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k-ZERO-DIVISOR HYPERGRAPHS OF FINITE COMMUTATIVE RINGS

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ปิ่นแก้ว ศิริวงศ์ : ไฮเพอร์กราฟเค-ตัวหารของศูนย์ของริงสลับที่จำกัด (k-ZERO-DIVISOR HYPERGRAPHS OF FINITE COMMUTATIVE RINGS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.ศจี เพียรสกุล, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : อ.ดร.รตินันท์ บุญเคลือบ, 46 หน้า.

ให้ R เป็นริงสลับที่ที่มีเอกลักษณ์ซึ่งไม่เป็นศูนย์ และ $k \geq 2$ เป็นจำนวนเต็มที่ตรึงไว้ ไฮเพอร์กราฟเค-ตัวหารของศูนย์ $\mathcal{H}_k(R)$ ของ R ประกอบด้วยเซตของจุดยอด $Z(R,k)$ ซึ่งคือ เซตของเค-ตัวหารของศูนย์ทั้งหมดใน R และเส้นเชื่อม (ไฮเพอร์) ในรูป $\{a_1, a_2, a_3, \dots, a_k\}$ โดยที่ $a_1, a_2, a_3, \dots, a_k$ เป็นสมาชิก k ตัวที่แตกต่างกันทั้งหมดใน $Z(R,k)$ ซึ่งหมายความว่า (i) $a_1 a_2 a_3 \cdots a_k = 0$ และ (ii) ผลคูณของสมาชิกทั้งหมดของสับเซตใดๆ ของ $\{a_1, a_2, a_3, \dots, a_k\}$ ที่มีสมาชิก $k-1$ ตัว ไม่เป็นศูนย์ วิทยานิพนธ์นี้ให้ (i) เงื่อนไขจำเป็นของริงสลับที่ที่ส่งผลให้เกิดความบริบูรณ์ของไฮเพอร์กราฟเค-ตัวหารของศูนย์ของริงเหล่านั้น (ii) เงื่อนไขจำเป็นของริงสลับที่ที่ส่งผลให้สามารถแบ่งเซตของเค-ตัวหารของศูนย์ทั้งหมดของริงเหล่านั้นออกเป็นผลแบ่งกั้น k เซต ซึ่งสามารถนำไปสร้างไฮเพอร์กราฟเค-ตัวหารของศูนย์เคส่วนได้ และเกิดความบริบูรณ์ของไฮเพอร์กราฟเค-ตัวหารของศูนย์เคส่วนของริงเหล่านั้น และ (iii) เงื่อนไขจำเป็นของริงสลับที่ที่ส่งผลให้สามารถแบ่งเซตของซิกมา-ตัวหารของศูนย์ทั้งหมดของริงเหล่านั้นออกเป็นผลแบ่งกั้น k เซต เมื่อ $\sigma \geq k$ เป็นจำนวนเต็มบางจำนวน ซึ่งสามารถนำไปสร้างไฮเพอร์กราฟซิกมา-ตัวหารของศูนย์เคส่วนได้ ยิ่งไปกว่านั้นยังได้กำหนดเส้นผ่านศูนย์กลางและความยาวสั้นสุดของวงของไฮเพอร์กราฟเหล่านี้

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Let R be a commutative ring with nonzero identity and $k \geq 2$ be a fixed integer. The k-zero-divisor hypergraph $\mathcal{H}_k(R)$ of R consists of the vertex set $Z(R, k)$, the set of all k-zero-divisors of R, and the (hyper)edges of the form $\{a_1, a_2, a_3, \dots, a_k\}$ where $a_1, a_2, a_3, \dots, a_k$ are k distinct elements in $Z(R, k)$, which means (i) $a_1 a_2 a_3 \cdots a_k = 0$ and (ii) the products of all elements of any $(k-1)$-subsets of $\{a_1, a_2, a_3, \dots, a_k\}$ are nonzero. This thesis provides (i) a necessary condition of commutative rings that implies the completeness of their k-zero-divisor hypergraphs; (ii) a necessary condition of commutative rings that implies the ability to partition their set of all k-zero-divisors into k partite sets and the completeness of that k-partite k-zero-divisor hypergraphs; and (iii) a necessary condition of commutative rings that implies the ability to partition their set of all σ-zero-divisors into k partite sets, for some integer $\sigma \geq k$. Moreover, the diameter and the minimum length of all cycles of those hypergraphs are determined.

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CONTENTS

CHAPTER I
INTRODUCTION

Graph structures and algebraic structures are closely related. For example, an element a and a nonzero element b, which are distinct, of a commutative ring R can be regarded as two vertices in a graph G and they can have an edge connecting between them whenever $ab = 0$, that is, a is a zero-divisor of R (a is allowed to be zero). This graph G is called a zero-divisor graph which has been extensively studied. According to Beck [2], the definition of zero-divisor graph was first introduced and the coloring and clique of such graphs were studied. Anderson and Livingston [1] changed the vertex set of the zero-divisor graph into the set of all nonzero zero-divisors of a commutative ring and investigated the completeness and automorphisms of such graph.

Later, researchers generalized the idea of a graph into a hypergraph. Chelvam et al. [4] said that Eslahchi and Rahimi were the first who defined the notion of k-zero-divisor and its k-zero-divisor hypergraph. In there, Chelvam et al. studied the planarity of k-zero-divisor hypergraphs.

In this thesis, we investigate the relationships between ring-theoretic properties of a commutative ring and graph-theoretic properties of its k-zero-divisor hypergraph.

We study the concept of k-zero-divisors of a commutative ring and k-zero-divisor hypergraphs in Chapter II. In the first part of that chapter, some definitions in ring theory and some interesting properties are mentioned. For the second part, we introduce the definition of k-zero-divisors of a commutative ring. Finally, we provide definitions of hypergraphs and k-zero-divisor hypergraphs along with some of their basic properties in the third section. Throughout this thesis, let k be a natural number that is greater than 1, R be a principal ideal domain and I be an ideal of R. Then, R/I is a commutative ring with nonzero identity. For certain ideals I of R, we give the necessary properties of R/I to prove the desired results.

In Chapter III, the method to construct complete k-zero-divisor hypergraphs is provided. We also give examples of such hypergraphs.

After that, the necessary condition of commutative rings that implies k-partite k-zero-divisor hypergraphs can be seen in Chapter IV. The completeness of the constructed k-partite k-zero-divisor hypergraphs is also proved. Moreover, examples of such hypergraphs are given.

Next, in Chapter V, we consider k-partite σ-zero-divisor hypergraphs with the integer $\sigma \geq k$. Then, we show how to construct k-partite σ-zero-divisor hypergraphs and examples of such hypergraphs are given.

Note that, the definition of k-partite hypergraph in Chapter IV is according to Kuhl and Schroeder [9] and in Chapter V is according to Jirimutu and Wang [8] which are a little bit different.

For each constructed hypergraph in Chapters III, IV and V, we determine its diameter, which is the maximum of distance between any two vertices, and its minimum length of all cycles.

Finally, we give conclusions and discussions on the future researches in Chapter VI.

CHAPTER II
PRELIMINARIES

We separate this chapter into three sections. In the first section, we review some definitions and some properties of the related algebraic structures. Then, we give the definition of k-zero-divisors of commutative rings in the second section. After that, we introduce hypergraph structures including k-zero-divisor hypergraphs with some relevant definitions in the last section.

2.1 Algebraic Structures

The algebraic structures studied in this thesis involve mainly with commutative rings. The study of rings deals with objects possessing two binary operations, called an *addition* (+) and a *multiplication* (×), related by the distributive laws.

Definition 2.1. ([5]) A *ring* $(R, +, \times)$ is a set together with two binary operations + and ×, called an *addition* and a *multiplication*, respectively, satisfying the following axioms:

(i) $(R, +)$ is an abelian group,

(ii) × is associative: $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in R$,

(iii) the distributive laws hold in R: $(a+b) \times c = (a \times c) + (b \times c)$ and $a \times (b+c) = (a \times b) + (a \times c)$ for all $a, b, c \in R$.

Remark. (i) The additive identity of a ring R is always denoted by 0.

(ii) We shall usually write simply ab rather than $a \times b$ for any elements a, b of a ring.

(iii) A ring R is said to have an identity (or contain 1) if there is an element $1 \in R$ with $1a = a1 = a$ for all $a \in R$.

(iv) Let R be a ring with nonzero identity. An element u of R is called a *unit* of R if there is an element v of R such that $uv = vu = 1$. The set of all units of R is denoted by $U(R)$.

(v) A ring is commutative if the multiplication is commutative.

(vi) A subring of a ring R is a subgroup of R that is closed under multiplication.

(vii) $r + I = \{r + a \mid a \in I\}$, $rI = \{ra \mid a \in I\}$ and $Ir = \{ar \mid a \in I\}$ for any element r and ideal I of a ring R.

In this thesis, we consider one special type of subrings of a commutative ring which is called an ideal.

Definition 2.2. ([5]) Let R be a ring, I be a nonempty subset of R and $r \in R$.

(i) A subset I of R is a *left (right) ideal* of R if

- I is a subring of R, and
- I is closed under left (right) multiplication by elements of R, i.e., $rI \subseteq I$ for all $r \in R$ ($Ir \subseteq I$ for all $r \in R$).

(ii) A subset I of R that is both a left ideal and a right ideal of R is called an *ideal* (or *a two-sided ideal*) of R.

Note that, for commutative rings, the notions of left, right and two-sided ideals coincide.

Definition 2.3. ([5]) Let R be a commutative ring with a nonzero identity and a be any element of R. Let $Ra = \{ra \mid r \in R\}$ denote the smallest ideal of R containing a, called the *ideal generated by* a or a *principal ideal.*

Some special types of rings are the following. Instead of considering any commutative rings R, we study the quotient ring $R/I = \{r + I \mid r \in R\}$ for any ideal I of R.

Proposition 2.4. *([5]) Let R be a ring and I be an ideal of R. Then, the (additive) quotient group $R/I = \{r + I \mid r \in R\}$ is a ring under the binary operations:*

$$(r + I) + (s + I) = (r + s) + I \text{ and } (r + I) \times (s + I) = (rs) + I$$

for all $r, s \in R$. The ring R/I with the prescribe operations is called the quotient ring of R by I.

Proposition 2.5. *Let Λ be any nonempty set. Let I be an ideal of a ring R and $\{J_\alpha\}_{\alpha \in \Lambda}$ be a collection of ideals of a ring R such that $I \subseteq J_\alpha$ for all $\alpha \in \Lambda$. Then, for each $\alpha \in \Lambda$,*

$$(J_\alpha/I) - \bigcup_{\beta \in \Lambda, \beta \neq \alpha} (J_\beta/I) = \left\{ a + I \,\middle|\, a \in J_\alpha - \bigcup_{\beta \in \Lambda, \beta \neq \alpha} J_\beta \right\}.$$

Proof. First, let $\alpha \in \Lambda$ and $x \in J_\alpha$ be such that $x + I \notin \bigcup_{\beta \in \Lambda, \beta \neq \alpha} (J_\beta/I)$. Then, $x + I \notin J_\beta/I$ for all $\beta \neq \alpha$. Thus, $x \notin J_\beta$ for all $\beta \neq \alpha$. Hence, $x \in J_\alpha - \bigcup_{\beta \in \Lambda, \beta \neq \alpha} J_\beta$.

Conversely, let $x \in J_\alpha - \bigcup_{\beta \in \Lambda, \beta \neq \alpha} J_\beta$. Then, $x + I \in J_\alpha/I$. Since $x \notin \bigcup_{\beta \in \Lambda, \beta \neq \alpha} J_\beta$, we obtain $x \notin J_\beta$ so that $x + I \notin J_\beta/I$ for all $\beta \neq \alpha$. Therefore, $x + I \notin \bigcup_{\beta \in \Lambda, \beta \neq \alpha} (J_\beta/I)$. Hence, $x + I \in (J_\alpha/I) - \bigcup_{\beta \in \Lambda, \beta \neq \alpha} (J_\beta/I)$. □

Definition 2.6. ([5]) An ideal M of a ring R is called a *maximal ideal* if $M \neq R$ and the only ideals containing M are M and R.

Definition 2.7. ([5]) Assume that R is a commutative ring. An ideal P of R is called a *prime ideal* if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

Definition 2.8. ([5]) Let R_1 and R_2 be rings. We shall denote by $R_1 \times R_2$ their *direct product* (as rings), that is, the set of ordered pairs (r_1, r_2) with $r_1 \in R_1$ and $r_2 \in R_2$ where addition and multiplication are performed componentwise:

(i) $(r_1, r_2) + (s_1, s_2) = (r_1 + s_1, r_2 + s_2)$; and

(ii) $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$

for all $r_1, s_1 \in R_1$ and $r_2, s_2 \in R_2$.

Definition 2.9. ([5]) Let R be a commutative ring with nonzero identity. Then, the distinct ideals A and B of R are said to be *comaximal* if $A + B = R$.

Proposition 2.10. *If A and B are distinct maximal ideals of a commutative ring with nonzero identity, then A and B are comaximal.*

Proof. Assume that A and B are maximal ideals of R. We know that $A \subset A + B$, $B \subset A + B$. Since A and B are maximal ideals, $A + B = R$. □

Theorem 2.11. *([5]) (Chinese Remainder Theorem) Let R be a commutative ring with nonzero identity and $A_1, A_2, A_3, \ldots, A_k$ be ideals of R. The map $R \to R/A_1 \times R/A_2 \times R/A_3 \times \cdots \times R/A_k$ defined by $r \mapsto (r + A_1, r + A_2, r + A_3, \ldots, r + A_k)$ is a ring homomorphism with kernel $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k$. If for each $1 \leq i, j \leq k$ with $i \neq j$, the ideals A_i and A_j are comaximal, then this map is surjective and $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k = A_1 A_2 A_3 \cdots A_k$, thus, $R/(A_1 A_2 A_3 \cdots A_k) = R/(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \cong R/A_1 \times R/A_2 \times R/A_3 \times \cdots \times R/A_k$.*

Next, we introduce zero-divisors of a ring and some definitions involving zero-divisors as follows. In this thesis, we follow the definition of zero-divisors given by Bourbaki [3].

Definition 2.12. ([3]) Let R be a ring. An element a of R is called a *zero-divisor* if there is a nonzero element b in R such that either $ab = 0$ or $ba = 0$.

Definition 2.13. ([5]) A commutative ring with identity $1 \neq 0$ is called an *integral domain* if it has no zero-divisors.

Definition 2.14. ([5]) A *principal ideal domain* (PID) is an integral domain in which every ideal is principal.

Example 2.15. The set $\mathbb{Z}$ of all integers and the set $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ of all Gaussian integers are PIDs.

There are some additional algebraic definitions and properties that are needed throughout this thesis.

Definition 2.16. ([5]) Let R be a commutative ring and let $a, b \in R$ with $b \neq 0$.

(i) An element a is said to be a *multiple* of b if there exists an element $x \in R$ with $a = bx$. In this case b is said to *divide* a or to be a *divisor* of a, written as $b \mid a$.

(ii) A *greatest common divisor* of a and b is a nonzero element d such that

- $d \mid a$ and $d \mid b$, and
- if $d' \mid a$ and $d' \mid b$, then $d' \mid d$ for any nonzero element $d' \in R$.

A greatest common divisor of a and b is denoted by $\gcd(a, b)$.

Note that, $b \mid a$ in a commutative ring with a nonzero identity R if and only if $a \in Rb$ if and only if $Ra \subseteq Rb$. In particular, if d is any divisor of both a and b, then there exist $r, s \in R$ such that $a = rd$ and $b = sd$. Thus, Rd must contain both a and b, and hence, it must contain the ideal generated by a and the ideal generated by b.

Proposition 2.17. *([5]) If a and b are nonzero elements of a commutative ring R such that the ideal generated by a and b is a principal ideal Rd, then d is a greatest common divisor of a and b.*

Proposition 2.18. *([5]) Let R be an integral domain. If two elements d and d' of R generate the same principal ideal, i.e., $Rd = Rd'$, then $d' = ud$ for some unit u of R. In particular, if d and d' are both greatest common divisors of a and b, then $d' = ud$ for some unit u.*

Proposition 2.19. *([5]) Let R be a PID and let a and b be nonzero elements of R. Moreover, let d be a generator of the principal ideal containing a and b. Then,*

(i) $d = \gcd(a, b)$;

(ii) d can be written as an R-linear combination of a and b, i.e., there are elements x and y of R such that $d = ax + by$; and

(iii) d is unique up to multiplication by a unit of R.

By Proposition 2.19, $\gcd(a, b) = ax + by$ for some $x, y \in R$, we have that if a is a nonzero element of R and $\gcd(a, b) = 1$, then $1 = ax + by \in ax + Rb = (a + Rb)(x + Rb)$. That is, $a + Rb$ is a unit of the quotient ring R/Rb.

Definition 2.20. ([5]) Let R be an integral domain. A nonzero element $p \in R$ is called *prime* in R if the ideal Rp generated by p is a prime ideal. In other words, a nonzero element p is prime if it is not a unit and whenever $p \mid ab$ for any $a, b \in R$, then either $p \mid a$ or $p \mid b$.

Example 2.21. The prime integers in $\mathbb{Z}$ are prime elements.

Example 2.22. ([6]) The prime elements in $\mathbb{Z}[i]$ are of the forms

(i) $a + bi$, where $a, b \neq 0$ and $a^2 + b^2 = p$ where p is a prime element of $\mathbb{Z}$,

(ii) up, where u is a unit of $\mathbb{Z}[i]$ and p is a prime element of $\mathbb{Z}$ such that $p \equiv 3 \pmod 4$.

