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# NON–UNIFORM BOUND OF NORMAL APPROXIMATION FOR COMBINATORIAL RANDOM SUMS

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In 2017, Frolov gave a uniform bound of normal approximation for combinatorial random sums. In this thesis, we find a non-uniform bound and investigate this bound for the random sums with random numbers having Poisson distributions.

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# CHAPTER I

# INTRODUCTION

Let $\{X_n\}_{n=2}^{\infty}$ be a sequence of random matrices such that $X_n = [X_n(i,j)],\ i,j = 1,2,\ldots,n$, is an $n \times n$ matrix of independent random variables. Let $\{\pi_n\}_{n=2}^{\infty}$ be a sequence of random permutations such that $\pi_n = (\pi_n(1), \pi_n(2), \ldots, \pi_n(n))$ is a uniformly distributed random permutation on $\{1,2,\ldots,n\}$. Assume that $X_n$ and $\pi_n$ are independent for all $n \geq 2$. For $n \geq 2$, define the **combinatorial sums** by

$$S_n = \sum_{i=1}^{n} X_n(i, \pi_n(i)).$$

It is well–known that the distribution of $S_n$ converges to the normal distribution under some conditions ([8], [9], [19]). This convergence is called a **combinatorial central limit theorem (CCLT)**. The bound in CCLT was investigated broadly such as in [1]–[3], [5], [7], [11]–[13], [15] and [18].

Random sums of random variables are very useful in probability theory and statistics and their applications. In particular, random sums of independent and identically distributed random variables which are called the compound random variables have many natural applications. For instance, they are used in insurance risk models for describing the aggregate claims. In statistics, they are also found in common tests and overdispersion modeling.

In this work, we investigate random sums called combinatorial random sums defined as the followings.

Let $N$ be an integer random variable such that $P(N \geq 2) = 1$ and $N$ is independent with $\{X_n\}_{n=2}^{\infty}$ and $\{\pi_n\}_{n=2}^{\infty}$. Define the **combinatorial random sums** by

$$S_N = \sum_{i=1}^{N} X_N(i, \pi_N(i)).$$

In our work, we study the distance between the distribution function of $S_N$ and the standard normal distribution function $\Phi$.

Let

$$\Delta_{N,z} := \left|P\Big(\frac{S_N}{\sqrt{VarS_N}} \le z\Big) - \Phi(z)\right| \text{ and } \Delta_N := \sup_{z\in\mathbb{R}} \Delta_{N,z}.$$

In 2017, Frolov ([6]) gave a bound of $\Delta_N$ under the $(2+\delta)$–th moment conditions where $\delta \in (0,1]$. The following is his result.

**Theorem 1.1.** *([6]) Assume that*

$$P(N \ge 2) = 1,$$

$$\exists\delta \in (0,1],\ E|X_n(i,j)|^{2+\delta} < \infty \ \ \textit{for all } 1 \le i,j \le n \textit{ and } n \ge 2,$$

$$\sum_{i=1}^{n} EX_n(i,k) = 0 \textit{ and } \sum_{j=1}^{n} EX_n(k,j) = 0 \textit{ for all } 1 \le k \le n \textit{ and } n \ge 2, \quad (1.1)$$

$$B_n := VarS_n > 0 \textit{ for all } n \ge 2 \ \ \textit{and} \quad (1.2)$$

$$EB_N < \infty. \quad (1.3)$$

*Then*

$$\Delta_N \le 45112EL_{N,\delta} + \frac{3\sqrt{VarB_N}}{EB_N},$$

*where* $L_{n,\delta} = \dfrac{1}{nB_n^{1+\frac{\delta}{2}}} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} E|X_n(i,j)|^{2+\delta}.$

In this work, we investigate a non–uniform bound of $\Delta_{N,z}$ under the third moment conditions. The followings are our main results.

**Theorem 1.2.** *Assume that*

$$P(N \ge 4) = 1, \quad (1.4)$$

$$E|X_n(i,j)|^3 < \infty \ \ \textit{for all } 1 \le i,j \le n \textit{ and } n \ge 2 \quad (1.5)$$

*and (1.1)–(1.3) hold. Then there exists a positive constant $C$ such that for fixed*

$z \in \mathbb{R}$ *with* $P(1+|z| \leq N^{\frac{1}{14}}) = 1$, *we have*

$$\Delta_{N,z} \leq \frac{C}{1+|z|}\Bigg[\frac{\sqrt{VarB_N}}{EB_N} + \frac{\sqrt{VarB_N}}{(EB_N)^2}\Big(EB_N^2\Big)^{\frac{1}{2}} + \frac{\sqrt{VarB_N}}{(EB_N)^4}\Big\{E\Big(B_N^6\gamma_N^4\Big)\Big\}^{\frac{1}{2}} + E\Big(\frac{1}{\sqrt{N}}\Big) + E\gamma_N + \frac{E\Big(B_N^6 N^{\frac{3}{2}}\gamma_N^4\Big)}{(EB_N)^6}\Bigg],$$

*where* $\gamma_n = L_{n,1}$.

In Theorem 1.2, we observe that if $N = n$ for some positive integer $n$, then

$$\Delta_{n,z} \leq \frac{C}{1+|z|}\Bigg(\frac{1}{\sqrt{n}} + \gamma_n + n^{\frac{3}{2}}\gamma_n^4\Bigg).$$

In the case of $\gamma_n = O(\frac{1}{n^\alpha})$ for some $\alpha > \frac{3}{8}$, we have $\Delta_{n,z} \to 0$ as $n \to \infty$. However, the bound in Theorem 1.2 may not tend to zero in general cases of $N$.

From the usefulness of the random sums when the random number is Poisson, we consider the case of $N$ having the Poisson distribution to show that our bound in this case tends to zero. Our result is stated in Theorem 1.3.

**Theorem 1.3.** *For* $k, n \in \mathbb{N}$, *let* $N_{k,n}$ *be a random variable such that*

$$P(N_{k,n} = m) = \frac{e^{-n}n^{m-k}}{(m-k)!} \quad \textit{where } m = k, k+1, \dots. \tag{1.6}$$

*Assume that (1.1), (1.2) and (1.5) hold,*

$$\gamma_n = O(\frac{1}{n^\alpha}) \textit{ as } n \to \infty \textit{ for some } \alpha > \frac{3}{8} \textit{ and} \tag{1.7}$$

$$B_n \sim cn^\beta \textit{ as } n \to \infty \textit{ for some } c, \beta > 0. \tag{1.8}$$

*Then for fixed* $z \in \mathbb{R}$ *and* $k \geq 4$ *such that* $1+|z| \leq k^{\frac{1}{14}}$, *we have*

$$\Delta_{N_{k,n},z} \to 0 \textit{ as } n \to \infty.$$

Note that $f(n) = O(g(n))$ as $n \to \infty$ represents that $\exists C > 0, \exists n_0 \in \mathbb{N}$ such that $|f(n)| \leq C|g(n)|$ for all $n \geq n_0$, and $f(n) \sim g(n)$ as $n \to \infty$ represents $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

The following is the case of $N$ having two possible values.

**Theorem 1.4.** *For $n \in \mathbb{N}$, let $N_n$ be a random variable such that*

$$P(N_n = n) = \frac{1}{n} \quad \textit{and} \quad P(N_n = 2n) = 1 - \frac{1}{n}.$$

*Assume that (1.1), (1.2), (1.5), (1.7) and (1.8) hold. Then for fixed $z \in \mathbb{R}$ and $n \geq 4$ such that $1 + |z| \leq n^{\frac{1}{14}}$, we have*

$$\Delta_{N_n,z} \to 0 \quad \textit{as } n \to \infty.$$

In this thesis, we organize as the follows. In Chapter II, we give a result for proving Theorem 1.2. The proofs of Theorem 1.2, Theorem 1.3 and Theorem 1.4 and some satisfying examples are in Chapter III.

# CHAPTER II

# NON-UNIFORM BOUND FOR COMBINATORIAL SUMS

For $n \geq 2$, let $Y(i,j),\ i,j = 1,2,\dots,n$ be independent random variables and $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ be a uniformly distributed random permutation on $\{1,2,\dots,n\}$ such that $\pi$ and $Y(i,j)$'s are independent. Define the **combinatorial sums** by

$$S_n = \sum_{i=1}^{n} Y(i,\pi(i)).$$

In 2016, Simcharoen and Neammanee ([15]) gave a non–uniform bound of normal approximation for $S_n$ under the third moment and variance one conditions. The following is their result.

**Theorem 2.1.** *([15]) Assume that*

$$\sum_{i=1}^{n} EY(i,k) = 0 \ and \ \sum_{j=1}^{n} EY(k,j) = 0 \ for\ all\ 1 \leq k \leq n, \tag{2.1}$$

$$E|Y(i,j)|^3 < \infty \ for\ all\ 1 \leq i,j \leq n \ \ and \tag{2.2}$$

$$VarS_n = 1. \tag{2.3}$$

*Then there exists $C > 0$ such that for fixed $z \in \mathbb{R}$ and a positive integer $n$ such that $1 + |z| \leq n^{\frac{1}{14}}$ and $n \geq 4$, we have*

$$|P(S_n \leq z) - \Phi(z)| \leq \frac{C}{1+|z|}\Big(\frac{1}{\sqrt{n}} + \sqrt{n}\delta_3^2\Big),$$

*where* $\quad \delta_3 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} E|Y(i,j)|^3.$

In this chapter, we generalize the result in Theorem 2.1 by reducing condition (2.3) for using in our main result in Chapter III. The following is our result.

**Theorem 2.2.** *Assume that (2.1) and (2.2) hold. Then there exists a positive constant $C$ such that for fixed $z \in \mathbb{R} \setminus (-1,1)$ and $n \geq 4$ such that $1+|z| \leq n^{\frac{1}{14}}$, we have*

$$|P(S_n \leq z) - \Phi(z)| \leq \frac{C}{1+|z|}\Big(|VarS_n - 1| + |VarS_n - 1|VarS_n + |VarS_n - 1|\delta_3^2 + \frac{1}{\sqrt{n}} + n^{\frac{3}{2}}\delta_3^4\Big).$$

**Remark 2.3.** In the case that $VarS_n = 1$, we see that our bound in Theorem 2.2 has the same order as in Theorem 2.1.

In this chapter, we organize as the follows. We give a useful lemma for proving Theorem 2.2 in Section 2.1. Then we use the Stein's method and the techniques from Simcharoen and Neammanee [15] to prove our theorem in the last Section.

Throughout this thesis, $C$ stands for a positive constant which may be different values in different places and we denote the double sums $\sum_{i=1}^{n}\sum_{j=1}^{n}$ by $\sum_{i,j=1}^{n}$ for every $n \in \mathbb{N}$.

## 2.1 Auxiliary Results

To obtain a non–uniform bound, we always use the technique called **truncation** of random variables. For $z \geq 0$ and $i, j \in \{1, 2, \ldots, n\}$, let

$$\hat{Y}_z(i,j) = Y(i,j)\mathbb{I}(|Y(i,j)| \leq 1+z) \quad \text{and} \quad Y_z(i,j) = Y(i,j)\mathbb{I}(|Y(i,j)| > 1+z)$$

where $\mathbb{I}$ is an indicator function. For a random permutation $\pi$, we let

$$\hat{Y}(\pi) = \sum_{i=1}^{n} \hat{Y}_z(i, \pi(i)).$$

Using the techniques from Neammanee and Rurkruthairat ([12]) and Simcharoen and Neammanee ([15]), we obtain the following lemma.

**Lemma 2.4.** *Assume that (2.1) and (2.2) hold. Then the followings hold.*

(i) $E\left(\sum_{i,j=1}^{n}\hat{Y}_z(i,j)\right)^2 \le C(n+n^2\delta_3^2)$ *for some* $C>0$.

(ii) $E\hat{Y}^2(\pi)\le C(VarS_n+1+\delta_3^2)$ *for some* $C>0$.

(iii) $E\hat{Y}^4(\pi)\le C\Big((1+z)\delta_3+n^{\frac{2}{3}}\delta_3^{\frac{4}{3}}+n^{\frac{1}{3}}\delta_3^{\frac{5}{3}}+n\delta_3^2+n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big)$ *for some* $C>0$ *and* $n\ge 4$.

