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นางสาวศศิกานต์ ยี่โป้

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POINTWISE CHARACTERIZATION OF SMOOTH FUNCTIONS WITHIN THE DUNKL-TYPE SEGAL-BARGMANN SPACE

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The Dunkl-type Segal-Bargmann space is the Hilbert space of holomorphic functions on $\mathbb{C}$ of which even parts and odd parts are square-integrable with respect to certain measures related to the modified Bessel functions of the second kind. The Dunkl-type Segal-Bargmann space is then the (internal) direct sum of the even and odd subspaces. In this work, we characterize smooth functions in the Dunkl-type Segal-Bargmann space associated with the Dunkl operator.

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CONTENTS

CHAPTER I

INTRODUCTION

Let U be a non-empty open subset in the complex plane $\mathbb{C}$ and denote by $\mathcal{H}L^2(U,\mu)$ the Hilbert space of holomorphic functions which are square-integrable with respect to the measure $d\mu(z)$. There exists a function $\kappa(z,w)$ on $U \times U$ such that

$$|f(z)|^2 \le \kappa(z,z)\|f\|^2_{L^2(U,\mu)}, \tag{1.1}$$

for all $f \in \mathcal{H}L^2(U,\mu)$ and $z \in U$. The function κ is known as the **reproducing kernel** for $\mathcal{H}L^2(U,\mu)$.

If β is the Gaussian function, i.e., $\beta(z) = \pi^{-1}e^{-|z|^2}$, then the space $\mathcal{H}L^2(\mathbb{C},\beta)$ is called the **Segal-Bargmann space**, introduced by Bargmann in [3]. It is well-known that the reproducing kernel for the Segal-Bargmann space is given by $\kappa(z,w) = e^{z\overline{w}}$. By (1.1), the pointwise bound for a function $f \in \mathcal{H}L^2(\mathbb{C},\beta)$ is given by

$$|f(z)|^2 \le e^{|z|^2}\|f\|^2_{L^2(\mathbb{C},\beta)}. \tag{1.2}$$

Let $\alpha > -\frac{1}{2}$ be a fixed real number. The **Dunkl-type Segal-Bargmann space** $\mathcal{H}L^2_\alpha(\mathbb{C})$ is the space of holomorphic functions f on $\mathbb{C}$ such that

$$\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} := \int_{\mathbb{C}} |f_e(z)|^2\, dm_{e,\alpha}(z) + \int_{\mathbb{C}} |f_o(z)|^2\, dm_{o,\alpha}(z) < \infty$$

where $f_e(z) = \frac{1}{2}(f(z) + f(-z)), f_o(z) = \frac{1}{2}(f(z) - f(-z))$ and the densities $m_{e,\alpha}$, $m_{o,\alpha}$ are related to the modified Bessel functions of the second kind. One can see [2] for the history of this space. In case $\alpha = -\frac{1}{2}$, we obtain the usual Segal-Bargmann space. According to [9], the pointwise bound for a function f in the

Dunkl-type Segal-Bargmann space is given by

$$|f(z)|^2 \ \leq \ e^{|z|^2}\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} \tag{1.3}$$

for all $z \in \mathbb{C}$. This result is due to the Cauchy-Bunyakovsky-Schwarz inequality. Note that the pointwise bound (1.3) is of the same form as the one in (1.2).

In this work, we improve a pointwise bound for a function in the Dunkl-type Segal-Bargmann space. Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. We show that there exists a constant $C > 0$ such that

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{1+|z|^{2\alpha+1}}$$

for all $z \in \mathbb{C}$. This pointwise bound is better than the one in (1.3). This result is obtained in Theorem 3.16.

The purpose of this work is to characterize certain smooth functions in the Dunkl-type Segal-Bargmann space. We use the idea of Chaiworn and Lewkeeratiyutkul [4] to obtain our main result. Consider the differential operator

$$Df(z) = \frac{df}{dz}(z).$$

For each $n \in \mathbb{N}$, define

$$\mathcal{D}(D^n) = \{f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \mid D^n f \in \mathcal{H}L^2_\alpha(\mathbb{C})\}$$

and let $C^\infty_\alpha(\mathbb{C})$ denote the set of functions in $\mathcal{H}L^2_\alpha(\mathbb{C})$ of which the derivatives of all orders are in $\mathcal{H}L^2_\alpha(\mathbb{C})$, i.e.,

$$C^\infty_\alpha(\mathbb{C}) = \bigcap_{n=1}^{\infty} \mathcal{D}(D^n).$$

Any function $f \in C^\infty_\alpha(\mathbb{C})$ is called a **smooth function** in $\mathcal{H}L^2_\alpha(\mathbb{C})$. We show that an element f in $\mathcal{D}(D^n)$ satisfies