Definition 2.23. ([5]) The *Euler φ-function* is defined as follows: for $n \in \mathbb{N}$, let $\varphi(n)$ be the number of positive integers $a \leq n$ with a relatively prime to n, i.e., $\gcd(a, n) = 1$.

For primes p, $\varphi(p) = p - 1$, and more generally, for all $a \geq 1$, we have formula $\varphi(p^a) = p^a - p^{a-1} = p^{a-1}(p - 1)$. The function φ is *multiplicative* in the sense that $\varphi(ab) = \varphi(a)\varphi(b)$ if $\gcd(a, b) = 1$. Therefore, if $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_s^{\alpha_s}$, then $\varphi(n) = \varphi(p_1^{\alpha_1})\varphi(p_2^{\alpha_2})\varphi(p_3^{\alpha_3}) \cdots \varphi(p_s^{\alpha_s})$.

Definition 2.24. ([5]) Let R be an integral domain. Two elements a and b of R differing by a unit (i.e., $a = ub$ for some unit u in R) are said to be *associate* in R.

Definition 2.25. ([5]) A *field* is a set F together with two commutative binary operations $+$ and $\cdot$ on F such that $(F, +)$ is an abelian group and $(F - \{0\}, \cdot)$ is also an abelian group, $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in F$, and the following *distributive* law holds:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \text{ for all } a, b, c \in F.$$

Proposition 2.26. *([5]) Let R be a commutative ring with nonzero identity. The ideal M is a maximal ideal if and only if the quotient ring R/M is a field.*

Corollary 2.27. *([5]) Every nonzero prime ideal in a PID is a maximal ideal.*

By Corollary 2.26 and Corollary 2.27, we obtain the following result.

Corollary 2.28. *Let R be a PID with a prime element p. Then, R/Rp is a field.*

Proposition 2.29. *([5]) Let I be an ideal of a ring R with nonzero identity.*

(i) $I = R$ if and only if I contains a unit.

(ii) If R is a commutative ring, then R is a field if and only if its only ideals are $\{0\}$ and R, that is, R contains only the zero element and unit elements.

2.2 k-Zero-Divisors of a Commutative Ring

In the previous section, zero-divisors of a ring are recalled. Chelvam et. al. [4] extended the definition of zero-divisors to k-zero-divisors where k is an integer such that $k \geq 2$.

Definition 2.30. ([4]) Let R be a commutative ring with nonzero identity. A nonzero and nonunit element z_1 of R is called a *k-zero-divisor* of R if there exist $k-1$ distinct nonunit elements $z_2, z_3, z_4, \ldots, z_k$ differ from z_1 and satisfy the following statements:

(i) $z_1 z_2 z_3 \cdots z_k = 0$; and

(ii) the products of all elements of any $(k-1)$-subsets of $\{z_1, z_2, z_3, \ldots, z_k\}$ are nonzero.

Moreover, we use $Z(R, k)$ to denote the set of all k-zero-divisors of R. Note that the elements $z_2, z_3, z_4, \ldots, z_k$ in Definition 2.30 must be nonzero elements.

Example 2.31. Consider the ring $\mathbb{Z}_{30}$. Since $\overline{2} \cdot \overline{3} \cdot \overline{5} = \overline{0} \in \mathbb{Z}_{30}$ and the products of any 2 elements of $\{\overline{2}, \overline{3}, \overline{5}\}$ are nonzero, $\overline{2}$ is one of the 3-zero-divisors of $\mathbb{Z}_{30}$. Actually,

$$Z\big(\mathbb{Z}_{30}, 3\big) = \big\{\overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{8}, \overline{9}, \overline{14}, \overline{16}, \overline{21}, \overline{22}, \overline{25}, \overline{26}, \overline{27}, \overline{28}\big\}.$$

We can see that 2-zero-divisors imply zero-divisors but not vice versa. For instance, $\overline{2}$ is a zero-divisor of $\mathbb{Z}_4$, but it is not a 2-zero-divisor of $\mathbb{Z}_4$.

2.3 Hypergraph Structures

Definition 2.32. ([10]) A *hypergraph* $\mathcal{H}(V, \mathcal{E})$ or $\mathcal{H}$ consists of a set of *vertices* or *vertex set* $V = V(\mathcal{H}) = \{v_1, v_2, v_3, \ldots, v_n\}$, and a set of *(hyper)edge* or *edge set* $\mathcal{E} = \mathcal{E}(\mathcal{H}) = \{E_1, E_2, E_3, \ldots, E_m\}$ where $E_i \subseteq V$ and $|E_i| \geq 0$ for all $n \geq 1$ and $1 \leq i \leq m$. Furthermore, let $l \geq 0$ be a fixed integer. If $|E_i| = l$ for all $E_i \in \mathcal{E}$, we say that $\mathcal{H}$ is an *l-uniform hypergraph*.

Example 2.33. Consider the vertex set $V = \{1, 2, 3, 4\}$. The edge set $\mathcal{E}$ of all 3-subsets of V is

$$\big\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\big\}$$

Such vertex set and edge set form a hypergraph $\mathcal{H}(V, \mathcal{E})$ which is a 3-uniform hypergraph.

Chelvam et al. link the idea between k-zero-divisors of a commutative ring R and k-uniform hypergraphs. Then, they can define k-zero-divisor hypergraphs of a commutative ring as follows.

Definition 2.34. ([4]) Let $k \geq 2$ be a fixed positive integer. A *k-zero-divisor hypergraph* of a commutative ring R with nonzero identity, denoted by $\mathcal{H}_k(R)$, is defined as a k-uniform hypergraph whose vertex set is $Z(R, k)$ and the set $\{a_1, a_2, a_3, \ldots, a_k\} \subseteq Z(R, k)$ is an edge if it satisfies the following statements:

(i) $a_1 a_2 a_3 \cdots a_k = 0$; and

(ii) the products of all elements of any $(k-1)$-subsets of $\{a_1, a_2, a_3, \ldots, a_k\}$ are nonzero.

Example 2.35. Consider the ring $\mathbb{Z}_{30}$. We know from Example 2.31 that $\overline{2}$ is a 3-zero-divisor of $\mathbb{Z}_{30}$. Not only $\overline{2} \cdot \overline{3} \cdot \overline{5} = \overline{0}$ in $\mathbb{Z}_{30}$, but also $\overline{2} \cdot \overline{3} \cdot \overline{25}$, $\overline{2} \cdot \overline{5} \cdot \overline{9}$, $\overline{2} \cdot \overline{9} \cdot \overline{25}$, $\overline{2} \cdot \overline{5} \cdot \overline{21}$, $\overline{2} \cdot \overline{21} \cdot \overline{25}$, $\overline{2} \cdot \overline{5} \cdot \overline{27}$, and $\overline{2} \cdot \overline{25} \cdot \overline{27}$ equal zero of $\mathbb{Z}_{30}$. We can construct edges between such elements as $\{\overline{2}, \overline{3}, \overline{5}\}$, $\{\overline{2}, \overline{3}, \overline{25}\}$, $\{\overline{2}, \overline{5}, \overline{9}\}$, $\{\overline{2}, \overline{9}, \overline{25}\}$, $\{\overline{2}, \overline{5}, \overline{21}\}$, $\{\overline{2}, \overline{21}, \overline{25}\}$, $\{\overline{2}, \overline{5}, \overline{27}\}$, and $\{\overline{2}, \overline{25}, \overline{27}\}$. These edges are some of all edges of the hypergraph $\mathcal{H}_3(\mathbb{Z}_{30})$. Later, in Chapter IV, we show all edges of $\mathcal{H}_3(\mathbb{Z}_{30})$.

To investigate a relationship between ring-theoretic properties of a commutative ring and graph-theoretic properties of its k-zero-divisor hypergraph, we introduce some interesting definitions of a hypergraph.

Definition 2.36. ([10]) Let $l \geq 2$ be an integer. The *complete l-uniform hypergraph* on n vertices is an l-uniform hypergraph which has all l-subsets of the n-set of vertices as edges.

Example 2.37. $\mathcal{H}(V, \mathcal{E})$ in Example 2.33 is a complete 3-uniform hypergraph.

Since hypergraph is still a new topic in mathematics, there are some researchers who gave slightly different definitions of the term k-partite hypergraph. Here, we use two definitions based on Kuhl and Schroeder [9], and Jirimutu and Wang [8] which are Definition 2.38 and Definition 2.40, respectively.

Definition 2.38. ([9]) Let $k \geq 2$ be a fixed positive integer. A *k-uniform k-partite hypergraph* has a vertex set V partitioned into k subsets $V_1, V_2, V_3, \ldots, V_k$, and the edge set $\mathcal{E}$ such that $\mathcal{E} = \big\{\{v_1, v_2, v_3, \ldots, v_k\} \mid v_j \in V_j \text{ for all } 1 \leq j \leq k\big\}$. A k-uniform k-partite hypergraph is said to be *complete* if $V_j = \{v_j^1, v_j^2, v_j^3, \ldots, v_j^{|V_j|}\}$ for all $1 \leq j \leq k$, and $\mathcal{E} = \big\{\{v_1^{i_1}, v_2^{i_2}, v_3^{i_3}, \ldots, v_k^{i_k}\} \mid v_j^{i_j} \in V_j \text{ for all } 1 \leq j \leq k \text{ and } 1 \leq i_j \leq \big|V_j\big|\big\}$.

Example 2.39. Let $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{a, b, c\}$, $V_2 = \{a', b', c'\}$ and $V_3 = \{\overline{a}, \overline{b}, \overline{c}\}$ are mutually disjoint. The complete 3-uniform 3-partite hypergraph given by Kuhl and Schroeder constructed from V has edges as follows.

$\{a,a',\overline{a}\},\{a,a',\overline{b}\},\{a,a',\overline{c}\},\{a,b',\overline{a}\},\{a,b',\overline{b}\},\{a,b',\overline{c}\},\{a,c',\overline{a}\},\{a,c',\overline{b}\},\{a,c',\overline{c}\},$
$\{b,a',\overline{a}\},\{b,a',\overline{b}\},\{b,a',\overline{c}\},\{b,b',\overline{a}\},\{b,b',\overline{b}\},\{b,b',\overline{c}\},\{b,c',\overline{a}\},\{b,c',\overline{b}\},\{b,c',\overline{c}\},$
$\{c,a',\overline{a}\},\{c,a',\overline{b}\},\{c,a',\overline{c}\},\{c,b',\overline{a}\},\{c,b',\overline{b}\},\{c,b',\overline{c}\},\{c,c',\overline{a}\},\{c,c',\overline{b}\},\{c,c',\overline{c}\}.$

Definition 2.40. ([8]) Let $k \geq 2$ and σ be fixed integers with $\sigma \geq k$. A *σ-uniform k-partite hypergraph* has the vertex set V partitioned into k subsets $V_1, V_2, V_3, \ldots, V_k$ and E is an edge if

$$\begin{aligned}&\big|E\big| = \sigma \text{ and } \big|E \cap V_i\big| \; < \sigma \text{ for all } 1 \leq i \leq k \\ &\quad\text{and there exist } 1 \leq m \neq n \leq k \text{ such that } E \cap V_m \neq \varnothing \text{ and } E \cap V_n \neq \varnothing. \quad (*)\end{aligned}$$

A σ-uniform k-partite hypergraph is said to be *complete* if the edge set $\mathcal{E}$ contains all edges satisfying $(*)$.

Example 2.41. Let $V = V_1 \cup V_2$ where $V_1 = \{a,b,c\}$ and $V_2 = \{a',b',c'\}$ and are mutually disjoint. The 3-uniform 2-partite hypergraph given by Jirimutu and Wang has edges as follows.

$\{a,b,a'\},\{a,c,a'\},\{b,c,a'\},\{a,b,b'\},\{a,c,b'\},\{b,c,b'\},\{a,b,c'\},\{a,c,c'\},$
$\{b,c,c'\},\{a',b',a\},\{a',c',a\},\{b',c',a\},\{a',b',b\},\{a',c',b\},\{b',c',b\},\{a',b',c\},$
$\{a',c',c\},\{b',c',c\}.$

Example 2.42. Let $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{a,b,c\}$, $V_2 = \{a',b',c'\}$ and $V_3 = \{\overline{a},\overline{b},\overline{c}\}$ are mutually disjoint. The 3-uniform 3-partite hypergraph given by Jirimutu and Wang has edges as follows.

$\{a,a',\overline{a}\},\{a,a',\overline{b}\},\{a,a',\overline{c}\},\{a,b',\overline{a}\},\{a,b',\overline{b}\},\{a,b',\overline{c}\},\{a,c',\overline{a}\},\{a,c',\overline{b}\},\{a,c',\overline{c}\},$
$\{b,a',\overline{a}\},\{b,a',\overline{b}\},\{b,a',\overline{c}\},\{b,b',\overline{a}\},\{b,b',\overline{b}\},\{b,b',\overline{c}\},\{b,c',\overline{a}\},\{b,c',\overline{b}\},\{b,c',\overline{c}\},$
$\{c,a',\overline{a}\},\{c,a',\overline{b}\},\{c,a',\overline{c}\},\{c,b',\overline{a}\},\{c,b',\overline{b}\},\{c,b',\overline{c}\},\{c,c',\overline{a}\},\{c,c',\overline{b}\},\{c,c',\overline{c}\},$
$\{a,b,a'\},\{a,c,a'\},\{b,c,a'\},\{a,b,b'\},\{a,c,b'\},\{b,c,b'\},\{a,b,c'\},\{a,c,c'\},\{b,c,c'\},$
$\{a,b,\overline{a}\},\{a,c,\overline{a}\},\{b,c,\overline{a}\},\{a,b,\overline{b}\},\{a,c,\overline{b}\},\{b,c,\overline{b}\},\{a,b,\overline{c}\},\{a,c,\overline{c}\},\{b,c,\overline{c}\}.$

Example 2.39 is an example of Definition 2.40.

In order to determine the diameter of each hypergraph, a path in a hypergraph and its length are needed.

Definition 2.43. ([12]) A *path* P in a hypergraph $\mathcal{H}$ from x_1 to x_{s+1} is a vertex-edge alternative sequence $x_1, E_1, x_2, E_2, \ldots, x_s, E_s, x_{s+1}$ such that $\{x_i, x_{i+1}\} \subseteq E_i$ for all $1 \leq i \leq s$ and $x_i \neq x_j$, $E_i \neq E_j$ with $i \neq j$, and s is called the *length* of the path P.

Example 2.44. From Example 2.41, some examples of paths from a to b are

(i) $a, \{a, b, a'\}, b,$

(ii) $a, \{a, c, a'\}, a', \{a, b, a'\}, b,$

(iii) $a, \{a, c, a'\}, c, \{a, c, c'\}, c', \{b, c, c'\}, b.$

Their lengths are 1, 2, and 3, respectively.

Definition 2.45. ([12]) The *distance* of distinct vertices x and y of a hypergraph, denoted by $d(x, y)$, is the minimum length of a path that connects x and y.

Example 2.46. From Example 2.41, we see that $d(a, b) = 1$ because in Example 2.44, we can find a path of length 1.

Definition 2.47. ([12]) The *diameter* of $\mathcal{H}(V, \mathcal{E})$, denoted by $d(\mathcal{H})$, is defined as $d(\mathcal{H}) = max\{d(x, y) \mid x, y \in V \text{ and } x \neq y\}$.

Finally, we need to introduce the definition of cycles to determine the minimum length of cycles.

Definition 2.48. ([7]) Let $s \geq 2$ be an integer. An s-cycle of a hypergraph is an alternating sequence, $C = x_1, E_1, x_2, E_2, \ldots, x_s, E_s$ of distinct vertices $x_1, x_2, x_3, \ldots, x_s$ and distinct edges $E_1, E_2, E_3, \ldots, E_s$ such that $x_1, x_s \in E_s$ and $x_i, x_{i+1} \in E_i$ for all $1 \leq i \leq s - 1$ and s is called the *length of cycle* C. If hypergraph has no cycles, we say that this hypergraph has a 0-*cycle* or *a cycle of length* 0.

Example 2.49. From Example 2.39, some examples of cycles that containing a are

(i) $a, \{a, a', \overline{a}\}, \overline{a}, \{a, b', \overline{a}\},$

(ii) $a, \{a, a', \overline{b}\}, a', \{b, a', \overline{b}\}, b, \{a, b, \overline{a}\},$

(iii) $a, \{a, c', \overline{c}\}, c', \{b, c', \overline{c}\}, \overline{c}, \{b, b', \overline{c}\}, b', \{a, b', \overline{a}\}.$

Their lengths are 2, 3, and 4, respectively.

CHAPTER III

COMPLETE k-ZERO-DIVISOR HYPERGRAPHS

Let $k \geq 2$ be a fixed integer. Our main objective of this chapter is to find a necessary condition of a commutative ring that implies completeness of its k-zero-divisor hypergraph.

Let R be a PID. Assume that R has a prime element, say p. Then, Rp^k is an ideal and, consequently, R/Rp^k is a commutative ring with nonzero identity.

By Proposition 2.5, we obtain the following result.

Proposition 3.1. *Let R be a commutative ring containing a prime element p and $k \geq 2$ be a fixed integer. Then,*

$$(Rp/Rp^k) - (Rp^2/Rp^k) = \left\{ a + Rp^k \,\middle|\, a \in Rp - Rp^2 \right\}.$$

Next, under some conditions on R/Rp^k, we can determine the set of all k-zero-divisors of the ring R/Rp^k, that is $Z(R/Rp^k, k)$.