(iv) *There exists a constant* $C>0$ *such that for* $n\ge 4$ *and* $1+z\le\sqrt{n}$,

$$E\hat{Y}^4(\pi)\le C(1+n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}).$$

*Proof.* *(i)* By the same argument as Neammanee and Rurkruthairat ([12], p.1592) and the fact that $|Y_z(i,j)|>1$ or $|Y_z(i,j)|=0$, we can show that

$$E\left(\sum_{i,j=1}^{n}\hat{Y}_z(i,j)\right)^2 \le \sum_{i,j=1}^{n}EY^2(i,j)+n^2\delta_3^2. \tag{2.4}$$

For positive real numbers $a_1,a_2,\ldots,a_n$ and real numbers $r,s\ne 0$ such that $r\le s$, we note from the power mean inequality that

$$\sum_{i=1}^{n}a_i^r\le n^{1-\frac{r}{s}}\left(\sum_{i=1}^{n}a_i^s\right)^{\frac{r}{s}} \quad ([10], \text{ p.32}).$$

From this fact and Hölder's inequality, we get that

$$\begin{aligned}\sum_{i,j=1}^{n}EY^2(i,j)&=E\left(\sum_{i,j=1}^{n}Y^2(i,j)\right)\\&\le n^{\frac{2}{3}}E\left(\sum_{i,j=1}^{n}|Y(i,j)|^3\right)^{\frac{2}{3}}\\&\le n^{\frac{2}{3}}\left\{E\left(\sum_{i,j=1}^{n}|Y(i,j)|^3\right)\right\}^{\frac{2}{3}}\end{aligned}$$

$$
\begin{aligned}
&= n^{\frac{2}{3}}\left(\sum_{i,j=1}^{n} E|Y(i,j)|^3\right)^{\frac{2}{3}} \\
&= n^{\frac{4}{3}}\delta_3^{\frac{2}{3}}. \qquad (2.5)
\end{aligned}
$$

Hence, by (2.4) and (2.5),

$$
E\left(\sum_{i,j=1}^{n} \hat{Y}_z(i,j)\right)^2 \le n^{\frac{4}{3}}\delta_3^{\frac{2}{3}} + n^2\delta_3^2.
$$

If $\delta_3 \le \frac{1}{\sqrt{n}}$, then $n^{\frac{4}{3}}\delta_3^{\frac{2}{3}} + n^2\delta_3^2 \le 2n$. In the case of $\delta_3 > \frac{1}{\sqrt{n}}$, we have $\frac{1}{\delta_3} < \sqrt{n}$ which implies that $n^{\frac{4}{3}}\delta_3^{\frac{2}{3}} \le n^2\delta_3^2$. Hence

$$
E\left(\sum_{i,j=1}^{n} \hat{Y}_z(i,j)\right)^2 \le C(n + n^2\delta_3^2). \qquad (2.6)
$$

*(ii)* Following [12] (pp.1592–1593), we can see that

$$
E\hat{Y}^2(\pi) \le VarS_n + \frac{2}{n(n-1)}\sum_{i_1,i_2=1}^{n} E|Y_z(i_1,i_2)|E|Y(i_1,i_2)| + \frac{n\delta_3^2}{n-1}. \qquad (2.7)
$$

By the fact that $|Y_z(i,j)| > 1$ or $|Y_z(i,j)| = 0$, we have

$$
\begin{aligned}
\sum_{i_1,i_2=1}^{n} E|Y_z(i_1,i_2)|E|Y(i_1,i_2)| &\le \sum_{i_1,i_2=1}^{n} EY_z^2(i_1,i_2)E|Y(i_1,i_2)| \\
&\le \sum_{i_1,i_2=1}^{n} E|Y(i_1,i_2)|^3 \\
&= n\delta_3.
\end{aligned}
$$

From this fact and (2.7), we obtain

$$
\begin{aligned}
E\hat{Y}^2(\pi) &\le VarS_n + \frac{2\delta_3}{n-1} + \frac{n\delta_3^2}{n-1} \\
&\le C\Big(VarS_n + \frac{\delta_3}{n} + \delta_3^2\Big).
\end{aligned}
$$

If $\delta_3 \le \frac{1}{\sqrt{n}}$, then $\frac{\delta_3}{n} + \delta_3^2 \le 2$ and if $\delta_3 > \frac{1}{\sqrt{n}}$, we have $\frac{\delta_3}{n} \le \delta_3^2$.

Hence

$$E\hat{Y}^2(\pi) \le C(VarS_n + 1 + \delta_3^2). \tag{2.8}$$

*(iii)* Note that

$$E\hat{Y}^4(\pi) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 \tag{2.9}$$

where

$$Q_1 = \sum_{i=1}^{n} E\hat{Y}_z^4(i,\pi(i)),$$
$$Q_2 = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E\hat{Y}_z^3(i,\pi(i))\hat{Y}_z(j,\pi(j)),$$
$$Q_3 = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E\hat{Y}_z^2(i,\pi(i))\hat{Y}_z^2(j,\pi(j)),$$
$$Q_4 = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i,j}}^{n} E\hat{Y}_z^2(i,\pi(i))\hat{Y}_z(j,\pi(j))\hat{Y}_z(k,\pi(k)),$$
$$Q_5 = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i,j}}^{n}\sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))\hat{Y}_z(k,\pi(k))\hat{Y}_z(l,\pi(l)).$$

Simcharoen and Neammanee ([15], pp.5520–5522) showed that

$$|Q_1| \le (1+z)\delta_3, \tag{2.10}$$
$$|Q_2| \le C\Big((1+z)\delta_3 + \delta_3^2\Big) \quad \text{and} \tag{2.11}$$
$$|Q_4| \le C\Big((1+z)\delta_3 + n\delta_3^2\Big). \tag{2.12}$$

By (2.5), we get that

$$\begin{aligned}|Q_3| &= \frac{1}{n(n-1)}\sum_{i_1,i_2=1}^{n}\sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} E\hat{Y}_z^2(i_1,i_2)E\hat{Y}_z^2(j_1,j_2)\\ &\le \frac{C}{n^2}\left(\sum_{i_1,i_2=1}^{n} EY^2(i_1,i_2)\right)^2\\ &\le Cn^{\frac{2}{3}}\delta_3^{\frac{4}{3}}.\end{aligned} \tag{2.13}$$

By (2.1) and the fact that $\hat{Y}_z(i,j) = Y(i,j) - Y_z(i,j)$, we note that

$$
\begin{aligned}
|Q_5| =& \Big| \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))\big(Y(k,\pi(k)) - Y_z(k,\pi(k))\big) \\
& \times \big(Y(l,\pi(l)) - Y_z(l,\pi(l))\big)\Big| \\
\leq& \Big| \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))Y(k,\pi(k))Y(l,\pi(l))\Big| \\
& + 2\Big| \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))Y_z(k,\pi(k))Y(l,\pi(l))\Big| \\
& + \Big| \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))Y_z(k,\pi(k))Y_z(l,\pi(l))\Big| \\
\leq& \frac{C}{n^4}\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\neq i_1,j_1\\ k_2\neq i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY(k_1,k_2) \sum_{\substack{l_1,l_2=1\\ l_1\neq i_1,j_1,k_1\\ l_2\neq i_2,j_2,k_2}}^{n} EY(l_1,l_2)\Big| \\
& + \frac{C}{n^4}\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\neq i_1,j_1\\ k_2\neq i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY_z(k_1,k_2) \sum_{\substack{l_1,l_2=1\\ l_1\neq i_1,j_1,k_1\\ l_2\neq i_2,j_2,k_2}}^{n} EY(l_1,l_2)\Big| \\
& + \Big| \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))Y_z(k,\pi(k))Y_z(l,\pi(l))\Big| \\
=& \frac{C}{n^4}\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\neq i_1,j_1\\ k_2\neq i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY(k_1,k_2) \sum_{\substack{p_1\in\{i_1,j_1,k_1\},\\ p_2\in\{i_2,j_2,k_2\}}} EY(p_1,p_2)\Big| \\
& + \frac{C}{n^4}\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\neq i_1,j_1\\ k_2\neq i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY_z(k_1,k_2) \sum_{\substack{p_1\in\{i_1,j_1,k_1\},\\ p_2\in\{i_2,j_2,k_2\}}} EY(p_1,p_2)\Big| \\
& + \Big| \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i,j}}^{n} \sum_{\substack{l=1\\ l\neq i,j,k}}^{n} E\hat{Y}_z(i,\pi(i))\hat{Y}_z(j,\pi(j))Y_z(k,\pi(k))Y_z(l,\pi(l))\Big| \\
& =: Q_{51} + Q_{52} + Q_{53}. \qquad (2.14)
\end{aligned}
$$

From [15] (p.5523), we get that

$$Q_{52} \le C\left(\frac{\delta_3}{n}\sum_{i_1,i_2=1}^{n} EY^2(i_1,i_2) + n\delta_3^2\right) \quad \text{and} \quad Q_{53} \le C\delta_3^2 \sum_{i_1,i_2=1}^{n} EY^2(i_1,i_2).$$

By this fact and (2.5),

$$Q_{52} + Q_{53} \le C\Big(n^{\frac{1}{3}}\delta_3^{\frac{5}{3}} + n\delta_3^2 + n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big). \tag{2.15}$$

Observe that

$$Q_{51} \le \frac{C}{n^4}\Big(R_1 + R_2 + R_3 + R_4\Big) \tag{2.16}$$

where

$$R_1 = \Big|\sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\ne i_1\\ j_2\ne i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\ne i_1,j_1\\ k_2\ne i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY(k_1,k_2) \sum_{\substack{p_1\in\{i_1,j_1\},\\ p_2\in\{i_2,j_2\}}} EY(p_1,p_2)\Big|$$

$$R_2 = \Big|\sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\ne i_1\\ j_2\ne i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\ne i_1,j_1\\ k_2\ne i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY(k_1,k_2) \sum_{p_2\in\{i_2,j_2\}} EY(k_1,p_2)\Big|$$

$$R_3 = \Big|\sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\ne i_1\\ j_2\ne i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\ne i_1,j_1\\ k_2\ne i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)EY(k_1,k_2) \sum_{p_1\in\{i_1,j_1\}} EY(p_1,k_2)\Big|$$

$$R_4 = \Big|\sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\ne i_1\\ j_2\ne i_2}}^{n} \sum_{\substack{k_1,k_2=1\\ k_1\ne i_1,j_1\\ k_2\ne i_2,j_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2)(EY(k_1,k_2))^2\Big|.$$

By (2.1), the fact that $|\hat{Y}_z(i,j)| \le 1+z$ and the inequality

$$a_1^{r_1}a_2^{r_2}\cdots a_n^{r_n} \le r_1a_1 + r_2a_2 + \cdots + r_na_n \tag{2.17}$$

for all $a_i, r_i > 0$ and $r_1 + \cdots + r_n = 1$, we have

$$\begin{aligned}
R_1 =&\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2) \sum_{\substack{p_1\in\{i_1,j_1\},\\ p_2\in\{i_2,j_2\}}} EY(p_1,p_2) \sum_{\substack{k_1,k_2=1\\ k_1\neq i_1,j_1\\ k_2\neq i_2,j_2}}^{n} EY(k_1,k_2)\Big| \\
=&\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2) \sum_{\substack{p_1\in\{i_1,j_1\},\\ p_2\in\{i_2,j_2\}}} EY(p_1,p_2) \sum_{\substack{q_1\in\{i_1,j_1\},\\ q_2\in\{i_2,j_2\}}} EY(q_1,q_2)\Big| \\
\leq& \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{n} \sum_{\substack{p_1,q_1\in\{i_1,j_1\},\\ p_2,q_2\in\{i_2,j_2\}}} E|\hat{Y}_z(i_1,i_2)|E|\hat{Y}_z(j_1,j_2)|E|Y(p_1,p_2)|E|Y(q_1,q_2)| \\
\leq&(1+z) \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{n} \sum_{\substack{p_1,q_1\in\{i_1,j_1\},\\ p_2,q_2\in\{i_2,j_2\}}} \{E|\hat{Y}_z(i_1,i_2)|^3\}^{\frac{1}{3}}\{E|Y(p_1,p_2)|^3\}^{\frac{1}{3}}\{E|Y(q_1,q_2)|^3\}^{\frac{1}{3}} \\
\leq&(1+z) \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{n} \sum_{\substack{p_1,q_1\in\{i_1,j_1\},\\ p_2,q_2\in\{i_2,j_2\}}} \Big(\frac{E|Y(i_1,i_2)|^3}{3} + \frac{E|Y(p_1,p_2)|^3}{3} + \frac{E|Y(q_1,q_2)|^3}{3}\Big) \\
=&16(1+z)n^2 \sum_{i_1,i_2=1}^{n} E|Y(i_1,i_2)|^3 \\
=&16(1+z)n^3\delta_3. \qquad (2.18)
\end{aligned}$$