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^n)^2}$$

for any $z \in \mathbb{C}$, where C is a positive constant depending only on f and n. Conversely, if a function f satisfies a pointwise bound of the form

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{n+2})^2}$$

for any $z \in \mathbb{C}$, then $f \in \mathcal{D}(D^n)$. Hence, we obtain the characterization of smooth functions in the Dunkl-type Segal-Bargmann space. Our main result which will be presented in the last chapter is the following.

Theorem 4.17. *Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. Then $f \in C^\infty_\alpha(\mathbb{C})$ if and only if for each $n \in \mathbb{N}$, there is $C_n > 0$ such that*

$$|f(z)|^2 \leq \frac{C_n e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^n)^2}$$

for any $z \in \mathbb{C}$.

An outline of our work is as follows. In Chapter II, after reviewing some definitions and properties of the Bessel functions, we then introduce a Hilbert space of holomorphic functions. In particular, we give a definition of the Dunkl-type Segal-Bargmann space. A pointwise bound for a function in the Dunkl-type Segal-Bargmann space is established in Chapter III. In the last chapter, we use the pointwise bound in Chapter III to characterize smooth functions in the Dunkl-type Segal-Bargmann space and its even and odd subspaces.

CHAPTER II

PRELIMINARIES

In this chapter, we review some definitions and properties of the Bessel functions and a Hilbert space of holomorphic functions. After that we introduce the Dunkl-type Segal-Bargmann space.

2.1 Bessel functions

We give a brief summary of the Bessel functions that will be used throughout this work. More details can be found in, e.g., [1], [8] and [10].

The second-order differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \alpha^2)y = 0,$$

where z is a complex variable and α is an arbitrary real or complex number, is called the **Bessel equation of order** α. This equation has two linearly independent solutions. One of them is called the **Bessel function of the first kind of order** α and is defined by

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{\alpha+2k}.$$

Then J_α is an analytic function of z in the plane cut along the negative real axis. The second linearly independent solution is defined by

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha\pi)}. \tag{2.1}$$

We observe that if $\alpha = n$ is an integer, then the right-hand side of (2.1) becomes indeterminate and in this case $Y_\alpha(z)$ is defined as a limit. By L'Hôpital's rule,

$$Y_n(z) = \lim_{\alpha \to n} Y_\alpha(z) = \frac{1}{\pi} \left\{ \frac{\partial J_\alpha}{\partial \alpha} - (-1)^n \frac{\partial J_{-\alpha}}{\partial \alpha} \right\} \Bigg|_{\alpha = n}.$$

We call $Y_\alpha(z)$ the **Bessel function of the second kind**. Next, we consider the **modified Bessel equation**

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} - (z^2 + \alpha^2) y = 0, \tag{2.2}$$

which differs from the Bessel equation only in the coefficient of y. Then we define the **modified Bessel function of the first kind** as

$$I_\alpha(z) = \begin{cases} e^{-\frac{\alpha\pi}{2} i} J_\alpha(z e^{\frac{\pi}{2} i}) & \text{if } -\pi < \arg(z) \le \frac{\pi}{2}; \\ e^{\frac{3\alpha\pi}{2} i} J_\alpha(z e^{-\frac{3\pi}{2} i}) & \text{if } \frac{\pi}{2} < \arg(z) \le \pi \end{cases}$$

and the **modified Bessel function of the second kind** as

$$K_\alpha(z) = \frac{\pi}{2} \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\alpha\pi)}.$$

However, for an integer n, we define

$$K_n(z) = \lim_{\alpha \to n} K_\alpha(z).$$

The functions $I_\alpha(z)$ and $K_\alpha(z)$ are analytic functions of z for all z in the plane cut along the negative real axis. Furthermore, they are linearly independent solutions of (2.2). Note that $K_\alpha(z)$ is also known as the **Macdonald function**. Next we give some properties of the modified Bessel functions of the first and second kinds.