Proposition 3.2. *Let R be a PID containing a prime element p and $k \geq 2$ be a fixed integer. Assume that R/Rp^k is finite and $\left| Rp/Rp^k - Rp^2/Rp^k \right| \geq k$. Then, $Z(R/Rp^k, k) = (Rp/Rp^k) - (Rp^2/Rp^k)$.*

Proof. Let $x_1 \in R$ be such that $x_1 + Rp^k \in (Rp/Rp^k) - (Rp^2/Rp^k)$. Since $\left| Rp/Rp^k - Rp^2/Rp^k \right| \geq k$, we can find distinct nonzero nonunit elements $x_2 + Rp^k, x_3 + Rp^k, x_4 + Rp^k, \dots, x_k + Rp^k \in (Rp/Rp^k) - (Rp^2/Rp^k)$ which differ from $x_1 + Rp^k$ such that $p \mid x_i$, but $p^2 \nmid x_i$ for all $1 \leq i \leq k$. Since $p \mid x_i$ for all $1 \leq i \leq k$, $(x_1 + Rp^k)(x_2 + Rp^k)(x_3 + Rp^k) \cdots (x_k + Rp^k) = \prod_{i=1}^{k} x_i + Rp^k = 0 + Rp^k$. Since $p^2 \nmid x_i$ for all $1 \leq i \leq k$, $(x_{i_1} + Rp^k)(x_{i_2} + Rp^k)(x_{i_3} + Rp^k) \cdots (x_{i_{k-1}} + Rp^k) = \prod_{j=1}^{k-1} x_{i_j} + Rp^k \neq 0 + Rp^k$ where $\{x_{i_j} \mid 1 \leq j \leq k-1\}$ is any $(k-1)$-subsets of $\{x_i \mid 1 \leq i \leq k\}$. Consequently, $x_1 + Rp^k$ is a k-zero-divisor of R/Rp^k. Then, $x_1 + Rp^k \in Z(R/Rp^k, k)$.

We need to show that $Z(R/Rp^k,k) \subseteq (Rp/Rp^k) - (Rp^2/Rp^k)$. First, let $x_1 + Rp^k \in Z(R/Rp^k,k)$. Then,

(i) $x_1 + Rp^k \neq 0 + Rp^k$ and $x_1 + Rp^k \notin U(R/Rp^k)$;

(ii) there exist $k-1$ distinct nonzero nonunit elements $x_2+Rp^k, x_3+Rp^k, x_4+Rp^k, \ldots, x_k + Rp^k$ of R/Rp^k, which differ from $x_1 + Rp^k$ such that $(x_1 + Rp^k)(x_2+Rp^k)(x_3+Rp^k)\cdots(x_k+Rp^k) = \prod_{i=1}^{k} x_i+Rp^k = 0+Rp^k$ and the products of all elements of any $(k-1)$-subsets of $\{x_1 + Rp^k, x_2 + Rp^k, x_3 + Rp^k, \ldots, x_k + Rp^k\}$ are nonzero.

From (ii), we have $\prod_{i=1}^{k} x_i \in Rp^k$. Since $x_1 + Rp^k \notin U(R/Rp^k)$, by the consequence of Proposition 2.19, $\gcd(x_1, p^k) \neq 1$. Thus, there exists $d_1 \in R - \{1\}$ such that $\gcd(x_1, p^k) = d_1$. Note that, since $x_1 + Rp^k$ is a nonzero element of R/Rp^k, we obtain $d_1 \neq p^k$. There are two possible cases for d_1 as follows.

Case 1. $d_1 = p$. Then, $x_1 \in Rp - Rp^2$. By Proposition 3.1, $x_1 + Rp^k \in (Rp/Rp^k) - (Rp^2/Rp^k)$.

Case 2. $d_1 = p^j$ for some $2 \leq j \leq k-1$. We have $x_1 \in Rp^j$. Since $x_l + Rp^k$ is not a unit of R/Rp^k for all $2 \leq l \leq k$, by the consequence of Proposition 2.19, $\gcd(x_l, p^k) \neq 1$, that is, there exist $d_l \in R - \{1\}$ such that $\gcd(x_l, p^k) = d_l$ for any $2 \leq l \leq k$. Note that, the prime element p must divide d_l for all $2 \leq l \leq k$. As a result, $x_l \in Rp$ for all $2 \leq l \leq k$. Therefore, $\prod_{s=1}^{k-1} x_s \in Rp^j Rp^{k-2}$. Since $j \geq 2$, $\prod_{s=1}^{k-1} x_s \in Rp^j Rp^{k-2} = Rp^{j-2} Rp^k \subseteq Rp^k$ which implies that $(x_1 + Rp^k)(x_2 + Rp^k)(x_3 + Rp^k)\cdots(x_{k-1} + Rp^k) = \prod_{s=1}^{k-1} x_s + Rp^k = 0 + Rp^k$ and contradicts the fact that $x_1 + Rp^k$ is a k-zero-divisor of R/Rp^k.

Thus, the only possible case is that $d_1 = p$. Hence, $x_1 + Rp^k \in (Rp/Rp^k) - (Rp^2/Rp^k)$ and we can determine $Z(R/Rp^k,k)$ by $(Rp/Rp^k) - (Rp^2/Rp^k)$.

□

Now, we know all elements in $Z(R/Rp^k,k)$. Then, we use Proposition 3.2 to prove the completeness of the k-zero-divisor hypergraph $\mathcal{H}_k(R/Rp^k)$. Since each element in R/Rp^k needs the other $k-1$ elements in R/Rp^k to be vertices, the set of all k-zero-divisors of R/Rp^k must have at least k elements.

Theorem 3.3. *Let $k \geq 2$ be a fixed integer. Assume that R is a PID containing a prime element p such that R/Rp^k is finite and $\left|Z(R/Rp^k, k)\right| \geq k$. Then, $\mathcal{H}_k(R/Rp^k)$ is a complete k-zero-divisor hypergraph.*

Proof. By Proposition 3.2, all elements in the vertex set are in the ring $(Rp/Rp^k) - (Rp^2/Rp^k)$. Then, p^k divides the product of any k elements of $Z(R/Rp^k, k)$ but p^k does not divide the product of any $k-1$ elements of such k elements. Therefore, all k-subsets of the vertex set form edges of $\mathcal{H}_k(R/Rp^k)$. Hence, $\mathcal{H}_k(R/Rp^k)$ is a complete k-uniform hypergraph. □

Corollary 3.4. *Let R be a PID containing a prime element p and $k \geq 2$ be a fixed integer. Then, R/Rp^k can be separated into four disjoint subsets: $U(R/Rp^k)$, $\{0 + Rp^k\}$, $Z(R/Rp^k, k)$ and $Rp^2/Rp^k - \{0 + Rp^k\}$.*

Proof. Let $A_1 = U(R/Rp^k)$, $A_2 = \{0 + Rp^k\}$, $A_3 = Z(R/Rp^k, k)$ and $A_4 = Rp^2/Rp^k - \{0 + Rp^k\}$. Obviously, A_1, A_2, A_3, and A_4 are mutually disjoint sets.

We need to show that $\bigcup_{i=1}^{4} A_i = R/Rp^k$.

Since $A_i \subset R/Rp^k$ for all $1 \leq i \leq 4$, $\cup_{i=1}^{4} A_i \subseteq R/Rp^k$.

Let $a + Rp^k \in R/Rp^k$. Asuume that $a + Rp^k \neq 0 + Rp^k$. Then, $\gcd(a, p^k) \neq p^k$. There are two possible cases.

Case 1. $\gcd(a, p^k) = 1$. By consequence of Proposition 2.19, we have $a + Rp^k$ is a unit of R/Rp^k. Thus, $a + Rp^k \in U(R/Rp^k) = A_1$.

Case 2. $\gcd(a, p^k) \neq 1$. Then, $p \big| a$. There are two subcases:

(i) $p^2 \nmid a$. Then, $a = p$. We have $a + Rp^k \in (Rp/Rp^k) - (Rp^2/Rp^k) = Z(R/Rp^k, k) = A_3$; or

(ii) $p^2 \big| a$. Then, $a + Rp^k \in Rp^2/Rp^k - \{0 + Rp^k\} = A_4$.

□

From Corollary 3.4, $\left|R/Rp^k\right| = \left|U(R/Rp^k)\right| + \left|\{0 + Rp^k\}\right| + \left|Z(R/Rp^k, k)\right| + \left|Rp^2/Rp^k - \{0 + Rp^k\}\right|$.

If $R = \mathbb{Z}$, then $\left|U(\mathbb{Z}/p^k\mathbb{Z})\right| = \varphi(p^k)$ and $\left|p^2\mathbb{Z}/p^k\mathbb{Z} - \{0 + p^k\mathbb{Z}\}\right| = p^{k-2} - 1$. Thus, $\left|\mathbb{Z}/p^k\mathbb{Z}\right| = \varphi(p^k) + 1 + \left|Z(\mathbb{Z}/p^k\mathbb{Z}, k)\right| + (p^{k-2} - 1)$. Therefore, $\left|Z(\mathbb{Z}/p^k\mathbb{Z}, k)\right| = \left|\mathbb{Z}/p^k\mathbb{Z}\right| - \varphi(p^k) - 1 - (p^{k-2} - 1)$.

We give an example of complete 3-uniform hypergraphs.

Example 3.5. Consider $\mathbb{Z}_{27} \cong \mathbb{Z}/3^3\mathbb{Z}$. We have $\left|U(\mathbb{Z}/3^3\mathbb{Z})\right| = \varphi(3^3)$ and $\left|3^2\mathbb{Z}/3^3\mathbb{Z} - \{0 + 3^3\mathbb{Z}\}\right| = 3^{3-2} - 1$. By previous remark, $\left|Z(\mathbb{Z}/3^3\mathbb{Z}, 3)\right| = \left|\mathbb{Z}/3^3\mathbb{Z}\right| - \varphi(3^3) - 1 - (3^{3-2} - 1)$. Then, $\left|Z(\mathbb{Z}/3^3\mathbb{Z}, 3)\right| = 27 - 18 - 1 - 2 = 6$. Moreover, $Z(\mathbb{Z}/3^3\mathbb{Z}, 3) = \{\overline{3}, \overline{6}, \overline{12}, \overline{15}, \overline{21}, \overline{24}\}$. All edges of $\mathcal{H}_3(\mathbb{Z}/3^3\mathbb{Z})$ are as follows.

$\{\overline{3}, \overline{6}, \overline{12}\}, \{\overline{3}, \overline{6}, \overline{15}\}, \{\overline{3}, \overline{6}, \overline{21}\}, \{\overline{3}, \overline{6}, \overline{24}\}, \{\overline{3}, \overline{12}, \overline{15}\}, \{\overline{3}, \overline{12}, \overline{21}\}, \{\overline{3}, \overline{12}, \overline{24}\},$

$\{\overline{3}, \overline{15}, \overline{21}\}, \{\overline{3}, \overline{15}, \overline{24}\}, \{\overline{3}, \overline{21}, \overline{24}\}, \{\overline{6}, \overline{12}, \overline{15}\}, \{\overline{6}, \overline{12}, \overline{21}\}, \{\overline{6}, \overline{12}, \overline{24}\}, \{\overline{6}, \overline{15}, \overline{21}\},$

$\{\overline{6}, \overline{15}, \overline{24}\}, \{\overline{6}, \overline{21}, \overline{24}\}, \{\overline{12}, \overline{15}, \overline{21}\}, \{\overline{12}, \overline{15}, \overline{24}\}, \{\overline{12}, \overline{21}, \overline{24}\},$ and $\{\overline{15}, \overline{21}, \overline{24}\}$.

Besides $k = 3$, we give an example of complete k-uniform hypergraphs when $k = 4$.

Example 3.6. Consider $\mathbb{Z}_{81} \cong \mathbb{Z}/3^4\mathbb{Z}$. By Corollary 3.4, $\left|Z(\mathbb{Z}/3^4\mathbb{Z}, 4)\right| = \left|\mathbb{Z}/3^4\mathbb{Z}\right| - \varphi(3^4) - 1 - (3^{4-2} - 1)$. We have $\left|Z(\mathbb{Z}/3^4\mathbb{Z}, 4)\right| = 81 - 54 - 1 - 8 = 18$. Furthermore, $Z(\mathbb{Z}/3^4\mathbb{Z}, 4) = \{\overline{3}, \overline{6}, \overline{12}, \overline{15}, \overline{21}, \overline{24}, \overline{30}, \overline{33}, \overline{39}, \overline{42}, \overline{48}, \overline{51}, \overline{57}, \overline{60}, \overline{66}, \overline{69}, \overline{72}, \overline{78}\}$. The cardinality of the edge set, $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{81}))$, is $\binom{18}{4} = 3,060$. These are some examples of elements of $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{81}))$,

$\{\overline{3}, \overline{6}, \overline{12}, \overline{15}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{21}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{24}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{30}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{33}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{39}\},$

$\{\overline{3}, \overline{6}, \overline{12}, \overline{42}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{48}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{51}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{57}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{60}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{66}\},$

$\{\overline{3}, \overline{6}, \overline{12}, \overline{69}\}, \{\overline{3}, \overline{6}, \overline{12}, \overline{72}\},$ and $\{\overline{3}, \overline{6}, \overline{12}, \overline{78}\}$.

Remark. The condition $\left|Z(R/Rp^k, k)\right| \geq k$ is necessary. For examples, since $Z(\mathbb{Z}/2^2\mathbb{Z}, 2) = \varnothing$, we cannot construct the 2-uniform hypergraph $\mathcal{H}_2(\mathbb{Z}/2^2\mathbb{Z})$ (in fact, the hypergraph $\mathcal{H}$ whose vertex set is $\mathbb{Z}/2^2\mathbb{Z}$ does not exist) and since only $\overline{2}, \overline{6}$ are candidates to be 3-zero-divisors of $\mathbb{Z}/2^3\mathbb{Z}$, we cannot construct the 3-uniform hypergraph $\mathcal{H}_3(\mathbb{Z}/2^3\mathbb{Z})$.

Here, we would like to determine the diameter and the minimum length of all cycles of the constructed complete k-zero-divisor hypergraph.

Proposition 3.7. *Let R be a PID containing a prime element p and $k \geq 2$ be a fixed integer. Then, the diameter of $\mathcal{H}_k(R/Rp^k)$ is 1.*

Proof. Let $x, y \in Z(R/Rp^k, k)$ be distinct. Since $\mathcal{H}_k(R/Rp^k)$ is complete, there exists an edge E such that $x, y \in E$. A path x, E, y is obtained. Then, $d(x, y) = 1$. Therefore, $d(\mathcal{H}_k(R/Rp^k)) = 1$. $\square$

In fact, the diameter of any complete hypergraphs is 1.

Proposition 3.8. *The minimum length of all cycles of $\mathcal{H}_k(R/Rp^k)$ is*

$$\begin{cases} 0, & \text{if } \left|Z(R/Rp^k, k)\right| = k, \\ 2, & \text{if } k \geq 3 \text{ and } \left|Z(R/Rp^k, k)\right| \geq k+1, \\ 3, & \text{if } k = 2 \text{ and } \left|Z(R/Rp^2, 2)\right| \geq 3. \end{cases}$$

Proof. First, we consider the case when $\left|Z(R/Rp^k, k)\right| = k$. Since $\mathcal{H}_k(R/Rp^k)$ is a k-uniform hypergraph, $\mathcal{H}_k(R/Rp^k)$ has only one edge. Then, $\mathcal{H}_k(R/Rp^k)$ has no cycles. Therefore, the minimum length of all cycles is 0.

After that, we consider $\left|Z(R/Rp^k, k)\right| \geq k+1$. Since $\mathcal{H}_k(R/Rp^k)$ has at least two edges and $\mathcal{H}_k(R/Rp^k)$ is complete, the desired hypergraph must have at least one cycle.

Next, assume that $k \geq 3$ and $\left|Z(R/Rp^k, k)\right| \geq k+1$. Let x_1 be a vertex of $\mathcal{H}_k(R/Rp^k)$. Since $\mathcal{H}_k(R/Rp^k)$ is complete and $\left|Z(R/Rp^k, k)\right| \geq k+1$, there exist a vertex x_2 differ from x_1, and two distinct edges $E_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ and $E_2 = \{x_1, x_2, x_3', \ldots, x_k'\}$ where $x_i \neq x_i'$ for some $3 \leq i \leq k$. That is, we have a cycle $C = x_1, E_1, x_2, E_2$ of length 2 which is a possible smallest cycle. Thus, the minimum length of all cycles is 2.

Finally, assume that $k = 2$ and $\left|Z(R/Rp^2, 2)\right| \geq 3$. Let $x_1 \in Z(R/Rp^2, 2)$. Suppose that $\mathcal{H}_2(R/Rp^2)$ has a 2-cycle. There exist distinct edges E_1 and E_2 with a vertex x_2 differ from x_1 such that $x_1, x_2 \in E_1$ and $x_1, x_2 \in E_2$. Since $k = 2$, we have $E_1 = E_2$, which is a contradiction. Then, $\mathcal{H}_2(R/Rp^2)$ has no 2-cycles. Since $\left|Z(R/Rp^2, 2)\right| \geq 3$ and $\mathcal{H}_2(R/Rp^2)$ is complete, there exist distinct vertices x_2 and x_3 and three edges $E_1 = \{x_1, x_2\}$, $E_2 = \{x_1, x_3\}$ and $E_3 = \{x_2, x_3\}$, which leads to a cycle $C = x_1, E_1, x_2, E_3, x_3, E_2$ of length 3. Then, the minimum length of all cycles is 3.

