite and positive means, and let $W = \sum_{i=1}^{n} X_i$. Denote $\mu_n = E(W)$ and $\sigma_n^2 = \text{Var}(W)$. By the result of Goldstein and Zhang [26], we get a Berry-Esseen bound for W expressed in the following theorem.

Theorem 1.1. *([26], p.880) If μ_n and σ_n^2 are finite and positive, and W^s is a random variable having the W-size biased distribution and satisfying $W \leq W^s \leq$*

$W+B$ with probability 1, for some $B>0$, then

$$\sup_{z\in\mathbb{R}}\Big|P\Big(\frac{W-\mu_n}{\sigma_n}\le z\Big)-\Phi(z)\Big|\le\frac{\mu_n}{\sigma_n^2}\Delta+0.82\frac{\mu_n B^2}{\sigma_n^3}+\frac{B}{\sigma_n},$$

where

$$\Delta=\sqrt{\mathrm{Var}(E(W^s-W\mid W))}.$$

In the present work, we obtain Theorem 1.1 in the case of a non-uniform exponential bound. Our main result is as follows.

Theorem 1.2. *Under the assumptions in Theorem 1.1, for $z\in\mathbb{R}$ such that $0<|z|\le 4\sigma_n$ and $c>0$, we have*

$$\Big|P\Big(\frac{W-\mu_n}{\sigma_n}\le z\Big)-\Phi(z)\Big|\le C_1(z)\frac{\mu_n}{\sigma_n^2}\Delta+C_2(z)\frac{\mu_n B^2}{\sigma_n^3}+C_3(z)\frac{B}{\sigma_n}$$

where

$$C_1(z)=\frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|}+e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)}+e^{\frac{\mu_n|z|^3}{192B\sigma_n^3}}e^{-\frac{z^2}{8B}\left(\frac{1}{1+c}-\frac{\mu_n}{8\sigma_n^2}\right)},$$

$$C_2(z)=\frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}}+\Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}}+e^{\frac{\mu_n|z|^3}{96B\sigma_n^3}+\frac{|z|}{4\sigma_n}}e^{-z^2\left(\frac{9}{40B}-\frac{\mu_n}{32B\sigma_n^2}\right)}\Big),$$

$$C_3(z)=2\Big(e^{\frac{\mu_n|z|^3}{192B\sigma_n^3}+\frac{|z|}{8\sigma_n}}\Big)e^{-\frac{z^2}{2}\left(\frac{1}{4B}-\frac{\mu_n}{32B\sigma_n^2}\right)}.$$

There are three examples to illustrate the usefulness of Theorem 1.2 consisting of sum of independent random variables, the number of bulbs on at the terminal time in the lightbulb process and the number of m runs.

In 2005, Chaidee [7] presented a non-uniform Berry-Esseen bound for sum of independent bounded random variables in the term of $e^{-\frac{|z|}{2}}$. A non-uniform bound for sum of independent random variables in the term of $e^{-\frac{z^2}{2}}$ with unknown constant which was suggested by Chen, Fang and Shao [8]. In this work, we apply Theorem 1.2 to obtain a non-uniform exponential bound for sum of independent random variables, especially, the bound is in the term of $e^{-\frac{z^2}{16}}$ with known constant when the approximated random variable is binomial with parameter $p = \frac{1}{2}$.

In 2011, Goldstein and Zhang [26] gave a uniform Berry-Esseen bound for the number of bulbs on at the terminal time in the lightbulb process by using a version of the Stein's method for bounded monotone size biased couplings. We obtain it in the case of a non-uniform bound. Application of the lightbulb process is also mentioned as the dermal patch problem.

In 2005, Goldstein derived a uniform bound for approximating the distribution of the number of m runs by normal distribution which can be found in [21]. Now, we attain the non-uniform exponential bound.

This dissertation is organized as follows. In Chapter II, we review some concept of the Stein's method and some properties of size biased couplings. In Chapter III, the proof of Theorem 1.2 is provided. The applications of Theorem 1.2 such as the sum of independent random variables, the lightbulb process and the number of m runs are presented in Chapter IV, Chapter V and Chapter VI, respectively.

CHAPTER II
STEIN'S METHOD AND SIZE BIASED COUPLINGS

In this chapter, we introduce the Stein's method for normal approximation and size biased couplings equipped with some involved properties.

2.1 Stein's Method for Normal Approximation

In 1972, Stein [44] introduced a method in finding a Berry-Esseen bound which is not concern with Fourier transformation and relied on the elementary differential equation. This method was extended from the normal distribution to other distributions, for example, Possion, Cauchy, binomial and clubbed binomial which can be seen in [3], [30], [43] and [25], respectively. In this dissertation, we interest in normal approximation. Consider the Stein's equation for normal distribution function

$$f'(w) - wf(w) = \mathbb{I}(w \leq z) - \Phi(z), \tag{2.1}$$

where $\mathbb{I}$ is the indicator function and Φ is the standard normal distribution function. The above linear differential equation has the unique solution f_z which is form

$$f_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(w)[1-\Phi(z)] & \text{if } w \leq z, \\ \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(z)[1-\Phi(w)] & \text{if } w > z. \end{cases} \tag{2.2}$$

Then

$$f_z'(w) = \begin{cases} [1-\Phi(z)][1+\sqrt{2\pi}we^{\frac{w^2}{2}}\Phi(w)] & \text{if } w < z, \\ \Phi(z)[-1+\sqrt{2\pi}we^{\frac{w^2}{2}}(1-\Phi(w))] & \text{if } w > z. \end{cases}$$

Note that the derivative of f_z at the point $w = z$ does not exist. From the Stein's equation (2.1) and the Stein's solution (2.2), we define f_z' at the point $w = z$ by

$$f_z'(z) = [1 - \Phi(z)][1 + \sqrt{2\pi} z e^{\frac{z^2}{2}} \Phi(z)].$$

Therefore,

$$f_z'(w) \quad = \begin{cases} [1 - \Phi(z)][1 + \sqrt{2\pi} w e^{\frac{w^2}{2}} \Phi(w)] & \text{if } w \leq z, \\ \Phi(z)[-1 + \sqrt{2\pi} w e^{\frac{w^2}{2}} (1 - \Phi(w))] & \text{if } w > z. \end{cases} \tag{2.3}$$

For any random variable W, by (2.1), we note that

$$E(f_z'(W)) - E(W f_z(W)) = P(W \leq z) - \Phi(z)$$

which implies that we can find a bound of $E(f_z'(W)) - E(W f_z(W))$ instead of $P(W \leq z) - \Phi(z)$. This technique is called the **Stein's method**.

In our work, we need the following properties of f_z':

$$|f_z'(w)| \leq 1 \quad \text{for all } w \in \mathbb{R} \tag{2.4}$$

([10], p.246).

For each $z > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(w) = (w f_z(w))'$. Then

$$g(w) \geq 0 \quad \text{for all } w \in \mathbb{R}, \tag{2.5}$$

$$g(w) \leq 2(1 - \Phi(z)) \quad \text{for } w \leq 0, \tag{2.6}$$

$$g(w) \leq \frac{2}{1 + w^3} \quad \text{for } w \geq z, \tag{2.7}$$

and

$$g(w) = \begin{cases} \left(\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}[1-\Phi(w)] - w\right)\Phi(z) & \text{if } w \geq z, \\ \left(\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\Phi(w) + w\right)(1-\Phi(z)) & \text{if } w < z \end{cases} \tag{2.8}$$

([10], pp.248–249).

In order to bound $|E(f_z'(W)) - E(Wf_z(W))|$, we rewrite $E(Wf_z(W))$ in a suitable form presented by many approaches, for example, the exchangeable pairs approach (see [33]–[36] and [45]), the zero bias transformations (see [21] and [23]) and the size bias transformations (see [5], [17]–[20], [22] and [24]). In this work, we focus on the size biased couplings approach to obtain the bound.

2.2 Size Biased Couplings

For a non-negative random variable Y with finite and positive mean μ_Y, the distribution of Y^s is said to be Y-**size biased** if

$$E(Yf(Y)) = \mu_Y Ef(Y^s) \tag{2.9}$$

for all functions f for which these expectations exist.

The distribution of Y^s exists with Radon Nikodym derivative which can be seen in [4] and [9].

The size biased coupling (Y, Y^s) is **bounded** if there exists a non-negative constant B such that

$$|Y^s - Y| \leq B$$

with probability 1 and it is **monotone** if

$$Y^s \geq Y$$

with probability 1. Therefore, if there exists a non-negative constant B such that

$$Y \leq Y^s \leq Y + B$$

with probability 1, (Y, Y^s) is called the **bounded monotone size biased coupling** (see [26]).

These are some examples of size biased couplings.

Example 2.1. If Y has a Bernoulli distribution with parameter $p \in (0, 1)$, then $Y^s = 1$ has the Y-size biased distribution (see [2]). Note that the coupling is monotone and bounded by $B = 1$.

Example 2.2. If Y has a Poisson distribution with parameter $\lambda > 0$, then $Y^s = Y + 1$ has the Y-size biased distribution (see [2]). It is easy to see that (Y, Y^s) is bounded monotone size biased coupling with $B = 1$.

Example 2.3. Let $X_1, X_2, \ldots, X_n$ be non-negative independent and identically distributed (i.i.d.) random variables with $E(X_i) > 0$ for $i = 1, 2, \ldots, n$. Consider $Y = X_1 + X_2 + \cdots + X_n$. Following the construction of size biased coupling in [1], for each $i = 1, 2, \ldots, n$,

$$Y^s = X_1 + \cdots + X_{i-1} + X_i^s + X_{i+1} + \cdots + X_n$$

has the Y-size biased distribution where X_i^s has the X_i-size biased distribution, independent of X_i and X_j^s for $j \neq i$. By choosing a random index I such that

$$P(I = i) = \frac{1}{n}$$

for $i = 1, 2, \ldots, n$ and $X_1, X_2, \ldots, X_n, I$ are independent,

$$Y^s = Y - X_I + X_I^s$$

has the Y-size biased distribution. We note that (Y, Y^s) is not necessary bounded or monotone.

Example 2.4. In the classical urn allocation model, n balls are thrown independently into one of m urns, with the probability of a ball being in any urn having equal probability. For $i = 1, 2, \ldots, n$, let X_i be the location of ball i, that is, the number of urn where ball i lands. The number of non-isolated balls is given by

$$Y = \sum_{i=1}^{n} \mathbb{I}(Z_i > 0) \quad \text{where } Z_i = -1 + \sum_{j=1}^{n} \mathbb{I}(X_j = X_i).$$

In 2011, Ghosh and Goldstein [17] gave the construction of size biased coupling of Y such that

$$|Y^s - Y| \leq 2,$$

but is not necessarily monotone in some situation.

Example 2.5. Let Y be a geometric random variable with parameter $p \in (0, 1)$. Arratia and Goldstein [2] showed that

$$Y^s = Y + G + 1$$

has the Y-size biased distribution where G is an independent copy of Y. This coupling is monotone but unbounded.

CHAPTER III

NON-UNIFORM BOUNDS FOR BOUNDED MONOTONE SIZE BIASED COUPLINGS

Let $X_1, X_2, \ldots, X_n$ be non-negative random variables with finite and positive means, and let $W = \sum_{i=1}^{n} X_i$ with mean μ_n and variance σ_n^2. A random variable W^s having the W-size biased distribution exists by using the size bias construction appearing in [4] and [9].

In 2011, Goldstein and Zhang [26] used the Stein's method and bounded monotone size biased couplings to give a uniform Berry-Esseen bound as the following result.

Theorem 3.1. *([26], p.880) If $W \leq W^s \leq W + B$ with probability 1, for some $B > 0$, then*

$$\sup_{z\in\mathbb{R}} \Big| P\Big(\frac{W-\mu_n}{\sigma_n} \leq z\Big) - \Phi(z)\Big| \leq \frac{\mu_n}{\sigma_n^2}\Delta + 0.82\frac{\mu_n B^2}{\sigma_n^3} + \frac{B}{\sigma_n},$$

where

$$\Delta = \sqrt{\mathrm{Var}(E(W^s - W \mid W))}.$$

In our work, we obtain Theorem 3.1 in the case of a non-uniform exponential bound.

Theorem 3.2. *If $W \leq W^s \leq W + B$ with probability 1, for some $B > 0$, then for $z \in \mathbb{R}$ such that $0 < |z| \leq 4\sigma_n$ and $c > 0$,*

$$\Big| P\Big(\frac{W-\mu_n}{\sigma_n} \leq z\Big) - \Phi(z)\Big| \leq C_1(z)\frac{\mu_n}{\sigma_n^2}\Delta + C_2(z)\frac{\mu_n B^2}{\sigma_n^3} + C_3(z)\frac{B}{\sigma_n}$$

where

$$C_1(z) = \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{\mu_n|z|^3}{192B\sigma_n^3}} e^{-\frac{z^2}{8B}\left(\frac{1}{1+c}-\frac{\mu_n}{8\sigma_n^2}\right)},$$

$$C_2(z) = \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \left(\frac{1+z^2}{2|z|}\right)\left(e^{-\frac{19z^2}{200}} + e^{\frac{\mu_n|z|^3}{96B\sigma_n^3}+\frac{|z|}{4\sigma_n}} e^{-z^2\left(\frac{9}{40B}-\frac{\mu_n}{32B\sigma_n^2}\right)}\right),$$

$$C_3(z) = 2\left(e^{\frac{\mu_n|z|^3}{192B\sigma_n^3}+\frac{|z|}{8\sigma_n}}\right)e^{-\frac{z^2}{2}\left(\frac{1}{4B}-\frac{\mu_n}{32B\sigma_n^2}\right)}.$$

This chapter is organized as follows. Auxiliary results are in Section 3.1 while the proof of the main result appears in Section 3.2.