By (2.1) again, we note that

$$\begin{aligned}
R_2 =&\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2) \sum_{\substack{k_1=1\\ k_1\neq i_1,j_1}}^{n} \sum_{p_2\in\{i_2,j_2\}} EY(k_1,p_2) \sum_{\substack{k_2=1\\ k_2\neq i_2,j_2}}^{n} EY(k_1,k_2)\Big| \\
=&\Big| \sum_{i_1,i_2=1}^{n} \sum_{\substack{j_1,j_2=1\\ j_1\neq i_1\\ j_2\neq i_2}}^{n} E\hat{Y}_z(i_1,i_2)E\hat{Y}_z(j_1,j_2) \sum_{\substack{k_1=1\\ k_1\neq i_1,j_1}}^{n} \sum_{p_2,q_2\in\{i_2,j_2\}} EY(k_1,p_2)EY(k_1,q_2)\Big| \\
\leq& \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{n} \sum_{k_1=1}^{n} \sum_{p_2,q_2\in\{i_2,j_2\}} E|\hat{Y}_z(i_1,i_2)|E|\hat{Y}_z(j_1,j_2)|E|Y(k_1,p_2)|E|Y(k_1,q_2)|.
\end{aligned}$$

From this fact and the same argument as (2.18), we have

$$R_2 \leq C(1+z)n^4\delta_3. \qquad (2.19)$$

In the same way as $R_2$, we can show that

$$R_3 \leq C(1+z)n^4\delta_3. \tag{2.20}$$

By (2.5) and (2.17),

$$\begin{aligned} R_4 &\leq \Big(\sum_{i_1,i_2=1}^{n}\sum_{j_1,j_2=1}^{n} E|\hat{Y}_z(i_1,i_2)|E|\hat{Y}_z(j_1,j_2)|\Big)\sum_{k_1,k_2=1}^{n}(EY(k_1,k_2))^2 \\ &\leq \Big(n^2\sum_{i_1,i_2=1}^{n} E\hat{Y}_z^2(i_1,i_2)\Big)\sum_{k_1,k_2=1}^{n} EY^2(k_1,k_2) \\ &\leq n^2\Big(\sum_{i_1,i_2=1}^{n} EY^2(i_1,i_2)\Big)^2 \\ &\leq n^{\frac{14}{3}}\delta_3^{\frac{4}{3}}. \end{aligned} \tag{2.21}$$

From (2.16) and (2.18)–(2.21), we have

$$Q_{51} \leq C\Big((1+z)\delta_3 + n^{\frac{2}{3}}\delta_3^{\frac{4}{3}}\Big).$$

This fact, (2.14) and (2.15) imply that

$$|Q_5| \leq C\Big((1+z)\delta_3 + n^{\frac{2}{3}}\delta_3^{\frac{4}{3}} + n^{\frac{1}{3}}\delta_3^{\frac{5}{3}} + n\delta_3^2 + n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big). \tag{2.22}$$

Hence, by (2.9)–(2.13) and (2.22), we obtain

$$E\hat{Y}^4(\pi) \leq C\Big((1+z)\delta_3 + n^{\frac{2}{3}}\delta_3^{\frac{4}{3}} + n^{\frac{1}{3}}\delta_3^{\frac{5}{3}} + n\delta_3^2 + n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big).$$

*(iv)* From (iii) and the fact that $1+z \leq \sqrt{n}$,

$$E\hat{Y}^4(\pi) \leq C\Big(\sqrt{n}\delta_3 + n^{\frac{2}{3}}\delta_3^{\frac{4}{3}} + n^{\frac{1}{3}}\delta_3^{\frac{5}{3}} + n\delta_3^2 + n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big).$$

Using the same technique of (2.6) and (2.8), we can show that

$$E\hat{Y}^4(\pi) \leq C\Big(1 + n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big).$$

□

Ho and Chen ([8]) constructed the following system to give an exchangeable pair. Let

$\{I, K, L, M, \rho, \tau\}$ is independent of $Y(i, j)$'s,

$(I, K), (L, M)$ uniformly distributed on $\{(i, k)|i, k = 1, \ldots, n, i \neq k\}$,

$(I, K), (L, M)$ and $\tau$ are mutually independent,

$(I, K)$ and $\rho$ are mutually independent, and

$$\rho(\alpha) = \begin{cases} \tau(\alpha) & \text{if } \alpha \neq I, K, \tau^{-1}(L), \tau^{-1}(M), \\ L & \text{if } \alpha = I, \\ M & \text{if } \alpha = K, \\ \tau(I) & \text{if } \alpha = \tau^{-1}(L), \\ \tau(K) & \text{if } \alpha = \tau^{-1}(M), \end{cases}$$

where $\rho(\rho^{-1}(\alpha)) = \rho^{-1}(\rho(\alpha)) = \alpha$. Let

$$\tilde{Y}(\rho) = \hat{Y}(\rho) - \widehat{S}_1 - \widehat{S}_2 + \widehat{S}_3 + \widehat{S}_4$$

where $\widehat{S}_1 = \hat{Y}_z(I, \rho(I))$, $\widehat{S}_2 = \hat{Y}_z(K, \rho(K))$, $\widehat{S}_3 = \hat{Y}_z(I, \rho(K))$ and $\widehat{S}_4 = \hat{Y}_z(K, \rho(I))$. Then

$$\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4 \quad \text{are identically distributed} \tag{2.23}$$

and $\tilde{Y}(\rho)$ and $\hat{Y}(\rho)$ are an exchangeable pair (see [11], [17] for more details).

In the case of $VarS_n = 1$, Neammanee and Rurkruthairat ([12]) showed that

$$E(\tilde{Y}(\rho) - \hat{Y}(\rho))^2 = \frac{4}{n} + \mathcal{R}$$

where $|\mathcal{R}| \leq \frac{4\delta_3}{n} + \frac{C}{n^2}$ for some $C > 0$. In Lemma 2.5, we compute $E(\tilde{Y}(\rho) - \hat{Y}(\rho))^2$ without using the condition that $VarS_n = 1$.

**Lemma 2.5.** *Assume that (2.1) and (2.2) hold. Then*

$$E(\tilde{Y}(\rho)-\hat{Y}(\rho))^2 = \frac{4VarS_n}{n} + \mathcal{R}$$

*where* $|\mathcal{R}| \leq C\left(\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{5}{3}}} + \frac{\delta_3}{n} + \frac{\delta_3^{\frac{4}{3}}}{n^{\frac{4}{3}}} + \frac{\delta_3^2}{n}\right)$ *for some* $C > 0$.

*Proof.* We can follow the argument of Lemma 3.7 in [14] (pp.30–33) to show that

$$E(\tilde{Y}(\rho)-\hat{Y}(\rho))^2 = \frac{4VarS_n}{n} + \mathcal{R} \tag{2.24}$$

where

$$\begin{aligned}|\mathcal{R}| \leq & C\Bigg[\frac{1}{n^2(n-1)}\sum_{i_1,i_2=1}^{n} EY^2(i_1,i_2) + \frac{\delta_3}{n-1} + \frac{1}{n^2(n-1)^2}\left(\sum_{i_1,i_2=1}^{n} EY^2(i_1,i_2)\right)^2 \\ & + \frac{\delta_3^2}{n-1}\Bigg].\end{aligned}$$

By (2.5), we get that

$$|\mathcal{R}| \leq C\left(\frac{n^{\frac{4}{3}}\delta^{\frac{2}{3}}}{n^3} + \frac{\delta_3}{n} + \frac{(n^{\frac{4}{3}}\delta^{\frac{2}{3}})^2}{n^4} + \frac{\delta_3^2}{n}\right) \leq C\left(\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{5}{3}}} + \frac{\delta_3}{n} + \frac{\delta_3^{\frac{4}{3}}}{n^{\frac{4}{3}}} + \frac{\delta_3^2}{n}\right).$$

From this fact and (2.24), the proof is complete. □

## 2.2 Proof of Theorem 2.2

*Proof.* From the fact that $\Phi(z) = 1 - \Phi(-z)$ for all $z \in \mathbb{R}$, it suffices to prove for the case that $z \geq 1$ as we can apply the result to $-S_n$ when $z \leq -1$.

Assume that $z \geq 1$. To prove our result, we use the Stein method which was introduced by Stein ([16]) in 1972. It begins from the Stein's equation for the normal distribution function

$$g'(w) - wg(w) = I_z(w) - \Phi(z) \tag{2.25}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and piecewise differentiable function and

$$I_z(w) = \begin{cases} 1 & \text{if } w \le z, \\ 0 & \text{if } w > z. \end{cases}$$

It is well–known that the solution $g_z$ of (2.25) is given by

$$g_z(w) = \begin{cases} \sqrt{2\pi} e^{\frac{1}{2}w^2} \Phi(w)[1 - \Phi(z)] & \text{if } w \le z, \\ \sqrt{2\pi} e^{\frac{1}{2}w^2} \Phi(z)[1 - \Phi(w)] & \text{if } w > z \end{cases}$$

and

$$0 \le g_z(w) \le \min\Big\{\frac{\sqrt{2\pi}}{4}, \ \frac{1}{|z|}\Big\} \quad \text{for all } w \in \mathbb{R} \quad ([17], \ \text{pp.22–23}). \tag{2.26}$$

Simcharoen and Neammanee ([15], pp.5519, 5526–5527) showed that

$$|P(S_n \le z) - \Phi(z)| \le \frac{\delta_3}{(1+z)^3} + T_1 + T_2 + T_3 + T_4 \tag{2.27}$$

where

$$\begin{aligned} T_1 &= \Big| E g_z'(\hat{Y}(\tau)) \int_{-\infty}^{\infty} K(t)dt - E \int_{-\infty}^{\infty} g_z'(\hat{Y}(\rho) + t) K(t) dt \Big|, \\ T_2 &= \Big| E g_z'(\hat{Y}(\tau)) E \int_{-\infty}^{\infty} K(t)dt - E g_z'(\hat{Y}(\tau)) \int_{-\infty}^{\infty} K(t)dt \Big|, \\ T_3 &= \Big| E g_z'(\hat{Y}(\tau)) \Big| \Big| 1 - \frac{n-1}{4} E(\tilde{Y}(\rho) - \hat{Y}(\rho))^2 \Big|, \\ T_4 &= \frac{1}{n} \Big\{ E g_z^2(\hat{Y}(\rho)) \Big\}^{\frac{1}{2}} \Big\{ E\Big( \sum_{i,j=1}^{n} \hat{Y}_z(i,j) \Big)^2 \Big\}^{\frac{1}{2}} \end{aligned}$$

and $K(t) = \dfrac{n-1}{4}[\tilde{Y}(\rho) - \hat{Y}(\rho)][\mathbb{I}(0 \le t \le \tilde{Y}(\rho) - \hat{Y}(\rho)) - \mathbb{I}(\tilde{Y}(\rho) - \hat{Y}(\rho) \le t < 0)]$.

By (2.26) and Lemma 2.4 (i), we get that

$$T_4 \le \frac{C}{nz}\Big(n + n^2 \delta_3^2\Big)^{\frac{1}{2}} \le \frac{C}{1+z}\left(\frac{1}{\sqrt{n}} + \delta_3\right). \tag{2.28}$$

To bound $T_3$, by following the argument of Lemma 5.1 in [4] (p.248), we can show that

$$E|g_z'(\hat{Y}(\tau))| \le \frac{C}{1+z}\Big(1 + E\hat{Y}^2(\tau)\Big).$$

By this fact and Lemma 2.4 (ii), we have

$$E|g_z'(\hat{Y}(\tau))| \le \frac{C}{1+z}\Big(1 + VarS_n + \delta_3^2\Big). \tag{2.29}$$

By (2.1), we observe that $ES_n = 0$ and

$$\begin{aligned} VarS_n =& \; ES_n^2 \\ =& \sum_{i=1}^{n} EY^2(i,\pi(i)) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} EY(i,\pi(i))Y(j,\pi(j)) \\ =& \; \frac{1}{n}\sum_{i,j=1}^{n} EY^2(i,j) + \frac{1}{n(n-1)}\sum_{i,j=1}^{n}[EY(i,j)]^2 \qquad (2.30) \\ \le& \; \frac{1}{n}\sum_{i,j=1}^{n} EY^2(i,j) + \frac{1}{n(n-1)}\sum_{i,j=1}^{n} EY^2(i,j) \\ \le& \; Cn^{\frac{1}{3}}\delta_3^{\frac{2}{3}} \qquad (2.31) \end{aligned}$$

where we have used (2.5) in the last inequality. By (2.31), Lemma 2.5 and the same teqnique as (2.6) and (2.8), we get that

$$\begin{aligned} \Big|1 - \frac{n-1}{4}E(\tilde{Y}(\rho) - \hat{Y}(\rho))^2\Big| &\le C\left(|VarS_n - 1| + \frac{VarS_n}{n} + \frac{\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}} + \delta_3 + \frac{\delta_3^{\frac{4}{3}}}{n^{\frac{1}{3}}} + \delta_3^2\right) \\ &\le C\left(|VarS_n - 1| + \frac{\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}} + \delta_3 + \frac{\delta_3^{\frac{4}{3}}}{n^{\frac{1}{3}}} + \delta_3^2\right) \\ &\le C\left(|VarS_n - 1| + \frac{1}{\sqrt{n}} + \sqrt{n}\delta_3^2\right). \end{aligned}$$