$\square$

Remark. Since $Z(\mathbb{Z}/3^2\mathbb{Z}, 2) = \{\overline{3}, \overline{6}\}$, the hypergraph $\mathcal{H}_2(\mathbb{Z}/3^2\mathbb{Z})$ has only one edge which is an example of the first case in Proposition 3.8.

CHAPTER IV
COMPLETE k-PARTITE k-ZERO-DIVISOR HYPERGRAPHS

Let $k \geq 2$ be a fixed integer. Our main objective of this chapter is to find a necessary condition of a commutative ring that implies the ability to partition its set of all k-zero-divisors into k partite sets and the completeness of that k-partite k-zero-divisor hypergraph. Note that, the definition of k-uniform k-partite hypergraph used in this chapter follows from Kuhl and Schroeder [9].

First, we recall the k-uniform k-partite hypergraph. The edge set $\mathcal{E}$ of a complete k-uniform k-partite hypergraph consists of all k-subsets of V such that each of those k-subsets of V contains exactly one element from each V_j for all $1 \leq j \leq k$.

Now, throughout this chapter, let R be a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$; moreover, let $\gamma = p_1 p_2 p_3 \cdots p_k$. We consider $R/R\gamma$ and use Chinese Remainder Theorem to explain how to construct k partite sets and a complete k-partite k-zero-divisor hypergraph, respectively.

Now, we consider the ring $R/R\gamma$. Since $p_1, p_2, p_3, \ldots, p_k$ are nonassociate distinct prime elements, $p_1 + R\gamma, p_2 + R\gamma, p_3 + R\gamma, \ldots, p_k + R\gamma$ are all distinct elements. Then, for each nonzero nonunit element $p_i + R\gamma$ where $1 \leq i \leq k$, $p_1 + R\gamma, p_2 + R\gamma, p_3 + R\gamma, \ldots, p_{i-1} + R\gamma, p_{i+1} \ldots, p_k + R\gamma$ which differ from $p_i + R\gamma$ are $k-1$ distinct nonunit elements such that $(p_1 + R\gamma)(p_2 + R\gamma)(p_3 + R\gamma) \cdots (p_k + R\gamma)$ is a zero element of $R/R\gamma$, the products of all elements of $k-1$ subsets of $\{p_1 + R\gamma, p_2 + R\gamma, p_3 + R\gamma, \ldots, p_k + R\gamma\}$ are nonzero. It follows that $p_i + R\gamma$ is a k-zero-divisor of $R/R\gamma$ for all $1 \leq i \leq k$. Thus, $R/R\gamma$ has at least k k-zero-divisors. Under some conditions on $R/R\gamma$, we can partition $Z(R/R\gamma, k)$ into k partite sets.

For each $1 \leq i \leq k$, since p_i is a prime element of R , Rp_i is a prime ideal of R. By Corollary 2.27, Rp_i is a maximal ideal. By Proposition 2.10, Rp_m and Rp_n are comaximal for all $1 \leq m \neq n \leq k$. By Theorem 2.11 (Chinese Remainder Theorem), we have $R/R\gamma \cong R/Rp_1 \times R/Rp_2 \times R/Rp_3 \times \cdots \times R/Rp_k$ whose each component is a field which contains either the zero element or unit elements by Corollary 2.28 and Proposition 2.29.

We know that $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. Then, $\mathbb{R}[x]/(x^2+1)$ is an infinitely quotient ring, but the vertex set in considered hypergraph structures has to be finite. Thus, the considered quotient ring R/I needs to be finite, for example, $\mathbb{Z}/n\mathbb{Z}$.

We know that $R/R\gamma \cong R/Rp_1 \times R/Rp_2 \times R/Rp_3 \times \cdots \times R/Rp_k$. Then, we can consider each element in $R/R\gamma$ as the form $(a_1 + Rp_1, a_2 + Rp_2, a_3 + Rp_3, \ldots, a_k + Rp_k)$ where $a_i + Rp_i \in R/Rp_i$ for all $1 \leq i \leq k$.

Proposition 4.1. *Let $k \geq 2$ be a fixed integer and R be a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$. Assume that $R/R\gamma$ is finite. Then, $Z\big(R/R\gamma, k\big)$ can be partitioned into k partite sets $V_1, V_2, V_3, \ldots, V_k$, where*

$$V_i = \big\{(u_1 + Rp_1, u_2 + Rp_2, u_3 + Rp_3, \ldots, u_{i-1} + Rp_{i-1}, 0 + Rp_i, u_{i+1} + Rp_{i+1}, \ldots, u_k + Rp_k) \;\big|\; u_j + Rp_j \in U(R/Rp_j) \text{ where } j \neq i\big\} \text{ for all } 1 \leq i \leq k.$$

Proof. Since $(1 + Rp_1, 1 + Rp_2, 1 + Rp_3, \ldots, 1 + Rp_{i-1}, 0 + Rp_i, 1 + Rp_{i+1}, \ldots, 1 + Rp_k) \in V_i$ for all $1 \leq i \leq k$, $V_i \neq \varnothing$. It is clear that V_m and V_n are disjoint for any $m \neq n$ from the definition of V_i

First, we show that $\bigcup_{j=1}^{k} V_j \subseteq Z(R/R\gamma, k)$. Let $x_1 + R\gamma \in \bigcup_{j=1}^{k} V_j$. Then, $x_1 + R\gamma \in V_j$ for some $1 \leq j \leq k$. Without loss of generality, assume that $j = 1$. Thus, we can write $x_1 + R\gamma = (0 + Rp_1, u_2 + Rp_2, u_3 + Rp_3, \ldots, u_k + Rp_k)$. For each $2 \leq m \leq k$, we can choose any $x_m \in V_m$ such that $(x_1 + R\gamma)(x_2 + R\gamma)(x_3 + R\gamma) \cdots (x_k + R\gamma) = (0 + Rp_1, 0 + Rp_2, 0 + Rp_3, \ldots, 0 + Rp_k)$. Next, we delete one element from $\{x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_k + R\gamma\}$. Without loss of generality, we delete $x_k + R\gamma$. Then, $(x_1 + R\gamma)(x_2 + R\gamma)(x_3 + R\gamma) \cdots (x_{k-1} + R\gamma) = (0 + Rp_1, 0 + Rp_2, 0 + Rp_3, \ldots, 0 + Rp_{k-1}, a + Rp_k)$ where $a + Rp_k = \prod_{l=1}^{k-1}(u_l + Rp_k) \neq 0 + Rp_k$. We can conclude that $\bigcup_{j=1}^{k} V_j \subseteq Z(R/R\gamma, k)$.

Next, we need to show that $Z(R/R\gamma, k) \subseteq \bigcup_{j=1}^{k} V_j$. Let $x_1+R\gamma \in Z(R/R\gamma, k)$. Then,

(i) $x_1 + R\gamma$ is a nonzero and nonunit element.

(ii) there exist distinct nonzero nonunit elements $x_2+R\gamma, x_3+R\gamma, x_4+R\gamma, \ldots, x_k+R\gamma \in R/R\gamma$ which differ from $x_1 + R\gamma$ such that $(x_1 + R\gamma)(x_2 + R\gamma)(x_3 + R\gamma)\cdots(x_k + R\gamma) = 0 + R\gamma$ and the products of all elements of any $(k-1)$-subsets of $\{x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_k + R\gamma\}$ are nonzero.

Since $x_1 + R\gamma \in R/R\gamma \cong R/Rp_1 \times R/Rp_2 \times R/Rp_3 \times \cdots \times R/Rp_k$, we write $x_1+R\gamma = (a_{1,1}+Rp_1, a_{1,2}+Rp_2, a_{1,3}+Rp_3, \ldots, a_{1,k}+Rp_k)$ where $a_{1,s}+Rp_s \in R/Rp_s$ for all $1 \leq s \leq k$. For each $1 \leq j \leq k$, since R/Rp_j is a field, $a_{1,j} + Rp_j$ is the zero element or a unit element of R/Rp_j. Since $x_1 + R\gamma$ is not a unit element, there exist $1 \leq m \leq k$ such that $a_{1,m} + Rp_m = 0 + Rp_m$. Without loss of generality, let $m = 1$

Case 1. $a_{1,1} + Rp_1$ is the unique element in $\{a_{1,1} + Rp_1, a_{1,2} + Rp_2, a_{1,3} + Rp_3, \ldots, a_{1,k} + Rp_k\}$ such that $a_{1,1} + Rp_1 = 0 + Rp_1$. Then, $x_1 + R\gamma = (0 + Rp_1, a_{1,2} + Rp_2, a_{1,3} + Rp_3, \ldots, a_{1,k} + Rp_k) \in V_1$.

Case 2. there exist $2 \leq l \leq k$ such that $a_{1,l} + Rp_l = 0 + Rp_l$. Without loss of generality, $l = 2$. Thus, $x_1+R\gamma = (0+Rp_1, 0+Rp_2, a_{1,3}+Rp_3, \ldots, a_{1,k}+Rp_k)$. We have for each $2 \leq q \leq k$, $x_q+R\gamma = (a_{q,1}+Rp_1, a_{q,2}+Rp_2, a_{q,3}+Rp_3, \ldots, a_{q,k}+Rp_k)$ for all $1 \leq s \leq k, a_{q,s} + Rp_s \in R/Rp_s$. Since $x_q + R\gamma$ is not a unit element, there exist $1 \leq m_q \leq k$ such that $a_{1,m_q} + Rp_{m_q} = 0 + Rp_{m_q}$. Then, For each $3 \leq s \leq k$, there exist at least one element of $\{x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_k + R\gamma\}$ having an entry which is the zero element in the s^{th}-component. The maximum number of distinct chosen elements which have desired properties in such $k-2$ components is $k-2$, say $x_2 + R\gamma, x_3 + R\gamma, x_4 + R\gamma \ldots, x_{k-1} + R\gamma$. Therefore, $(x_1+R\gamma)(x_2+R\gamma)(x_3+R\gamma)\cdots(x_{k-1}+R\gamma) = (0+Rp_1, 0+Rp_2, 0+Rp_3, \ldots, 0+Rp_k)$ which is a contradiction.

Hence, Case 1 is the only possible case. Thus, $x_1 + R\gamma \in \bigcup_{i=1}^{k} V_i$. Therefore, $Z(R/R\gamma, k)$ can be partitioned into k partite sets. $\square$

Finally, we can use the partite sets obtained from Proposition 4.1 to construct our k-partite k-zero-divisor hypergraph of the ring $R/R\gamma$ via its k-zero-divisors. Under the same assumptions as in Proposition 4.1, we can also prove that our constructed k-uniform k-partite k-zero-divisor hypergraph is complete.

Proposition 4.2. *Let $k \geq 2$ be a fixed integer and R be a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$. Assume that $R/R\gamma$ is finite. Then, we can construct a complete k-partite k-zero-divisor hypergraph whose vertex set is $Z\big(R/R\gamma, k\big)$.*

Proof. By Proposition 4.1, we can partition $Z\big(R/R\gamma, k\big)$ into k partite sets V_1, V_2, V_3, …, V_k. Then, without loss of generality, consider element $x_1 + R\gamma \in V_1$, we can find $x_2 + R\gamma \in V_2, x_3 + R\gamma \in V_3, x_4 + R\gamma \in V_4, \ldots, x_k + R\gamma \in V_k$ which cause $(x_1 + R\gamma)(x_2 + R\gamma)(x_3 + R\gamma)\cdots(x_k + R\gamma) = (0 + Rp_1, 0 + Rp_2, 0 + Rp_3, \ldots, 0 + Rp_k)$ and the product of all elements of $k-1$ subsets of $\{x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_k + R\gamma\}$ are nonzero. Therefore, $\{x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_k + R\gamma\}$ is an edge, that is, we can connect k elements from k partite sets. Next, we prove that no k-subsets in each partite set V_i form an edge.

Fix $1 \leq i \leq k$. Let $x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_k + R\gamma \in V_i$. Hence, $x_j + R\gamma = (u_{j,1} + Rp_1, u_{j,2} + Rp_2, u_{j,3} + Rp_3, \ldots, u_{j,i-1} + Rp_{i-1}, 0 + Rp_i, u_{j,i+1} + Rp_{i+1}, \ldots, u_{j,k} + Rp_k)$ where $u_{j,r} + Rp_r \in U(R/Rp_r)$ for each $1 \leq j \leq k, 1 \leq r \leq k$ and then, $(x_1 + R\gamma)(x_2 + R\gamma)(x_3 + R\gamma)\cdots(x_k + R\gamma) = (a_1 + Rp_1, a_2 + Rp_2, a_3 + Rp_3, \ldots, a_{i-1} + Rp_{i-1}, 0 + Rp_i, a_{i+1} + Rp_{i+1}, \ldots, a_k + Rp_k)$ where $a_m + Rp_m = \prod_{s=1}^{k}(u_{s,m} + Rp_m) \neq 0 + Rp_m$ for all $1 \leq m \neq i \leq k$. Thus, $\prod_{i=1}^{k}(x_i + R\gamma) \neq (0 + Rp_1, 0 + Rp_2, 0 + Rp_3, \ldots, 0 + Rp_{i-1}, 0 + Rp_i, 0 + Rp_{i+1}, \ldots, 0 + Rp_k)$. Then, this hypergraph has no edge containing any k elements of V_i. Now, we have a k-partite k-zero-divisor hypergraph $\mathcal{H}_k(R/R\gamma)$ whose vertex set is $Z\big(R/R\gamma, k\big)$ and each edge is of the form $\{x_1 + R\gamma,\ x_2 + R\gamma,\ x_3 + R\gamma, \ldots, x_k + R\gamma\}$, where $x_j + R\gamma \in V_j$ for all $1 \leq j \leq k$.

It remains to show that $\mathcal{H}_k(R/R\gamma)$ is complete. By the proof of Proposition 4.1, for each element $x_i + R\gamma$ in V_i with $1 \leq i \leq k$, there exists an element $x_j + R\gamma$ from each partite set V_j for which $1 \leq j \leq k$ and $j \neq i$ such that $\{x_1 + R\gamma, x_2 + R\gamma, x_3 + R\gamma, \ldots, x_i + R\gamma, \ldots, x_k + R\gamma\}$ is an edge of $\mathcal{H}$. Since V_i and V_j are disjoint sets when $i \neq j$, such edge contains exactly one element from each V_i with $1 \leq i \leq k$. Therefore, we obtain a complete k-partite k-zero-divisor hypergraph. □

In summary, we rephrase Proposition 4.2 and obtain the following result:

Theorem 4.3. *If k is a positive integer greater than 1 and R is a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$, such that $R/R\gamma$ is finite where $\gamma = \prod_{i=1}^{k} p_i$, then there exists a complete k-partite k-zero-divisor hypergraph whose vertex set is $Z\big(R/R\gamma, k\big)$.*

Finally, we provide examples of finite commutative rings that imply a complete 3-partite 3-zero-divisor hypergraph as follows.

Example 4.4. Consider the ring $\mathbb{Z}_{30} \cong \mathbb{Z}/(2 \cdot 3 \cdot 5)\mathbb{Z}$. By Theorem 2.11 (Chinese Remainder Theorem), $\mathbb{Z}/(2 \cdot 3 \cdot 5)\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. From Proposition 4.1 and $\big|U(\mathbb{Z}/p_i\mathbb{Z})\big| = \big|\mathbb{Z}/p_i\mathbb{Z}\big| - 1$, we obtain

$V_1 = \{(\overline{0}, \overline{u_2}, \overline{u_3}) \mid \overline{0} \in \mathbb{Z}/2\mathbb{Z}, \overline{u_2} \in U(\mathbb{Z}/3\mathbb{Z}), \overline{u_3} \in U(\mathbb{Z}/5\mathbb{Z})\}$ with $\big|V_1\big| = 1 \cdot (3-1) \cdot (5-1) = 8$,

$V_2 = \{(\overline{u_1}, \overline{0}, \overline{u_3}) \mid \overline{0} \in \mathbb{Z}/3\mathbb{Z}, \overline{u_1} \in U(\mathbb{Z}/2\mathbb{Z}), \overline{u_3} \in U(\mathbb{Z}/5\mathbb{Z})\}$ with $\big|V_2\big| = (2-1) \cdot 1 \cdot (5-1) = 4$,

$V_3 = \{(\overline{u_1}, \overline{u_2}, \overline{0}) \mid \overline{0} \in \mathbb{Z}/5\mathbb{Z}, \overline{u_1} \in U(\mathbb{Z}/2\mathbb{Z}), \overline{u_2} \in U(\mathbb{Z}/3\mathbb{Z})\}$ with $\big|V_3\big| = (2-1) \cdot (3-1) \cdot 1 = 2$.