Proposition 2.1. *1. For $z \in \mathbb{C}$ with $|\arg(z)| < \pi$ and $\alpha \in \mathbb{C}$,*

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k + \alpha + 1)} \left(\frac{z}{2}\right)^{\alpha + 2k}.$$

2. For $\mathrm{Re}(z) > 0$ and $\alpha \in \mathbb{R}$,

$$K_\alpha(z) = \int_0^\infty e^{-z \cosh u} \cosh(\alpha u)\, du.$$

3. *For* $\alpha, \beta \in \mathbb{R}$ *with* $|\alpha| < |\beta|$,

$$K_\alpha(z) < K_\beta(z) \quad \text{for all } z > 0.$$

4. *For* α, $\beta \in \mathbb{R}$ *such that* $\beta > |\alpha|$, *we have*

$$\int_0^\infty K_\alpha(s) s^{\beta-1}\, ds = 2^{\beta-2}\Gamma\left(\frac{\beta-\alpha}{2}\right)\Gamma\left(\frac{\beta+\alpha}{2}\right).$$

5. *For* $z \in \mathbb{C}$ *and* $|\arg(z)| < \pi$, $K_{-\alpha}(z) = K_\alpha(z)$.

6. *For* $z \in \mathbb{C}$ *and* $|\arg(z)| < \pi$, $K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}$.

7. *For* $z > 0$ *and* $z \to 0^+$, *the asymptotic behavior of* $I_\alpha(z)$ *and* $K_\alpha(z)$ *is given by*

$$\begin{aligned} I_\alpha(z) &\approx \frac{z^{|\alpha|}}{2^{|\alpha|}\Gamma(1+|\alpha|)}, \\ K_\alpha(z) &\approx \frac{2^{|\alpha|-1}\Gamma(|\alpha|)}{z^{|\alpha|}}, \\ K_0(z) &\approx \log\frac{2}{z}. \end{aligned}$$

Furthermore, for $z > 0$ *and* $z \to \infty$, *the asymptotic behavior of these functions is given by*

$$\begin{aligned} I_\alpha(z) &\approx \frac{e^z}{\sqrt{2\pi z}}, \\ K_\alpha(z) &\approx \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \end{aligned}$$

for all $\alpha \in \mathbb{R}$. *Here we use the notation* $f(z) \approx g(z)$ *as* $z \to a$ *when* $\lim_{z\to a} f(z)/g(z) = 1$ *and in this case we say that* $f(z)$ *and* $g(z)$ *are asymptotic as* $z \to a$.

2.2 Hilbert space of holomorphic functions

In this section, we define a Hilbert space of holomorphic functions and basic theorems about this space. The details of the proof can be found in [7].

Let U be a non-empty open set in complex plane $\mathbb{C}$ and μ a strictly positive continuous function on U. We denote by

- $\mathcal{H}(U)$ the space of holomorphic functions on U, and
- $L^2(U,\mu)$ the space of square-integrable functions with respect to measure $d\mu(z)$, that is,

$$L^2(U,\mu) = \left\{ f : U \to \mathbb{C} \;\middle|\; \int_U |f(z)|^2 \, d\mu(z) < \infty \right\}.$$

Then $L^2(U,\mu)$ is a Hilbert space. We write $\mathcal{H}L^2(U,\mu) = \mathcal{H}(U) \cap L^2(U,\mu)$ the space of holomorphic functions on U which are square-integrable with respect to measure $d\mu(z)$, that is,

$$\mathcal{H}L^2(U,\mu) = \left\{ f \in \mathcal{H}(U) \;\middle|\; \int_U |f(z)|^2 \, d\mu(z) < \infty \right\}.$$

Here dz denotes Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$.

Remark. If f and g are continuous functions and $f = g$ μ-a.e., then $f = g$ everywhere. *Therefore, we can consider the space $\mathcal{H}L^2(U,\mu)$ as a subspace of $L^2(U,\mu)$.* To prove this, let $G = \{z \in \mathbb{C} \mid f(z) \neq g(z)\}$. Then G has μ-measure zero. Since f and g are continuous functions, $G = (f-g)^{-1}(\mathbb{C}\backslash\{0\})$ is an open set. If $G \neq \varnothing$, then there exists $z_0 \in G$ and $\varepsilon > 0$ such that $D(z_0,\varepsilon) \subseteq G$. Since μ is a strictly positive function, μ-measure of $D(z_0,\varepsilon)$ is not zero, so is G which is a contradiction. Hence, $G = \varnothing$. This show that $f = g$ everywhere on $\mathbb{C}$.