By this fact, (2.29), (2.31) and the same technique as (2.6) and (2.8), we have

$$\begin{aligned}
T_3 \leq & \frac{C}{1+z}\Bigg(|VarS_n - 1| + |VarS_n - 1|VarS_n + |VarS_n - 1|\delta_3^2 + \frac{1}{\sqrt{n}} + \frac{VarS_n}{\sqrt{n}} \\
& + \sqrt{n}\delta_3^2 + \sqrt{n}\delta_3^2 VarS_n + \sqrt{n}\delta_3^4\Bigg) \\
\leq & \frac{C}{1+z}\Bigg(|VarS_n - 1| + |VarS_n - 1|VarS_n + |VarS_n - 1|\delta_3^2 + \frac{1}{\sqrt{n}} + \frac{\delta_3^{\frac{2}{3}}}{n^{\frac{1}{6}}} \\
& + \sqrt{n}\delta_3^2 + n^{\frac{5}{6}}\delta_3^{\frac{8}{3}} + \sqrt{n}\delta_3^4\Bigg) \\
\leq & \frac{C}{1+z}\Bigg(|VarS_n - 1| + |VarS_n - 1|VarS_n + |VarS_n - 1|\delta_3^2 + \frac{1}{\sqrt{n}} + n^{\frac{3}{2}}\delta_3^4\Bigg).
\end{aligned} \tag{2.32}$$

To bound $T_2$, let

$$\begin{aligned}
G = & \hat{Y}_z(I, M) + \hat{Y}_z(K, L) - \hat{Y}_z(I, L) - \hat{Y}_z(K, M), \\
A = & \{\tau(I) \neq L, \tau(K) \neq M, \tau(I) \neq M, \tau(K) \neq L\}
\end{aligned}$$

and $\mathcal{B}$ be the $\sigma$-algebra generated by $\{I, K, L, M, Y(i,j) : 1 \leq i, j \leq n\}$. Neammanee and Rattanawong ([11], p.40) showed that

$$E^{\mathcal{B}}\mathbb{I}(A^C) \leq \frac{C}{n}, \tag{2.33}$$

$$EG^2\mathbb{I}(A^C) \leq \frac{C}{n^3}\sum_{i,j=1}^{n} EY^2(i,j) \quad \text{and} \tag{2.34}$$

$$E|G|^3\mathbb{I}(A^C) \leq \frac{C\delta_3}{n^2}. \tag{2.35}$$

From [15] (p.5527), we have that

$$T_2 \leq \frac{Cn}{1+z}EG^2\mathbb{I}(A^C) + Cn\left(\frac{E\hat{Y}^4(\tau)}{z^4}\right)^{\frac{1}{3}}\left(E|G|^3\mathbb{I}(A^C)\right)^{\frac{2}{3}}.$$

Thus, by (2.5), (2.34), (2.35), Lemma 2.4 (iv) and the same technique as (2.6) and

(2.8), we have

$$
\begin{aligned}
T_2 \leq & \frac{C}{(1+z)n^2}\sum_{i,j=1}^{n} EY^2(i,j) + \frac{C\delta_3^{\frac{2}{3}}}{z^{\frac{4}{3}}n^{\frac{1}{3}}}\Big(E\hat{Y}^4(\tau)\Big)^{\frac{1}{3}} \\
\leq & \frac{C\delta_3^{\frac{2}{3}}}{(1+z)n^{\frac{2}{3}}} + \frac{C\delta_3^{\frac{2}{3}}}{(1+z)n^{\frac{1}{3}}}\Big(1+n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big)^{\frac{1}{3}} \\
\leq & \frac{C}{1+z}\left(\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{1}{3}}} + n^{\frac{1}{9}}\delta_3^{\frac{14}{9}}\right) \\
\leq & \frac{C}{1+z}\left(\frac{1}{n^{\frac{2}{3}}} + n^{\frac{1}{3}}\delta_3^{2}\right). \qquad (2.36)
\end{aligned}
$$

To bound $T_1$, let $\Delta\hat{Y} = \hat{Y}(\rho) - \hat{Y}(\tau)$ and $\delta = |\Delta\hat{Y}| + |\tilde{Y}(\rho) - \hat{Y}(\rho)|$. From [15] (p.5528), we have

$$
\begin{aligned}
E|\tilde{Y}(\rho) - \hat{Y}(\rho)|^3 &\leq \frac{C\delta_3}{n}, & (2.37)\\
E(\tilde{Y}(\rho) - \hat{Y}(\rho))^4 &\leq \frac{C(1+z)\delta_3}{n}, & (2.38)\\
E\delta^3 &\leq \frac{C\delta_3}{n} \quad \text{and} & (2.39)\\
E\delta^4 &\leq \frac{C(1+z)\delta_3}{n}. & (2.40)
\end{aligned}
$$

By (2.5) and (2.23), we note that

$$
\begin{aligned}
E\delta^2 &\leq C(E|\Delta\hat{Y}|^2 + E|\tilde{Y}(\rho) - \hat{Y}(\rho)|^2) \\
&\leq CE\hat{Y}_z^2(I,\rho(I)) \\
&\leq \frac{C}{n^2}\sum_{i,j=1}^{n} EY^2(i,j) \\
&\leq \frac{C\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}}. \qquad (2.41)
\end{aligned}
$$

Let

$$f_\delta(t) = \begin{cases} 0 & \text{if } t < z - 2\delta, \\ (1+t+\delta)(t-z+2\delta) & \text{if } z-2\delta \le t \le z+2\delta, \\ 4\delta(1+t+\delta) & \text{if } t > z+2\delta. \end{cases}$$

From [15] (pp.5528–5530), we have

$$T_1 \le T_{11} + (1+z)(M_1+M_2) + \frac{C\delta_3}{1+z}, \tag{2.42}$$

where

$$\begin{aligned} T_{11} &\le \frac{C}{1+z}\left[|E\hat{Y}(\tau)f_\delta(\hat{Y}(\tau))| + \frac{1}{n}\left\{E\left(\sum_{i,j=1}^{n}\hat{Y}_z(i,j)\right)^2\right\}^{\frac{1}{2}}\left\{Ef_\delta^2(\hat{Y}(\tau))\right\}^{\frac{1}{2}}\right], \\ M_1 &\le \frac{Cn}{z^2}(E\delta^3)^{\frac{1}{3}}\left(E\hat{Y}^4(\tau)\right)^{\frac{1}{2}}\left(E|\tilde{Y}(\rho)-\hat{Y}(\rho)|^3\right)^{\frac{2}{3}}, \\ M_2 &\le \frac{Cn}{z}\left(E\hat{Y}^4(\tau)\right)^{\frac{1}{4}}\left(E\delta^4\right)^{\frac{1}{4}}\left(E|\tilde{Y}(\rho)-\hat{Y}(\rho)|^4E^{\mathcal{B}}\mathbb{I}(A^C)\right)^{\frac{1}{2}}. \end{aligned}$$

By (2.31) and Lemma 2.4 (ii), we have

$$E\hat{Y}^2(\pi) \le C(1 + n^{\frac{1}{3}}\delta_3^{\frac{2}{3}} + \delta_3^2)$$

and note that $|f_\delta(t)| \le 4\delta(1+|t|+\delta)$ for all $t \in \mathbb{R}$.

From these facts, (2.39)–(2.41), Lemma 2.4 (iv) and the fact that $1+z \le n^{\frac{1}{14}}$, we have

$$\begin{aligned} &E|\hat{Y}(\tau)f_\delta(\hat{Y}(\tau))| \\ &\quad\le C\Big(E|\hat{Y}(\tau)|\delta + E\hat{Y}^2(\tau)\delta + E|\hat{Y}(\tau)|\delta^2\Big) \\ &\quad\le C\Big(\{E\hat{Y}^2(\tau)\}^{\frac{1}{2}}\{E\delta^2\}^{\frac{1}{2}} + \{E\hat{Y}^4(\tau)\}^{\frac{1}{2}}\{E\delta^2\}^{\frac{1}{2}} + \{E\hat{Y}^4(\tau)\}^{\frac{1}{4}}\{E\delta^3\}^{\frac{2}{3}}\Big) \\ &\quad\le C\left(\frac{\delta_3^{\frac{1}{3}}}{n^{\frac{1}{3}}} + \frac{\delta_3^{\frac{2}{3}}}{n^{\frac{1}{6}}} + \frac{\delta_3^{\frac{4}{3}}}{n^{\frac{1}{3}}} + n^{\frac{1}{3}}\delta_3^{\frac{5}{3}}\right) \end{aligned}$$

and

$$
\begin{aligned}
&Ef_\delta^2(\hat{Y}(\rho))\\
\le& CE(\delta+\delta|\hat{Y}(\rho)|+\delta^2)^2\\
\le& C\Big(E\delta^2+\{E\delta^4\}^{\frac{1}{2}}\{E\hat{Y}^4(\rho)\}^{\frac{1}{2}}+E\delta^4\Big)\\
\le& C\left(\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}}+\frac{(1+z)^{\frac{1}{2}}\delta_3^{\frac{1}{2}}}{\sqrt{n}}+(1+z)^{\frac{1}{2}}n^{\frac{1}{6}}\delta_3^{\frac{11}{6}}+\frac{(1+z)\delta_3}{n}\right)\\
\le& C\left(\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}}+\frac{\delta_3^{\frac{1}{2}}}{n^{\frac{13}{28}}}+n^{\frac{17}{84}}\delta_3^{\frac{11}{6}}+\frac{\delta_3}{n^{\frac{13}{14}}}\right).
\end{aligned}
$$

From these facts and the same technique as (2.6) and (2.8), we have

$$
\begin{aligned}
T_{11}\le&\frac{C}{1+z}\Bigg[\frac{\delta_3^{\frac{1}{3}}}{n^{\frac{1}{3}}}+\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}}+\frac{\delta_3^{\frac{4}{3}}}{n^{\frac{1}{3}}}+n^{\frac{1}{3}}\delta_3^{\frac{5}{3}}\\
&+\frac{1}{n}\Big(n+n^2\delta_3^2\Big)^{\frac{1}{2}}\Big(\frac{\delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}}+\frac{\delta_3^{\frac{1}{2}}}{n^{\frac{13}{28}}}+n^{\frac{17}{84}}\delta_3^{\frac{11}{6}}+\frac{\delta_3}{n^{\frac{13}{14}}}\Big)^{\frac{1}{2}}\Bigg]\\
\le&\frac{C}{1+z}\Big(\frac{1}{\sqrt{n}}+\sqrt{n}\delta_3^2\Big).
\end{aligned}
\tag{2.43}
$$

By (2.33), (2.37)–(2.40), Lemma 2.4 (iv) and the fact that $1+z\le n^{\frac{1}{14}}$, we get that

$$
M_1\le\frac{C\delta_3}{(1+z)^2}\Big(E\hat{Y}^4(\tau)\Big)^{\frac{1}{2}}\le\frac{C}{(1+z)^2}\Big(\delta_3+n^{\frac{2}{3}}\delta_3^{\frac{7}{3}}\Big)
\tag{2.44}
$$

and

$$
\begin{aligned}
M_2\le&\frac{C(1+z)^{\frac{3}{4}}\delta_3^{\frac{3}{4}}}{zn^{\frac{1}{4}}}\Big(E\hat{Y}^4(\tau)\Big)^{\frac{1}{4}}\\
\le&\frac{C(1+z)^{\frac{7}{4}}\delta_3^{\frac{3}{4}}}{(1+z)^2n^{\frac{1}{4}}}\Big(1+n^{\frac{4}{3}}\delta_3^{\frac{8}{3}}\Big)^{\frac{1}{4}}\\
\le&\frac{C}{(1+z)^2}\Big(\frac{\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}}+n^{\frac{5}{24}}\delta_3^{\frac{17}{12}}\Big).
\end{aligned}
\tag{2.45}
$$

From (2.42)–(2.45), and the same technique as (2.6) and (2.8), we get that

$$T_1 \leq \frac{C}{1+z}\Big(\frac{1}{\sqrt{n}} + n^{\frac{3}{2}}\delta_3^4\Big). \tag{2.46}$$

From (2.27), (2.28), (2.32), (2.36) and (2.46), we obtain that

$$\begin{aligned}
&|P(S_n \leq z) - \Phi(z)| \\
&\quad\leq \frac{C}{1+z}\Bigg(|VarS_n - 1| + |VarS_n - 1|VarS_n + |VarS_n - 1|\delta_3^2 + \frac{1}{\sqrt{n}} + \delta_3 \\
&\qquad + \sqrt{n}\delta_3^2 + n^{\frac{3}{2}}\delta_3^4\Bigg) \\
&\quad\leq \frac{C}{1+z}\Bigg(|VarS_n - 1| + |VarS_n - 1|VarS_n + |VarS_n - 1|\delta_3^2 + \frac{1}{\sqrt{n}} + n^{\frac{3}{2}}\delta_3^4\Bigg).
\end{aligned}$$

□

# CHAPTER III

# NON-UNIFORM BOUND FOR COMBINATORIAL RANDOM SUMS

Let $\{X_n\}_{n=2}^{\infty}$ be a sequence of random matrices such that $X_n = [X_n(i,j)],\ i,j = 1,2,\ldots,n,$ is an $n \times n$ matrix of independent random variables. Let $\{\pi_n\}_{n=2}^{\infty}$ be a sequence of random permutations such that $\pi_n = (\pi_n(1), \pi_n(2), \ldots, \pi_n(n))$ is a uniformly distributed random permutation on $\{1,2,\ldots,n\}$. Assume that $X_n$ and $\pi_n$ are independent for all $n \geq 2$. Let $N$ be an integer random variable such that $P(N \geq 2) = 1$ and $N$ is independent with $\{X_n\}_{n=2}^{\infty}$ and $\{\pi_n\}_{n=2}^{\infty}$. Define the **combinatorial random sums** by

$$S_N = \sum_{i=1}^{N} X_N(i, \pi_N(i)).$$

In this chapter, we study the distance between the distribution function of $S_N$ and the standard normal distribution function $\Phi$.