In fact, $V_1 = \{(\overline{0}, \overline{1}, \overline{1}), (\overline{0}, \overline{1}, \overline{2}), (\overline{0}, \overline{1}, \overline{3}), (\overline{0}, \overline{1}, \overline{4}), (\overline{0}, \overline{2}, \overline{1}), (\overline{0}, \overline{2}, \overline{2}), (\overline{0}, \overline{2}, \overline{3}), (\overline{0}, \overline{2}, \overline{4})\} \cong \{\overline{2}, \overline{4}, \overline{8}, \overline{14}, \overline{16}, \overline{22}, \overline{26}, \overline{28}\}$, $V_2 = \{(\overline{1}, \overline{0}, \overline{1}), (\overline{1}, \overline{0}, \overline{2}), (\overline{1}, \overline{0}, \overline{3}), (\overline{1}, \overline{0}, \overline{4})\} \cong \{\overline{3}, \overline{9}, \overline{21}, \overline{27}\}$ and $V_3 = \{(\overline{1}, \overline{1}, \overline{0}), (\overline{1}, \overline{2}, \overline{0})\} \cong \{\overline{5}, \overline{25}\}$. Then, there exists a complete 3-partite 3-zero-divisor hypergraph whose vertex set is $Z\big(\mathbb{Z}/(2 \cdot 3 \cdot 5)\mathbb{Z}, 3\big) = V_1 \cup V_2 \cup V_3$ and edges are as follows.

$\{\overline{2}, \overline{3}, \overline{5}\}, \{\overline{2}, \overline{3}, \overline{25}\}, \{\overline{2}, \overline{9}, \overline{5}\}, \{\overline{2}, \overline{9}, \overline{25}\}, \{\overline{2}, \overline{21}, \overline{5}\}, \{\overline{2}, \overline{21}, \overline{25}\}, \{\overline{2}, \overline{27}, \overline{5}\}, \{\overline{2}, \overline{27}, \overline{25}\},$

$\{\overline{4},\overline{3},\overline{5}\},\{\overline{4},\overline{3},\overline{25}\},\{\overline{4},\overline{9},\overline{5}\},\{\overline{4},\overline{9},\overline{25}\},\{\overline{4},\overline{21},\overline{5}\},\{\overline{4},\overline{21},\overline{25}\},\{\overline{4},\overline{27},\overline{5}\},\{\overline{4},\overline{27},\overline{25}\},$

$\{\overline{8},\overline{3},\overline{5}\},\{\overline{8},\overline{3},\overline{25}\},\{\overline{8},\overline{9},\overline{5}\},\{\overline{8},\overline{9},\overline{25}\},\{\overline{8},\overline{21},\overline{5}\},\{\overline{8},\overline{21},\overline{25}\},\{\overline{8},\overline{27},\overline{5}\},\{\overline{8},\overline{27},\overline{25}\},$

$\{\overline{14},\overline{3},\overline{5}\},\{\overline{14},\overline{3},\overline{25}\},\{\overline{14},\overline{9},\overline{5}\},\{\overline{14},\overline{9},\overline{25}\},\{\overline{14},\overline{21},\overline{5}\},\{\overline{14},\overline{21},\overline{25}\},\{\overline{14},\overline{27},\overline{5}\},$

$\{\overline{14},\overline{27},\overline{25}\},\{\overline{16},\overline{3},\overline{5}\},\{\overline{16},\overline{3},\overline{25}\},\{\overline{16},\overline{9},\overline{5}\},\{\overline{16},\overline{9},\overline{25}\},\{\overline{16},\overline{21},\overline{5}\},\{\overline{16},\overline{21},\overline{25}\},$

$\{\overline{16},\overline{27},\overline{5}\},\{\overline{16},\overline{27},\overline{25}\},\{\overline{22},\overline{3},\overline{5}\},\{\overline{22},\overline{3},\overline{25}\},\{\overline{22},\overline{9},\overline{5}\},\{\overline{22},\overline{9},\overline{25}\},\{\overline{22},\overline{21},\overline{5}\},$

$\{\overline{22},\overline{21},\overline{25}\},\{\overline{22},\overline{27},\overline{5}\},\{\overline{22},\overline{27},\overline{25}\},\{\overline{26},\overline{3},\overline{5}\},\{\overline{26},\overline{3},\overline{25}\},\{\overline{26},\overline{9},\overline{5}\},\{\overline{26},\overline{9},\overline{25}\},$

$\{\overline{26},\overline{21},\overline{5}\},\{\overline{26},\overline{21},\overline{25}\},\{\overline{26},\overline{27},\overline{5}\},\{\overline{26},\overline{27},\overline{25}\},\{\overline{28},\overline{3},\overline{5}\},\{\overline{28},\overline{3},\overline{25}\},\{\overline{28},\overline{9},\overline{5}\},$

$\{\overline{28},\overline{9},\overline{25}\},\{\overline{28},\overline{21},\overline{5}\},\{\overline{28},\overline{21},\overline{25}\},\{\overline{28},\overline{27},\overline{5}\},$ and $\{\overline{28},\overline{27},\overline{25}\}$.

Besides the examples of the integer modulo n ($\mathbb{Z}_n$), we give the example of the set of all Gaussian integers, $\mathbb{Z}[i]$. Recall that the Gaussian integer $\mathbb{Z}[i]$ is a PID, which has a set of prime elements of the form up, where u is a unit of $\mathbb{Z}[i]$ and p is a prime element of $\mathbb{Z}$ such that $p \equiv 3 \pmod 4$. Since there are infinitely many primes p such that $p \equiv 3 \pmod 4$, we can find k prime numbers $p_1, p_2, p_3, \ldots, p_k \in \mathbb{Z}$ where $p_i \equiv 3 \pmod 4$ for all $k \geq 2, 1 \leq i \leq k$. Then, this ring has at least k nonassociate distinct primes , see [5]. If $u = 1$ and $\beta = p_1p_2p_3\cdots p_k$, where p_j are all nonassociate distinct primes such that $p_j \equiv 3 \pmod 4$ for all $1 \leq j \leq k$, then we can see that $\mathbb{Z}[i]/\beta\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}_\beta\}$ is finite.

Example 4.5. Consider $\mathbb{Z}[i]/(-231i)\mathbb{Z}[i] \cong \mathbb{Z}[i]/\big(3i \cdot 7i \cdot 11i\big)\mathbb{Z}[i]$. By Theorem 2.11 (Chinese Remainder Theorem), $\mathbb{Z}[i]/\big(3i \cdot 7i \cdot 11i\big)\mathbb{Z}[i] \cong \mathbb{Z}[i]/(3i)\mathbb{Z}[i] \times \mathbb{Z}[i]/(7i)\mathbb{Z}[i] \times \mathbb{Z}[i]/(11i)\mathbb{Z}[i]$. By Proposition 4.1 and same idea as Example 4.4, we obtain

$V_1 = \{(\overline{0}, \overline{u_2} + \overline{u_2'}i, \overline{u_3} + \overline{u_3'}i) \mid \overline{0} \in \mathbb{Z}[i]/(3i)\mathbb{Z}[i], \overline{u_2} + \overline{u_2'}i \in U(\mathbb{Z}[i]/(7i)\mathbb{Z}[i]), \overline{u_3} +$

$\overline{u_3'}i \in U(\mathbb{Z}[i]/(11i)\mathbb{Z}[i])\}$ with $|V_1| = 1\cdot\big((7-1)\cdot(7-1)\big)\cdot\big((11-1)\cdot(11-1)\big) = 3,600$,

$V_2 = \{(\overline{u_1}+\overline{u_1'}i, \overline{0}, \overline{u_3}+\overline{u'3}i) \mid \overline{0} \in \mathbb{Z}[i]/(7i)\mathbb{Z}[i], \overline{u_1}+\overline{u_1'}i \in U(\mathbb{Z}[i]/(3i)\mathbb{Z}[i]), \overline{u_3}+ \overline{u_3'}i \in U(\mathbb{Z}[i]/(11i)\mathbb{Z}[i])\}$ with $|V_2| = 1\cdot\big((3-1)\cdot(3-1)\big)\cdot\big((11-1)\cdot(11-1)\big) = 400$,

$V_3 = \{(\overline{u_1}+\overline{u_1'}i, \overline{u_2}+\overline{u_2'}i, \overline{0}) \mid \overline{0} \in \mathbb{Z}[i]/(11i)\mathbb{Z}[i], \overline{u_1}+\overline{u_1'}i \in U(\mathbb{Z}[i]/(3i)\mathbb{Z}[i]), \overline{u_2}+ \overline{u_2'}i \in U(\mathbb{Z}[i]/(7i)\mathbb{Z}[i])\}$ with $|V_3| = 1\cdot\big((3-1)\cdot(3-1)\big)\cdot\big((7-1)\cdot(7-1)\big) = 144$.

In fact,

$V_1 \cong \big\{3a+3bi \mid a \in \mathbb{Z}_{77} - (7\mathbb{Z}_{77} \cup 11\mathbb{Z}_{77}), b \in \mathbb{Z}_{77} - (7\mathbb{Z}_{77} \cup 11\mathbb{Z}_{77})\big\}$,

$V_2 \cong \big\{7a+7bi \mid a \in \mathbb{Z}_{33} - (3\mathbb{Z}_{33} \cup 11\mathbb{Z}_{33}), b \in \mathbb{Z}_{33} - (3\mathbb{Z}_{33} \cup 11\mathbb{Z}_{33})\big\}$,

$V_3 \cong \big\{11a+11bi \mid a \in \mathbb{Z}_{21} - (3\mathbb{Z}_{21} \cup 7\mathbb{Z}_{21}), b \in \mathbb{Z}_{21} - (3\mathbb{Z}_{21} \cup 7\mathbb{Z}_{21})\big\}$.

The cardinality of edge set, $\mathcal{E}(\mathcal{H}_3(\mathbb{Z}[i]/(-231i)\mathbb{Z}[i]))$, is $3,600 \times 400 \times 144 = 207,360,000$. These are some edges of $\mathcal{H}_3(\mathbb{Z}[i]/(-231i)\mathbb{Z}[i])$,

$\{3,7,11\}$, $\{3,-7,11\}$, $\{3,7i,11\}$, $\{3,-7i,11\}$, $\{3,7+7i,11\}$, $\{3,7-7i,11\}$,

$\{3,-7+7i,11\}$, $\{3,-7-7i,11\}$, $\{3,14,11\}$, $\{3,-14,11\}$, $\{3,14i,11\}$,

$\{3,-14i,11\}$, $\{3,7+14i,11\}$, $\{3,7-14i,11\}$, $\{3,-7+14i,11\}$, $\{3,-7-14i,11\}$,

$\{3,14+7i,11\}$, $\{3,14-7i,11\}$, $\{3,-14+7i,11\}$, and $\{3,-14-7i,11\}$.

Finally, we compute the diameter of $\mathcal{H}_k(R/R\gamma)$ and the minimum length of its cycles.

Proposition 4.6. *Let $k \geq 2$ be a fixed integer and R be a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$. Then, the diameter of $\mathcal{H}_k(R/R\gamma)$ is 2.*

Proof. Let x and y be distinct vertices of $\mathcal{H}_k(R/R\gamma)$.

Case 1. x and y are in different partite sets. By the completeness of such hypergraph, there exists an edge E such that $x, y \in E$. Then, x, E, y forms a path. As a result, the distance between x and y is 1.

Case 2. x and y are in the same partite set. By the definition of $\mathcal{H}_k(R/R\gamma)$, there is no edge E such that $x, y \in E$. However, there exist two distinct edges E_1 and E_2 such that $x \in E_1$ and $y \in E_2$. Since $\mathcal{H}_k(R/R\gamma)$ is complete, there exists a vertex v in the other partite sets such that $v \in E_1 \cap E_2$. Then, a path x, E_1, v, E_2, y is obtained. Therefore, $d(x, y) = 2$.

From these two cases, the maximum distance between x and y is 2, that is, the diameter of $\mathcal{H}_k(R/R\gamma)$ is 2. □

We need to seperate the possible cases to find the minimum length of all cycles in $\mathcal{H}_k(R/R\gamma)$ into two cases by size of $Z(R/R\gamma)$.

Proposition 4.7. *The minimum length of all cycles in $\mathcal{H}_k(R/R\gamma)$ is 0 when $\left|Z(R/R\gamma, k)\right| = k$.*

Proof. Since $\mathcal{H}_k(R/R\gamma)$ has only one edge, it has no cycles. □

Next, we split the second possible case into two cases with $k = 2$ and $k \geq 3$.

Proposition 4.8. *The minimum length of all cycles in $\mathcal{H}_k(R/R\gamma)$ is 2 when $k \geq 3$ and $\left|Z(R/R\gamma, k)\right| \geq k + 1$.*

Proof. Since each partite set is a nonempty set, there exists at least one element in each set. Since $\left|Z(R/R\gamma, k)\right| \geq k + 1$ and $\mathcal{H}_k(R/R\gamma)$ has k partite sets, there are at least two elements x_1 and x_1' in one of the partite sets, say V_1. By the completeness of $\mathcal{H}_k(R/R\gamma)$, there exist two distinct edges $E_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ and $E_2 = \{x_1', x_2, x_3, \ldots, x_k\}$. Thus, $\mathcal{H}_k(R/R\gamma)$ has at least one cycle. One of the constructed cycles is a cycle $C = x_2, E_1, x_3, E_2$ of length 2. Then, the minimum length of all cycles is 2. □

Proposition 4.9. *Assume that $\left|Z(R/R\gamma, 2)\right| \geq 3$. The minimum length of all cycles in $\mathcal{H}_2(R/R\gamma)$ is*

$$\begin{cases} 0, & \text{if there exists one partite set that has only one element,} \\ 4, & \text{if each partite set has more than one element.} \end{cases}$$

Proof. We consider all possible cases in each partite set.

Case 1. Let V_1 be a partite set such that $\left|V_1\right| = 1$. Suppose that $\mathcal{H}_2(R/R\gamma)$ has a cycle, say $x_1, E_1, x_2, E_2, x_3, \ldots, x_{r-1}, E_r$ where $r \geq 2$.

If $x_1 \in V_1$, then from a cycle C, we have $x_3 = x_1$. Since $k = 2$, $E_1 = \{x_1, x_2\}$ and $E_2 = \{x_2, x_3\}$. Since $x_1 = x_3$, $E_1 = E_2$, which is a contradiction.

If x_1 is in another partite set rather than V_1, then we have $x_2 \in V_1$. By the same argument, we obtain $E_2 = E_3$ which is also a contradiction.

Therefore, $\mathcal{H}_2(R/R\gamma)$ has no cycles.

Case 2. Each partite set has more than one element. Since $\mathcal{H}_2(R/R\gamma)$ is complete, there are at least one cycle. In this case, we have $\left|Z(R/R\gamma, 2)\right| \geq 4$. Let x_1 be a vertex of $\mathcal{H}_2(R/R\gamma)$ and V_1 be a partite set containing x_1. Suppose that $\mathcal{H}_2(R/R\gamma)$ has 2-cycle. There exist two distinct edges E_1 and E_2 and a vertex x_2 differ from x_1 such that $x_1, x_2 \in E_1$ and $x_1, x_2 \in E_2$. Since $k = 2$, we have $E_1 = E_2$, which is a contradiction. Thus, $\mathcal{H}_2(R/R\gamma)$ does not have any 2-cycles. By the definition of 2-partite hypergraph in Definition 2.38, it is a bipartite graph. We know that bipartite graphs have no odd cycles, see [11]. Since $k = 2$, $\left|Z(R/R\gamma, 2)\right| \geq 4$ and the completeness of $\mathcal{H}_2(R/R\gamma)$, there exist three distinct vertices x_2, x_3 and x_4 with $x_2 \in V_1$ and x_3, x_4 are in other partite set such that $E_1 = \{x_1, x_3\}$, $E_2 = \{x_1, x_4\}$, $E_3 = \{x_2, x_3\}$ and $E_4 = \{x_2, x_4\}$ form edges. We have a cycle $C = x_1, E_1, x_3, E_3, x_2, E_4, x_4, E_2$ of length 4. Then, the minimum of length of all cycles is 4. $\square$

CHAPTER V

k-PARTITE σ-ZERO-DIVISOR HYPERGRAPHS

First of all, let $k \geq 2$ be a fixed integer. Throughout this chapter, let R be a PID with at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$ and let $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Then, we consider $R/Rp_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}$ and its $\sum_{i=1}^{k}\alpha_i$-zero-divisors are considered. Throughout this chapter, let $\sigma = \sum_{i=1}^{k}\alpha_i$ and $\pi = \prod_{i=1}^{k} p_i^{\alpha_i}$.

The definition of σ-partite hypergraph used in this chapter differs from the definition used in Chapter IV. From Jirimutu and Wang [8], a k-partite σ-uniform hypergraph with σ is a fixed integer greater or equal k consists of the vertex set V partitioned into k subsets $V_1, V_2, V_3, \ldots, V_k$ and E is an edge if $|E| = \sigma$ and $|E \cap V_i| < \sigma$ for all $1 \leq i \leq k$ and there exist $1 \leq m \neq n \leq k$ such that $E \cap V_m \neq \varnothing$ and $E \cap V_n \neq \varnothing$. With this definition, each edge of such k-partite hypergraph can have more or equal one element from some partite sets, but no more than σ elements from such partite sets.

We can generalize Proposition 2.5 to the following propositions.

Proposition 5.1. *We have*

$$(Rp_i/R\pi) - \bigcup_{j=1, j\neq i}^{k} (Rp_j/R\pi) = \left\{ a + R\pi \,\middle|\, a \in Rp_i - \bigcup_{j=1, j\neq i}^{k} Rp_j \right\}.$$

Next, we use the idea of Proposition 3.1 to obtain the following result.