Theorem 2.2. *1. Let $z \in U$. Then there exists a constant c_z such that*

$$|f(z)|^2 \leq c_z \|f\|^2_{L^2(U,\mu)}$$

for any $f \in \mathcal{H}L^2(U,\mu)$.

2. $\mathcal{H}L^2(U,\mu)$ is a closed subspace of $L^2(U,\mu)$ and therefore it is a Hilbert space.

It follows by Theorem 2.2 that the pointwise evaluation is continuous. This means that for each $z \in U$, the evaluation map $T_z : \mathcal{H}L^2(U,\mu) \longrightarrow \mathbb{C}$ defined by

$$T_z(f) = f(z)$$

for any $f \in \mathcal{H}L^2(U,\mu)$ is a continuous linear functional on $\mathcal{H}L^2(U,\mu)$. Thus, by the Riesz representation theorem, for each $z \in \mathbb{C}$ there exists a unique function $\kappa_z \in \mathcal{H}L^2(U,\mu)$ such that

$$f(z) = \langle f, \kappa_z \rangle_{L^2(U,\mu)}$$

for any $f \in \mathcal{H}L^2(U,\mu)$. We define $\kappa : U \times U \longrightarrow \mathbb{C}$ by

$$\kappa(w,z) = \kappa_z(w)$$

for any $w, z \in U$. The function κ is called the **reproducing kernel** for $\mathcal{H}L^2(U,\mu)$.

Theorem 2.3. *The reproducing kernel κ satisfies the following properties :*

1. *$\kappa(w,z)$ is holomorphic in w and anti-holomorphic in z, and satisfies*

$$\kappa(w,z) = \overline{\kappa(z,w)}.$$

2. *For each $z \in U, \kappa(w,z)$ is square-integrable with respect to measure $d\mu(w)$.*

3. *For each $z \in U$,*

$$|f(z)|^2 \leq \kappa(z,z)\|f\|^2_{L^2(U,\mu)},$$

and the constant $\kappa(z,z)$ is optimal in the sense that for each $z \in U$ there exists a non-zero $f_z \in \mathcal{H}L^2(U,\mu)$ for which equality holds.

Theorem 2.4. *Let $\{e_j\}_{j=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}L^2(U,\mu)$. Then for all $w, z \in U$,*

$$\sum_{j=0}^{\infty} \left| e_j(w)\overline{e_j(z)} \right| < \infty$$

and the reproducing kernel for $\mathcal{H}L^2(U,\mu)$ is given by

$$\kappa(w,z)=\sum_{j=0}^{\infty}e_j(w)\overline{e_j(z)}.$$

The series converges absolutely and uniformly on compact subsets of $U\times U$.

Definition 2.5. The **Segal-Bargmann space** is the space $\mathcal{H}L^2(\mathbb{C},\beta)$, where

$$\beta(z)=\pi^{-1}e^{-|z|^2}.$$

Theorem 2.6. $\left\{\frac{z^n}{\sqrt{n!}}\right\}_{n=0}^{\infty}$ *is an orthonomal basis for the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C},\beta)$. Therefore, the reproducing kernel for this space is given by*

$$\kappa(w,z)=e^{w\overline{z}}$$

for all $w,z\in\mathbb{C}$.

Theorem 2.7. *For any $f\in\mathcal{H}L^2(\mathbb{C},\beta)$ and for any $z\in\mathbb{C}$,*

$$|f(z)|^2\leq e^{|z|^2}\|f\|^2_{L^2(\mathbb{C},\beta)}.$$

Next, we introduce the Dunkl-type Segal-Bargmann space. The definition of this space can be found in [2] and [9]. We start by defining measures on this space. These measures are related to the modified Bessel functions of the second kind.

Here and throughout this work, let $\alpha>-\frac{1}{2}$ be a fixed real number.