3.1 Auxiliary Results

In this section, we give auxiliary results to prove our main result in Section 3.2. For convenience, we let

$$U = \frac{W-\mu_n}{\sigma_n} \quad \text{and} \quad \widetilde{U} = \frac{W^s-\mu_n}{\sigma_n}.$$

To obtain a non-uniform Berry-Esseen bound for sum of independent random variables W, we can apply the Bennett-Hoeffding inequality

$$Ee^{kU} \leq \exp(e^k - 1 - k)$$

for $k > 0$ ([11], p.40). But in this work, W is a sum of not necessarily independent random variables. Therefore, we cannot apply the Bennett-Hoeffding inequality and thus Lemma 3.3 is required to get the bound.

First, we need to bound Ee^{kU} seen in Lemma 3.3. To do this, we use some ideas from Lemma 5.1 of Chen et al. [8] and apply the technique in Lemma 3.1 of Chuntee and Neammanee [13].

Lemma 3.3. *If $W \le W^s \le W + B$ with probability 1, for some $B > 0$, then for any $k > 0$ such that $kB \le \sigma_n$, we have*

$$Ee^{kU} \le e^{\frac{\mu_n k^2 B}{2\sigma_n^2} + \frac{2\mu_n k^3 B^2}{3\sigma_n^3}}.$$

Proof. Let k be a positive real number such that $kB \le \sigma_n$. Let $f : \mathbb{R} \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$ be defined by $f(u) = e^{ku}$ and $h(t) = Ee^{tU}$. Note that

$$|\widetilde{U} - U| = \Big|\frac{W^s - W}{\sigma_n}\Big| \le \frac{B}{\sigma_n}. \tag{3.1}$$

By the fact that

$$E(Ug(U)) = \frac{\mu_n}{\sigma_n} E(g(\widetilde{U}) - g(U)) \tag{3.2}$$

for all functions g for which these expectations exist (see [9], p.31) equipped with (3.1), we have

$$\begin{aligned}
h'(k) &= E\big(Ue^{kU}\big) \\
&= E(Uf(U)) \\
&= \frac{\mu_n}{\sigma_n} E(f(\widetilde{U}) - f(U)) \\
&= \frac{\mu_n}{\sigma_n} E \int_0^{\widetilde{U}-U} f'(U+t)dt \\
&= \frac{\mu_n}{\sigma_n} E \int_0^{\widetilde{U}-U} ke^{k(U+t)}dt \\
&\le \frac{\mu_n k}{\sigma_n} E \int_0^{\frac{B}{\sigma_n}} e^{k(U+\frac{B}{\sigma_n})}dt \\
&= \frac{\mu_n kB}{\sigma_n^2} e^{\frac{kB}{\sigma_n}} Ee^{kU} \\
&\le \frac{\mu_n kB}{\sigma_n^2} Ee^{kU} + \frac{\mu_n kB}{\sigma_n^2} Ee^{kU}\big|e^{\frac{kB}{\sigma_n}} - 1\big| \\
&\le \frac{\mu_n kB}{\sigma_n^2} Ee^{kU} + \frac{2\mu_n k^2 B^2}{\sigma_n^3} Ee^{kU}
\end{aligned}$$

where the last inequality holds by the fact that $|e^w - 1| \le 2|w|$ for any $|w| \le 1$.

Thus,

$$\frac{h'(k)}{h(k)} \leq \frac{\mu_n k B}{\sigma_n^2} + \frac{2\mu_n k^2 B^2}{\sigma_n^3}$$

which implies that

$$\ln h(k) \leq \frac{\mu_n k^2 B}{2\sigma_n^2} + \frac{2\mu_n k^3 B^2}{3\sigma_n^3}.$$

Therefore, we have the lemma. □

We use the idea from Lemma 2.1 of Goldstein and Zhang [26] to obtain the lemma below.

Lemma 3.4. *If $W \leq W^s$ with probability 1, then for any $z > 0$ and $\lambda, r > 0$,*

$$\frac{\mu_n}{\sigma_n} E(\widetilde{U} - U)\mathbb{I}(\widetilde{U} - U \leq \lambda)\mathbb{I}(z - \lambda \leq U \leq z) \leq 2\lambda(Ee^{2rzU})^{\frac{1}{2}} e^{-rz(z-\lambda)}.$$

Proof. Let $z > 0$, we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0 & \text{if } t < z - \lambda, \\ (t - z + \lambda)e^{rzt} & \text{if } z - \lambda \leq t \leq z + \lambda, \\ 2\lambda e^{rzt} & \text{if } t > z + \lambda. \end{cases}$$

Then

$$|f(t)| \leq 2\lambda e^{rzt} \quad \text{for all } t \in \mathbb{R} \tag{3.3}$$

and

$$f'(t) = \begin{cases} 0 & \text{if } t < z - \lambda, \\ (rz(t - z + \lambda) + 1)e^{rzt} & \text{if } z - \lambda < t < z + \lambda, \\ 2\lambda rze^{rzt} & \text{if } t > z + \lambda. \end{cases}$$

These imply that

$$f'(t) \geq \begin{cases} e^{rzt} & \text{for } z-\lambda < t < z+\lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

By (3.2)–(3.4) equipped with $EU^2 = 1$ and the Hölder inequality, we get

$$\begin{aligned} 2\lambda(Ee^{2rzU})^{\frac{1}{2}} &= 2\lambda(EU^2)^{\frac{1}{2}}(Ee^{2rzU})^{\frac{1}{2}} \\ &\geq E(Uf(U)) \\ &= \frac{\mu_n}{\sigma_n} E\big(f(\widetilde{U}) - f(U)\big) \\ &= \frac{\mu_n}{\sigma_n} E \int_0^{\widetilde{U}-U} f'(U+t)dt \\ &\geq \frac{\mu_n}{\sigma_n} E\mathbb{I}(z-\lambda \leq U \leq z) \int_0^{\widetilde{U}-U} \mathbb{I}(0 \leq t \leq \lambda) f'(U+t)dt \\ &\geq \frac{\mu_n}{\sigma_n} E\mathbb{I}(z-\lambda \leq U \leq z) \int_0^{\widetilde{U}-U} \mathbb{I}(0 \leq t \leq \lambda) e^{rz(U+t)}dt \\ &\geq \frac{\mu_n}{\sigma_n} e^{rz(z-\lambda)} E\mathbb{I}(z-\lambda \leq U \leq z) \min(\lambda, \widetilde{U}-U) \\ &\geq \frac{\mu_n}{\sigma_n} e^{rz(z-\lambda)} E(\widetilde{U}-U)\mathbb{I}(\widetilde{U}-U \leq \lambda)\mathbb{I}(z-\lambda \leq U \leq z). \end{aligned}$$

Therefore,

$$\frac{\mu_n}{\sigma_n} E(\widetilde{U}-U)\mathbb{I}(\widetilde{U}-U \leq \lambda)\mathbb{I}(z-\lambda \leq U \leq z) \leq 2\lambda(Ee^{2rzU})^{\frac{1}{2}} e^{-rz(z-\lambda)}.$$

□

We use the idea from Chuntee and Neammanee ([13]) to prove the following lemmas.

Lemma 3.5. *Let f_z be the solution of the Stein's equation (2.1). Then for any*

$z, c > 0$, we have

$$|f_z'(w)| \le \begin{cases} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} & \text{if } w < 0, \\ \left(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\right)\left[1 + \sqrt{2\pi}ze^{\frac{z^2}{2(1+c)^2}}\right] & \text{if } 0 \le w \le \frac{z}{1+c}, \\ 1 & \text{if } w > \frac{z}{1+c}. \end{cases}$$

Proof. Let $z, c > 0$ and $w \in \mathbb{R}$ be given. From (2.4) and [12] (p.43), we have $|f_z'(w)| \le 1$ for $w > \frac{z}{1+c}$ and $|f_z'(w)| \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}$ for $w < 0$, respectively. Assume that $0 \le w \le \frac{z}{1+c}$. From (2.3) and the fact that

$$1 - \Phi(z) \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} \qquad \text{for} \quad z > 0 \tag{3.5}$$

([11], p.11), we get

$$\begin{aligned} f_z'(w) &= [1 - \Phi(z)][1 + \sqrt{2\pi}we^{\frac{w^2}{2}}\Phi(w)] \\ &\le [1 - \Phi(z)][1 + \left(\frac{\sqrt{2\pi}z}{1+c}\right)e^{\frac{z^2}{2(1+c)^2}}] \\ &\le \left(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\right)\left[1 + \sqrt{2\pi}ze^{\frac{z^2}{2(1+c)^2}}\right]. \end{aligned}$$

□

Lemma 3.6. *Let $g(w) = (wf_z(w))'$. Then for any $z > 0$, we have*

$$g(w) \le \begin{cases} \frac{(1+z^2)}{z}e^{-\frac{19z^2}{200}} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} & \text{if } w \le \frac{9z}{10}, \\ \frac{1+z^2}{z} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} & \text{if } w > \frac{9z}{10}. \end{cases}$$

Proof. Let $z > 0$ and $w \in \mathbb{R}$ be given.

Case $w \le \frac{9z}{10}$. For $w \le 0$, by (2.6) and (3.5), we have

$$g(w) \le 2(1 - \Phi(z)) \le \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} \le \frac{e^{-\frac{19z^2}{200}}}{z}.$$

For $0 < w \le \frac{9z}{10}$, we use (2.8) and (3.5) to obtain

$$\begin{aligned} g(w) &= \left(\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\Phi(w) + w\right)(1-\Phi(z)) \\ &\le \left(\sqrt{2\pi}(1+z^2)e^{\frac{81z^2}{200}} + z\right)\left(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\right) \\ &= \frac{(1+z^2)}{z}e^{-\frac{19z^2}{200}} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}. \end{aligned}$$

Therefore, we get the bound as desired.

Case $w > \frac{9z}{10}$. If $w < z$, by (2.8) and (3.5), we get

$$\begin{aligned} g(w) &= \left(\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\Phi(w) + w\right)(1-\Phi(z)) \\ &\le \left(\sqrt{2\pi}(1+z^2)e^{\frac{z^2}{2}} + z\right)\left(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\right) \\ &= \frac{1+z^2}{z} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}. \end{aligned}$$

For $w \ge z$, from (2.7), we have

$$g(w) \le \frac{2}{1+w^3} \le \frac{2}{1+z^3} \le \frac{1+z^2}{z}$$

where the last inequality holds by using the fact that

$$\frac{2}{1+z^3} - \frac{1+z^2}{z} = \frac{-(z-1)^2 - z^3 - z^5}{z(1+z^3)} \le 0.$$

These lead us to obtain the bound. $\square$

3.2 Proof of Theorem 3.2

Assume that $0 < z \le 4\sigma_n$. In the case $-4\sigma_n \le z < 0$, we use the symmetry of $\Phi(z)$ that $\Phi(z) = 1 - \Phi(-z)$ and apply the result to $-U$.

Goldstein and Zhang [26] used the Stein's method and the property of the monotone size biased coupling for W in (3.2) to obtain

$$\begin{aligned}
P(U \le z) - \Phi(z) &= E\Big(f_z'(U)\Big(1 - \frac{\mu_n}{\sigma_n}(\widetilde{U} - U)\Big)\Big) \\
&\quad - E\Big(\frac{\mu_n}{\sigma_n}\int_0^{\widetilde{U}-U}\big((U+t)f_z(U+t) - Uf_z(U))\big)dt\Big) \\
&\quad - E\Big(\frac{\mu_n}{\sigma_n}\int_0^{\widetilde{U}-U}\big(\mathbb{I}(U+t \le z) - \mathbb{I}(U \le z)\big)dt\Big) \\
&=: R_1 - R_2 - R_3
\end{aligned}$$

([26], p.881). By using (2.9) with the identity function, we have

$$EW^s = \frac{EW^2}{\mu_n}$$

which leads us to the property that

$$\frac{\mu_n}{\sigma_n}E(\widetilde{U} - U) = \frac{\mu_n}{\sigma_n^2}E(W^s - W) = \frac{\mu_n}{\sigma_n^2}\Big(\frac{EW^2}{\mu_n} - \mu_n\Big) = \frac{1}{\sigma_n^2}\Big(EW^2 - (EW)^2\Big) = 1.$$

Therefore,

$$\begin{aligned}
E\Big|E\big(1 - \frac{\mu_n}{\sigma_n}(\widetilde{U} - U) \mid U\big)\Big|^2 &= \frac{\mu_n^2}{\sigma_n^2}E\Big|E(\widetilde{U} - U) - E(\widetilde{U} - U \mid U)\Big|^2 \\
&= \frac{\mu_n^2}{\sigma_n^2}E\Big|E(\widetilde{U} - U \mid U) - E(E(\widetilde{U} - U \mid U))\Big|^2 \\
&= \frac{\mu_n^2}{\sigma_n^2}\mathrm{Var}(E(\widetilde{U} - U \mid U))
\end{aligned}$$

which implies that

$$\begin{aligned}
\sqrt{E\Big|E\big(1 - \frac{\mu_n}{\sigma_n}(\widetilde{U} - U) \mid U\big)\Big|^2} &= \frac{\mu_n}{\sigma_n}\sqrt{\mathrm{Var}(E(\widetilde{U} - U \mid U))} \\
&= \frac{\mu_n}{\sigma_n^2}\sqrt{\mathrm{Var}(E(W^s - W \mid W))} \\
&= \frac{\mu_n}{\sigma_n^2}\Delta. \qquad (3.6)
\end{aligned}$$