Proposition 5.2. *We have*

$$(Rp_i/R\pi) - \left((Rp_i^2/R\pi) \cup \bigcup_{j=1, j\neq i}^{k} (Rp_i/R\pi)\right)$$
$$= \left\{ a + R\pi \,\middle|\, a \in Rp_i - \left(Rp_i^2 \cup \bigcup_{j=1, j\neq i}^{k} Rp_j\right)\right\}.$$

Next, under some conditions on $R/R\pi$, we can separate the set of all σ-zero-divisors of $R/R\pi$ into k partite sets.

Proposition 5.3. *Let $k \geq 2$ be a fixed integer and R be a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$. Assume that $R/R\pi$ is finite and for all $1 \leq i \leq k$, $\left|(Rp_i/R\pi) - \left((Rp_j^2/R\pi) \cup \bigcup_{j=1, j\neq i}^{k}(Rp_j/R\pi)\right)\right| \geq \alpha_i$. Then, $Z(R/R\pi, \sigma) = \bigcup_{j=1}^{k} V_j$ where*

$$V_j = \begin{cases} (Rp_j/R\pi) - \displaystyle\bigcup_{l=1, l\neq j}^{k} (Rp_l/R\pi), & \text{if } \alpha_j = 1, \\ (Rp_j/R\pi) - \left((Rp_j^2/R\pi) \cup \displaystyle\bigcup_{l=1, l\neq j}^{k} Rp_l/R\pi\right), & \text{if } \alpha_j \geq 2, \end{cases}$$

and $\{V_j \mid 1 \leq j \leq k\}$ is mutually disjoint.

Proof. Let $x_1 + R\pi \in \bigcup_{j=1}^{k} V_j$. Then, $x_1 + R\pi \in V_j$ for some $1 \leq j \leq k$. Without loss of generality, assume that $j = 1$.

Case 1. Assume that $\alpha_1 = 1$. Then, $x_1 \in Rp_1 - \bigcup_{l=1, l\neq j}^{k} Rp_l$.

We choose $x_2 + R\pi, x_3 + R\pi, x_4 + R\pi, \ldots, x_{\alpha_2+1} + R\pi \in V_2$,

$x_{\alpha_2+2} + R\pi, x_{\alpha_2+3} + R\pi, x_{\alpha_2+4} + R\pi, \ldots, x_{\alpha_2+\alpha_3+1} + R\pi \in V_3, \ldots,$

$x_{\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+2} + R\pi, x_{\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+3} + R\pi, x_{\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+4} + R\pi, \ldots,$

$x_{\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+\alpha_k+1} + R\pi \in V_k$.

For each $2 \leq \beta \leq k$, since $x_{j_\beta} + R\pi \in V_\beta$ for all $\alpha_2 + \alpha_3 + \cdots + \alpha_{\beta-1} + 2 \leq j_\beta \leq \alpha_2 + \alpha_3 + \cdots + \alpha_\beta + 1$, we obtain

$$x_{j_\beta} \in Rp_\beta - \bigcup_{l=1, l\neq\beta}^{k} Rp_l \text{ or } x_{j_\beta} \in Rp_\beta - \left(Rp_\beta^2 \cup \bigcup_{l=1, l\neq\beta}^{k} Rp_l\right).$$

Since $x_1 \in Rp_1$, $\prod_{m_2=2}^{\alpha_2+1} x_{m_2} \in Rp_2^{\alpha_2}$, $\ldots$, $\prod_{m_k=\sum_{m'=2}^{k-1}\alpha_{m'}+2}^{\sigma} x_{m_k} \in Rp_k^{\alpha_k}$, we have $(x_1 + R\pi)(x_2 + R\pi)(x_3 + R\pi)\cdots(x_\sigma + R\pi) = \prod_{m=1}^{\sigma} x_m + R\pi = 0 + R\pi$.

After that, we delete one element $x_s + R\pi$ for some $1 \leq s \leq \sigma$. Without loss of generality, assume that $x_s + R\pi \in V_{s'}$ for some $1 \leq s' \leq k$.

If $x_s + R\pi \in V_1$, then $x_s + R\pi = x_1 + R\pi$ and $p_1 \nmid \prod_{l_2=2}^{\sigma} x_{l_2}$. Thus, $\prod_{l_2=2}^{\sum_{i=2}^{k} \alpha_i + 1} x_{l_2} \notin R\pi$. Thus, $\prod_{l_2=2}^{\sigma} x_{l_2} + R\pi \neq 0 + R\pi$.

If $x_s + R\pi \in V_t$ for some $t \neq 1$, in either case of V_t, we have $p_t^{\alpha_t - 1} \mid \prod_{l=1, l\neq s}^{\sigma} x_l$, but $p_t^{\alpha_t} \nmid \prod_{l=1, l\neq s}^{\sigma} x_l$. Thus, $\prod_{l=1, l\neq s}^{\sigma} x_l \notin R\pi$ and $(x_1 + R\pi)(x_2 + R\pi)(x_3 + R\pi) \cdots (x_{s-1} + R\pi)(x_{s+1} + R\pi) \cdots (x_\sigma + R\pi) = \prod_{l=1, l\neq s}^{\sigma} x_l + R\pi \neq 0 + R\pi$.

That is, $x_1 + R\pi$ is a σ-zero-divisor of $R/R\pi$.

Case 2. Assume that $\alpha_1 \geq 2$. Then, $x_1 \in Rp_1 - [Rp_1^2 \cup \bigcup_{l=1, l\neq j}^{k} Rp_l]$.

We choose $x_2 + R\pi, x_3 + R\pi, x_4 + R\pi, \ldots, x_{\alpha_1} + R\pi \in V_1$,

$x_{\alpha_1+1} + R\pi, x_{\alpha_1+2} + R\pi, x_{\alpha_1+3} + R\pi, \ldots, x_{\alpha_1+\alpha_2} + R\pi \in V_2$,

$x_{\alpha_1+\alpha_2+1} + R\pi, x_{\alpha_1+\alpha_2+2} + R\pi, x_{\alpha_1+\alpha_2+3} + R\pi, \ldots, x_{\alpha_1+\alpha_2+\alpha_3} + R\pi \in V_3, \ldots,$

$x_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+1} + R\pi, x_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+2} + R\pi, x_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+3} + R\pi,$

$\ldots, x_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{k-1}+\alpha_k} + R\pi \in V_k$.

For each $2 \leq \beta \leq k$, since $x_{j_\beta} + R\pi \in V_\beta$ for all $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{\beta-1} + 1 \leq j_\beta \leq \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_\beta$, we obtain $x_{j_\beta} \in Rp_\beta - \bigcup_{l=1, l\neq \beta}^{k} Rp_l$ or $a_{j_\beta} \in Rp_\beta - \left(Rp_\beta^2 \cup \bigcup_{l=1, l\neq \beta}^{k} Rp_l\right)$.

Since $\prod_{m_1=1}^{\alpha_1} x_{m_1} \in Rp_1^{\alpha_1}, \prod_{m_2=\alpha_1+1}^{\alpha_1+\alpha_2} x_{m_2} \in Rp_2^{\alpha_2}, \ldots, \prod_{m_k=\sum_{m'=1}^{k-1} \alpha_{m'}+1}^{\sigma} x_{m_k} \in Rp_k^{\alpha_k}$, we have $(x_1 + R\pi)(x_2 + R\pi)(x_3 + R\pi) \cdots (x_\sigma + R\pi) = \prod_{m=1}^{\sigma} x_m + R\pi = 0 + R\pi$.

After that, we delete one element $x_s + R\pi$ for some $1 \leq s \leq \sigma$. Without loss of generality, assume that $x_s + R\pi \in V_t$ for some $1 \leq t \leq k$. In either case of V_t, we have $p_t^{\alpha_t - 1} \mid \prod_{l=1, l\neq s}^{\sigma} x_l$, but $p_t^{\alpha_t} \nmid \prod_{l=1, l\neq s}^{\sigma} x_l$. Thus, $\prod_{l=1, l\neq s}^{\sigma} x_l \notin R\pi$ and $(x_1 + R\pi)(x_2 + R\pi)(x_3 + R\pi) \cdots (x_{s-1} + R\pi)(x_{s+1} + R\pi) \cdots (x_\sigma + R\pi) = \prod_{l=1, l\neq s}^{\sigma} x_l + R\pi \neq 0 + R\pi$. We can conclude that $\bigcup_{j=1}^{k} V_j \subseteq Z(R/R\pi, \sigma)$.

On the other hand, let $x_1 + R\pi \in Z(R/R\pi, \sigma)$. Then,

(i) $x_1 + R\pi \neq 0 + R\pi$ and $x_1 + R\pi \notin U(R/R\pi)$.

(ii) there exist nonzero nonunit distinct elements $x_2 + R\pi, x_3 + R\pi, x_4 + R\pi, \ldots, x_\sigma + R\pi \in R/R\pi$ which differ from $x_1 + R\pi$ such that $(x_1 + R\pi)(x_2 + R\pi)$

$(x_3+R\pi)\cdots(x_\sigma+R\pi)=\prod_{m=1}^{\sigma}x_m+R\pi=0+R\pi$, and the products of all elements of any $(\sigma-1)$-subsets of $\{x_1+R\pi, x_2+R\pi, x_3+R\pi, \ldots, x_\sigma+R\pi\}$ are nonzero.

Since $x_1+R\pi$ is not a unit of $R/R\pi$, by the consequence of Proposition 2.19, $\gcd(x_1,\pi)\neq 1$. There exists $d_1\in R-\{1\}$ such that $\gcd(x_1,\pi)=d_1$. Therefore, we consider the possible cases for d_1 as follows.

Case 1. $d_1=p_i$ for some $1\le i\le k$. Then,

$$x_1\in\begin{cases} Rp_i-\bigcup\limits_{j=1,j\neq i}^{k}Rp_j, \text{if } \alpha_i=1\\ Rp_i-\left(Rp_l^2\cup\bigcup\limits_{j=1,j\neq i}^{k}Rp_j\right), \text{if } \alpha_i\ge 2\end{cases}$$

By Proposition 5.1 and Proposition 5.2, $x_1+R\pi\in V_i$.

Case 2. $d_1=p_i^j$ for some $1\le i\le k$ and $2\le j\le\alpha_i$. Without loss of generality, assume that $i=1$. For all $1\le t\le k$, let $n_t(x)$ be the maximum power of prime p_t such that $p_t^{n_t(x)}\mid x$. For each $2\le s\le\sigma$, since $x_s+R\pi\notin U(R/R\pi)$, by the consequence of Proposition 2.19, $\gcd(x_s,\pi)\neq 1$. There exists $d_s\in R-\{1\}$ such that $\gcd(x_s,\pi)=d_s$.

Case 2.1. $\sum_{q=2}^{\sigma}n_1(d_q)<\alpha_1-j$. Then, $p_1^{\alpha_1}\nmid\prod_{m=1}^{\sigma}d_m$. Thus, $\prod_{m=1}^{\sigma}x_m\notin R\pi$. Therefore, $(x_1+R\pi)(x_2+R\pi)(x_3+R\pi)\cdots(x_\sigma+R\pi)=\prod_{m=1}^{\sigma}x_m+R\pi\neq 0+R\pi$, which is a contradiction.

Case 2.2. $\sum_{q=2}^{\sigma}n_1(d_q)\ge\alpha_1-j$.

Case 2.2.1. There exist $2\le r\le k$ such that $\sum_{q=2}^{\sigma}n_r(d_q)<\alpha_r$. Then, $p_r^{\alpha_r}\nmid\prod_{m=1}^{\sigma}d_m$. Thus, $\prod_{m=1}^{\sigma}x_m\notin R\pi$. Therefore, $(x_1+R\pi)(x_2+R\pi)(x_3+R\pi)\cdots(x_\sigma+R\pi)=\prod_{m=1}^{\sigma}x_m+R\pi\neq 0+R\pi$, which is a contradiction.

Case 2.2.2. $\sum_{q=2}^{\sigma}n_r(d_q)\ge\alpha_r$ for all $2\le r\le k$. Let $\Gamma=\{x_2,x_3,x_4,\ldots,x_\sigma\}$.

Since $\sum_{q=2}^{\sigma}n_1(d_q)\ge\alpha_1-j$ and $j\ge 2$, we can choose $\Gamma_1\subset\Gamma$ with $|\Gamma_1|=\alpha_1-2$ and $x_1\cdot\prod_{r_1\in\Gamma_1}r_1\in Rp_1^{\alpha_1}$.

Since $\sum_{q=2}^{\sigma}n_2(d_q)\ge\alpha_2$, we can choose $\Gamma_2\subset\Gamma-\Gamma_1$ with $|\Gamma_2|=\alpha_2$ and $\prod_{r_2\in\Gamma_2}r_2\in Rp_2^{\alpha_2}$.

Since $\sum_{q=2}^{\sigma} n_3(d_q) \geq \alpha_3$, we can choose $\Gamma_3 \subset \Gamma - (\Gamma_1 \cup \Gamma_2)$ with $|\Gamma_3| = \alpha_3$ and $\prod_{r_3 \in \Gamma_3} r_3 \in Rp_3^{\alpha_3}$.

In general, for each $2 \leq s \leq k$, since $\sum_{q=2}^{\sigma} n_s(d_q) \geq \alpha_s$, we can choose $\Gamma_s \subset \Gamma - \bigcup_{\gamma=1}^{s-1} \Gamma_\gamma$ with $|\Gamma_s| = \alpha_s$ and $\prod_{r_s \in \Gamma_s} r_s \in Rp_s^{\alpha_s}$.

Therefore, $x_1 \cdot \prod_{r \in \bigcup_{\gamma=1}^{k} \Gamma_\gamma} r \in R\pi$ and $\left|\bigcup_{\gamma=1}^{k} \Gamma_\gamma\right| = \sigma - 2$.

Notice that $\bigcup_{\gamma=1}^{k} \Gamma_\gamma$ misses one element out from Γ. Without loss of generality, let x_σ be the missing element. Thus, $\prod_{j'=1}^{\sigma-1} x_{j'} \in R\pi$ and consequently, $(x_1 + R\pi)(x_2 + R\pi)\ (x_3 + R\pi) \cdots (x_{\sigma-1} + R\pi) = \prod_{j'=1}^{\sigma-1} x_{j'} + R\pi = 0 + R\pi$, which is a contradiction.

Case 3. $d_1 = p_1^{j_1} p_2^{j_2} p_3^{j_3} \cdots p_k^{j_k}$ where $0 \leq j_m \leq \alpha_m$ for all $1 \leq m \leq k$ with at least nonzero j_b and j_c $(b \neq c)$. If $j_v = \alpha_v$, for all $1 \leq v \leq k$, then $d_1 = \pi$. Then, $x_1 + R\pi$ is a zero element of $R/R\pi$ which is a contradiction. Otherwise, we split our considerations into two cases.

Case 3.1. there exist $1 \leq r \leq k$ such that $\sum_{q=2}^{\sigma} n_r(d_q) < \alpha_r - j_r$. Without loss of generality, assume that $r = 1$. Then, $p_1^{\alpha_1} \nmid \prod_{m=1}^{\sigma} d_m$. Thus, $\prod_{m=1}^{\sigma} x_m \notin R\pi$. Therefore, $(x_1 + R\pi)(x_2 + R\pi)(x_3 + R\pi) \cdots (x_\sigma + R\pi) = \prod_{m=1}^{\sigma} x_m + R\pi \neq 0 + R\pi$, which is a contradiction.

Case 3.2. $\sum_{q=2}^{\sigma} n_r(d_q) \geq \alpha_r - j_r$ for all $1 \leq r \leq k$. Without loss of generality, assume that $p_1 p_2 \mid d_1$. Let $\Gamma = \{x_2, x_3, x_4, \ldots, x_\sigma\}$.

Since $\sum_{q=2}^{\sigma} n_1(d_q) \geq \alpha_1 - j_1$, we can choose $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| = \alpha_1 - 1$ and $x_1 \cdot \prod_{r_1 \in \Gamma_1} r_1 \in Rp_1^{\alpha_1}$.

Since $\sum_{q=2}^{\sigma} n_2(d_q) \geq \alpha_2 - j_2$, we can choose $\Gamma_2 \subset \Gamma - \Gamma_1$ with $|\Gamma_2| = \alpha_2 - 1$ and $x_1 \cdot \prod_{r_2 \in \Gamma_2} r_2 \in Rp_2^{\alpha_2}$.

Since $\sum_{q=2}^{\sigma} n_3(d_q) \geq \alpha_3 - j_3$, we can choose $\Gamma_3 \subset \Gamma - (\Gamma_1 \cup \Gamma_2)$ with $|\Gamma_3| = \alpha_3$ and $\prod_{r_3 \in \Gamma_3} r_3 \in Rp_3^{\alpha_3}$.

Since $\sum_{q=2}^{\sigma} n_4(d_q) \geq \alpha_4 - j_4$, we can choose $\Gamma_4 \subset \Gamma - (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ with $|\Gamma_4| = \alpha_4$ and $\prod_{r_4 \in \Gamma_4} r_4 \in Rp_4^{\alpha_4}$.

In general, for each $3 \leq s \leq k$, since $\sum_{q=2}^{\sigma} n_s(d_q) \geq \alpha_s - j_s$, we can choose $\Gamma_s \subset \Gamma - \bigcup_{\gamma=1}^{s-1} \Gamma_\gamma$ with $|\Gamma_s| = \alpha_s$ and $\prod_{r_s \in \Gamma_s} r_s \in Rp_s^{\alpha_s}$.

Therefore, $x_1 \cdot \prod_{r \in \bigcup_{\gamma=1}^{k} \Gamma_\gamma} r \in R\pi$ and $\left|\bigcup_{\gamma=1}^{k} \Gamma_\gamma\right| = \sigma - 2$.