Definition 2.8. We define measures on the complex plane $\mathbb{C}$ by

$$dm_{e,\alpha}(z):=m_{e,\alpha}(z)dz$$

and

$$dm_{o,\alpha}(z):=m_{o,\alpha}(z)dz$$

whose densities are defined by

$$m_{e,\alpha}(z):=\frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)}K_{\alpha}(|z|^2)|z|^{2\alpha+2}$$

and

$$m_{o,\alpha}(z) := \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha+1}(|z|^2)|z|^{2\alpha+2}$$

for all $0 \neq z \in \mathbb{C}$, where Γ is the gamma function and K_α is the modified Bessel function of the second kind of order α.

Note that $K_\alpha(x)$ is a strictly positive continuous function for all $x > 0$. Thus, $m_{e,\alpha}$ and $m_{o,\alpha}$ are nonnegative functions on $\mathbb{C}$. However, $K_\alpha(x)$ diverges at $x = 0$. Hence we consider the asymptotic behavior of $m_{e,\alpha}$ and $m_{o,\alpha}$ near zero. We write $m_{e,\alpha}$ in the polar coordinate ($z = re^{i\theta}$) with respect to $drd\theta$:

$$m_{e,\alpha}(z) = \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha}(r^2) r^{2\alpha+3}.$$

Since the asymptotic behavior of $K_\alpha(r^2)$ as $r \to 0$ is

$$K_\alpha(r^2) \approx \frac{2^{|\alpha|-1}\Gamma(|\alpha|)}{(r^2)^{|\alpha|}},$$

we divide α into three cases as follow:

Case 1 For $\alpha \in \left(-\frac{1}{2}, 0\right)$, we have

$$K_\alpha(r^2) \approx \frac{2^{|\alpha|-1}\Gamma(|\alpha|)}{(r^2)^{|\alpha|}} = \frac{2^{-\alpha-1}\Gamma(-\alpha)}{(r^2)^{-\alpha}}.$$

Thus, we obtain that

$$\begin{aligned} m_{e,\alpha}(z) &\approx \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} \frac{2^{-\alpha-1}\Gamma(-\alpha)}{(r^2)^{-\alpha}} r^{2\alpha+3} \\ &= \frac{2^{-2\alpha-1}\Gamma(-\alpha)}{\pi\Gamma(\alpha+1)} r^{4\alpha+3}. \end{aligned}$$

Case 2 For $\alpha = 0$, we have

$$m_{e,\alpha}(z) = \frac{1}{\pi\Gamma(1)} K_0(r^2) r^3 \approx \frac{1}{\pi} \log\left(\frac{2}{r^2}\right) r^3.$$

Case 3 For $\alpha \in (0, \infty)$, we have

$$K_\alpha(r^2) \approx \frac{2^{|\alpha|-1}\Gamma(|\alpha|)}{(r^2)^{|\alpha|}} = \frac{2^{\alpha-1}\Gamma(\alpha)}{(r^2)^{\alpha}}.$$

It follows that

$$\begin{aligned} m_{e,\alpha}(z) &\approx \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} \frac{2^{\alpha-1}\Gamma(\alpha)}{(r^2)^{\alpha}} r^{2\alpha+3} \\ &= \frac{\Gamma(\alpha)}{2\pi\Gamma(\alpha+1)} r^3. \end{aligned}$$

Similarly, we write $m_{o,\alpha}$ in the polar coordinate with respect to $drd\theta$. The asymptotic behavior of $m_{o,\alpha}$ near zero is given by

$$\begin{aligned} m_{o,\alpha}(z) &= \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha+1}(r^2) r^{2\alpha+3} \\ &\approx \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} \frac{2^{|\alpha+1|-1}\Gamma(|\alpha+1|)}{(r^2)^{|\alpha+1|}} r^{2\alpha+3} \\ &= \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} \frac{2^{\alpha+1-1}\Gamma(\alpha+1)}{(r^2)^{\alpha+1}} r^{2\alpha+3} \\ &= \frac{1}{\pi} r, \end{aligned}$$

here we use the fact that $\alpha + 1 > \frac{1}{2} > 0$.

In either case $\lim_{|z|\to 0} m_{e,\alpha}(z) = 0$ and $\lim_{|z|\to 0} m_{o,\alpha}(z) = 0$. Thus, the densities $m_{e,\alpha}(z)$ and $m_{o,\alpha}(z)$ are finite.