Let

$$\Delta_{N,z} := \Big| P\Big(\frac{S_N}{\sqrt{VarS_N}} \leq z\Big) - \Phi(z) \Big| \text{ and } \ \Delta_N := \sup_{z \in \mathbb{R}} \Delta_{N,z}.$$

In 2017, Frolov ([6]) gave a bound of $\Delta_N$ under the $(2+\delta)$–th moment conditions where $\delta \in (0,1]$. The following is his result.

**Theorem 3.1.** *([6]) Assume that*

$$P(N \geq 2) = 1,$$

$$\exists \delta \in (0,1],\ E|X_n(i,j)|^{2+\delta} < \infty \quad \textit{for all } 1 \leq i,j \leq n \textit{ and } n \geq 2,$$

$$\sum_{i=1}^{n} EX_n(i,k) = 0 \textit{ and } \sum_{j=1}^{n} EX_n(k,j) = 0 \textit{ for all } 1 \leq k \leq n \textit{ and } n \geq 2, \tag{3.1}$$

$$B_n := VarS_n > 0 \textit{ for all } \ n \geq 2 \ \textit{ and} \tag{3.2}$$

$$EB_N < \infty. \tag{3.3}$$

*Then*

$$\Delta_N \leq 45112EL_{N,\delta} + \frac{3\sqrt{VarB_N}}{EB_N},$$

*where* $L_{n,\delta} = \dfrac{1}{nB_n^{1+\frac{\delta}{2}}} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} E|X_n(i,j)|^{2+\delta}.$

In this chapter, we investigate a non–uniform bound of $\Delta_{N,z}$ under the third moment conditions. The followings are our main results.

**Theorem 3.2.** *Assume that*

$$P(N \geq 4) = 1, \tag{3.4}$$

$$E|X_n(i,j)|^3 < \infty \quad \textit{for all } 1 \leq i,j \leq n \textit{ and } n \geq 2 \tag{3.5}$$

*and (3.1)–(3.3) hold. Then there exists a positive constant $C$ such that for fixed $z \in \mathbb{R}$ with $P(1+|z| \leq N^{\frac{1}{14}}) = 1$, we have*

$$\Delta_{N,z} \leq \frac{C}{1+|z|}\Bigg[\frac{\sqrt{VarB_N}}{EB_N} + \frac{\sqrt{VarB_N}}{(EB_N)^2}\Big(EB_N^2\Big)^{\frac{1}{2}} + \frac{\sqrt{VarB_N}}{(EB_N)^4}\Big\{E\Big(B_N^6\gamma_N^4\Big)\Big\}^{\frac{1}{2}} + E\Big(\frac{1}{\sqrt{N}}\Big) + E\gamma_N + \frac{E\Big(B_N^6 N^{\frac{3}{2}}\gamma_N^4\Big)}{(EB_N)^6}\Bigg],$$

*where* $\gamma_n = L_{n,1}.$

To show that there exists a situation such that the bound in Theorem 3.2 tends to zero, we give some examples of the random number $N$ in Theorem 3.3 and Theorem 3.4.

**Theorem 3.3.** *For $k, n \in \mathbb{N}$, let $N_{k,n}$ be a random variable such that*

$$P(N_{k,n} = m) = \frac{e^{-n} n^{m-k}}{(m-k)!} \quad \text{where } m = k, k+1, \dots. \tag{3.6}$$

*Assume that (3.1), (3.2) and (3.5) hold,*

$$\gamma_n = O(\frac{1}{n^{\alpha}}) \text{ as } n \to \infty \text{ for some } \alpha > \frac{3}{8} \text{ and} \tag{3.7}$$

$$B_n \sim cn^{\beta} \text{ as } n \to \infty \text{ for some } c, \beta > 0. \tag{3.8}$$

*Then for fixed $z \in \mathbb{R}$ and $k \geq 4$ such that $1 + |z| \leq k^{\frac{1}{14}}$, we have*

$$\Delta_{N_{k,n},z} \to 0 \text{ as } n \to \infty.$$

**Theorem 3.4.** *For $n \in \mathbb{N}$, let $N_n$ be a random variable such that*

$$P(N_n = n) = \frac{1}{n} \text{ and } P(N_n = 2n) = 1 - \frac{1}{n}. \tag{3.9}$$

*Assume that (3.1), (3.2), (3.5), (3.7) and (3.8) hold. Then for fixed $z \in \mathbb{R}$ and $n \geq 4$ such that $1 + |z| \leq n^{\frac{1}{14}}$, we have*

$$\Delta_{N_n,z} \to 0 \text{ as } n \to \infty.$$

In this chapter, we organize as the follows. In Section 3.1, we prove Theorem 3.2 by using the result of Theorem 2.2. In Section 3.2, we prove Theorem 3.3 by using the idea from [6]. A proof of Theorem 3.4 is in Section 3.3. In the last section, we give some examples satisfying our conditions.

## 3.1 Proof of Theorem 3.2

*Proof.* Without loss of generality, it suffices to prove the theorem in the case of $z \geq 0$ as we can apply the result to $-\dfrac{S_N}{\sqrt{VarS_N}}$ when $z < 0$.
If $0 \leq z < 1$, then $1 < \dfrac{2}{1+z}$. By Theorem 3.1, we have

$$\Delta_{N,z} \leq 45112E\gamma_N + \frac{3\sqrt{VarB_N}}{EB_N} \leq \frac{2}{1+z}\Bigg(45112E\gamma_N + \frac{3\sqrt{VarB_N}}{EB_N}\Bigg). \quad (3.10)$$

Next, we will prove the theorem in the case of $z \geq 1$.
Note that $Var\Big(\dfrac{S_n}{\sqrt{EB_N}}\Big) = \dfrac{B_n}{EB_N}$ and $E\left|\dfrac{X_n(i,j)}{\sqrt{EB_N}}\right|^3 = \dfrac{E|X_n(i,j)|^3}{(EB_N)^{\frac{3}{2}}}$. By Theorem 2.2, there exists a constant $C > 0$ such that for all $n \geq 4$ such that $1 + z \leq n^{\frac{1}{14}}$, we have

$$\begin{aligned}
&\Big|P\Big(\frac{S_n}{\sqrt{EB_N}} \leq z\Big) - \Phi(z)\Big| \\
&\leq \frac{C}{1+z}\Bigg(\Big|\frac{B_n}{EB_N} - 1\Big| + \Big|\frac{B_n}{EB_N} - 1\Big|\frac{B_n}{EB_N} + \Big|\frac{B_n}{EB_N} - 1\Big|\frac{\beta_3^2}{(EB_N)^3} + \frac{1}{\sqrt{n}} + \frac{n^{\frac{3}{2}}\beta_3^4}{(EB_N)^6}\Bigg) \\
&= \frac{C}{1+z}\Bigg(\frac{|B_n - EB_N|}{EB_N} + \frac{|B_n - EB_N|B_n}{(EB_N)^2} + \frac{|B_n - EB_N|B_n^3\gamma_n^2}{(EB_N)^4} + \frac{1}{\sqrt{n}} + \frac{B_n^6 n^{\frac{3}{2}}\gamma_n^4}{(EB_N)^6}\Bigg),
\end{aligned}$$

where $\beta_3 = \dfrac{1}{n}\displaystyle\sum_{i,j=1}^{n} E|X_n(i,j)|^3$. Since $VarS_N = EB_N > 0$ ([6], p.5934) and $E|B_N - EB_N| \leq \sqrt{VarB_N}$, we get that

$$\begin{aligned}
\Delta_{N,z} &= \Big|P\Big(\frac{S_N}{\sqrt{EB_N}} \leq z\Big) - \Phi(z)\Big| \\
&= \sum_{n=4}^{\infty} P(N=n)\Big|P\Big(\frac{S_n}{\sqrt{EB_N}} \leq z\Big) - \Phi(z)\Big| \\
&\leq \frac{C}{1+z}\Bigg[\frac{E|B_N - EB_N|}{EB_N} + \frac{E\Big(|B_N - EB_N|B_N\Big)}{(EB_N)^2} + \frac{E\Big(|B_N - EB_N|B_N^3\gamma_N^2\Big)}{(EB_N)^4} \\
&\quad + E\Big(\frac{1}{\sqrt{N}}\Big) + \frac{E\Big(B_N^6 N^{\frac{3}{2}}\gamma_N^4\Big)}{(EB_N)^6}\Bigg]
\end{aligned}$$

$$\leq \frac{C}{1+z}\left[\frac{\sqrt{VarB_N}}{EB_N} + \frac{\sqrt{VarB_N}}{(EB_N)^2}\Big(EB_N^2\Big)^{\frac{1}{2}} + \frac{\sqrt{VarB_N}}{(EB_N)^4}\Big\{E\Big(B_N^6\gamma_N^4\Big)\Big\}^{\frac{1}{2}} + E\Big(\frac{1}{\sqrt{N}}\Big) + \frac{E\Big(B_N^6 N^{\frac{3}{2}}\gamma_N^4\Big)}{(EB_N)^6}\right].$$

From this fact and (3.10), the proof is complete. □

## 3.2 Proof of Theorem 3.3

In this section, we consider the bound in Theorem 3.2 in the case of the random number having the Poisson distribution. Throughout this section, for $k, n \in \mathbb{N}$, we define a Poisson random number $N_{k,n}$ by

$$P(N_{k,n} = m) = \frac{e^{-n}n^{m-k}}{(m-k)!} \text{ where } m = k, k+1, \ldots .$$

In the following proposition, we will show that $EB^r_{N_{k,n}} \sim (EB_{N_{k,n}})^r$ as $n \to \infty$ for all $k \in \mathbb{N}$ and $r > 0$. To prove this proposition, we use a technique from [6].