For $c > 0$, we can see that

$$\begin{aligned}|R_1| &\le E\Big|f_z'(U)E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|\mathbb{I}(U<0)\\&\quad+E\Big|f_z'(U)E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|\mathbb{I}\Big(0\le U\le\frac{z}{1+c}\Big)\\&\quad+E\Big|f_z'(U)E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|\mathbb{I}\Big(U>\frac{z}{1+c}\Big)\\&=: R_{11}+R_{12}+R_{13}.\end{aligned}$$

By Lamma 3.5 equipped with (3.6), we obtain

$$\begin{aligned}R_{11} &\le \Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\Big)E\Big|E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|\\&\le \Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\Big)\sqrt{E\Big|E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|^2}\\&= \Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\Big)\frac{\mu_n}{\sigma_n^2}\Delta.\end{aligned}\tag{3.7}$$

Similar to (3.7), we get

$$\begin{aligned}R_{12} &\le \Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\Big)\Big[1+\sqrt{2\pi}ze^{\frac{z^2}{2(1+c)^2}}\Big]E\Big|E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|\\&\le \Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\Big)\Big[1+\sqrt{2\pi}ze^{\frac{z^2}{2(1+c)^2}}\Big]\sqrt{E\Big|E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|^2}\\&= \Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\Big)\Big[1+\sqrt{2\pi}ze^{\frac{z^2}{2(1+c)^2}}\Big]\frac{\mu_n}{\sigma_n^2}\Delta\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}R_{13} &\le E\Big|E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|\mathbb{I}\Big(U>\frac{z}{1+c}\Big)\\&\le \sqrt{P\Big(U>\frac{z}{1+c}\Big)}\sqrt{E\Big|E\Big(1-\frac{\mu_n}{\sigma_n}(\widetilde{U}-U)\mid U\Big)\Big|^2}\\&= \sqrt{P\Big(U>\frac{z}{1+c}\Big)}\frac{\mu_n}{\sigma_n^2}\Delta.\end{aligned}\tag{3.9}$$

Since $0 < z \le 4\sigma_n$, we can apply Lemma 3.3 with $k = \frac{z}{4B}$ and the Markov's inequality to get

$$P\Big(U > \frac{z}{1+c}\Big) \le \frac{Ee^{\frac{z}{4B}U}}{e^{\frac{z^2}{4B(1+c)}}} \le \frac{e^{\frac{\mu_n z^2}{32B\sigma_n^2}+\frac{\mu_n z^3}{96B\sigma_n^3}}}{e^{\frac{z^2}{4B(1+c)}}} = e^{\frac{\mu_n z^3}{96B\sigma_n^3}} e^{-\frac{z^2}{4B}\left(\frac{1}{1+c}-\frac{\mu_n}{8\sigma_n^2}\right)}. \tag{3.10}$$

By this fact and (3.7)–(3.9), we conclude that

$$|R_1| \le \Big[\frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{\mu_n z^3}{192B\sigma_n^3}} e^{-\frac{z^2}{8B}\left(\frac{1}{1+c}-\frac{\mu_n}{8\sigma_n^2}\right)}\Big]\frac{\mu_n}{\sigma_n^2}\Delta. \tag{3.11}$$

Next, we will bound the term R_2. Let $g(w) = (wf_z(w))'$. We use (2.5), Lemma 3.6 and the fact that $0 \le U^s - U \le \frac{B}{\sigma_n}$ to obtain

$$\begin{aligned}
|R_2| &\le E\Big|\frac{\mu_n}{\sigma_n}\int_0^{\widetilde{U}-U}\big[(U+t)f_z(U+t) - Uf_z(U))\big]dt\Big| \\
&= \frac{\mu_n}{\sigma_n}E\int_0^{\widetilde{U}-U}\Big(\int_0^t g(U+u)du\Big)dt \\
&\le \frac{\mu_n}{\sigma_n}E\int_0^{\frac{B}{\sigma_n}}\Big(\int_0^t g(U+u)du\Big)dt \\
&\le \frac{\mu_n}{\sigma_n}E\int_0^{\frac{B}{\sigma_n}}\int_0^t\Big(\frac{1+z^2}{z}e^{-\frac{19z^2}{200}} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\Big)\mathbb{I}\Big(U+u \le \frac{9z}{10}\Big) \\
&\quad + \Big(\frac{1+z^2}{z} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\Big)\mathbb{I}\Big(U+u > \frac{9z}{10}\Big)dudt \\
&\le \frac{\mu_n}{\sigma_n}\int_0^{\frac{B}{\sigma_n}}\int_0^t\Big(\frac{1+z^2}{z}e^{-\frac{19z^2}{200}} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} + \frac{1+z^2}{z}P\Big(U+u > \frac{9z}{10}\Big)\Big)dudt.
\end{aligned}$$

Applying Lemma 3.3 which is the same argument as (3.10),

$$\begin{aligned}
P\Big(U+u > \frac{9z}{10}\Big) &\le P\Big(U + \frac{B}{\sigma_n} > \frac{9z}{10}\Big) \\
&= P\Big(\frac{z}{4B}U > \frac{9z^2}{40B} - \frac{z}{4\sigma_n}\Big) \\
&\le \frac{Ee^{\frac{z}{4B}U}}{e^{\frac{9z^2}{40B}}}e^{\frac{z}{4\sigma_n}} \\
&\le e^{\frac{\mu_n z^3}{96B\sigma_n^3}+\frac{z}{4\sigma_n}} e^{-z^2\left(\frac{9}{40B}-\frac{\mu_n}{32B\sigma_n^2}\right)}
\end{aligned}$$

for $0 \le u \le t$ whenever $0 \le t \le \frac{B}{\sigma_n}$. Therefore,

$$
\begin{aligned}
|R_2| &\le \frac{\mu_n}{\sigma_n}\int_0^{\frac{B}{\sigma_n}}\int_0^t \Big(\frac{1+z^2}{z}e^{-\frac{19z^2}{200}} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} + \frac{1+z^2}{z}e^{\frac{\mu_n z^3}{96B\sigma_n^3}+\frac{z}{4\sigma_n}}e^{-z^2\left(\frac{9}{40B}-\frac{\mu_n}{32B\sigma_n^2}\right)}\Big)dudt \\
&= \frac{B^2\mu_n}{2\sigma_n^3}\left[\Big(\frac{1+z^2}{z}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{\mu_n z^3}{96B\sigma_n^3}+\frac{z}{4\sigma_n}}e^{-z^2\left(\frac{9}{40B}-\frac{\mu_n}{32B\sigma_n^2}\right)}\Big) + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\right] \\
&= \frac{B^2\mu_n}{2\sigma_n^3}\Big(\frac{1+z^2}{z}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{\mu_n z^3}{96B\sigma_n^3}+\frac{z}{4\sigma_n}}e^{-z^2\left(\frac{9}{40B}-\frac{\mu_n}{32B\sigma_n^2}\right)}\Big) + \frac{B^2\mu_n e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}\sigma_n^3}. \qquad (3.12)
\end{aligned}
$$

To finish the proof, it remains to bound R_3. Goldstein and Zhang [26] showed that

$$
|R_3| \le \frac{\mu_n}{\sigma_n}E(\widetilde{U}-U)\mathbb{I}\Big(z-\frac{B}{\sigma_n} \le U \le z\Big) \qquad (3.13)
$$

([26], p.882). Applying Lemma 3.4 when $\lambda = \frac{B}{\sigma_n}$ and $r = \frac{1}{8B}$,

$$
\frac{\mu_n}{\sigma_n}E(\widetilde{U}-U)\mathbb{I}\Big(\widetilde{U}-U \le \frac{B}{\sigma_n}\Big)\mathbb{I}\Big(z-\frac{B}{\sigma_n} \le U \le z\Big) \le \frac{2B}{\sigma_n}\Big(Ee^{\frac{z}{4B}U}\Big)^{\frac{1}{2}}e^{-\frac{z}{8B}\left(z-\frac{B}{\sigma_n}\right)}. \qquad (3.14)
$$

By (3.13), (3.14) and the fact that $0 < \widetilde{U}-U \le \frac{B}{\sigma_n}$, we obtain

$$
\begin{aligned}
|R_3| &\le \frac{2B}{\sigma_n}\Big(Ee^{\frac{z}{4B}U}\Big)^{\frac{1}{2}}e^{-\frac{z}{8B}\left(z-\frac{B}{\sigma_n}\right)} \\
&\le \frac{2B}{\sigma_n}\Big(e^{\frac{\mu_n z^3}{192B\sigma_n^3}+\frac{\mu_n z^2}{64B\sigma_n^2}}\Big)e^{-\frac{z^2}{8B}+\frac{z}{8\sigma_n}} \\
&= \frac{2B}{\sigma_n}e^{\frac{\mu_n z^3}{192B\sigma_n^3}+\frac{z}{8\sigma_n}}e^{-\frac{z^2}{2}\left(\frac{1}{4B}-\frac{\mu_n}{32B\sigma_n^2}\right)} \qquad (3.15)
\end{aligned}
$$

where the last inequality holds by using Lemma 3.3 with $k = \frac{z}{4B}$.

Combining (3.11), (3.12) and (3.15), the proof is completed. □

CHAPTER IV

NON-UNIFORM BOUNDS FOR SUM OF INDEPENDENT RANDOM VARIABLES

Let $X_1, X_2, \ldots, X_n$ be independent and not necessary identically distributed random variables with zero means and finite variances. Let

$$W = \sum_{i=1}^{n} X_i$$

and assume that $\text{Var}(W) = 1$.

Under the finiteness of the third moment, we have the uniform and non-uniform Berry-Esseen theorem

$$|P(W \leq z) - \Phi(z)| \leq C_0 \sum_{i=1}^{n} E|X_i|^3$$

and

$$|P(W \leq z) - \Phi(z)| \leq \frac{C_1}{1 + |z|^3} \sum_{i=1}^{n} E|X_i|^3,$$

respectively, where C_0 and C_1 are absolute constants. If X_i's are identically distributed, then both inequalities shown above can be found in the original works of Esseen [15] and Nagaev [29], respectively. The upper estimate of the constant C_0 was studied for a long history seen in Van Beek [47], Shiganov [42], Shevtsova [39], Korolev and Shevtsova [28] and Tyurin [46]. In 2011, Shevtsova [41] obtained the bound as the following

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq 0.4748 n E|X_1|^3.$$

The result in the case that X_i's are non-identically distributed was generalized by Bikelis [6] and the constant is 0.5600 obtained by Shevtsova [40]. For the non-uniform version, the constant C_1 was calculated to be 31.395 by Paditz [31] for non-identically distributed random variables. In 2001, Chen and Shao [10] gave the non-uniform bound without assuming the existence of the third moment. In the case of bounded random variables, that is, $|X_i| \leq \delta$ for $i = 1, \ldots, n$, they obtained the uniform bound as

$$|P(W \leq z) - \Phi(z)| \leq 3.3\delta$$

([11], p.23). Next, Chaidee [7] used the idea of Chen and Shao [11] to prove a non-uniform exponential bound for independent bounded random variables

$$|P(W \leq z) - \Phi(z)| \leq C_\delta e^{-\frac{|z|}{2}} \delta,$$

where $C_\delta = 4.45 + 2.21 e^{2\delta + \delta^{-2}(e^{2\delta} - 1 - 2\delta)}$. Furthermore, a non-uniform exponential bound in the term of $e^{-\frac{z^2}{2}}$ with unknown constant was proved by Chen, Fang and Shao [8].

In this chapter, we concentrate on finding a non-uniform exponential bound for sum of independent random variables in the term of $e^{-\frac{z^2}{k}}$ equipped with some known constants by using Stein's method and bounded monotone size biased coupling.

Now, we consider

$$W = \sum_{i=1}^{n} X_i$$

where X_i's are non-negative independent random variables with finite and positive mean $\tilde{\mu}_i$ for $i = 1, \ldots, n$, and let $\mu_n = E(W)$ and $\sigma_n^2 = \mathrm{Var}(W)$.

For each $i = 1, \ldots, n$, let X_i^s have the X_i-size biased distribution, independent of X_j and X_j^s for $j \neq i$, and $0 \leq X_i^s - X_i \leq l$ with probability 1, for some $l > 0$. By the size bias construction appearing in [9] and [37], W^s can be constructed as follows. For a random index I such that

$$P(I = i) = \frac{E(X_i)}{\mu_n}$$

for $i = 1, \dots, n$ and $X_1, \dots, X_n, I$ are independent,

$$W^s = W - X_I + X_I^s$$

has the W-size biased distribution satisfying

$$W \le W^s \le W + l$$

with probability 1. Thus, Theorem 3.2 can be applied with $B = l$, and then we get a non-uniform exponential bound for independent random variables as below.