Moreover, $K_\alpha(x)$ is a monotone decreasing function on $(0, \infty)$ and decay as $x \to \infty$. By Proposition 2.1 (7), we have

$$K_\alpha(r^2) \approx \left(\frac{\pi}{2r^2}\right)^{\frac{1}{2}} e^{-r^2}$$

as $r \to \infty$. Thus, the asymptotic behavior of $m_{e,\alpha}$ as $r \to \infty$ is

$$\begin{aligned} m_{e,\alpha}(z) &= \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha}(r^2) r^{2\alpha+3} \\ &\approx \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} \left(\frac{\pi}{2r^2}\right)^{\frac{1}{2}} e^{-r^2} r^{2\alpha+3} \\ &= \frac{1}{\pi^{\frac{1}{2}} 2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} e^{-r^2} r^{2\alpha+2}. \end{aligned}$$

Since the asymptotic behavior of $K_\alpha(x)$ as $x \to \infty$ does not depend on α, the asymptotic behaviors of $m_{o,\alpha}$ and $m_{e,\alpha}$ as $r \to \infty$ are the same as follows:

$$\begin{aligned} m_{o,\alpha}(z) &= \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha+1}(r^2) r^{2\alpha+3} \\ &\approx \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} \left(\frac{\pi}{2r^2}\right)^{\frac{1}{2}} e^{-r^2} r^{2\alpha+3} \\ &= \frac{1}{\pi^{\frac{1}{2}} 2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} e^{-r^2} r^{2\alpha+2}. \end{aligned}$$

Notation. We denote by

- $\mathcal{H}_e(\mathbb{C})$ the set of all holomorphic even functions on $\mathbb{C}$, i.e.,

$$\mathcal{H}_e(\mathbb{C}) = \{f \in \mathcal{H}(\mathbb{C}) \mid f(z) = f(-z) \text{ for any } z \in \mathbb{C}\},$$

- $\mathcal{H}_o(\mathbb{C})$ the set of all holomorphic odd functions on $\mathbb{C}$, i.e.,

$$\mathcal{H}_o(\mathbb{C}) = \{f \in \mathcal{H}(\mathbb{C}) \mid f(z) = -f(-z) \text{ for any } z \in \mathbb{C}\}.$$

For any function $f : \mathbb{C} \longrightarrow \mathbb{C}$, we define $f_e, f_o : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$f_e(z) = \frac{f(z) + f(-z)}{2}, \qquad z \in \mathbb{C}$$

and

$$f_o(z) = \frac{f(z) - f(-z)}{2}, \qquad z \in \mathbb{C}.$$

We call f_e and f_o the **even part** and the **odd part** of f, respectively. Note that $f = f_e + f_o$ and $f_e = f_o$ if and only if $f \equiv 0$. Moreover, a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ can be written as a sum of the even part and the odd part in a unique way.

If f is a holomorphic function on $\mathbb{C}$, f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for any $z \in \mathbb{C}$. In this case, we see that

$$f_e(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$$

and

$$f_o(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$$

for any $z \in \mathbb{C}$.

Definition 2.9. The **Dunkl-type Segal-Bargmann space** $\mathcal{H}L^2_\alpha(\mathbb{C})$ is the space of holomorphic functions such that $f_e \in L^2(\mathbb{C}, m_{e,\alpha})$ and $f_o \in L^2(\mathbb{C}, m_{o,\alpha})$. We define the inner product on $\mathcal{H}L^2_\alpha(\mathbb{C})$ by

$$\langle f, g\rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \langle f_e, g_e\rangle_{L^2(\mathbb{C}, m_{e,\alpha})} + \langle f_o, g_o\rangle_{L^2(\mathbb{C}, m_{o,\alpha})}$$

for all $f, g \in \mathcal{H}L^2_\alpha(\mathbb{C})$. We also define the even subspace of $\mathcal{H}L^2_\alpha(\mathbb{C})$ by

$$\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}) = \mathcal{H}_e(\mathbb{C}) \cap L^2(\mathbb{C}, m_{e,\alpha})$$

and the odd subspace of $\mathcal{H}L^2_\alpha(\mathbb{C})$ by

$$\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}) = \mathcal{H}_o(\mathbb{C}) \cap L^2(\mathbb{C}, m_{o,\alpha}).$$

Then $(\mathcal{H}L^2_\alpha(\mathbb{C}), \langle \cdot, \cdot\rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})})$ is an inner product space over $\mathbb{C}$. In fact, it is the (internal) direct sum of $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ and $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ since the zero function is the only function that is both even and odd.