**Proposition 3.5.** *Assume that (3.2) and (3.5) hold and*

$$B_n \sim cn^\beta \text{ as } n \to \infty \text{ for some } c, \beta > 0. \tag{3.11}$$

*Then for all $k \geq 2$ and $r > 0$, we have $0 < EB^r_{N_{k,n}} < \infty$ for all $n \in \mathbb{N}$ and*

$$EB^r_{N_{k,n}} \sim (EB_{N_{k,n}})^r \text{ as } n \to \infty.$$

*Proof.* Let $k \geq 2$, $r > 0$ and $l = \lfloor r\beta \rfloor + 1$ where $\lfloor \cdot \rfloor$ denotes the integer part of the number in the bracket. First, we will show that $0 < EB^r_{N_{k,n}} < \infty$ for all $n \in \mathbb{N}$. Since

$$\frac{m^l}{(m-k)(m-k-1)\cdots(m-k-l+1)} \to 1 \text{ as } m \to \infty, \tag{3.12}$$

there exists $m_0 \geq k + l$ such that

$$\frac{m^l}{(m-k)(m-k-1)\cdots(m-k-l+1)} < 2 \quad \text{for } m \geq m_0. \tag{3.13}$$

By (3.11), there exists $m_1 \geq \max\{m_0, k+1\}$ such that

$$B_m < 2cm^\beta \quad \text{for } m \geq m_1.$$

From this fact, (3.2) and (3.13), we have

$$\begin{aligned}
0 < EB^r_{N_{k,n}} &= \sum_{m=k}^{\infty} P(N_{k,n} = m) B^r_m \\
&= \sum_{m=k}^{m_1-1} P(N_{k,n} = m) B^r_m + \sum_{m=m_1}^{\infty} P(N_{k,n} = m) B^r_m \\
&< \sum_{m=k}^{m_1-1} \frac{e^{-n} n^{m-k}}{(m-k)!} \max_{k \leq m \leq m_1-1} B^r_m + 2^r c^r \sum_{m=m_1}^{\infty} \frac{e^{-n} n^{m-k} m^l}{(m-k)!} \\
&< e^{-n} n^{m_1-k-1} \max_{k \leq m \leq m_1-1} B^r_m + 2^{r+1} c^r \sum_{m=m_1}^{\infty} \frac{e^{-n} n^{m-k}}{(m-k-l)!} \\
&= e^{-n} n^{m_1-k-1} \max_{k \leq m \leq m_1-1} B^r_m + 2^{r+1} c^r n^l \sum_{m=m_1-l}^{\infty} \frac{e^{-n} n^{m-k}}{(m-k)!} \\
&= e^{-n} n^{m_1-k-1} \max_{k \leq m \leq m_1-1} B^r_m + 2^{r+1} c^r n^l P(N_{k,n} \geq m_1 - l) \\
&< \infty \quad \text{for every } n \in \mathbb{N}.
\end{aligned}$$

Next, we will show that $EB^r_{N_{k,n}} \sim c^r n^{r\beta}$ as $n \to \infty$.
Let $\epsilon$ and $\delta$ be positive real numbers such that $\epsilon, \delta \in (0,1)$. We write

$$\begin{aligned}
EB^r_{N_{k,n}} &= \sum_{m=k}^{\infty} P(N_{k,n} = m) B^r_m \\
&= \sum_{m=k}^{\lfloor (1-\delta)n \rfloor} P(N_{k,n} = m) B^r_m + \sum_{m=\lfloor (1-\delta)n \rfloor + 1}^{\lfloor (1+\delta)n \rfloor - 1} P(N_{k,n} = m) B^r_m \\
&\quad + \sum_{m=\lfloor (1+\delta)n \rfloor}^{\infty} P(N_{k,n} = m) B^r_m \\
&=: s_1 + s_2 + s_3.
\end{aligned}$$

By (3.11), there exists $n_0 \geq k+1$ such that

$$(1-\delta)cn^{\beta} < B_n < (1+\delta)cn^{\beta} \quad \text{for all } n \geq n_0. \tag{3.14}$$

From this fact, we note that for $n \geq \dfrac{n_0}{1-\delta}$,

$$\begin{aligned}
s_1 &= \sum_{m=k}^{n_0-1} P(N_{k,n}=m)B_m^r + \sum_{m=n_0}^{\lfloor (1-\delta)n \rfloor} P(N_{k,n}=m)B_m^r \\
&< \sum_{m=k}^{n_0-1} \frac{e^{-n}n^{m-k}}{(m-k)!}B_m^r + (1+\delta)^r c^r \sum_{m=n_0}^{\lfloor (1-\delta)n \rfloor} P(N_{k,n}=m)m^{r\beta} \\
&\leq e^{-n}n^{n_0-k-1} \max_{k\leq m\leq n_0-1} B_m^r + (1+\delta)^r c^r (\lfloor (1-\delta)n \rfloor)^{r\beta} P(N_{k,n} \leq \lfloor (1-\delta)n \rfloor) \\
&\leq e^{-n}n^{n_0-k-1} \max_{k\leq m\leq n_0-1} B_m^r + (1+\delta)^r c^r (1-\delta)^{r\beta} n^{r\beta} P(N_{k,n} \leq \lfloor (1-\delta)n \rfloor).
\end{aligned} \tag{3.15}$$

Observe that $N_{k,n} = X_n + k$ where $X_n$ is a Poisson random variable with mean $n$. Then $EN_{k,n} = EX_n + k = n + k$ and $VarN_{k,n} = VarX_n = n$. By Chebyshev's inequality, we have

$$\begin{aligned}
P(N_{k,n} \leq \lfloor (1-\delta)n \rfloor) &\leq P(N_{k,n} \leq (1-\delta)n) \\
&= P(N_{k,n} - EN_{k,n} \leq -\delta n - k) \\
&\leq P(|N_{k,n} - EN_{k,n}| \geq \delta n + k) \\
&\leq \frac{VarN_{k,n}}{(\delta n + k)^2} \\
&= \frac{n}{(\delta n + k)^2} \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{3.16}$$

Thus there exists $n_1 \geq \dfrac{n_0}{1-\delta}$ such that

$$P(N_{k,n} \leq \lfloor (1-\delta)n \rfloor) < \frac{\delta}{2(1+\delta)^r c^r (1-\delta)^{r\beta}} \quad \text{for } n \geq n_1. \tag{3.17}$$

Since $e^{-n}n^{n_0-k-1} \max\limits_{k\leq m\leq n_0-1} B_m^r$ tends to zero for large $n$, there exists $n_2 \geq n_1$ such that

$$e^{-n} n^{n_0-k-1} \max_{k \le m \le n_0-1} B_m^r < \frac{\delta}{4} \quad \text{for } n \ge n_2. \tag{3.18}$$

From (3.15), (3.17) and (3.18), we have that

$$0 < s_1 < \frac{\delta}{4} + \frac{\delta n^{r\beta}}{2} \quad \text{for } n \ge n_2. \tag{3.19}$$

By (3.14), we have that for $n \ge \dfrac{n_0}{1-\delta}$,

$$\begin{aligned} s_2 &< (1+\delta)^r c^r \Big( \lfloor (1+\delta)n \rfloor - 1 \Big)^{r\beta} P\Big( \lfloor (1-\delta)n \rfloor < N_{k,n} < \lfloor (1+\delta)n \rfloor \Big) \\ &< (1+\delta)^{r+r\beta} c^r n^{r\beta} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} s_2 &> (1-\delta)^r c^r \Big( \lfloor (1-\delta)n \rfloor + 1 \Big)^{r\beta} P\Big( \lfloor (1-\delta)n \rfloor < N_{k,n} < \lfloor (1+\delta)n \rfloor \Big) \\ &> (1-\delta)^{r+r\beta} c^r n^{r\beta} P\Big( \lfloor (1-\delta)n \rfloor < N_{k,n} < \lfloor (1+\delta)n \rfloor \Big) \\ &= (1-\delta)^{r+r\beta} c^r n^{r\beta} \Big( 1 - P(N_{k,n} \ge \lfloor (1+\delta)n \rfloor) - P(N_{k,n} \le \lfloor (1-\delta)n \rfloor) \Big). \end{aligned} \tag{3.21}$$

In the same way as (3.16), we get that

$$\begin{aligned} P(N_{k,n} \ge \lfloor (1+\delta)n \rfloor) &\le P(N_{k,n} \ge (1+\delta)n - 1) \\ &= P(N_{k,n} - EN_{k,n} \ge \delta n - k - 1) \\ &\le P(|N_{k,n} - EN_{k,n}| \ge \delta n - k - 1) \\ &\le \frac{Var N_{k,n}}{(\delta n - k - 1)^2} \\ &= \frac{n}{(\delta n - k - 1)^2} \quad \text{for } n > \frac{k+1}{\delta}. \end{aligned} \tag{3.22}$$

From (3.16) and (3.22), there exists $n_3 > \max\left\{ \dfrac{n_0}{1-\delta}, \dfrac{k+1}{\delta} \right\}$ such that

$$P(N_{k,n} \ge \lfloor (1+\delta)n \rfloor) < \frac{\delta}{2} \quad \text{and} \quad P(N_{k,n} \le \lfloor (1-\delta)n \rfloor) < \frac{\delta}{2} \quad \text{for } n \ge n_3. \tag{3.23}$$

Thus by (3.20), (3.21) and (3.23),

$$(1-\delta)^{r+r\beta+1}c^r n^{r\beta} < s_2 < (1+\delta)^{r+r\beta}c^r n^{r\beta} \quad \text{for } n \geq n_3. \tag{3.24}$$

By (3.14), we note that for $n \geq \dfrac{n_0}{1-\delta}$,

$$s_3 < (1+\delta)^r c^r \sum_{m=\lfloor(1+\delta)n\rfloor}^{\infty} P(N_{k,n}=m)m^{r\beta} < (1+\delta)^r c^r \sum_{m=\lfloor(1+\delta)n\rfloor}^{\infty} \frac{e^{-n}n^{m-k}m^l}{(m-k)!}.$$

By (3.12), there exists $n_4 \in \mathbb{N}$ such that

$$\frac{m^l}{(m-k)(m-k-1)\cdots(m-k-l+1)} < 1+\delta \quad \text{for } m \geq n_4.$$

Then for all $n \geq n_4$,

$$\begin{aligned} s_3 &< (1+\delta)^{r+1}c^r \sum_{m=\lfloor(1+\delta)n\rfloor}^{\infty} \frac{e^{-n}n^{m-k}}{(m-k-l)!} \\ &= (1+\delta)^{r+1}c^r n^l \sum_{m=\lfloor(1+\delta)n\rfloor-l}^{\infty} \frac{e^{-n}n^{m-k}}{(m-k)!} \\ &= (1+\delta)^{r+1}c^r n^l P\Big(N_{k,n} \geq \lfloor(1+\delta)n\rfloor - l\Big). \end{aligned} \tag{3.25}$$

Note that

$$\begin{aligned} P\Big(N_{k,n} \geq \lfloor(1+\delta)n\rfloor - l\Big) &\leq P\Big(N_{k,n} \geq \Big(1+\frac{\delta}{2}\Big)n + \frac{\delta n}{2} - l - 1\Big) \\ &\leq P\Big(N_{k,n} \geq \Big(1+\frac{\delta}{2}\Big)n\Big) \quad \text{for } n \geq \frac{2(l+1)}{\delta}. \end{aligned} \tag{3.26}$$

By the same argument as [6] (p.5938), there exists a positive real number $t$ such that

$$P\Big(N_{k,n} \geq \Big(1+\frac{\delta}{2}\Big)n\Big) \leq e^{-tn} \quad \text{for } n \geq \frac{4k}{\delta}. \tag{3.27}$$

From (3.25)–(3.27), there exists $n_5 \geq \max\Big\{n_4, \ \dfrac{2(l+1)}{\delta}, \ \dfrac{4k}{\delta}\Big\}$ such that

$$0 < s_3 < (1+\delta)^{r+1} c^r n^{\lfloor r\beta \rfloor + 1} e^{-tn} < \frac{\delta}{4} \quad \text{for } n \geq n_5. \tag{3.28}$$

Let $n_\epsilon \in \mathbb{N}$ be such that the conditions of (3.19), (3.24) and (3.28) hold for $n \geq n_\epsilon$. Hence for all $n \geq n_\epsilon$,

$$(1-\delta)^{r+r\beta+1} c^r n^{r\beta} < s_1 + s_2 + s_3 < \left(\frac{\delta}{c^r} + (1+\delta)^{r+r\beta}\right) c^r n^{r\beta}.$$

By choosing $\delta = \min\left\{\frac{\epsilon c^r}{2},\ 1-(1-\epsilon)^{\frac{1}{r+r\beta+1}},\ \left(1+\frac{\epsilon}{2}\right)^{\frac{1}{r+r\beta}} - 1\right\}$, we have that $\delta \in (0,1)$ and

$$(1-\epsilon) c^r n^{r\beta} < EB^r_{N_{k,n}} = s_1 + s_2 + s_3 < (1+\epsilon) c^r n^{r\beta} \quad \text{for } n \geq n_\epsilon.$$

So $\left|\frac{EB^r_{N_{k,n}}}{c^r n^{r\beta}} - 1\right| < \epsilon$ for all $n \geq n_\epsilon$. This implies that

$$EB^r_{N_{k,n}} \sim c^r n^{r\beta} \text{ as } n \to \infty. \tag{3.29}$$

From (3.29), we have

$$EB_{N_{k,n}} \sim c n^\beta \text{ as } n \to \infty.$$

Hence

$$(EB_{N_{k,n}})^r \sim c^r n^{r\beta} \text{ as } n \to \infty.$$

From this fact and (3.29), we have

$$EB^r_{N_{k,n}} \sim (EB_{N_{k,n}})^r \text{ as } n \to \infty. \tag{3.30}$$

□

***Proof of Theorem 3.3.***
Let $z \in \mathbb{R}$ and $k \geq 4$ such that $1+|z| \leq k^{\frac{1}{14}}$. Then $P(N_{k,n} \geq 4) = 1$ and $P(1+|z| \leq N_{k,n}^{\frac{1}{14}}) = 1$ for all $n \in \mathbb{N}$. By Proposition 3.5, we have (3.3) holds.