Theorem 4.1. *Let X_i^s have the X_i-size biased distribution, independent of X_j and X_j^s for $j \neq i$. Assume that $0 \le X_i^s - X_i \le l$ with probability 1, for some $l > 0$ and for $i = 1, \dots, n$. Then for $z \in \mathbb{R}$ such that $0 < |z| \le 4\sigma_n$ and $c > 0$,*

$$\Big|P\Big(\frac{W - \mu_n}{\sigma_n} \le z\Big) - \Phi(z)\Big| \le C_1(z)\frac{\mu_n}{\sigma_n^2}\Delta + C_2(z)\frac{\mu_n l^2}{\sigma_n^3} + C_3(z)\frac{l}{\sigma_n}$$

where

$$C_1(z) = \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{\mu_n|z|^3}{192 l\sigma_n^3}} e^{-\frac{z^2}{8l}\left(\frac{1}{1+c}-\frac{\mu_n}{8\sigma_n^2}\right)},$$

$$C_2(z) = \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{\mu_n|z|^3}{96 l\sigma_n^3}+\frac{|z|}{4\sigma_n}} e^{-z^2\left(\frac{9}{40l}-\frac{\mu_n}{32l\sigma_n^2}\right)}\Big),$$

$$C_3(z) = 2\Big(e^{\frac{\mu_n|z|^3}{192 l\sigma_n^3}+\frac{|z|}{8\sigma_n}}\Big)e^{-\frac{z^2}{2}\left(\frac{1}{4l}-\frac{\mu_n}{32l\sigma_n^2}\right)}$$

and

$$\Delta = \sqrt{\mathrm{Var}(E(W^s - W \mid W))}.$$

If $X_1, \dots, X_n$ are independent Bernoulli random variables with parameter p, then $W = \sum_{i=1}^{n} X_i$ is a binomial random variable with

$$\mu_n = np \quad \text{and} \quad \sigma_n^2 = npq$$

where $q = 1 - p$. Since Baldi et al. [5] showed that $X_i^s = 1$ has the X_i-size biased distribution satisfying

$$X_i \le X_i^s \le X_i + 1 \quad \text{and} \quad \operatorname{Var}(E(W^s - W \mid W)) = \frac{pq}{n}$$

which imply that

$$l = 1 \quad \text{and} \quad \Delta = \sqrt{\operatorname{Var}(E(W^s - W \mid W))} = \sqrt{\frac{pq}{n}},$$

we can apply Theorem 4.1 to obtain a non-uniform exponential bound for W shown in the following theorem.

Theorem 4.2. *Let $W \sim$ Binomial(n,p). For large n such that $0 < |z| \le 4\sqrt{npq}$ and $c > 0$,*

$$\left| P\Big(\frac{W - np}{\sqrt{npq}} \le z\Big) - \Phi(z) \right| \le C_1(z)\sqrt{\frac{p}{nq}} + C_2(z)\frac{1}{q\sqrt{npq}} + C_3(z)\frac{1}{\sqrt{npq}}$$

where

$$\begin{aligned}
C_1(z) &= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1 - \frac{1}{(1+c)^2}\right)} + e^{\frac{|z|^3}{192q\sqrt{npq}}} e^{-\frac{z^2}{8}\left(\frac{1}{1+c} - \frac{1}{8q}\right)}, \\
C_2(z) &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{|z|^3}{96q\sqrt{npq}} + \frac{|z|}{4\sqrt{npq}}} e^{-z^2\left(\frac{9}{40} - \frac{1}{32q}\right)}\Big), \\
C_3(z) &= 2\Big(e^{\frac{|z|^3}{192q\sqrt{npq}} + \frac{|z|}{8\sqrt{npq}}}\Big) e^{-\frac{z^2}{2}\left(\frac{1}{4} - \frac{1}{32q}\right)}.
\end{aligned}$$

Theorem 4.3. *If $W \sim$ Binomial$(n, \frac{1}{2})$, then for all real numbers z such that*

$|z| \le \sqrt{n}$,

$$\Big|P\Big(\frac{W-n/2}{\sqrt{n/4}} \le z\Big) - \Phi(z)\Big| \le \frac{1}{\sqrt{n}e^{\frac{z^2}{2}}} + \frac{17.9278}{\sqrt{n}e^{\frac{z^2}{16}}}.$$

Remark 4.4. Since $W \sim \text{Binomial}(n, \frac{1}{2})$, $0 \le W \le n$. Then $P\big(\frac{W-n/2}{\sqrt{n/4}} \le z\big) = 1$ for all $z > \sqrt{n}$ and $P\big(\frac{W-n/2}{\sqrt{n/4}} \le z\big) = 0$ for all $z < -\sqrt{n}$. Therefore, it is reasonable considering the bound when $|z| \le \sqrt{n}$.

Proof of Theorem 4.3

Let $W \sim \text{Binomial}(n, \frac{1}{2})$ and $|z| \le \sqrt{n}$.

Case $|z| \le 4$. Since $e^{\frac{z^2}{16}} \le e \le 2.7183$, we apply Theorem 3.1 with

$$B = 1, \quad \mu_n = \frac{n}{2}, \quad \sigma_n = \frac{n}{4} \quad \text{and} \quad \Delta = \frac{1}{2\sqrt{n}}$$

to get that

$$\Big|P\Big(\frac{W-n/2}{\sqrt{n/4}} \le z\Big) - \Phi(z)\Big| \le \frac{2.7183}{e^{\frac{z^2}{16}}}\Big(\frac{1}{\sqrt{n}} + \frac{3.28}{\sqrt{n}} + \frac{2}{\sqrt{n}}\Big) \le \frac{17.071}{\sqrt{n}e^{\frac{z^2}{16}}}.$$

Case $|z| > 4$. Applying Theorem 4.2 with $p = \frac{1}{2}$, we obtain

$$\Big|P\Big(\frac{W-n/2}{\sqrt{n/4}} \le z\Big) - \Phi(z)\Big| \le C_1(z)\frac{1}{\sqrt{n}} + C_2(z)\frac{4}{\sqrt{n}} + C_3(z)\frac{2}{\sqrt{n}} \tag{4.1}$$

where

$$\begin{aligned}
C_1(z) &= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{|z|^3}{48\sqrt{n}}}e^{-\frac{z^2}{8}\left(\frac{1}{1+c}-\frac{1}{4}\right)},\\
C_2(z) &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{|z|^3}{24\sqrt{n}}+\frac{|z|}{2\sqrt{n}}}e^{-z^2\left(\frac{9}{40}-\frac{1}{16}\right)}\Big),\\
C_3(z) &= 2\Big(e^{\frac{|z|^3}{48\sqrt{n}}+\frac{|z|}{4\sqrt{n}}}\Big)e^{-\frac{z^2}{2}\left(\frac{1}{4}-\frac{1}{16}\right)}.
\end{aligned}$$

By the assumption that $|z| \leq \sqrt{n}$, we can bound the terms of $C_1(z)$, $C_2(z)$ and $C_3(z)$ as follows. First, we consider the bound of $C_1(z)$ and see that

$$\begin{aligned} C_1(z) &\leq \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{z^2}{48}} e^{-\frac{z^2}{8}\left(\frac{1}{1+c}-\frac{1}{4}\right)} \\ &= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{-\frac{z^2}{8}\left(\frac{1}{1+c}-\frac{1}{4}-\frac{1}{6}\right)}. \end{aligned}$$

Next, we will find a positive constant c such that $e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} = e^{-\frac{z^2}{8}\left(\frac{1}{1+c}-\frac{1}{4}-\frac{1}{6}\right)}$. Assume that

$$1 - \frac{1}{(1+c)^2} = \frac{1}{4}\Big(\frac{1}{1+c} - \frac{1}{4} - \frac{1}{6}\Big).$$

Then

$$\frac{1}{(1+c)^2} + \frac{1}{4(1+c)} = \frac{53}{48},$$

that is,

$$53(1+c)^2 - 12(1+c) - 48 = 0.$$

Thus,

$$53c^2 + 94c - 7 = 0$$

which implies that

$$c = \frac{-94 \pm \sqrt{94^2 - 4(53)(-7)}}{2(53)} = \frac{-47 \pm 2\sqrt{645}}{53}.$$

Choose $c = \frac{-47+2\sqrt{645}}{53}$. This leads us to obtain that

$$e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} = e^{-\frac{z^2}{8}\left(\frac{1}{1+c}-\frac{1}{4}-\frac{1}{6}\right)} \leq e^{-\frac{807z^2}{12500}}.$$

Therefore,

$$\begin{aligned} C_1(z) &\le \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + 2e^{-\frac{807z^2}{12500}} \\ &= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + 2e^{-\frac{103z^2}{50000}} e^{-\frac{z^2}{16}} \\ &\le 0.2e^{-\frac{z^2}{2}} + 1.9352e^{-\frac{z^2}{16}}. \end{aligned}$$

To estimate the term $C_2(z)$, we use the fact that $\frac{1}{|z|}e^{-\frac{13z^2}{400}} \le 0.1487$, $|z|e^{-\frac{13z^2}{400}} \le 2.3781$, $\frac{1}{|z|}e^{-\frac{7z^2}{120}} \le 0.0984$ and $|z|e^{-\frac{7z^2}{120}} \le 1.573$ for $|z| \ge 4$ to get

$$\begin{aligned} C_2(z) &\le \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{z^2}{24}+\frac{1}{2}}e^{-z^2\left(\frac{9}{40}-\frac{1}{16}\right)}\Big) \\ &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{1}{2}}e^{-z^2\left(\frac{9}{40}-\frac{1}{16}-\frac{1}{24}\right)}\Big) \\ &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{1}{2}}e^{-\frac{29z^2}{240}}\Big) \\ &\le 0.2e^{-\frac{z^2}{2}} + \Big(\frac{1+z^2}{2|z|}\Big)e^{-\frac{z^2}{16}}\Big(e^{-\frac{13z^2}{400}} + 1.6488e^{-\frac{7z^2}{120}}\Big) \\ &\le 0.2e^{-\frac{z^2}{2}} + 2.6412e^{-\frac{z^2}{16}}. \end{aligned}$$

For the last term $C_3(z)$, we see that

$$\begin{aligned} C_3(z) &\le 2\big(e^{\frac{z^2}{48}+\frac{1}{4}}\big)e^{-\frac{z^2}{2}\left(\frac{1}{4}-\frac{1}{16}\right)} \\ &= 2e^{\frac{1}{4}}e^{-\frac{z^2}{2}\left(\frac{1}{4}-\frac{1}{16}-\frac{1}{24}\right)} \\ &= 2e^{\frac{1}{4}}e^{-\frac{7z^2}{96}} \\ &= 2e^{\frac{1}{4}}e^{-\frac{z^2}{16}}e^{-\frac{z^2}{96}} \\ &\le 2.1739e^{-\frac{z^2}{16}}. \end{aligned}$$

Including the bounds of $C_1(z)$, $C_2(z)$ and $C_3(z)$ as above into (4.1),

$$\begin{aligned}\Big|P\Big(\frac{W-n/2}{\sqrt{n/4}}\le z\Big)-\Phi(z)\Big| &\le \frac{1}{\sqrt{n}e^{\frac{z^2}{2}}}(0.2+4(0.2))\\ &\quad+\frac{1}{\sqrt{n}e^{\frac{z^2}{16}}}(1.9352+4(2.6412)+2(2.7139))\\ &=\frac{1}{\sqrt{n}e^{\frac{z^2}{2}}}+\frac{17.9278}{\sqrt{n}e^{\frac{z^2}{16}}}.\end{aligned}$$

From the both cases, for $|z|\le\sqrt{n}$,

$$\Big|P\Big(\frac{W-n/2}{\sqrt{n/4}}\le z\Big)-\Phi(z)\Big|\le\frac{1}{\sqrt{n}e^{\frac{z^2}{2}}}+\frac{17.9278}{\sqrt{n}e^{\frac{z^2}{16}}}.$$

$\square$

CHAPTER V

BERRY-ESSEEN BOUNDS FOR THE LIGHTBULB PROCESS

A medicated adhesive patch which delivers the medication through the skin by placing on the skin is called a dermal patch or skin patch. In pharmaceutical studies, the dermal patch method is a way to measure the amount of pesticide residue directly deposited on the skin in various body area. It is applied to estimate the true exposure of pesticide by Clough et al. [14]. Moreover, this method is also developed in exposure studies using video imaging techniques and in lyphocyte transformation testing for quantifying metal hypersensitivity in Houghton et al. [27] and Garino and Beredjiklian [16], respectively. In addition, a study in the pharmaceutical industry, see Rao, Rao and Zhang [32] and Sharma et al. [38], the potential of using the dermal patch has been recognized.

The problem from a study in the pharmaceutical industry, how to check the patches before going to market, on the effects of dermal patches designed to activate targeted receptors continues to receive attention. This system includes inactive receptors which is active if it receives a dose of medicine released from the dermal patch and inactive if it receives once again. The doses are increased progressively for a limited number of days as follows: for the number of receptors n, on each day $r = 1, \dots, n$ of the study, there exist r randomly selected receptors for which each receives one dose of medicine from the patch. Thus, their status will be changed between the active and inactive states. By this problem, we can simulate it into the lightbulb process given by the following argument.

Let n be any positive integer. We have n switches and n individuals, both serially numbered from 1 to n. Each switch is attached to a lightbulb.
Initially, all the bulbs are off.

Day 1: Individual 1 selects one switch randomly and presses it. The corresponding lightbulb lights up.

Day 2: Individual 2 selects two switches randomly and presses them.

Day 3: Individual 3 selects three switches randomly and presses them.

$$\vdots$$

Day n: Individual n presses all switches.