In particular, if $\alpha = -\frac{1}{2}$, we obtain

$$m_{e,-\frac{1}{2}}(z) = \frac{2^{\frac{1}{2}}}{\pi\Gamma(\frac{1}{2})} K_{-\frac{1}{2}}(|z|^2)|z| = \frac{2^{\frac{1}{2}}}{\pi(\pi)^{\frac{1}{2}}} \left(\frac{\pi}{2|z|^2}\right)^{\frac{1}{2}} e^{-|z|^2}|z| = \frac{1}{\pi} e^{-|z|^2}$$

and

$$m_{o,-\frac{1}{2}}(z) = \frac{2^{\frac{1}{2}}}{\pi\Gamma(\frac{1}{2})} K_{\frac{1}{2}}(|z|^2)|z| = \frac{2^{\frac{1}{2}}}{\pi(\pi)^{\frac{1}{2}}} \left(\frac{\pi}{2|z|^2}\right)^{\frac{1}{2}} e^{-|z|^2}|z| = \frac{1}{\pi} e^{-|z|^2},$$

which follow from Proposition 2.1 (5) and (6). Thus, the Dunkl-type Segal-Bargmann space becomes to the usual Segal-Bargmann space.

Theorem 2.10. *$\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ and $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ are Hilbert spaces.*

Proof. Let

$$\mathcal{H}L^2(\mathbb{C}, m_{e,\alpha}) = \mathcal{H}(\mathbb{C}) \cap L^2(\mathbb{C}, m_{e,\alpha})$$

and

$$\mathcal{H}L^2(\mathbb{C}, m_{o,\alpha}) = \mathcal{H}(\mathbb{C}) \cap L^2(\mathbb{C}, m_{o,\alpha}).$$

By Theorem 2.2, $\mathcal{H}L^2(\mathbb{C}, m_{e,\alpha})$ and $\mathcal{H}L^2(\mathbb{C}, m_{o,\alpha})$ are Hilbert spaces. Thus, it suffices to show that $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ and $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ are closed subspaces of $\mathcal{H}L^2(\mathbb{C}, m_{e,\alpha})$ and $\mathcal{H}L^2(\mathbb{C}, m_{o,\alpha})$, respectively. Let $(f_n) \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ and $f \in \mathcal{H}L^2(\mathbb{C}, m_{e,\alpha})$ such that $f_n \to f$ in $\mathcal{H}L^2(\mathbb{C}, m_{e,\alpha})$. It remains to show that f is an even function. Let $z \in \mathbb{C}$. By Theorem 2.2, there exists a constant c_z such that

$$|f_n(z) - f(z)| \leq c_z \|f_n - f\|_{L^2(\mathbb{C}, m_{e,\alpha})} \to 0 \quad \text{as } n \to \infty.$$

This shows that $f_n(z) \to f(z)$. Since f_n is an even function, we have

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} f_n(-z) = f(-z).$$

Hence, f is an even function. The proof for $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ is similar to $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ hence we omit the proof. □

Corollary 2.11. *The Dunkl-type Segal-Bargmann space $\mathcal{H}L^2_\alpha(\mathbb{C})$ is a Hilbert space.*

Proof. The proof follows from the fact that the direct sum of two Hilbert spaces is also a Hilbert space. □

CHAPTER III

POINTWISE BOUNDS

In this chapter, we first obtain a pointwise bound for a function in the even and odd subspaces of the Dunkl-type Segal-Bargmann space. Then a pointwise bound for a function in the Dunkl-type Segal-Bargmann space is obtained from the pointwise bound for its even part and odd part.

Proposition 3.1. *If f is an entire function with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$, then*

$$\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=0}^{\infty} |a_n|^2 d_n(\alpha)$$

where

$$d_n(\alpha) = \frac{2^n \lfloor \frac{n}{2} \rfloor! \, \Gamma\left(\lfloor \frac{n+1}{2} \rfloor + \alpha + 1\right)}{\Gamma(\alpha + 1)}.$$

Proof. Note that

$$\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \|f_e\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} + \|f_o\|^2_{L^2(\mathbb{C}, m_{o,\alpha})}.$$