Hence, by Theorem 3.2, Hölder's inequality and the fact that

$$\frac{\sqrt{VarB_{N_{k,n}}}}{EB_{N_{k,n}}} = \frac{\sqrt{EB^2_{N_{k,n}} - (EB_{N_{k,n}})^2}}{EB_{N_{k,n}}} = \left(\frac{EB^2_{N_{k,n}}}{(EB_{N_{k,n}})^2} - 1\right)^{\frac{1}{2}},$$

we have a positive constant $C$ such that

$$\begin{aligned}\Delta_{N_{k,n},z} \leq & \frac{C}{1+|z|}\Bigg[\left\{\frac{EB^2_{N_{k,n}}}{(EB_{N_{k,n}})^2} - 1\right\}^{\frac{1}{2}} + \left\{\frac{EB^2_{N_{k,n}}}{(EB_{N_{k,n}})^2} - 1\right\}^{\frac{1}{2}}\left\{\frac{EB^2_{N_{k,n}}}{(EB_{N_{k,n}})^2}\right\}^{\frac{1}{2}} \\ & + \left\{\frac{EB^2_{N_{k,n}}}{(EB_{N_{k,n}})^2} - 1\right\}^{\frac{1}{2}}\left\{\frac{EB^{12}_{N_{k,n}}}{(EB_{N_{k,n}})^{12}}\right\}^{\frac{1}{4}}\left\{E\gamma^8_{N_{k,n}}\right\}^{\frac{1}{4}} + E\Big(\frac{1}{\sqrt{N_{k,n}}}\Big) \\ & + E\gamma_{N_{k,n}} + \left\{\frac{EB^{12}_{N_{k,n}}}{(EB_{N_{k,n}})^{12}}\right\}^{\frac{1}{2}}\left\{E\Big(N^3_{k,n}\gamma^8_{N_{k,n}}\Big)\right\}^{\frac{1}{2}}\Bigg] \end{aligned} \tag{3.31}$$

for all $n \in \mathbb{N}$. From this fact and Proposition 3.5, to complete the proof it suffices to show that

$$\begin{aligned} &\lim_{n\to\infty} E\Big(\frac{1}{N^r_{k,n}}\Big) = 0 \quad \text{for all } r > 0 \quad \text{and} \\ &\lim_{n\to\infty} E\Big(N^l_{k,n}\gamma^t_{N_{k,n}}\Big) = 0 \quad \text{for all } l \geq 0, t > 0 \text{ such that } t > \frac{l}{\alpha}. \end{aligned} \tag{3.32}$$

We devide the proof into 2 steps.

**Step 1 :** we will show that $\lim_{n\to\infty} E\Big(\frac{1}{N^r_{k,n}}\Big) = 0$ for all $r > 0$.

Let $r$ and $\epsilon$ be positive real numbers such that $\epsilon \in (0, 1)$.

For $n \geq \max\left\{2k, \frac{2^{\frac{r+1}{r}}}{\epsilon^{\frac{1}{r}}}\right\}$, we note that

$$\begin{aligned} E\Big(\frac{1}{N^r_{k,n}}\Big) &= \sum_{m=k}^{\lfloor\frac{n}{2}\rfloor}\frac{P(N_{k,n}=m)}{m^r} + \sum_{m=\lfloor\frac{n}{2}\rfloor+1}^{\infty}\frac{P(N_{k,n}=m)}{m^r} \\ &\leq \frac{P(N_{k,n} \leq \lfloor\frac{n}{2}\rfloor)}{k^r} + \frac{P(N_{k,n} \geq \lfloor\frac{n}{2}\rfloor + 1)}{(\lfloor\frac{n}{2}\rfloor+1)^r} \\ &\leq P\Big(N_{k,n} \leq \Big\lfloor\frac{n}{2}\Big\rfloor\Big) + \frac{2^r}{n^r} \\ &\leq P\Big(N_{k,n} \leq \frac{n}{2}\Big) + \frac{\epsilon}{2}. \end{aligned} \tag{3.33}$$

By Chebyshev's inequality, we have that

$$\begin{aligned} P\Big(N_{k,n} \leq \frac{n}{2}\Big) &= P\Big(N_{k,n} - EN_{k,n} \leq -\frac{n}{2} - k\Big) \\ &\leq P\Big(|N_{k,n} - EN_{k,n}| \geq \frac{n}{2} + k\Big) \\ &\leq \frac{VarN_{k,n}}{(\frac{n}{2}+k)^2} \\ &= \frac{n}{(\frac{n}{2}+k)^2} \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

From this fact and (3.33), there exists $n_0 \geq \max\Big\{2k,\ \frac{2^{\frac{r+1}{r}}}{\epsilon^{\frac{1}{r}}}\Big\}$ such that

$$E\Big(\frac{1}{N^r_{k,n}}\Big) \ \leq \ \frac{n}{(\frac{n}{2}+k)^2} + \frac{\epsilon}{2} < \epsilon \quad \text{for } n \geq n_0.$$

This implies

$$\lim_{n\to\infty} E\Big(\frac{1}{N^r_{k,n}}\Big) = 0 \quad \text{for all } r > 0. \tag{3.34}$$

**Step 2 :** we will show that $\lim_{n\to\infty} E\Big(N^l_{k,n}\gamma^t_{N_{k,n}}\Big) = 0$ for all $l \geq 0,\ t > 0$ such that $t > \frac{l}{\alpha}$. Let $l \geq 0$ and $t > 0$ such that $t > \frac{l}{\alpha}$ and let $\epsilon > 0$.
By (3.7), there exist $d > 0$ and $n_1 \geq k+1$ such that

$$\gamma_m \leq \frac{d}{m^\alpha} \quad \text{for } m \geq n_1.$$

By this fact and the fact that $t > \frac{l}{\alpha}$, we have that for $m \geq n_1$,

$$m^l\gamma^t_m \leq \frac{m^l d^t}{m^{\alpha t}} = \frac{d^t}{m^{\alpha t - l}} = \frac{d^t}{m^s}$$

where $s = \alpha t - l > 0$. From this fact, we have that

$$\begin{aligned} E\Big(N^l_{k,n}\gamma^t_{N_{k,n}}\Big) &= \sum_{m=k}^{\infty} P(N_{k,n} = m)m^l\gamma^t_m \\ &= \sum_{m=k}^{n_1-1} P(N_{k,n} = m)m^l\gamma^t_m + \sum_{m=n_1}^{\infty} P(N_{k,n} = m)m^l\gamma^t_m \end{aligned}$$

$$< \sum_{m=k}^{n_1-1} \frac{e^{-n}n^{m-k}}{(m-k)!} m^l \gamma_m^t + \sum_{m=n_1}^{\infty} P(N_{k,n}=m)\frac{d^t}{m^s}$$
$$\le e^{-n}n^{n_1-k-1}(n_1-1)^l \max_{k\le m\le n_1-1} \gamma_m^t + d^t E\Big(\frac{1}{N_{k,n}^s}\Big). \tag{3.35}$$

By (3.34) and the fact that $e^{-n}n^{n_1-k-1}(n_1-1)^l \max_{k\le m\le n_1-1} \gamma_m^t$ tends to zero for large $n$, there exists $n_2 \ge n_1$ such that

$$e^{-n}n^{n_1-k-1}(n_1-1)^l \max_{k\le m\le n_1-1} \gamma_m^t < \frac{\epsilon}{2} \quad \text{and}$$
$$E\Big(\frac{1}{N_{k,n}^s}\Big) < \frac{\epsilon}{2d^t} \quad \text{for } n \ge n_2.$$

From these facts and (3.35), we obtain that

$$E\Big(N_{k,n}^l \gamma_{N_{k,n}}^t\Big) < \epsilon \quad \text{for } n \ge n_2.$$

This implies

$$\lim_{n\to\infty} E\Big(N_{k,n}^l \gamma_{N_{k,n}}^t\Big) = 0. \tag{3.36}$$

By (3.31), (3.32), (3.34) and (3.36), we obtain

$$\Delta_{N_{k,n},z} \to 0 \text{ as } n \to \infty.$$

□

## 3.3 Proof of Theorem 3.4

*Proof.* Let $z \in \mathbb{R}$. By (2.30), (3.2) and (3.5), we note that

$$\begin{aligned} 0 < B_m = &\ \frac{1}{m}\sum_{i,j=1}^{m} EX_m^2(i,j) + \frac{1}{m(m-1)}\sum_{i,j=1}^{m}[EX_m(i,j)]^2 \\ \le &\ \frac{1}{m}\sum_{i,j=1}^{m}\Big\{E|X_m(i,j)|^3\Big\}^{\frac{2}{3}} + \frac{1}{m(m-1)}\sum_{i,j=1}^{m}\Big\{E|X_m(i,j)|^3\Big\}^{\frac{2}{3}} \\ < &\ \infty \quad \text{for all } m \ge 2. \end{aligned}$$

From this fact, we have

$$\begin{aligned}0 < EB^r_{N_n} &= B^r_n P(N_n = n) + B^r_{2n} P(N_n = 2n)\\ &\le B^r_n + B^r_{2n}\\ &< \infty \quad \text{for all } r > 0 \text{ and } n \ge 2.\end{aligned}$$

So (3.3) holds for $n \ge 2$. For $n \ge \max\{4,\ (1+|z|)^{14}\}$, we note that $P(N_n \ge 4) = 1$ and $P(1 + |z| \le N_n^{\frac{1}{14}}) = 1$. Similar to (3.31), we have a positive constant $C$ such that

$$\begin{aligned}\Delta_{N_n,z} \le & \frac{C}{1+|z|}\Bigg[\bigg\{\frac{EB^2_{N_n}}{(EB_{N_n})^2} - 1\bigg\}^{\frac{1}{2}} + \bigg\{\frac{EB^2_{N_n}}{(EB_{N_n})^2} - 1\bigg\}^{\frac{1}{2}}\bigg\{\frac{EB^2_{N_n}}{(EB_{N_n})^2}\bigg\}^{\frac{1}{2}}\\ & + \bigg\{\frac{EB^2_{N_n}}{(EB_{N_n})^2} - 1\bigg\}^{\frac{1}{2}}\bigg\{\frac{EB^{12}_{N_n}}{(EB_{N_n})^{12}}\bigg\}^{\frac{1}{4}}\Big\{E\gamma^8_{N_n}\Big\}^{\frac{1}{4}} + E\Big(\frac{1}{\sqrt{N_n}}\Big) + E\gamma_{N_n}\\ & + \bigg\{\frac{EB^{12}_{N_n}}{(EB_{N_n})^{12}}\bigg\}^{\frac{1}{2}}\Big\{E\Big(N_n^3\gamma^8_{N_n}\Big)\Big\}^{\frac{1}{2}}\Bigg] \qquad (3.37)\end{aligned}$$

for $n \ge n_0 := \max\{4,\ (1+|z|)^{14}\}$. To complete the proof, it suffices to show that

$$\begin{aligned}\lim_{n\to\infty} E\Big(\frac{1}{\sqrt{N_n}}\Big) = \lim_{n\to\infty} E\gamma^8_{N_n} = \lim_{n\to\infty} E\Big(N_n^3\gamma^8_{N_n}\Big) = 0 \quad \text{and}\\ \lim_{n\to\infty}\frac{EB^r_{N_n}}{(EB_{N_n})^r} = 1 \quad \text{for all } r > 0. \qquad (3.38)\end{aligned}$$

Since $E\Big(\frac{1}{\sqrt{N_n}}\Big) = \frac{1}{n\sqrt{n}} + \frac{1}{\sqrt{2n}}\Big(1 - \frac{1}{n}\Big) < \frac{2}{\sqrt{n}}$,

$$\lim_{n\to\infty} E\Big(\frac{1}{\sqrt{N_n}}\Big) = 0. \qquad (3.39)$$

By (3.7), there exist $d > 0$ and $n_1 \ge n_0$ such that

$$0 \le \gamma_m \le \frac{d}{m^\alpha} \quad \text{for } m \ge n_1.$$

From this fact, we have that

$$0 \leq E\Big(N_n^3 \gamma_{N_n}^8\Big) \leq n^3\gamma_n^8 + 8n^3\gamma_{2n}^8 \leq \frac{d^8}{n^{8\alpha-3}} + \frac{d^8}{2^{8\alpha-3}n^{8\alpha-3}} \quad \text{for } n \geq n_1.$$