Therefore, for the number of bulbs n, on each day $r = 1, \ldots, n$ of the study, there exist r randomly selected bulbs which each is pressed the toggle switch connected to a bulb, thus their status will be changed between on and off, we consider the number of bulbs on at the terminal time n. Let

$$\mathbf{X} = \{X_{rk} \mid r = 0, 1, \ldots, n; k = 1, \ldots, n\},$$

a collection of Bernoulli random variables such that for $r \geq 1$,

$$X_{rk} = \begin{cases} 1 & \text{if the status of bulb } k \text{ is changed on Day } r, \\ 0 & \text{otherwise.} \end{cases}$$

In the initial state, all bulbs are off, that is, $X_{0k} = 0$ for all $k = 1, 2, \ldots, n$. At stage r, the bulbs are chosen uniformly to have their status changed and the stages are independent of each other. Let

$$X_k = \Big(\sum_{r=0}^{n} X_{rk}\Big) \bmod 2 \quad \text{and} \quad W = \sum_{k=1}^{n} X_k,$$

the random variable X_k is the indicator that bulb k is on at the terminal time and the random variable W is the number of bulbs on at the terminal time.

In 2007, Rao et al. [32] gave the mean and the variance of W as follows:

$$\mu_n = E(W) = \frac{n}{2}\Big(1 - \prod_{s=1}^{n} \lambda_{n,1,s}\Big) \tag{5.1}$$

and

$$\sigma_n^2 = \mathrm{Var}(W) = \frac{n}{4}\Big(1 - \prod_{s=1}^{n} \lambda_{n,2,s}\Big) + \frac{n^2}{4}\Big(\prod_{s=1}^{n} \lambda_{n,2,s} - \prod_{s=1}^{n} \lambda_{n,1,s}^2\Big), \tag{5.2}$$

where

$$\lambda_{n,1,s} = 1 - \frac{2s}{n} \quad \text{and} \quad \lambda_{n,2,s} = 1 - \frac{4s}{n} + \frac{4s(s-1)}{n(n-1)} \quad \text{for } s = 1, \ldots, n. \tag{5.3}$$

In addition, they proved the distribution of W can be approximated by the standard normal distribution.

In the case that n is an even number, $\lambda_{n,1,\frac{n}{2}} = 0$ which implies

$$\prod_{s=1}^{n} \lambda_{n,1,s} = \prod_{s=1}^{n} \lambda_{n,1,s}^2 = 0,$$

and then, from (5.1) and (5.2),

$$\mu_n = \frac{n}{2} \quad \text{and} \quad \sigma_n^2 = \frac{n}{4}\Big(1 + (n-1)\prod_{s=1}^{n} \lambda_{n,2,s}\Big), \tag{5.4}$$

respectively.

In 2011, Goldstein and Zhang [26] described that W can be coupled to a variable W^s having the W-size biased distribution. For every $i = 1, \ldots, n$, the collection of variables $\mathbf{X}^i$ is constructed from $\mathbf{X}$ as follows. If $X_i = 1$, then $\mathbf{X}^i = \mathbf{X}$. Otherwise, with J^i uniformly chosen from $\{j \mid X_{n/2,j} = 1 - X_{n/2,i}\}$, let $\mathbf{X}^i = \{X_{rk}^i \mid r, k = 1, \ldots, n\}$, where

$$X_{rk}^i = \begin{cases} X_{rk} & \text{if } r \neq n/2, \\ X_{n/2,k} & \text{if } r = n/2, k \notin \{i, J^i\}, \\ X_{n/2,J^i} & \text{if } r = n/2, k = i, \\ X_{n/2,i} & \text{if } r = n/2, j = J^i, \end{cases}$$

and let

$$X_k^i = \Big(\sum_{r=1}^{n} X_{rk}^i\Big) \bmod 2 \quad \text{and} \quad W^i = \sum_{k=1}^{n} X_k^i.$$

By choosing a random index I uniformly from $\{1, \ldots, n\}$ and independent of all other variables, $W^s = W^I$ has the W-size biased distribution. This lead them to obtain that

$$W \leq W^s \leq W + 2.$$

They also provided a Berry-Esseen bound for W by using Theorem 3.1 which is shown in the following theorem.

Theorem 5.1. *([26], p.877) For any even number $n \geq 6$,*

$$\sup_{z \in \mathbb{R}} \Big| P\Big(\frac{W - n/2}{\sigma_n} \leq z\Big) - \Phi(z)\Big| \leq \frac{n}{2\sigma_n^2}\Delta_0 + 1.64\frac{n}{\sigma_n^3} + \frac{2}{\sigma_n}$$

where

$$\Delta_0 = \sqrt{\mathrm{Var}(E(W^s - W|W))}.$$

For the odd case, $n = 2m + 1$, Goldstein and Zhang [26] formed a symmetrical random variable V closely W in the following proceeds. Set $\mathcal{J}_r = \{j \mid X_{rj} = 0\}$ for $r = m, m + 1$, so that $\mathcal{J}_m$ and $\mathcal{J}_{m+1}$ are the bulbs that do not get toggled in stages m and $m + 1$, respectively. Let B_m and B_{m+1} be uniformly chosen from $\mathcal{J}_m$ and $\mathcal{J}_{m+1}^c$, respectively, which are independent of $\mathbf{X}$ and of each other. Let C_m and C_{m+1} be symmetric Bernouli variables, independent of $\mathbf{X}$ and of B_m and B_{m+1}. Now, let the collection of variables

$$\mathbf{V} = \{V_{rj} \mid r, j = 1, \ldots, n\}$$

be given by

$$V_{rj} = \begin{cases} X_{rj} & \text{if } r \notin \{m, m+1\}, \\ X_{mj} & \text{if } r = m, j \neq B_m, \\ C_m & \text{if } r = m, j = B_m, \\ X_{m+1,j} & \text{if } r = m+1, j \neq B_{m+1}, \\ C_{m+1} & \text{if } r = m+1, j = B_{m+1}, \end{cases}$$

and let

$$V_j = \Big(\sum_{r=1}^{n} V_{rj}\Big) \bmod 2 \quad \text{and} \quad V = \sum_{j=1}^{n} V_j.$$

From this construction, they attained

$$|W - V| \leq 2$$

with the mean and the variance of V given, respectively, by

$$\mu_V = \frac{n}{2}\Big(1 - \prod_{r=1}^{n} \bar{\lambda}_{n,1,r}\Big) = \frac{n}{2}$$

and

$$\sigma_V^2 = \frac{n}{4}\Big(1 - (1-n)\prod_{r=1}^{n} \bar{\lambda}_{n,2,r}\Big)$$

where

$$\bar{\lambda}_{n,b,r} = \begin{cases} \frac{1}{2}(\lambda_{n,b,m} + \lambda_{n,b,m+1}) & \text{if } r \in \{m, m+1\}, \\ \lambda_{n,b,r} & \text{otherwise,} \end{cases} \tag{5.5}$$

for $b = 1, 2$.

To obtain the bound for W when n is odd, they constructed a random variable

V^s with the V-size biased distribution satisfying

$$V \leq V^s \leq V + 2.$$

Applying Theorem 3.1, their result is presented as follows.

Theorem 5.2. *([26], p.877) For any odd number $n \geq 7$,*

$$\sup_{z\in\mathbb{R}} \Big| P\Big(\frac{W - n/2}{\sigma_V} \leq z\Big) - \Phi(z) \Big| \leq \frac{n}{2\sigma_V^2}\Delta_1 + 1.64\frac{n}{\sigma_V^3} + \frac{2}{\sigma_V}\Big(1 + \frac{1}{\sqrt{2\pi}}\Big)$$

where

$$\Delta_1 = \sqrt{\mathrm{Var}(E(V^s - V|V))}.$$

Theorem 5.1 and Theorem 5.2 contain the terms Δ_0 and Δ_1 respectively, which is not accurate values. The bounds above can be expressed in the easier forms as the following.

Theorem 5.3. *For any even number $n \geq 6$,*

$$\sup_{z\in\mathbb{R}} \Big| P\Big(\frac{W - n/2}{\sigma_n} \leq z\Big) - \Phi(z) \Big| \leq \frac{18.72}{\sqrt{n}}.$$

Theorem 5.4. *For any odd number $n \geq 7$,*

$$\sup_{z\in\mathbb{R}} \Big| P\Big(\frac{W - n/2}{\sigma_V} \leq z\Big) - \Phi(z) \Big| \leq \frac{21.15}{\sqrt{n}}.$$

In this chapter, we generalize Theorem 5.1 and Theorem 5.2 to the case of a non-uniform bounds by using the non-uniform bound for a binomial random variable and the result of the clubbed binomial approximation for W. These are our results.

Theorem 5.5. *For all even numbers $n \geq 6$ and for any real number z such that $0 < |z| \leq \sqrt{n}$,*

$$\Big| P\Big(\frac{W - n/2}{\sigma_n} \leq z\Big) - \Phi(z) \Big| \leq \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}} e^{\frac{z^2+1}{4}}} + \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{34.3131}{\sqrt{n-1}e^{\frac{z^2}{16}}}.$$

Theorem 5.6. *For all odd numbers $n \geq 7$ and for any real number z such that $0 < |z| \leq \sqrt{n}$,*

$$\Big|P\Big(\frac{W - n/2}{\sigma_V} \leq z\Big) - \Phi(z)\Big| \leq \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}} e^{\frac{z^2+1}{4}}} + \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{34.3131}{\sqrt{n-1}e^{\frac{z^2}{16}}}.$$

Remark 5.7. Note that $0 \leq W \leq n$ and $\sigma_n^2 \approx \frac{n}{4}$ (see (5.7)). These implies that $P\big(\frac{W-n/2}{\sigma_n} \leq z\big) = 1$ for all $z > \sqrt{n}$ and $P\big(\frac{W-n/2}{\sigma_n} \leq z\big) = 0$ for all $z < -\sqrt{n}$. Therefore, it suffices to consider the bounds when $0 < |z| \leq \sqrt{n}$.

Table 5.1 compares the constants of the uniform bounds (Theorem 5.1 and Theorem 5.2) and the non-uniform bounds (Theorem 5.5 and Theorem 5.6).

Statistics		$\frac{W_n-n/2}{\sigma_n}$	$\frac{W_n-n/2}{\sigma_V}$
Uniform bounds		$\frac{18.72}{\sqrt{n}}$	$\frac{21.15}{\sqrt{n}}$
Non-uniform bounds	$z = 2$	$\frac{0.79\sqrt{n}}{e^{\frac{n+1}{12}}} + \frac{26.86}{\sqrt{n-1}}$	
	$z = 6$	$\frac{2.63\times10^{-4}\sqrt{n}}{e^{\frac{n+1}{12}}} + \frac{3.62}{\sqrt{n-1}}$	
	$z = 8$	$\frac{2.4\times10^{-7}\sqrt{n}}{e^{\frac{n+1}{12}}} + \frac{0.63}{\sqrt{n-1}}$	
	$z = 10$	$\frac{2.96\times10^{-11}\sqrt{n}}{e^{\frac{n+1}{12}}} + \frac{0.07}{\sqrt{n-1}}$	
	$z = 100$	$\frac{3.91\times10^{-1086}\sqrt{n}}{e^{\frac{n+1}{12}}} + \frac{1.27\times10^{-270}}{\sqrt{n-1}}$	

Table 5.1: Constants of uniform and non-uniform exponential bounds.

This chapter is organized into 2 sections as follows. Section 5.1 includes the proofs of Theorem 5.3 and Theorem 5.4. The proof of Theorem 5.5 and Theorem 5.6 are given in Section 5.2.

5.1 Uniform Bounds

In this section, we wil prove Theorem 5.3 and Theorem 5.4 by estimating σ_n^2 and σ_V^2, and using the bounds of Δ_0 and Δ_1 which are given by Goldstein and

Zhang in [26].

5.1.1 Proof of Theorem 5.3

In the case that n is an even number, by (5.3), we see that

$$\lambda_{n,2,n/2} = 1 - \frac{4(n/2)}{n} + \frac{4(n/2)(n/2-1)}{n(n-1)} = -\frac{1}{n-1}. \tag{5.6}$$

Goldstein and Zhang ([26], pp.896) showed that for $n \geq 6$,

$$-e^{-n} \leq \prod_{\substack{s=1 \\ s \neq n/2}}^{n} \lambda_{n,2,s} \leq e^{-n},$$

and then, by (5.6),

$$-\frac{e^{-n}}{n-1} \leq \prod_{s=1}^{n} \lambda_{n,2,s} \leq \frac{e^{-n}}{n-1}.$$

From this fact and (5.4), we have

$$\frac{n}{4}(1-e^{-n}) \leq \sigma_n^2 \leq \frac{n}{4}(1+e^{-n}). \tag{5.7}$$

Therefore,

$$0.249n \leq \frac{n}{4}(1-e^{-6}) \leq \sigma_n^2 \leq \frac{n}{4}(1+e^{-6}) \leq 0.251n \quad \text{for } n \geq 6. \tag{5.8}$$

By Theorem 5.1, (5.8) and the fact that

$$\Delta_0 \leq \frac{1}{2\sqrt{n}} + \frac{1}{2n} + \frac{1}{3}e^{-n/2}$$

([26], p.888), we have

$$\sup_{z\in\mathbb{R}}\Big|P\Big(\frac{W-n/2}{\sigma_n}\le z\Big)-\Phi(z)\Big|$$
$$\le \frac{n}{2\sigma_n^2}\Big(\frac{1}{2\sqrt{n}}+\frac{1}{2n}+\frac{1}{3}e^{-n/2}\Big)+1.64\frac{n}{\sigma_n^3}+\frac{2}{\sigma_n}$$
$$\le \frac{1}{2(0.249)}\Big(\frac{1}{2\sqrt{n}}+\frac{1}{2n}+\frac{1}{3}e^{-n/2}\Big)+\frac{1.64}{0.249\sqrt{0.249n}}+\frac{2}{\sqrt{0.249n}}$$
$$\le \frac{18.72}{\sqrt{n}}$$

where we use the fact that $e^{\frac{n}{2}} \ge 8.19\sqrt{n}$ and $n \ge 2.4\sqrt{n}$ for all $n \ge 6$ in the last inequality. □