Notice that $\bigcup_{\gamma=1}^{k}\Gamma_\gamma$ misses one element out from Γ. Without loss of generality, let x_σ be the missing element. Thus, $\prod_{j'=1}^{\sigma-1}x_{j'}\in R\pi$ and consequently, $(x_1+R\pi)$ $(x_2+R\pi)(x_3+R\pi)\cdots(x_{\sigma-1}+R\pi)=\prod_{j'=1}^{\sigma-1}x_{j'}+R\pi=0+R\pi$, which is a contradiction.

Hence, the Case 1 is the only possible case for d_1. Thus, $x_1+R\pi\in\bigcup_{j=1}^{k}V_j$. □

Proposition 5.4. *Let $k\geq 2$ be a fixed integer and R be a PID containing at least k nonassociate distinct prime elements, say $p_1,p_2,p_3,\dots,p_k$. Assume that $R/R\pi$ is finite. Then, we can construct a k-partite σ-zero-divisor hypergraph whose vertex set is $Z(R/R\pi,\sigma)$.*

Proof. By Proposition 5.3, we can partition the vertex set into k partite sets, say $V_1,V_2,V_3,\dots,V_k$. Then, for each element $v_i\in V_i$, we can find

$v_1,v_2,v_3,\dots,v_{\alpha_1}\in V_1$,

$v_{\alpha_1+1},v_{\alpha_1+2},v_{\alpha_1+3},\dots,v_{\alpha_1+\alpha_2}\in V_2$,

$v_{\alpha_1+\alpha_2+1},v_{\alpha_1+\alpha_2+2},v_{\alpha_1+\alpha_2+3},\dots,v_{\alpha_1+\alpha_2+\alpha_3}\in V_3,\ \dots,$

$v_{\sum_{l=1}^{i-1}\alpha_l+1},v_{\sum_{l=1}^{i-1}\alpha_l+2},v_{\sum_{l=1}^{i-1}\alpha_l+3},\dots,v_{\sum_{l=1}^{i-1}\alpha_l+\alpha_i-1}\in V_i,\ \dots,\ v_{\sum_{l'=1}^{k-1}\alpha_{l'}+1},$

$v_{\sum_{l'=1}^{k-1}\alpha_{l'}+2},v_{\sum_{l'=1}^{k-1}\alpha_{l'}+3},\dots,v_\sigma\in V_k$

which $\{v_1,v_2,v_3,\dots,v_i,\dots,v_\sigma\}$ forms an edge.

We need to prove that no σ-subsets in each partite set V_i form an edge.

Fix $1\leq j\leq k$. Let $x_1+R\pi,x_2+R\pi,x_3+R\pi,\dots,x_\sigma+R\pi\in V_j$.

If $\alpha_j=1$, then, for all $1\leq l\leq\sigma$, $x_l\in Rp_j-\bigcup_{l'\neq j}Rp_{l'}$. Therefore, $\prod_{m=1}^{\sigma}x_m\in Rp_j-R\pi$ and thus, $(x_1+R\pi)$ $(x_2+R\pi)(x_3+R\pi)\cdots(x_\sigma+R\pi)=\prod_{m=1}^{\sigma}x_m+R\pi\neq 0+R\pi$. We cannot obtain an edge constructed by any σ elements from V_j. Now, we have a k-partite σ-zero-divisor hypergraph $\mathcal{H}_\sigma(R/R\pi)$.

If $\alpha_j\geq 2$, then, for all $1\leq l\leq\sigma$, $x_l\in Rp_j-\left(Rp_j^2\cup\bigcup_{l'\neq j}Rp_{l'}\right)$. Therefore, $\prod_{m=1}^{\sigma}x_m\in Rp_j-R\pi$ and thus, $(x_1+R\pi)(x_2+R\pi)(x_3+R\pi)\cdots(x_\sigma+R\pi)=\prod_{m=1}^{\sigma}x_m+R\pi\neq 0+R\pi$. We cannot obtain an edge constructed by any σ elements from V_j. Now, we have a k-partite σ-zero-divisor hypergraph $\mathcal{H}_\sigma(R/R\pi)$. □

In summary, we rephrase Proposition 5.4 as follows.

Theorem 5.5. *If k is a positive integer greater than 1 and R is a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$, such that $R/R\pi$ is finite with $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq k$ where $\pi = \prod_{i=1}^{k} p_i^{\alpha_i}$, then there exists a k-partite σ-zero-divisor hypergraph whose vertex set is $Z\big(R/R\pi, \sigma\big)$ where $\sigma = \sum_{i=1}^{k} \alpha_i$.*

In this chapter, any k-partite σ-zero-divisor hypergraph is not complete because we have only the edge of the form $\{a_{1,1}, a_{1,2}, a_{1,3}, \ldots, a_{1,\alpha_1}, a_{2,1}, a_{2,2}, a_{2,3}, \ldots, a_{2,\alpha_2}, a_{3,1}, a_{3,2}, a_{3,3}, \ldots, a_{3,\alpha_3}, \ldots, a_{k,1}, a_{k,2}, a_{k,3}, \ldots, a_{k,\alpha_k}\}$ where $a_{i,j} \in V_i$ for all $1 \leq i \leq k$ and $1 \leq j \leq \alpha_i$. However, we cannot find the edge of the form $\{a_{1,1}, a_{1,2}, a_{1,3}, \ldots, a_{1,\alpha_1+1}, a_{2,1}, a_{2,2}, a_{2,3}, \ldots, a_{2,\alpha_2}, a_{3,1}, a_{3,2}, a_{3,3}, \ldots, a_{3,\alpha_3}, \ldots, a_{k,1}, a_{k,2}, a_{k,3}, \ldots, a_{k,\alpha_k - 1}\}$ where $a_{i,j} \in V_i$ for all $1 \leq i \leq k$ and $1 \leq j \leq \alpha_i + 1$ from $\mathcal{H}_\sigma(R/R\pi)$, that is, if such hypergraph is complete, we need to construct an edge by α_{m_1} vertices from V_1, α_{m_2} vertices from V_2, α_{m_3} vertices from V_3, ..., α_{m_k} vertices from V_k for all $1 \leq i \leq k$, $1 \leq m_i \leq k$.

Example 5.6. Consider the ring $\mathbb{Z}_{60} \cong \mathbb{Z}/(2^2 \cdot 3 \cdot 5)\mathbb{Z}$. We can compute
$V_1 = \{\overline{2}, \overline{14}, \overline{22}, \overline{26}, \overline{34}, \overline{38}, \overline{46}, \overline{58}\}$, $V_2 = \{\overline{3}, \overline{9}, \overline{21}, \overline{27}, \overline{33}, \overline{39}, \overline{51}, \overline{57}\}$ and
$V_3 = \{\overline{5}, \overline{25}, \overline{35}, \overline{55}\}$. Then, there exist a 3-partite 4-zero-divisor hypergraph whose vetex set is $Z\big(\mathbb{Z}/(2^2 \cdot 3 \cdot 5)\mathbb{Z}, 4\big) = V_1 \cup V_2 \cup V_3$. The cardinality of the edge set, $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{60}))$, is 256. These are some examples of $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{60}))$,

$\{\overline{2}, \overline{14}, \overline{3}, \overline{5}\}$,$\{\overline{2}, \overline{14}, \overline{3}, \overline{25}\}$,$\{\overline{2}, \overline{14}, \overline{3}, \overline{35}\}$,$\{\overline{2}, \overline{14}, \overline{3}, \overline{55}\}$,$\{\overline{2}, \overline{14}, \overline{9}, \overline{5}\}$,$\{\overline{2}, \overline{14}, \overline{9}, \overline{25}\}$,

$\{\overline{2}, \overline{14}, \overline{9}, \overline{35}\}$,$\{\overline{2}, \overline{14}, \overline{9}, \overline{55}\}$,$\{\overline{2}, \overline{14}, \overline{21}, \overline{5}\}$,$\{\overline{2}, \overline{14}, \overline{21}, \overline{25}\}$,$\{\overline{2}, \overline{14}, \overline{21}, \overline{35}\}$,

$\{\overline{2}, \overline{14}, \overline{2155}\}$,$\{\overline{2}, \overline{14}, \overline{27}, \overline{5}\}$,$\{\overline{2}, \overline{14}, \overline{27}, \overline{25}\}$,$\{\overline{2}, \overline{14}, \overline{27}, \overline{35}\}$,$\{\overline{2}, \overline{14}, \overline{27}, \overline{25}\}$,

$\{\overline{2}, \overline{14}, \overline{33}, \overline{5}\}$,$\{\overline{2}, \overline{14}, \overline{33}, \overline{25}\}$,$\{\overline{2}, \overline{14}, \overline{33}, \overline{35}\}$,$\{\overline{2}, \overline{14}, \overline{33}, \overline{55}\}$,$\{\overline{2}, \overline{14}, \overline{39}, \overline{5}\}$,

$\{\overline{2}, \overline{14}, \overline{39}, \overline{25}\}$,$\{\overline{2}, \overline{14}, \overline{39}, \overline{35}\}$,$\{\overline{2}, \overline{14}, \overline{39}, \overline{55}\}$,$\{\overline{2}, \overline{14}, \overline{51}, \overline{5}\}$,$\{\overline{2}, \overline{14}, \overline{51}, \overline{25}\}$,

$\{\overline{2},\overline{14},\overline{51},\overline{35}\}$,$\{\overline{2},\overline{14},\overline{51},\overline{55}\}$,$\{\overline{2},\overline{14},\overline{57},\overline{5}\}$,$\{\overline{2},\overline{14},\overline{57},\overline{25}\}$,$\{\overline{2},\overline{14},\overline{57},\overline{35}\}$, and

$\{\overline{2},\overline{14},\overline{57},\overline{55}\}$.

We consider an example of the Gaussian integers in $\mathbb{Z}[i]$.

Example 5.7. Consider $\mathbb{Z}[i]/(693)\mathbb{Z}[i] \cong \mathbb{Z}[i]/\big((3i)^2\cdot 7i\cdot 11i\big)\mathbb{Z}[i]$. We can compute

$V_1 = \big\{3a+3bi \;\big|\; a \in \mathbb{Z}_{231} - (3\mathbb{Z}_{231} \cup 7\mathbb{Z}_{231} \cup 11\mathbb{Z}_{231}), b \in \mathbb{Z}_{231} - (3\mathbb{Z}_{231} \cup 7\mathbb{Z}_{231} \cup 11\mathbb{Z}_{231})\big\}$,

$V_2 = \big\{7a+7bi \;\big|\; a \in \mathbb{Z}_{99} - (3\mathbb{Z}_{99} \cup 11\mathbb{Z}_{99}), b \in \mathbb{Z}_{99} - (3\mathbb{Z}_{99} \cup 11\mathbb{Z}_{99})\big\}$,

$V_3 = \big\{11a+11bi \;\big|\; a \in \mathbb{Z}_{63} - (3\mathbb{Z}_{63} \cup 7\mathbb{Z}_{63}), b \in \mathbb{Z}_{63} - (3\mathbb{Z}_{63} \cup 7\mathbb{Z}_{63})\big\}$.

We have $|V_1| = \big(231 - \big(\frac{231}{3} + \frac{231}{7} + \frac{231}{11} - \frac{231}{21} - \frac{231}{33} - \frac{231}{77} + \frac{231}{231}\big)\big)^2 = 14,400$, $|V_2| = \big(99 - \big(\frac{99}{3} + \frac{99}{11} - \frac{99}{33}\big)\big)^2 = 3,600$, and $|V_3| = \big(63 - \big(\frac{63}{3} + \frac{63}{7} - \frac{63}{21}\big)\big)^2 = 1,296$. The cardinality of the edge set, $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}[i]/(693)\mathbb{Z}[i]))$, is $14,400 \times 3,600 \times 1,296 = 67,184,640,000$. Here are some examples of $\mathcal{E}(\mathcal{H}_4(\mathbb{Z}[i]/(693)\mathbb{Z}[i]))$,

$\{3,6,7,11\}$, $\{3,6,-7,11\}$, $\{3,6,7i,11\}$, $\{3,6,-7i,11\}$, $\{3,6,7+7i,11\}$,

$\{3,6,7-7i,11\}$, $\{3,6,-7+7i,11\}$, $\{3,6,-7-7i,11\}$, $\{3,6,14,11\}$,

$\{3,6,-14,11\}$, $\{3,6,14i,11\}$, $\{3,6,-14i,11\}$, $\{3,6,7+14i,11\}$,

$\{3,6,7-14i,11\}$, $\{3,6,-7+14i,11\}$, $\{3,6,-7-14i,11\}$,

$\{3,6,14+7i,11\}$, $\{3,6,14-7i,11\}$, $\{3,6,-14+7i,11\}$, and

$\{3,6,-14-7i,11\}$.

Remark 1. Let $1 \leq i \neq j \leq k$ and p_i and p_j are nonassociate distinct prime element. Since $\gcd(p_i,p_j) = 1$, by Proposition 2.19, there exist nonzero element $a,b \in R$ such that $1 = ap_i + bp_j \in Rp_i + Rp_j$. By Proposition 2.29, $Rp_i + Rp_j = R$, that is, Rp_i and Rp_j are comaximal. By Theorem 2.11 (Chinese Remainder

Theorem), $R/Rp_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k} \cong R/Rp_1^{\alpha_1} \times R/Rp_2^{\alpha_2} \times R/Rp_3^{\alpha_3} \times \cdots \times R/Rp_k^{\alpha_k}$. Then, we can define

$$V_j = \{(u_1+Rp_1^{\alpha_1}, u_2+Rp_2^{\alpha_2}, u_3+Rp_3^{\alpha_3}, \ldots, 0+Rp_j^{\alpha_j}, \ldots, u_k+Rp_k^{\alpha_k}) \mid u_m+Rp_m^{\alpha_m} \in U(R/Rp_m^{\alpha_m}) \text{ for all } m \neq j\}, \text{ if } \alpha_j = 1,$$

$$V_j = \{(u_1+Rp_1^{\alpha_1}, u_2+Rp_2^{\alpha_2}, u_3+Rp_3^{\alpha_3}, \ldots, a_j+Rp_j^{\alpha_j}, \ldots, u_k+Rp_k^{\alpha_k}) \mid u_m+Rp_m^{\alpha_m} \in U(R/Rp_m^{\alpha_m}) \text{ for all } m \neq j \text{ and } a_j + Rp_j^{\alpha_j} \in Z(R/Rp_j^{\alpha_j}, \alpha_j)\}, \text{ if } \alpha_j \geq 2.$$

When $R = \mathbb{Z}$, we can compute the number of elements in V_i.

(i) If $\alpha_i \geq 2$, then by Corollary 3.4, we obtain $\left|U(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})\right| = \phi(p_i^{\alpha_i})$ and $\left|Z(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}, k)\right| = \left|\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}\right| - \phi(p_i^{\alpha_i}) - 1 - (p^{\alpha_i - 2} - 1)$

(ii) If $\alpha_i = 1$, then by Corollary 3.4, we obtain $\left|U(\mathbb{Z}/p_i\mathbb{Z})\right| = \left|\mathbb{Z}/p_i\mathbb{Z}\right| - 1$

Remark 2. We know that $\mathbb{Z}/(2^2)\mathbb{Z}$ has no 2-zero-divisors. If one component of $R/Rp_1^{\alpha_1} \times R/Rp_2^{\alpha_2} \times R/Rp_3^{\alpha_3} \times \cdots \times R/Rp_k^{\alpha_k}$ is $\mathbb{Z}/(2^2)\mathbb{Z} \cong \mathbb{Z}_4$, we define the partite set corresponding to $\mathbb{Z}_4$ by

$$V_j = \{(u_1 + Rp_1^{\alpha_1}, u_2 + Rp_2^{\alpha_2}, u_3 + Rp_3^{\alpha_3}, \ldots, \overline{2}_j, \ldots, u_k + Rp_k^{\alpha_k}) \mid u_m + Rp_m^{\alpha_m} \in U(R/Rp_m^{\alpha_m}) \text{ for all } m \neq j \text{ and } \overline{2} \in \mathbb{Z}_4\}.$$

Example 5.8.

(i) From Example 5.6, in $\mathbb{Z}/60\mathbb{Z} \cong \mathbb{Z}/(2^2 \cdot 3 \cdot 5)\mathbb{Z}$, we compute $\left|V_1\right| = \left|Z(\mathbb{Z}/2^2\mathbb{Z}, 2)\right| \cdot \left|U(\mathbb{Z}/3\mathbb{Z})\right| \cdot \left|U(\mathbb{Z}/5\mathbb{Z})\right| = (4-2-1-0)\cdot(3-1)\cdot(5-1) = 1\cdot 2\cdot 4 = 8$, $\left|V_2\right| = \left|U(\mathbb{Z}/2^2\mathbb{Z})\right| \cdot 1 \cdot \left|U(\mathbb{Z}/\mathbb{Z})\right| = (4-2)\cdot 1\cdot(5-1) = 2\cdot 1\cdot 4 = 8$, and $\left|V_3\right| = \left|U(\mathbb{Z}/2^2\mathbb{Z})\right| \cdot \left|U(\mathbb{Z}/3\mathbb{Z})\right| \cdot 1 = (4-2)\cdot(3-1)\cdot 1 = 2\cdot 2\cdot 1 = 4$. Then, $\left|Z\big(\mathbb{Z}/(2^2 \cdot 3 \cdot 5)\mathbb{Z}, 4\big)\right| = 8 + 8 + 4 = 20$ and $\left|\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{60}))\right| = 8 \cdot 8 \cdot 4 = 256$.