This fact and the fact that $8\alpha - 3 > 0$ imply

$$\lim_{n\to\infty} E\Big(N_n^3 \gamma_{N_n}^8\Big) = 0. \tag{3.40}$$

Note that $0 \leq E\gamma_{N_n}^8 \leq E\Big(N_n^3\gamma_{N_n}^8\Big)$ for all $n \geq 2$. Then, by (3.40), we get

$$\lim_{n\to\infty} E\gamma_{N_n}^8 = 0. \tag{3.41}$$

Last, we will show that $\lim_{n\to\infty} \dfrac{EB_{N_n}^r}{(EB_{N_n})^r} = 1$ for all $r > 0$.
Let $r, \epsilon$ and $\delta$ be positive real numbers such that $\epsilon, \delta \in (0, 1)$. By (3.8), there exists $n_2 \geq n_1$ such that

$$(1-\delta)cm^\beta < B_m < (1+\delta)cm^\beta \quad \text{for } m \geq n_2.$$

By this fact, we note that for $n \geq \max\Big\{n_2,\ \dfrac{(1+\delta)^r}{\delta}\Big\}$,

$$\begin{aligned}
EB_{N_n}^r =& B_n^r P(N_n = n) + B_{2n}^r P(N_n = 2n) \\
\leq& \frac{B_n^r}{n} + B_{2n}^r \\
<& \frac{(1+\delta)^r (cn^\beta)^r}{n} + (1+\delta)^r (c2^\beta n^\beta)^r \\
=& \Big(\frac{(1+\delta)^r}{2^{\beta r} n} + (1+\delta)^r\Big)(c2^\beta n^\beta)^r \\
\leq& \Big(\delta + (1+\delta)^r\Big)(c2^\beta n^\beta)^r
\end{aligned}$$

and

$$\begin{aligned}
EB_{N_n}^r \geq& B_{2n}^r P(N_n = 2n) \\
=& B_{2n}^r - \frac{B_{2n}^r}{n} \\
>& (1-\delta)^r (c2^\beta n^\beta)^r - \frac{(1+\delta)^r (c2^\beta n^\beta)^r}{n} \\
=& \Big((1-\delta)^r - \frac{(1+\delta)^r}{n}\Big)(c2^\beta n^\beta)^r \\
\geq& \Big((1-\delta)^r - \delta\Big)(c2^\beta n^\beta)^r.
\end{aligned}$$

Then

$$(1-\delta)^r - \delta < \frac{EB^r_{N_n}}{(c2^\beta n^\beta)^r} < \delta + (1+\delta)^r \quad \text{for } n \geq \max\Big\{n_2,\ \frac{(1+\delta)^r}{\delta}\Big\}.$$

By choosing $\delta = \min\Big\{\frac{\epsilon}{2},\ 1-\Big(1-\frac{\epsilon}{2}\Big)^{\frac{1}{r}},\ \Big(1+\frac{\epsilon}{2}\Big)^{\frac{1}{r}}-1\Big\}$, we get that

$$1-\epsilon < \frac{EB^r_{N_n}}{(c2^\beta n^\beta)^r} < 1+\epsilon \quad \text{for } n \geq \max\Big\{n_2,\ \frac{(1+\delta)^r}{\delta}\Big\}.$$

This implies

$$EB^r_{N_n} \sim (c2^\beta n^\beta)^r \text{ as } n \to \infty.$$

By this fact and the same technique as (3.30), we get that

$$EB^r_{N_n} \sim (EB_{N_n})^r \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \frac{EB^r_{N_n}}{(EB_{N_n})^r} = 1 \text{ for all } r > 0.$$

From this fact and (3.37)–(3.41), we obtain that

$$\Delta_{N_n,z} \to 0 \text{ as } n \to \infty.$$

□

## 3.4 Examples

Let $N_{k,n}$ and $N_n$ be defined as in (3.6) and (3.9), respectively. In this section, we give some examples to show that the conditions of Theorem 3.3 and Theorem 3.4 hold. The following are our examples.

**Example 3.6.** Let $X_n(i,j)$ be a random variable such that

$$P(X_n(i,j) = i) = P(X_n(i,j) = -i) = \frac{1}{2}$$

for all $i, j = 1, 2, \ldots, n$ and $n \geq 2$. Note that $EX_n(i,j) = 0$, $VarX_n(i,j) = i^2$ and $E|X_n(i,j)|^3 = i^3$ for all $i, j = 1, 2, \ldots, n$ and $n \geq 2$. Then (3.1) and (3.5) hold.

From (2.30), we observe that

$$\begin{aligned} B_n &= \frac{1}{n}\sum_{i,j=1}^{n} EX_n^2(i,j) + \frac{1}{n(n-1)}\sum_{i,j=1}^{n}[EX_n(i,j)]^2 \\ &= \frac{1}{n}\sum_{i,j=1}^{n} VarX_n(i,j) + \frac{1}{n-1}\sum_{i,j=1}^{n}[EX_n(i,j)]^2 \qquad (3.42) \\ &= \frac{1}{n}\sum_{i,j=1}^{n} i^2 \\ &= \frac{n^3}{3}\Big(1+\frac{3}{2n}+\frac{1}{2n^2}\Big) \quad \text{for } n \geq 2. \qquad (3.43) \end{aligned}$$

Thus (3.2) holds and $B_n \sim \dfrac{n^3}{3}$ as $n \to \infty$. Hence (3.8) holds by choosing $c = \dfrac{1}{3}$ and $\beta = 3$. By (3.43), we have that

$$\begin{aligned} \gamma_n &= \frac{1}{nB_n^{\frac{3}{2}}}\sum_{i,j=1}^{n} E|X_n(i,j)|^3 \\ &= \frac{1}{nB_n^{\frac{3}{2}}}\sum_{i,j=1}^{n} i^3 \\ &= \frac{1}{B_n^{\frac{3}{2}}}\sum_{i=1}^{n} i^3 \\ &< \frac{n^4}{(\frac{n^3}{3})^{\frac{3}{2}}} \\ &= \frac{3^{\frac{3}{2}}}{\sqrt{n}} \quad \text{for } n \geq 2. \end{aligned}$$

This implies that $\gamma_n = O\Big(\dfrac{1}{\sqrt{n}}\Big)$ as $n \to \infty$.
So (3.7) holds. Therefore the conditions of Theorem 3.3 and Theorem 3.4 hold. □

By using the idea from [6], we obtain Example 3.7 and Example 3.8.

**Example 3.7.** Let $a, b > 0$ such that $b < \dfrac{1}{4}$. For every $i \in \mathbb{N}$, let $Y_i$ be a random variable such that

$$P(Y_i = i^a) = P(Y_i = -i^a) = \frac{1}{2i^b} \text{ and } P(Y_i = 0) = 1 - \frac{1}{i^b}. \qquad (3.44)$$

Note that $EY_i = 0$, $VarY_i = EY_i^2 = i^{2a-b}$ and $E|Y_i|^3 = i^{3a-b}$ for all $i \in \mathbb{N}$.

For every $n \geq 2$, let $X_n(i,j) = Y_i$ for $i,j = 1,2,\dots,n$. Then (3.1) and (3.5) hold. From (3.42), it follows that (3.2) holds. From [6] (p.5939), we have (3.8) holds and

$$B_n \sim C_1 n^{2a-b+1} \text{ as } n \to \infty \text{ for some } C_1 > 0.$$

By this fact, there exists $n_0 \geq 2$ such that

$$B_n > 2C_1 n^{2a-b+1} \quad \text{for } n \geq n_0. \tag{3.45}$$

Note that

$$\frac{1}{n}\sum_{i,j=1}^{n} E|X_n(i,j)|^3 = \sum_{i=1}^{n} E|Y_i|^3 = \sum_{i=1}^{n} i^{3a-b} < n^{3a-b+1}.$$

From this fact and (3.45), we have

$$\gamma_n = \frac{1}{nB_n^{\frac{3}{2}}}\sum_{i,j=1}^{n} E|X_n(i,j)|^3 < \frac{n^{3a-b+1}}{(2C_1 n^{2a-b+1})^{\frac{3}{2}}} = \frac{1}{(2C_1)^{\frac{3}{2}} n^{\frac{1-b}{2}}} \quad \text{for } n \geq n_0.$$

So $\gamma_n = O\Big(\frac{1}{n^{\frac{1-b}{2}}}\Big)$ as $n \to \infty$ and $\dfrac{1-b}{2} > \dfrac{3}{8}$. Thus (3.7) holds.
Hence the conditions of Theorem 3.3 and Theorem 3.4 hold. □

**Example 3.8.** Let $a,b,r > 0$ such that $b < \dfrac{1}{4}$ and $r < a + \dfrac{1-b}{3}$. For every $n \geq 2$ such that $n = 2k$ or $n = 2k+1$, define an $n \times n$ matrix $[c_{ij}]$ by

$$c_{ii} = c_{i+1,i+1} = i^r \text{ and } c_{i,i+1} = c_{i+1,i} = -i^r \quad \text{for } i = 1,3,\dots,2k-1$$

and the other entries equal zero.

Let $X_n(i,j) = Y_i + c_{ij}$ where $Y_i$ is defined as in (3.44) for all $i,j = 1,2,\dots,n$ and $n \geq 2$. Note that

$$E|X_n(i,j)|^3 = E|Y_i + c_{ij}|^3 \leq 4(E|Y_i|^3 + |c_{ij}|^3) = 4i^{3a-b} + 4|c_{ij}|^3 \tag{3.46}$$

and, from (3.42),

$$B_n = \frac{1}{n}\sum_{i,j=1}^{n} VarY_i + \frac{1}{n-1}\sum_{i,j=1}^{n} c_{ij}^2 = \sum_{i=1}^{n} VarY_i + \frac{1}{n-1}\sum_{i,j=1}^{n} c_{ij}^2. \tag{3.47}$$

Then (3.2) and (3.5) hold. From [6] (p.5939), we have that (3.1) holds,

$$\sum_{i=1}^{n} VarY_i \sim C_1 n^{2a-b+1} \text{ as } n \to \infty, \tag{3.48}$$

$$\frac{1}{n-1}\sum_{i,j=1}^{n} c_{ij}^2 \sim C_2 n^{2r} \text{ as } n \to \infty \quad \text{and} \tag{3.49}$$

$$\frac{1}{n}\sum_{i,j=1}^{n} |c_{ij}|^3 \sim C_3 n^{3r} \text{ as } n \to \infty \tag{3.50}$$

for some $C_1, C_2, C_3 > 0$. By (3.47)–(3.49) and the fact that $2r < 2a - b + 1$, we get

$$B_n \sim C_1 n^{2a-b+1} + C_2 n^{2r} \sim C_4 n^{2a-b+1} \text{ as } n \to \infty \quad \text{for some } C_4 > 0. \tag{3.51}$$

Then (3.8) holds. From (3.50) and (3.51), there exist $n_1 \in \mathbb{N}$ and $C_5, C_6 > 0$ such that

$$\frac{1}{n}\sum_{i,j=1}^{n} |c_{ij}|^3 < C_5 n^{3r} \quad \text{and} \tag{3.52}$$

$$B_n > C_6 n^{2a-b+1} \quad \text{for } n \geq n_1. \tag{3.53}$$

From (3.46), (3.52) and the fact that $3r < 3a - b + 1$, there exists $n_2 \geq n_1$ and $C_7 > 0$ such that

$$\begin{aligned}\frac{1}{n}\sum_{i,j=1}^{n} E|X_n(i,j)|^3 &\leq 4\sum_{i=1}^{n} i^{3a-b} + \frac{4}{n}\sum_{i,j=1}^{n} |c_{ij}|^3 \\ &< 4n^{3a-b+1} + 4C_5 n^{3r} \\ &< C_7 n^{3a-b+1} \quad \text{for } n \geq n_2.\end{aligned}$$

By this fact and (3.53), we have that

$$\begin{aligned}\gamma_n &= \frac{1}{nB_n^{\frac{3}{2}}} \sum_{i,j=1}^{n} E|X_n(i,j)|^3 \\ &< \frac{C_7 n^{3a-b+1}}{(C_6 n^{2a-b+1})^{\frac{3}{2}}} \\ &= \frac{C_7}{C_6^{\frac{3}{2}} n^{\frac{1-b}{2}}} \quad \text{for } n \geq n_2.\end{aligned}$$

So $\gamma_n = O\left(\frac{1}{n^{\frac{1-b}{2}}}\right)$ as $n \to \infty$ and $\dfrac{1-b}{2} > \dfrac{3}{8}$. Thus (3.7) holds.
Hence the conditions of Theorem 3.3 and Theorem 3.4 hold. □

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