5.1.2 Proof of Theorem 5.4

Let $n = 2m + 1$ be an odd number. By (5.3) and (5.5), we have

$$\bar\lambda_{n,2,m} = \frac{1}{2}\Big(1-\frac{4m}{n}+\frac{4m(m-1)}{n(n-1)}+1-\frac{4(m+1)}{n}+\frac{4m(m+1)}{n(n-1)}\Big) = -\frac{1}{n}$$

which implies

$$\sigma_V^2 = \frac{n}{4}\Big(1+\Big(\frac{1-n}{n}\Big)\prod_{\substack{r=1\\ r\ne m}}^{n}\bar\lambda_{n,2,r}\Big).$$

Goldstein and Zhang ([26], p.888) showed that for $n \ge 7$,

$$-e^{-n} \le \prod_{\substack{r=1\\ r\ne m}}^{n}\bar\lambda_{n,2,r} \le e^{-n}.$$

These lead us to see that

$$\frac{n}{4}-\frac{(n-1)}{4}e^{-n} \le \sigma_V^2 \le \frac{n}{4}+\frac{(n-1)}{4}e^{-n}. \tag{5.9}$$

Then

$$0.249n \leq \frac{n}{4} - \frac{(n-1)}{4}e^{-7} \leq \sigma_V^2 \leq \frac{n}{4} + \frac{(n-1)}{4}e^{-7} \leq 0.251n \quad \text{for } n \geq 7. \quad (5.10)$$

By (5.10) and the fact that

$$\Delta_1 \leq \frac{1}{\sqrt{n}} + \frac{1}{2\sqrt{2}}e^{-n/4}$$

for $n \geq 7$ ([26], p.890), the bound in Theorem 5.2 is expressed as follows.

$$\begin{aligned}
&\sup_{z\in\mathbb{R}} \left| P\Big(\frac{W - n/2}{\sigma_V} \leq z\Big) - \Phi(z)\right| \\
&\leq \frac{n}{2\sigma_V^2}\Big(\frac{1}{\sqrt{n}} + \frac{1}{2\sqrt{2}}e^{-n/4}\Big) + 1.64\frac{n}{\sigma_V^3} + \frac{2}{\sigma_V}\Big(1 + \frac{1}{\sqrt{2\pi}}\Big) \\
&\leq \frac{1}{2(0.249)}\Big(\frac{1}{\sqrt{n}} + \frac{1}{2\sqrt{2}}e^{-n/4}\Big) + \frac{1.64}{0.249\sqrt{0.249n}} + \frac{2}{\sqrt{0.249n}}\Big(1 + \frac{1}{\sqrt{2\pi}}\Big) \\
&\leq \frac{21.15}{\sqrt{n}}
\end{aligned}$$

where the last inequality holds by using the fact that $e^{\frac{n}{4}} \geq 2.17\sqrt{n}$ for all $n \geq 7$.□

5.2 Non-uniform Bounds

It is known that a uniform Berry-Esseen bound for W is discussed by Goldstein and Zhang [26] in 2011. Next, in 2012, Goldstein and Xia [25] gave a uniform bound for W in the clubbed binomial approximation. In this work, we use the result of the clubbed binomial approximation and Theorem 4.3 to get a non-uniform exponential bound for W.

We say that the random variable C_n has the clubbed binomial distribution if C_n has the following distribution. If n mod $4 \in \{0, 3\}$,

$$P(C_n = i) = \begin{cases} \binom{n}{i}\left(\frac{1}{2}\right)^{n-1} & \text{if } i \text{ is an even number in } \{0, 1, \ldots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and if $n \bmod 4 \in \{1,2\}$,

$$P(C_n = i) = \begin{cases} \binom{n}{i}\left(\frac{1}{2}\right)^{n-1} & \text{if } i \text{ is an odd number in } \{0,1,\dots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Goldstein and Xia ([25], p.32) showed that

$$\sup_{A\subseteq\mathbb{Z}} \Big| P(W\in A) - P(C_n \in A) \Big| \le \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{3}}}. \tag{5.11}$$

5.2.1 Proof of Theorem 5.5

Starting with a non-uniform concentration inequality for binomial random variable, it is a tool to prove Theorem 5.5 and Theorem 5.6. We use Lemma 3.3 to obtain the inequality as mentioned above. The concentration inequality is presented as follows.

Lemma 5.8. *Let $B_n \sim$ Binomial$(n, \frac{1}{2})$. Then for all $z \in \mathbb{R}$ and $\lambda, s > 0$ such that $2s \le \sqrt{\frac{n}{4}}$,*

$$P\Big(z \le \frac{B_n - n/2}{\sqrt{n/4}} \le z+\lambda\Big) \le 2e^{-sz} e^{2s^2 + \frac{32s^3}{3\sqrt{n}}} \Big(e^{\frac{s}{2\sqrt{n}}} \Big(\lambda + \frac{1}{\sqrt{n}}\Big) + \frac{1}{2\sqrt{n}}\Big).$$

Proof. Let $z \in \mathbb{R}$ and $\lambda, s > 0$ be such that $2s \le \sqrt{\frac{n}{4}}$. Since $B_n = \sum_{i=1}^{n} Y_i$ where Y_i's are independent Bernoulli random variables with $E(Y_i) = \frac{1}{2}$ and $\mathrm{Var}(Y_i) = \frac{1}{4}$ for $i = 1, \dots, n$. Thus,

$$\gamma := \sum_{i=1}^{n} E\Big|\frac{Y_i - E(Y_i)}{\sqrt{\mathrm{Var}(B_n)}}\Big|^3 = nE\Big|\frac{Y_1 - 1/2}{\sqrt{n/4}}\Big|^3 = \frac{1}{\sqrt{n}}.$$

We follow the proof of a non-uniform concentration inequality in [11] by choosing a function f defined by

$$f(w) = \begin{cases} 0 & \text{if } w < z - \frac{\gamma}{2}, \\ e^{sw}(w - z + \frac{\gamma}{2}) & \text{if } z - \frac{\gamma}{2} \leq w \leq z + \lambda + \frac{\gamma}{2}, \\ e^{sw}(\lambda + \gamma) & \text{if } w > z + \lambda + \frac{\gamma}{2}, \end{cases}$$

to obtain that

$$P\Big(z \leq \frac{B_n - n/2}{\sqrt{n/4}} \leq z + \lambda\Big) \leq 2e^{-sz}\Big(Ee^{2s\left(\frac{B_n - n/2}{\sqrt{n/4}}\right)}\Big)^{\frac{1}{2}}\Big(e^{\frac{s\gamma}{2}}(\lambda + \gamma) + \frac{\gamma}{2}\Big). \quad (5.12)$$

In Chapter IV, we see that $Y_i^s = 1$ for all $i = 1, \ldots, n$ and the size bias coupling for the binomial random variable B_n, can be constructed. Following the size bias construction as in Example 2.3, for a random index I such that

$$P(I = i) = \frac{1}{n}$$

for $i = 1, \ldots, n$ and $Y_1, \ldots, Y_n, I$ are independent,

$$B_n^s = B_n - Y_I + Y_I^s$$

has the B_n-size biased distribution satisfying

$$B_n \leq B_n^s \leq B_n + 1.$$

Under the assumption that $2s \leq \sqrt{\frac{n}{4}}$, we use (5.12) and Lemma 3.3 with $k = 2s$, $B = 1$, $E(B_n) = \frac{n}{2}$ and $\mathrm{Var}(B_n) = \frac{n}{4}$ to get

$$\begin{aligned} P\Big(z \leq \frac{B_n - n/2}{\sqrt{n/4}} \leq z + \lambda\Big) &\leq 2e^{-sz}\Big(e^{4s^2 + \frac{64s^3}{3\sqrt{n}}}\Big)^{\frac{1}{2}}\Big(e^{\frac{s\gamma}{2}}(\lambda + \gamma) + \frac{\gamma}{2}\Big) \\ &= 2e^{-sz}e^{2s^2 + \frac{32s^3}{3\sqrt{n}}}\Big(e^{\frac{s}{2\sqrt{n}}}\Big(\lambda + \frac{1}{\sqrt{n}}\Big) + \frac{1}{2\sqrt{n}}\Big). \end{aligned}$$

□

From the result of Rao et al. [32], we know that the difference between the cumulative distribution of C_n and B_{n-1} converges to zero pointwise. This leads us to find a bound for approximating the distribution of binomial by clubbed binomial distribution seen in the following lemma.

Lemma 5.9. *Let* $B_n \sim Binomial(n, \frac{1}{2})$ *and* $t = 1, 2, \ldots, n$. *Then*

$$\Big|P(C_n \leq t) - P(B_{n-1} \leq t)\Big| \leq P(B_{n-1} = t).$$

Proof. It suffices to assume that n is even. In the case that n is odd, we can follow the proof in the even case to obtain the same result. Note that the inequality holds when $t = n$. Therefore, it remains to prove the lemma for $t = 1, 2, \ldots, n-1$. Since n is even, $n \bmod 4 \in \{0, 2\}$. Suppose that $n \bmod 4 = 0$. If t is even, then

$$\begin{aligned}
&\Big|P(C_n \leq t) - P(B_{n-1} \leq t)\Big| \\
&= \Big|\sum_{i=0}^{t} P(C_n = i) - \sum_{i=0}^{t} P(B_{n-1} = i)\Big| \\
&= \left(\frac{1}{2}\right)^{n-1} \Big|\sum_{i=0}^{\frac{t}{2}} \binom{n}{2i} - \sum_{i=0}^{t} \binom{n-1}{i}\Big| \\
&= \left(\frac{1}{2}\right)^{n-1} \Big|\sum_{i=1}^{\frac{t}{2}} \left[\binom{n}{2i} - \binom{n-1}{2i}\right] - \sum_{i=1}^{\frac{t}{2}} \binom{n-1}{2i-1}\Big| \\
&= 0
\end{aligned}$$

where the last equality is true by using the property that $\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$ for all integers n, k such that $1 \leq k \leq n-1$.

If t is odd, then for $t = 1$,

$$\Big|P(C_n \leq 1) - P(B_{n-1} \leq 1)\Big| = \Big|P(C_n = 0) - \sum_{i=0}^{1} P(B_{n-1} = i)\Big| = \left(\frac{1}{2}\right)^{n-1} \binom{n-1}{1}$$

and for $t \geq 3$, we have

$$\begin{aligned}
&\Big|P(C_n \leq t) - P(B_{n-1} \leq t)\Big| \\
&= \Big|\sum_{i=0}^{t-1} P(C_n = i) - \sum_{i=0}^{t} P(B_{n-1} = i)\Big| \\
&= \Big(\frac{1}{2}\Big)^{n-1} \Big|\sum_{i=0}^{\frac{t-1}{2}} \binom{n}{2i} - \sum_{i=0}^{t} \binom{n-1}{i}\Big| \\
&= \Big(\frac{1}{2}\Big)^{n-1} \Big|\sum_{i=1}^{\frac{t-1}{2}} \Big[\binom{n}{2i} - \binom{n-1}{2i}\Big] - \sum_{i=1}^{\frac{t-1}{2}} \binom{n-1}{2i-1} - \binom{n-1}{t}\Big| \\
&= \Big(\frac{1}{2}\Big)^{n-1} \binom{n-1}{t}.
\end{aligned}$$

Therefore, from both cases, we get

$$\Big|P(C_n \leq t) - P(B_{n-1} \leq t)\Big| = \begin{cases} \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{t} & \text{if } t \text{ is odd,} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

In the case that $n \bmod 4 = 2$, we use the same argument to obtain

$$\Big|P(C_n \leq t) - P(B_{n-1} \leq t)\Big| = \begin{cases} 0 & \text{if } t \text{ is odd,} \\ \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{t} & \text{if } t \text{ is even.} \end{cases}$$

Hence,

$$\Big|P(C_n \leq t) - P(B_{n-1} \leq t)\Big| \leq P(B_{n-1} = t).$$

$\square$

Proof of Theorem 5.5. Assume that $0 < z \leq \sqrt{n}$. Let $B_n \sim \text{Binomial}(n, \frac{1}{2})$. We note that

$$
\begin{aligned}
&\Big|P\Big(\frac{W-n/2}{\sigma_n}\le z\Big)-\Phi(z)\Big| \\
&\le \Big|P\Big(\frac{W-n/2}{\sigma_n}\le z\Big)-P\Big(\frac{C_n-n/2}{\sigma_n}\le z\Big)\Big| \\
&\quad+\Big|P\Big(\frac{C_n-n/2}{\sigma_n}\le z\Big)-P\Big(\frac{B_{n-1}-n/2}{\sigma_n}\le z\Big)\Big| \\
&\quad+\Big|P\Big(\frac{B_{n-1}-n/2}{\sigma_n}\le z\Big)-P\Big(\frac{B_{n-1}-(n-1)/2}{\sqrt{(n-1)/4}}\le z\Big)\Big| \\
&\quad+\Big|P\Big(\frac{B_{n-1}-(n-1)/2}{\sqrt{(n-1)/4}}\le z\Big)-\Phi(z)\Big| \\
&=: A_1+A_2+A_3+A_4. \qquad (5.13)
\end{aligned}
$$

By the fact that $\frac{n+1}{3}=\frac{n+1}{12}+\frac{n+1}{4}\ge\frac{n+1}{12}+\frac{z^2+1}{4}$ for $0<z\le\sqrt{n}$ and (5.11), we have

$$A_1\le\frac{2.7314\sqrt{n}}{e^{\frac{n+1}{3}}}\le\frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}}e^{\frac{z^2+1}{4}}}. \qquad (5.14)$$

From (5.7) and $0<z\le\sqrt{n}$, we have

$$\frac{n}{2}<\sigma_n z+\frac{n}{2}\le\big(\sqrt{1+e^{-n}}\big)\frac{n}{2}+\frac{n}{2}\le\big(1+e^{-\frac{n}{2}}\big)\frac{n}{2}+\frac{n}{2}=n+\frac{n}{2}e^{-\frac{n}{2}}<n+1.$$

Hence,

$$3\le\Big\lfloor\sigma_n z+\frac{n}{2}\Big\rfloor\le n \quad \text{for } n\ge 6.$$

Applying Lemma 5.9,

$$
\begin{aligned}
A_2&=\Big|P\Big(C_n\le\Big\lfloor\sigma_n z+\frac{n}{2}\Big\rfloor\Big)-P\Big(B_{n-1}\le\Big\lfloor\sigma_n z+\frac{n}{2}\Big\rfloor\Big)\Big| \\
&\le P\Big(B_{n-1}=\Big\lfloor\sigma_n z+\frac{n}{2}\Big\rfloor\Big) \\
&\le P\Big(\sigma_n z+\frac{n}{2}-1\le B_{n-1}\le\sigma_n z+\frac{n}{2}\Big) \qquad (5.15)
\end{aligned}
$$

where we use the fact that $x-1\le\lfloor x\rfloor\le x$ for any $x\in\mathbb{R}$ in the last inequality.