(ii) From Example 5.7, in $\mathbb{Z}[i]/(693)\mathbb{Z}[i] \cong \mathbb{Z}[i]/((3i)^2 \cdot 7i \cdot 11i)\mathbb{Z}[i]$, we compute $\left|V_1\right| = \big(\left|Z(\mathbb{Z}[i]/(3i)^2\mathbb{Z}[i], 2)\right| \cdot \left|Z(\mathbb{Z}[i]/(3i)^2\mathbb{Z}[i], 2)\right|\big) \cdot \big(\left|U(\mathbb{Z}[i]/(7i)\mathbb{Z}[i])\right| \cdot \left|U(\mathbb{Z}[i]/(7i)\mathbb{Z}[i])\right|\big) \cdot \big(\left|U(\mathbb{Z}[i]/(11i)\mathbb{Z}[i])\right| \cdot \left|U(\mathbb{Z}[i]/(11i)\mathbb{Z}[i])\right|\big) = \big((9 - 6 - 1 - 0) \cdot (9 - 6 - 1 - 0)\big) \cdot \big((7 - 1) \cdot (7 - 1)\big) \cdot \big((11 - 1) \cdot (11 - 1)\big) = 14,400$, $\left|V_2\right| = \big(\left|U(\mathbb{Z}[i]/(3i)^2\mathbb{Z}[i])\right| \cdot \left|U(\mathbb{Z}[i]/(3i)^2\mathbb{Z}[i])\right|\big) \cdot 1 \cdot \big(\left|U(\mathbb{Z}[i]/(11i)\mathbb{Z}[i])\right| \cdot$

$\big|U(\mathbb{Z}[i]/(11i)\mathbb{Z}[i])\big|\big) = \big((9-3)\cdot(9-3)\big)\cdot 1\cdot\big((11-1)\cdot(11-1)\big) = 36\cdot 1\cdot 100 = 3,600$, and

$\big|V_3\big| = \big(\big|U(\mathbb{Z}[i]/(3i)^2\mathbb{Z}[i])\big|\cdot\big|U(\mathbb{Z}[i]/(3i)^2\mathbb{Z}[i])\big|\big)\cdot\big(\big|U(\mathbb{Z}[i]/(7i)\mathbb{Z}[i])\big|\cdot\big|U(\mathbb{Z}[i]/(7i)\mathbb{Z}[i])\big|\big)\cdot 1 = \big((9-3)\cdot(9-3)\big)\cdot\big((7-1)\cdot(7-1)\big)\cdot 1 = 36\cdot 36\cdot 1 = 1,296.$

Then, $\big|Z\big(\mathbb{Z}/(2^2\cdot 3\cdot 5)\mathbb{Z}, 4\big)\big| = 14,400+3,600+1,296 = 19,296$ and $\big|\mathcal{E}(\mathcal{H}_4(\mathbb{Z}_{60}))\big| = 14,400\times 3,600\times 1,296 = 67,184,640,000.$

Now, we consider $\mathcal{H}_\sigma(R/R\pi)$ with its diameter and its minimum length of all cycles.

Proposition 5.9. *Let $k \geq 2$ be a fixed integer and R be a PID containing at least k nonassociate distinct prime elements, say $p_1, p_2, p_3, \ldots, p_k$. Then, the diameter of $\mathcal{H}_\sigma(R/R\pi)$ is 2.*

Proof. Let x and y be distinct vertices of $\mathcal{H}_\sigma(R/R\pi)$.

Case 1. x and y are in different partites. By definition of σ-uniform k-partite hypergraph, there exists an edge E such that $x, y \in E$. A path x, E, y is obtained. Then, the distance between x and y is 1.

Case 2. x and y are in the same partite V_i with $\alpha_i = 1$. By the definition of $\mathcal{H}_\sigma(R/R\pi)$, there are no edges E such that $x, y \in E$. However, there exist edges E_1 and E_2 such that $x \in E_1$ and $y \in E_2$. By the definition of $\mathcal{H}_\sigma(R/R\pi)$, there exists a vertex v in the other partite sets such that $v \in E_1 \cap E_2$. Then, a path x, E_1, v, E_2, y is obtained. Therefore, $d(x, y) = 2$.

Case 3. x and y are in the same partite V_i with $\alpha_i \geq 2$. By the definition of $\mathcal{H}_\sigma(R/R\pi)$, there exists an edge E such that $x, y \in E$. We have a path x, E, y. Then, $d(x, y) = 1$.

From these three cases, the maximum of distance between x and y is 2, that is, the diameter of $\mathcal{H}_\sigma(R/R\pi)$ is 2. □

To consider the minimum length of all cycles, we need to split into two cases of $\Big|Z(R/R\pi, \sigma)\Big|$, that is, $\Big|Z(R/R\pi, \sigma)\Big| = \sigma$ and $\Big|Z(R/R\pi, \sigma)\Big| \geq \sigma + 1$, which can be seperated into two subcases with cases of k.

Proposition 5.10. *The minimum length of all cycles in* $\mathcal{H}_\sigma(R/R\pi)$ *is* 0 *when* $\left|Z(R/R\pi,\sigma)\right| = \sigma$.

Proof. Since $\mathcal{H}_\sigma(R/R\pi)$ has only one edge, such hypergraph has no cycles. □

Next, we consider the case $\left|Z(R/R\pi,\sigma)\right| \geq \sigma + 1$.

Proposition 5.11. *The minimum length of all cycles in* $\mathcal{H}_\sigma(R/R\pi)$ *is* 2 *when* $k \geq 3$ *and* $\left|Z(R/R\pi,\sigma)\right| \geq \sigma + 1$.

Proof. We separate into two cases.

Case 1. $\alpha_i = 1$ for all $1 \leq i \leq k$. Since each partite set is a nonempty set, there exists at least one element in each set. Since $\left|Z(R/R\pi,\sigma)\right| \geq \sigma+1 = \sum_{i=1}^{k} 1+1 = k+1$ and there are k partite sets in $\mathcal{H}_\sigma(R/R\pi)$, there are at least two elements, say x_1 and x_1', in one of the partite sets, say V_1. By the definition of $\mathcal{H}_\sigma(R/R\pi)$, there exist two distinct edges $E_1 = \{x_1, x_2, x_3, \ldots, x_\sigma\}$ and $E_2 = \{x_1', x_2, x_3, \ldots, x_\sigma\}$. Thus, $\mathcal{H}_\sigma(R/R\pi)$ has a 2-cycle $C = x_2, E_1, x_3, E_2$. Then, the minimum length of all cycles is 2.

Case 2. There exists $1 \leq i \leq k$ such that $\alpha_i \geq 2$. Without loss of generality, $\alpha_1 \geq 2$. Then, $\left|V_1\right| \geq 2$.

(i) if $\left|V_1\right| = 2$, say $V_1 = \{x_1, x_1'\}$, then there exists one of the other partite sets has two elements x_2 and x_2', say V_2. We have two distinct edges $E_1 = \{x_1, x_1', x_2, x_3, \ldots, x_\sigma\}$ and $E_2 = \{x_1, x_1', x_2', x_3, \ldots, x_\sigma\}$. Thus, we obtain one cycle. That is, we have a 2-cycle $C = x_1, E_1, x_1', E_2$. Then, the minimum length of all cycles is 2.

(ii) if $\left|V_1\right| \geq 3$, say $V_1 = \{x_1, x_1', x_1'', \ldots\}$ then there exist two distinct edges $E_1 = \{x_1, x_1', x_2, x_3, \ldots, x_\sigma\}$ and $E_2 = \{x_1, x_1'', x_2, x_3, \ldots, x_\sigma\}$. Thus, there exists one cycle in $\mathcal{H}_\sigma(R/R\pi)$. Therefore, we have a 2-cycle $C = x_1, E_1, x_2, E_2$. Then, the minimum length of all cycles is 2.

From these two cases, the minimum length of all cycles is 2. □

Proposition 5.12. *Assume that $k = 2$ and $\left|Z(R/R\pi, \alpha_1 + \alpha_2)\right| \geq \alpha_1 + \alpha_2 + 1$. The minimum length of all cycles is*

$$\begin{cases} 0, & \text{if } \left|V_1\right| = 1 \text{ and } \alpha_2 = 1, \\ 2, & \text{if } \left|V_1\right| = 1 \text{ and } \alpha_2 \geq 2, \\ 2, & \text{if } \left|V_i\right| \geq 2 \text{ for all } i \in \{1, 2\} \text{ and there exists } i \in \{1, 2\} \text{ such that } \alpha_i \geq 2, \\ 4, & \text{if } \left|V_i\right| \geq 2 \text{ with } \alpha_i = 1 \text{ for all } i \in \{1, 2\}. \end{cases}$$

Proof. The four possible cases are considered as follows. Let V_1 and V_2 be partite sets of $\mathcal{H}_{\alpha_1+\alpha_2}(R/R\pi)$.

Case 1. $\left|V_1\right| = 1$ and $\alpha_2 = 1$. Suppose that $\mathcal{H}_2(R/R\pi)$ has a cycle $C = x_1, E_1, x_2, E_2, x_3, \ldots, x_{r-1}, E_r$ where $r \geq 2$.

If $x_1 \in V_1$, then from a cycle C, we have $x_3 = x_1$. Since $k = 2$, $E_1 = \{x_1, x_2\}$ and $E_2 = \{x_2, x_3\}$. Since $x_1 = x_3$, $E_1 = E_2$, which is a contradiction.

If x_1 is in another partite set rather than V_1, then we have $x_2 \in V_1$. By the same argument, we obtain $E_2 = E_3$, which is also a contradiction.

Therefore, $\mathcal{H}_2(R/R\pi)$ has no cycles.

Case 2. $\left|V_1\right| = 1$ and $\alpha_2 \geq 2$. Let $x_1 \in V_1$. Since $\left|Z(R/R\pi, \alpha_1 + \alpha_2)\right| \geq \alpha_1 + \alpha_2 + 1$, $\left|V_2\right| \geq 3$. There exist two distinct edges $E_1 = \{x_1, x_2, x_3, \ldots, x_{\alpha_1+\alpha_2}\}$ and $E_2 = \{x_1, x_2', x_3, \ldots, x_{\alpha_1+\alpha_2}\}$ where $x_2, x_2', x_3, \ldots, x_{\alpha_1+\alpha_2} \in V_2$. Thus, $\mathcal{H}_\sigma(R/R\pi)$ has a 2-cycle $C = x_1, E_1, x_3, E_2$. Then, the minimum length of all cycles is 2.

Case 3. $\left|V_i\right| \geq 2$ for all $i \in \{1, 2\}$ and there exists $i \in \{1, 2\}$, $\alpha_i \geq 2$. Without loss of generality, let $\alpha_1 \geq 2$. Since $\left|Z(R/R\pi, \alpha_1 + \alpha_2)\right| \geq \alpha_1 + \alpha_2 + 1$, we have $\left|V_1\right| \geq 3$, say $V = \{x_1, x_1', x_1'', \ldots\}$. There exist two distinct edges $E_1 = \{x_1, x_1', x_2, \ldots, x_{\alpha_1+\alpha_2}\}$ and $E_2 = \{x_1, x_1'', x_2, x_3, \ldots, x_{\alpha_1+\alpha_2}\}$. Thus, $\mathcal{H}_\sigma(R/R\pi)$ has a cycle, that is, $C = x_1, E_1, x_2, E_2$ of length 2. Then, the minimum length of all cycles is 2.

Case 4. $\left|V_i\right| \geq 2$ with $\alpha_i = 1$ for all $i \in \{1, 2\}$. Let x_1 be a vertex of $\mathcal{H}_2(R/R\pi)$ and V_1 be a partite set containing x_1. Since $\left|V_i\right| \geq 2$ with $\alpha_i = 1$ for all $i \in \{1, 2\}$, there are at least distinct four edges in $\mathcal{H}_2(R/R\pi)$ which form one cycle. Suppose that $\mathcal{H}_2(R/R\pi)$ has a 2-cycle. There exist two distinct edges E_1 and E_2 and a vertex x_2 differ from x_1 such that $x_1, x_2 \in E_1$ and $x_1, x_2 \in E_2$. Since $k = 2$, we have $E_1 = E_2$, which is a contradiction. Since $\alpha_i = 1$ for all $i \in \{1, 2\}$, $\mathcal{H}_2(R/R\pi)$ is a bipartite graph. We know that bipartite graphs have no odd cycle, see [11]. There exist distinct vertices x_2, x_3 and x_4 with $x_2 \in V_1$ and x_3, x_4 are in other partite set such that $E_1 = \{x_1, x_3\}$, $E_2 = \{x_1, x_4\}$, $E_3 = \{x_2, x_3\}$ and $E_4 = \{x_2, x_4\}$ form edges. We have a 4-cycle $C = x_1, E_1, x_3, E_3, x_2, E_4, x_4, E_2$. Then, the minimum of length of all cycles is 4. $\square$

CHAPTER VI

CONCLUSION AND DISCUSSION

We can see throughout this thesis that instead of considering directly a commutative ring R, we consider a commutative ring R/I where R is a PID and I is an appropriate ideal of R. First, in Chapter III, we assume that the existence of prime element p and the finiteness of R/Rp^k together with the cardinality of $(Rp/Rp^k) - (Rp^2/Rp^k)$ to be greater or equal to k. We can construct $\mathcal{H}_k(R/Rp^k)$ and it is complete. We can compute $\left|Z(R/Rp^k, k)\right| = \left|\mathbb{Z}/p^k\mathbb{Z}\right| - \varphi(p^k) - 1 - (p^{k-2} - 1)$. By the completeness, the diameter of $\mathcal{H}_k(R/Rp^k)$ is 1. However, k and the cardinality of $Z(R/Rp^k, k)$ determine the minimum length of all cycles of $\mathcal{H}_k(R/Rp^k)$ which can be either $3, 2$ or 0.

Next, in Chapter IV, we assume that the existence of nonassociate distinct prime elements $p_1, p_2, p_3, \ldots, p_k$ and the finiteness of $R/R\gamma$ where $\gamma = \prod_{i=1}^{k} p_i$. Instead of considering directly a commutative ring $R/R\gamma$, by the Chinese Remainder Theorem, we consider $R/Rp_1 \times R/Rp_2 \times R/Rp_3 \times \cdots \times R/Rp_k$. We can partition $Z(R/R\gamma, k)$ into k partite sets and then, $\mathcal{H}_k(R/R\gamma)$ is a k-partite hypergraph which is complete according to Kuhl and Schroeder [9]. Moreover, $\left|Z(R/R\gamma, k)\right| = \sum_{i=1}^{k} \left|V_i\right|$ where $\left|V_i\right| = \prod_{j=1, j\neq i}^{k} \left|U(R/Rp_j)\right|$. By the completeness, we can easily obtain its diameter as 2. Similar to the previous constructed hypergraph, the minimum length of all cycles of $H_k(R/R\gamma)$ depend on k and the cardinality of $Z(R/R\gamma, k)$. However, for $\mathcal{H}_k(R/R\gamma)$, the cardinality of each partite set also an important factor to determine its minimum length of all cycles which can be either $4, 2$ or 0.

Finally, by using the idea in Chapter III, we assume again that the existence of nonassociate distinct prime elements $p_1, p_2, p_3, \ldots, p_k$, the finiteness of $R/R\pi$, where $\pi = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq k$, and the cardinality of $(Rp_i/R\pi) -$

$\Big((Rp_i^2/R\pi) \cup \bigcup_{j=1, j\neq i}^{k}(Rp_j/R\pi)\Big)$ to be greater or equal to α_i for all $1 \leq i \leq k$. We can partition $Z(R/R\pi, \sigma)$, where $\sigma = \sum_{i=1}^{k} \alpha_i$, into k partite sets and from these partite sets, we can construct k-partite σ-zero-divisor hypergraph of $Z(R/R\pi, \sigma)$ according to Jirimutu and Wang [8].

Moreover, by the Chinese Remainder Theorem, we can see that $R/R\pi \cong R/Rp_1^{\alpha_1} \times R/Rp_2^{\alpha_2} \times R/Rp_3^{\alpha_3} \times \cdots \times R/Rp_k^{\alpha_k}$ and we can compute $\left|Z(R/R\pi, \sigma)\right| = \sum_{i=1}^{k} \left|V_i\right|$ where $\left|V_i\right| = \left|Z(R/Rp_i^{\alpha_i}, \alpha_i)\right| \cdot \prod_{j=1, j\neq i}^{k} \left|U(R/Rp_j)\right|$ if $\alpha_i \geq 2$ or $\prod_{j=1, j\neq i}^{k} \left|U(R/Rp_j)\right|$ if $\alpha_i = 1$.

Unfortunately, according to Jirimutu and Wang [8], our constructed k-partite hypergraph is not complete. We, then, find the diameter of $\mathcal{H}_\sigma(R/R\pi)$ to be 2.

Here, σ, the cardinality of $Z(R/R\pi, \sigma)$ and the cardinality of each partition set determines the minimum length of all cycles of $\mathcal{H}_\sigma(R/R\pi)$ which can be either $4, 2$ or 0.

As for the future research, we suggest one to investigate the way to construct k-partite σ-zero-divisor hypergraphs to be a complete k-partite σ-uniform hypergraph according to Jirimutu and Wang's definition [8].

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