Next, consider A_3. By (5.7), and the fact that $e^n \geq n$ for any $n \in \mathbb{N}$, we have

$$\sigma_n \geq \sqrt{\frac{n}{4}(1-e^{-n})} \geq \sqrt{\frac{n-1}{4}}$$

and for $z \leq \sqrt{n}$,

$$\begin{aligned}
\left(\sigma_n - \sqrt{\frac{n-1}{4}}\right)z &\leq \frac{1}{2}\Big(\sqrt{n(1+e^{-n})} - \sqrt{n-1}\Big)z \\
&= \frac{1}{2}\left(\frac{n(1+e^{-n})-(n-1)}{\sqrt{n(1+e^{-n})}+\sqrt{n-1}}\right)z \\
&\leq \frac{1}{2}\left(\frac{ne^{-n}+1}{\sqrt{n(1+e^{-n})}+\sqrt{n-1}}\right)\sqrt{n} \\
&= \frac{1}{2}\left(\frac{ne^{-n}+1}{\sqrt{1+e^{-n}}+\sqrt{1-\frac{1}{n}}}\right) \\
&< \frac{1}{2}
\end{aligned}$$

where we use the fact that $ne^{-n}+1 \leq 1.37$ and $\sqrt{1+e^{-n}} + \sqrt{1-\frac{1}{n}} \geq 1.91$ for $n \geq 6$ in the last inequality.

Hence,

$$\frac{1}{2} \leq \left(\sigma_n - \sqrt{\frac{n-1}{4}}\right)z + \frac{1}{2} < 1.$$

This leads us to

$$\sigma_n z + \frac{n}{2} - 1 < \sqrt{\frac{n-1}{4}}z + \frac{n-1}{2}$$

and

$$\begin{aligned}
A_3 &= P\left(\sqrt{\frac{n-1}{4}}z + \frac{n-1}{2} < B_{n-1} \leq \sigma_n z + \frac{n}{2}\right) \\
&\leq P\Big(\sigma_n z + \frac{n}{2} - 1 \leq B_{n-1} \leq \sigma_n z + \frac{n}{2}\Big). \qquad (5.16)
\end{aligned}$$

From (5.15) and (5.16), we obtain

$$\begin{aligned}A_2 + A_3 &\le 2P\Big(\sigma_n z + \frac{n}{2} - 1 \le B_{n-1} \le \sigma_n z + \frac{n}{2}\Big)\\ &= 2P\Big(\frac{2\sigma_n}{\sqrt{n-1}}z - \frac{1}{\sqrt{n-1}} \le \frac{B_{n-1} - (n-1)/2}{\sqrt{(n-1)/4}} \le \frac{2\sigma_n}{\sqrt{n-1}}z + \frac{1}{\sqrt{n-1}}\Big).\end{aligned}$$

Since $\frac{z}{5} \le \frac{\sqrt{n}}{5} \le \sqrt{\frac{n-1}{4}}$, we can apply Lemma 5.8 to B_{n-1} with $s = \frac{z}{10}$, $\lambda = \frac{2}{\sqrt{n-1}}$ and z replaced by $\frac{2\sigma_n}{\sqrt{n-1}}z - \frac{1}{\sqrt{n-1}}$,

$$\begin{aligned}A_2 + A_3 &\le 4e^{-\frac{z}{10}\left(\frac{2\sigma_n}{\sqrt{n-1}}z - \frac{1}{\sqrt{n-1}}\right)} e^{\frac{z^2}{50} + \frac{4z^3}{375\sqrt{n-1}}}\Big(e^{\frac{z}{20\sqrt{n-1}}}\Big(\frac{3}{\sqrt{n-1}}\Big) + \frac{1}{2\sqrt{n-1}}\Big)\\ &\le 4e^{-\frac{\sigma_n}{5\sqrt{n-1}}z^2} e^{\frac{1.1}{10}} e^{\frac{z^2}{50} + \frac{4(1.1)z^2}{375}}\Big(e^{\frac{1.1}{20}}\Big(\frac{3}{\sqrt{n-1}}\Big) + \frac{1}{2\sqrt{n-1}}\Big)\\ &\le 4e^{-\frac{997}{10^4}z^2} e^{\frac{1.1}{10}} e^{\frac{3174}{10^5}z^2}\Big(e^{\frac{1.1}{20}}\Big(\frac{3}{\sqrt{n-1}}\Big) + \frac{1}{2\sqrt{n-1}}\Big)\\ &\le \frac{16.3853}{\sqrt{n-1}e^{\frac{z^2}{15}}} \qquad (5.17)\end{aligned}$$

where we use the fact that $\frac{n}{n-1} = 1 + \frac{1}{n-1} \le 1.2$ which implies $z \le \sqrt{n} \le \sqrt{1.2}\sqrt{n-1} \le 1.1\sqrt{n-1}$ for $n \ge 6$ in the second inequality and (5.8) in the third inequality.

It remains to bound the error term A_4. Applying Theorem 4.3 to B_{n-1},

$$A_4 \le \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{17.9278}{\sqrt{n-1}e^{\frac{z^2}{16}}}. \qquad (5.18)$$

Therefore, by (5.13), (5.14), (5.17) and (5.18),

$$\begin{aligned}\Big|P\Big(\frac{W - n/2}{\sigma_n} \le z\Big) - \Phi(z)\Big| &\le \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}} e^{\frac{z^2+1}{4}}} + \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{16.3853}{\sqrt{n-1}e^{\frac{z^2}{15}}} + \frac{17.9278}{\sqrt{n-1}e^{\frac{z^2}{16}}}\\ &\le \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}} e^{\frac{z^2+1}{4}}} + \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{34.3131}{\sqrt{n-1}e^{\frac{z^2}{16}}},\end{aligned}$$

completing the proof. □

5.2.2 Proof of Theorem 5.6

Assume that $0 < z \le \sqrt{n}$. To prove the theorem, we use the same technique in the proof of Theorem 5.5 by considering

$$\begin{aligned}
&\Big|P\Big(\frac{W-n/2}{\sigma_V}\le z\Big)-\Phi(z)\Big| \\
&\le \Big|P\Big(\frac{W-n/2}{\sigma_V}\le z\Big)-P\Big(\frac{C_n-n/2}{\sigma_V}\le z\Big)\Big| \\
&\quad+\Big|P\Big(\frac{C_n-n/2}{\sigma_V}\le z\Big)-P\Big(\frac{B_{n-1}-n/2}{\sigma_V}\le z\Big)\Big| \\
&\quad+\Big|P\Big(\frac{B_{n-1}-n/2}{\sigma_V}\le z\Big)-P\Big(\frac{B_{n-1}-(n-1)/2}{\sqrt{(n-1)/4}}\le z\Big)\Big| \\
&\quad+\Big|P\Big(\frac{B_{n-1}-(n-1)/2}{\sqrt{(n-1)/4}}\le z\Big)-\Phi(z)\Big| \\
&=: R_1+R_2+R_3+R_4. \qquad (5.19)
\end{aligned}$$

Applying (5.11), since $z \le \sqrt{n}$,

$$R_1 \le \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{3}}} \le \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}}e^{\frac{z^2+1}{4}}}. \qquad (5.20)$$

By using (5.9) and $0 < z \le \sqrt{n}$, we have

$$\frac{n}{2} < \sigma_V z + \frac{n}{2} \le \Big(\sqrt{\frac{n}{4}+\Big(\frac{n-1}{4}\Big)e^{-n}}\Big)\sqrt{n}+\frac{n}{2} \le \frac{n}{2}+\sqrt{\frac{n(n-1)}{4}}e^{-\frac{n}{2}}+\frac{n}{2} < n+1.$$

Therefore,

$$3 \le \Big\lfloor \sigma_V z + \frac{n}{2}\Big\rfloor \le n \quad \text{for } n \ge 7.$$

From Lemma 5.9, we obtain

$$\begin{aligned}
R_2 &\le P\Big(B_{n-1} = \Big\lfloor \sigma_V z + \frac{n}{2}\Big\rfloor\Big) \\
&\le P\Big(\sigma_V z + \frac{n}{2} - 1 \le B_{n-1} \le \sigma_V z + \frac{n}{2}\Big). \qquad (5.21)
\end{aligned}$$

Next, we will estimate the term R_3. Using (5.9) and the fact that $e^n \geq n-1$ for $n \in \mathbb{N}$,

$$\sigma_V \geq \sqrt{\frac{n}{4} - \Big(\frac{n-1}{4}\Big)e^{-n}} \geq \sqrt{\frac{n-1}{4}}$$

and for $z \leq \sqrt{n}$,

$$\begin{aligned}\left(\sigma_V - \sqrt{\frac{n-1}{4}}\right)z &\leq \frac{1}{2}\Big(\sqrt{n+(n-1)e^{-n}} - \sqrt{n-1}\Big)z \\ &= \frac{1}{2}\left(\frac{n+(n-1)e^{-n}-(n-1)}{\sqrt{n+(n-1)e^{-n}}+\sqrt{n-1}}\right)z \\ &\leq \frac{1}{2}\left(\frac{(n-1)e^{-n}+1}{\sqrt{n+(n-1)e^{-n}}+\sqrt{n-1}}\right)\sqrt{n} \\ &= \frac{1}{2}\left(\frac{(n-1)e^{-n}+1}{\sqrt{1+(1-\frac{1}{n})e^{-n}}+\sqrt{1-\frac{1}{n}}}\right) \\ &< \frac{1}{2}\end{aligned}$$

where the last inequality holds by applying the fact that $(n-1)e^{-n}+1 \leq 1.01$ and $\sqrt{1+(1-\frac{1}{n})e^{-n}} + \sqrt{1-\frac{1}{n}} \geq 1.92$ for $n \geq 7$.
Thus,

$$\frac{1}{2} \leq \left(\sigma_V - \sqrt{\frac{n-1}{4}}\right)z + \frac{1}{2} < 1.$$

This implies that

$$\sigma_V z + \frac{n}{2} - 1 < \sqrt{\frac{n-1}{4}}z + \frac{n-1}{2}$$

and

$$\begin{aligned}R_3 &= P\Big(\sqrt{\frac{n-1}{4}}z + \frac{n-1}{2} < B_{n-1} \leq \sigma_V z + \frac{n}{2}\Big) \\ &\leq P\Big(\sigma_V z + \frac{n}{2} - 1 \leq B_{n-1} \leq \sigma_V z + \frac{n}{2}\Big). \qquad (5.22)\end{aligned}$$

By (5.21) and (5.22), we see that

$$R_2 + R_3 \leq 2P\Big(\sigma_V z + \frac{n}{2} - 1 \leq B_{n-1} \leq \sigma_V z + \frac{n}{2}\Big).$$

Next, we use Lemma 5.8 which is the same argument as estimating $A_2 + A_3$ with (5.10) and z replaced by $\frac{2\sigma_V}{\sqrt{n-1}}z - \frac{1}{\sqrt{n-1}}$, to bound the term $R_2 + R_3$ as follows.

$$R_2 + R_3 \leq \frac{16.3853}{\sqrt{n-1}e^{\frac{z^2}{15}}}. \tag{5.23}$$

Applying Theorem 4.3 with B_{n-1},

$$R_4 \leq \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{17.9278}{\sqrt{n-1}e^{\frac{z^2}{16}}} \tag{5.24}$$

Combining (5.19), (5.20), (5.23) and (5.24),

$$\begin{aligned}\Big|P\Big(\frac{W - n/2}{\sigma_V} \leq z\Big) - \Phi(z)\Big| &\leq \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}}e^{\frac{z^2+1}{4}}} + \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{16.3853}{\sqrt{n-1}e^{\frac{z^2}{15}}} + \frac{17.9278}{\sqrt{n-1}e^{\frac{z^2}{16}}} \\ &\leq \frac{2.7314\sqrt{n}}{e^{\frac{n+1}{12}}e^{\frac{z^2+1}{4}}} + \frac{1}{\sqrt{n-1}e^{\frac{z^2}{2}}} + \frac{34.3131}{\sqrt{n-1}e^{\frac{z^2}{16}}}.\end{aligned}$$

□

CHAPTER VI

NON-UNIFORM BOUNDS FOR m RUNS

The m runs is an important model for applications such as sensors or stock market measurements, where items arrive one at a time, and only most recent m items remain active for some fixed parameter m. It can be represent as the following mathematical model.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent and identically distributed Bernoulli random variables with parameter $p \in (0, 1)$. Let W be the number of m runs of the sequence $\xi_1, \xi_2, \ldots, \xi_n$ given by

$$W = \sum_{i=1}^{n} X_i \quad \text{where } X_i = \xi_i \xi_{i+1} \cdots \xi_{i+m-1}$$

with the periodic convention $\xi_{n+k} = \xi_k$.

Ghosh and Goldstein [17] gave the mean μ_n and the variance σ_n^2 of W for $n \geq 2m$ in the following form

$$\mu_n = np^m \quad \text{and} \quad \sigma_n^2 = np^m\Big(1 + \frac{2(p - p^m)}{1 - p} - (2m - 1)p^m\Big).$$

They also constructed the size bias W^s of W as

$$W^s = \sum_{i=1}^{n} \xi'_i \xi'_{i+1} \cdots \xi'_{i+m-1}$$

where

$$\xi'_j = \begin{cases} \xi_j & \text{if } j \notin \{i, \ldots, i + m - 1\}, \\ 1 & \text{if } j \in \{i, \ldots, i + m - 1\}, \end{cases}$$

which satisfies the following property

$$0 \leq W^s - W \leq 2m - 1$$

([17], p.76).

Moreover, Goldstein [21] bounded the term $\Delta = \sqrt{\mathrm{Var}(E(W^s - W|W))}$ in the form of

$$\Delta \leq n^{-1/2}(2m-1)(6m-5)^{1/2}$$

to give a Berry-Esseen theorem for W as follows.

Theorem 6.1. *([21], p.664) If* $2m - 1 \leq \frac{\sigma_n^{\frac{3}{2}}}{\sqrt{6np^m}}$*, then for* $z \in \mathbb{R}$,

$$\Big|P\Big(\frac{W - np^m}{\sigma_n} \leq z\Big) - \Phi(z)\Big| \leq 0.4A + \frac{np^m}{\sigma_n}(2m-1)(64A^2 + 4A^3) + \frac{23np^m}{\sigma_n^2}\Delta$$

where $A = \frac{2m-1}{\sigma_n}$.

Note that a non-uniform exponential bound for the number of m runs of the sequence $\xi_1, \xi_2, \ldots, \xi_n$ can be obtained by using Theorem 3.2. In particular, we consider the bound in the case that $m = 2$ to attain a concrete example.

Now, we use Theorem 3.2 equipped with

$$B = 3, \quad \mu_n = np^2, \quad \sigma_n^2 = np^2(1 + 2p - 3p^2) \quad \text{and} \quad \Delta \leq \frac{3\sqrt{7}}{\sqrt{n}}$$

to get a bound for the number of 2 runs presented in Theorem 6.2.

Theorem 6.2. *For large* n *such that* $0 < |z| \leq 4p\sqrt{nr}$ *and* $c > 0$,

$$\Big|P\Big(\frac{W - np^2}{\sqrt{np^2r}} \leq z\Big) - \Phi(z)\Big| \leq C_1(z)\frac{3\sqrt{7}}{r\sqrt{n}} + C_2(z)\frac{9}{pr\sqrt{nr}} + C_3(z)\frac{3}{p\sqrt{nr}}$$

where

$$C_1(z) = \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{|z|^3}{576pr\sqrt{nr}}}e^{-\frac{z^2}{24}\left(\frac{1}{1+c}-\frac{1}{8r}\right)},$$

$$C_2(z) = \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{|z|^3}{288pr\sqrt{nr}}+\frac{|z|}{4p\sqrt{nr}}}e^{-z^2\left(\frac{3}{40}-\frac{1}{96r}\right)}\Big),$$

$$C_3(z) = 2\Big(e^{\frac{|z|^3}{576pr\sqrt{nr}}+\frac{|z|}{8p\sqrt{nr}}}\Big)e^{-\frac{z^2}{2}\left(\frac{1}{12}-\frac{1}{96r}\right)}$$

and $$r = 1 + 2p - 3p^2.$$

For a symmetric case, $p = \frac{1}{2}$, we give the exponential bound with a known constant in the easier form as follows.

Theorem 6.3. *If $p = \frac{1}{2}$, then for all real numbers z such that $|z| \leq \frac{3\sqrt{5n}}{5}$,*

$$\Big|P\Big(\frac{W-n/4}{\sqrt{5n}/4} \leq z\Big) - \Phi(z)\Big| \leq \frac{3.7545}{\sqrt{n}e^{\frac{z^2}{2}}} + \frac{67.5622}{\sqrt{n}e^{\frac{z^2}{31}}}.$$

Remark 6.4. Since $0 \leq W \leq n$, $P\big(\frac{W-n/4}{\sqrt{5n}/4} \leq z\big) = 1$ for all $z > \frac{3\sqrt{5n}}{5}$ and $P\big(\frac{W-n/4}{\sqrt{5n}/4} \leq z\big) = 0$ for all $z < -\frac{3\sqrt{5n}}{5}$. Therefore, we regard the bound with $|z| \leq \frac{3\sqrt{5n}}{5}$.

Proof of Theorem 6.3

Let $p = \frac{1}{2}$ and $z \in \mathbb{R}$ be such that $|z| \leq \frac{3\sqrt{5n}}{5}$.

Case $|z| \leq 4.3$. Note that $e^{\frac{z^2}{31}} \leq 1.8157$. Applying Theorem 3.1 with

$$B = 3, \quad \mu_n = \frac{n}{4}, \quad \sigma_n = \frac{5n}{16} \quad \text{and} \quad \Delta = \frac{3\sqrt{7}}{\sqrt{n}},$$

we see that

$$\Big|P\Big(\frac{W-n/4}{\sqrt{5n}/4} \leq z\Big) - \Phi(z)\Big| \leq \frac{1.8157}{e^{\frac{z^2}{31}}}\Big(\frac{12\sqrt{7}}{5\sqrt{n}} + \frac{118.08\sqrt{5}}{25\sqrt{n}} + \frac{12\sqrt{5}}{5\sqrt{n}}\Big) \leq \frac{40.4498}{\sqrt{n}e^{\frac{z^2}{31}}}.$$

Case $|z| > 4.3$. Since $p = \frac{1}{2}$, $r = \frac{5}{4}$. By Theorem 6.2, we have

$$\left|P\Big(\frac{W - n/4}{\sqrt{5n/4}} \leq z\Big) - \Phi(z)\right| \leq C_1(z)\frac{12\sqrt{7}}{5\sqrt{n}} + C_2(z)\frac{144\sqrt{5}}{25\sqrt{n}} + C_3(z)\frac{12\sqrt{5}}{5\sqrt{n}} \quad (6.1)$$

where

$$\begin{aligned}
C_1(z) &= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{|z|^3}{180\sqrt{5n}}}e^{-\frac{z^2}{24}\left(\frac{1}{1+c}-\frac{1}{10}\right)},\\
C_2(z) &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \Big(\frac{1+z^2}{2|z|}\Big)\Big(e^{-\frac{19z^2}{200}} + e^{\frac{|z|^3}{90\sqrt{5n}}+\frac{|z|}{\sqrt{5n}}}e^{-z^2\left(\frac{3}{40}-\frac{1}{120}\right)}\Big),\\
C_3(z) &= 2\Big(e^{\frac{|z|^3}{180\sqrt{5n}}+\frac{|z|}{2\sqrt{5n}}}\Big)e^{-\frac{z^2}{2}\left(\frac{1}{12}-\frac{1}{120}\right)}.
\end{aligned}$$

Under the assumption that $|z| \leq \frac{3\sqrt{5n}}{5}$, $C_1(z)$, $C_2(z)$ and $C_3(z)$ can be bounded as follows. Starting with $C_1(z)$, we have

$$\begin{aligned}
C_1(z) &\leq \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{\frac{z^2}{300}}e^{-\frac{z^2}{24}\left(\frac{1}{1+c}-\frac{1}{10}\right)}\\
&= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} + e^{-\frac{z^2}{24}\left(\frac{1}{1+c}-\frac{1}{10}-\frac{2}{25}\right)}.
\end{aligned}$$

We want to find a positive constant c such that $e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} = e^{-\frac{z^2}{24}\left(\frac{1}{1+c}-\frac{1}{10}-\frac{2}{25}\right)}$. Suppose that

$$1 - \frac{1}{(1+c)^2} = \frac{1}{12}\Big(\frac{1}{1+c} - \frac{1}{10} - \frac{2}{25}\Big).$$

Then

$$609(1+c)^2 - 50(1+c) - 600 = 0$$

which is equivalent to

$$609c^2 + 1168c - 41 = 0.$$

Hence,

$$c = \frac{-1168 \pm \sqrt{1168^2 - 4(609)(-41)}}{2(609)} = \frac{-584 \pm 605}{609}.$$

Choose $c = \frac{-584+605}{609} = \frac{1}{29}$. This lead us to see that

$$e^{-\frac{z^2}{2}\left(1-\frac{1}{(1+c)^2}\right)} = e^{-\frac{z^2}{24}\left(\frac{1}{1+c}-\frac{1}{10}-\frac{2}{25}\right)} = e^{-\frac{59z^2}{1800}}.$$

Therefore,

$$\begin{aligned} C_1(z) &\leq \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + 2e^{-\frac{59z^2}{1800}} \\ &= \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} + 2e^{-\frac{29z^2}{55800}}e^{-\frac{z^2}{31}} \\ &\leq 0.1856e^{-\frac{z^2}{2}} + 1.9809e^{-\frac{z^2}{31}}. \end{aligned}$$

Now, we bound the term $C_2(z)$ as below.

$$\begin{aligned} C_2(z) &\leq \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \left(\frac{1+z^2}{2|z|}\right)\left(e^{-\frac{19z^2}{200}} + e^{\frac{z^2}{150}+\frac{3}{5}}e^{-z^2\left(\frac{3}{40}-\frac{1}{120}\right)}\right) \\ &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \left(\frac{1+z^2}{2|z|}\right)\left(e^{-\frac{19z^2}{200}} + e^{\frac{3}{5}}e^{-z^2\left(\frac{3}{40}-\frac{1}{120}-\frac{1}{150}\right)}\right) \\ &\leq \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \left(\frac{1+z^2}{2|z|}\right)\left(e^{-\frac{19z^2}{200}} + 1.8222e^{-\frac{3z^2}{50}}\right) \\ &= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}} + \left(\frac{1+z^2}{2|z|}\right)e^{-\frac{z^2}{31}}\left(e^{-\frac{389z^2}{6200}} + 1.8222e^{-\frac{43z^2}{1550}}\right) \\ &\leq 0.2e^{-\frac{z^2}{2}} + 3.1831e^{-\frac{z^2}{31}} \end{aligned}$$

where the last inequality is true by using the fact that $\frac{1}{|z|}e^{-\frac{389z^2}{6200}} \leq 0.0729$, $|z|e^{-\frac{389z^2}{6200}} \leq 1.3479$, $\frac{1}{|z|}e^{-\frac{43z^2}{1550}} \leq 0.1393$ and $|z|e^{-\frac{43z^2}{1550}} \leq 2.5746$ for $|z| \geq 4.3$. Consider estimating in the last term, $C_3(z)$, we see that

$$\begin{aligned} C_3(z) &\leq 2\left(e^{\frac{z^2}{300}+\frac{3}{10}}\right)e^{-\frac{z^2}{2}\left(\frac{1}{12}-\frac{1}{120}\right)} \\ &= 2e^{\frac{3}{10}}e^{-\frac{41z^2}{1200}} \\ &\leq 2.6998e^{-\frac{z^2}{31}}e^{-\frac{71z^2}{37200}} \\ &\leq 2.6062e^{-\frac{z^2}{31}}. \end{aligned}$$

Applying the bounds of $C_1(z)$, $C_2(z)$ and $C_3(z)$, to (6.1), for $|z| > 4.3$,

$$\begin{aligned}
&\Big|P\Big(\frac{W - n/4}{\sqrt{5n}/4} \le z\Big) - \Phi(z)\Big| \\
&\le \frac{1}{\sqrt{n}e^{\frac{z^2}{2}}}\Big(\frac{12\sqrt{7}}{5}(0.1856) + \frac{144\sqrt{5}}{25}(0.2)\Big) \\
&\quad + \frac{1}{\sqrt{n}e^{\frac{z^2}{31}}}\Big(\frac{12\sqrt{7}}{5}(1.9809) + \frac{144\sqrt{5}}{25}(3.1831) + \frac{12\sqrt{5}}{5}(2.6062)\Big) \\
&\le \frac{3.7545}{\sqrt{n}e^{\frac{z^2}{2}}} + \frac{67.5622}{\sqrt{n}e^{\frac{z^2}{31}}}.
\end{aligned}$$

Therefore, from the both cases,

$$\Big|P\Big(\frac{W - n/4}{\sqrt{5n}/4} \le z\Big) - \Phi(z)\Big| \le \frac{3.7545}{\sqrt{n}e^{\frac{z^2}{2}}} + \frac{67.5622}{\sqrt{n}e^{\frac{z^2}{31}}}.$$

□

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