We will show that

$$\|f_e\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} = \sum_{n=0}^{\infty} |a_{2n}|^2 d_{2n}(\alpha)$$

and

$$\|f_o\|^2_{L^2(\mathbb{C}, m_{o,\alpha})} = \sum_{n=0}^{\infty} |a_{2n+1}|^2 d_{2n+1}(\alpha).$$

Let $R \in [0, \infty)$ and

$$M(R) = \int_{|z| \le R} |f_e(z)|^2 \, dm_{e,\alpha}(z).$$

Then

$$\begin{aligned} M(R) &= \int_{|z|\le R}\left(\sum_{n=0}^{\infty} a_{2n} z^{2n}\right)\left(\overline{\sum_{m=0}^{\infty} a_{2m} z^{2m}}\right)\, dm_{e,\alpha}(z) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_{|z|\le R} z^{2n}\overline{z}^{2m}\, dm_{e,\alpha}(z), \end{aligned}$$

by the uniform convergence of the power series of f on compact subsets of $\mathbb{C}$. We write $z = re^{i\theta}$. Then

$$\begin{aligned} M(R) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_{|z|\le R} z^{2n}\overline{z}^{2m} \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha}(|z|^2)|z|^{2\alpha+2}\, dz \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_0^{2\pi}\int_0^{R} r^{2n+2m} e^{i2(n-m)\theta} \frac{1}{\pi 2^{\alpha}\Gamma(\alpha+1)} K_{\alpha}(r^2) r^{2\alpha+2} r\, dr d\theta \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_{2n}\overline{a_{2m}}}{\pi 2^{\alpha}\Gamma(\alpha+1)} \int_0^{2\pi} e^{i2(n-m)\theta} d\theta \int_0^{R} r^{2n+2m+2\alpha+3} K_{\alpha}(r^2)\, dr \\ &= \sum_{n=0}^{\infty} \frac{2\pi a_{2n}\overline{a_{2n}}}{\pi 2^{\alpha}\Gamma(\alpha+1)} \int_0^{R} r^{4n+2\alpha+3} K_{\alpha}(r^2)\, dr \\ &= \sum_{n=0}^{\infty} \frac{|a_{2n}|^2}{2^{\alpha}\Gamma(\alpha+1)} \int_0^{R} (r^2)^{2n+\alpha+1} K_{\alpha}(r^2) 2r\, dr. \end{aligned}$$

Now, using the monotone convergence theorem twice, we have

$$\begin{aligned} \|f_e\|^2_{L^2(\mathbb{C},m_{e,\alpha})} &= \lim_{R\to\infty} M(R) \\ &= \lim_{R\to\infty} \sum_{n=0}^{\infty} \frac{|a_{2n}|^2}{2^{\alpha}\Gamma(\alpha+1)} \int_0^{R} (r^2)^{2n+\alpha+1} K_{\alpha}(r^2) 2r\, dr \\ &= \sum_{n=0}^{\infty} \frac{|a_{2n}|^2}{2^{\alpha}\Gamma(\alpha+1)} \int_0^{\infty} (r^2)^{2n+\alpha+1} K_{\alpha}(r^2) 2r\, dr. \end{aligned}$$

By Proposition 2.1 (4), we obtain that

$$\int_0^{\infty} r^{4n+2\alpha+2} K_{\alpha}(r^2) 2r\, dr = 2^{2n+\alpha}\Gamma(n+1)\Gamma(n+\alpha+1).$$

Hence,

$$\|f_e\|^2_{L^2(\mathbb{C},m_{e,\alpha})} = \sum_{n=0}^{\infty} |a_{2n}|^2 d_{2n}(\alpha).$$

Similarly, we have

$$\|f_o\|^2_{L^2(\mathbb{C},m_{o,\alpha})} = \sum_{n=0}^{\infty} |a_{2n+1}|^2 d_{2n+1}(\alpha).$$

It follows that

$$\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=0}^{\infty} |a_n|^2 d_n(\alpha). \qquad \square$$

Lemma 3.2. *$d_n(\alpha) \geq n!$ for all $n \in \mathbb{N} \cup \{0\}$.*

Proof. For $n = 0$, we have $0! = 1 = d_0(\alpha)$. Let $k \in \mathbb{N}$. Assume that $d_k(\alpha) \geq k!$. We consider

$$d_{k+1}(\alpha) = \frac{2^{k+1}\lfloor \frac{k+1}{2} \rfloor! \, \Gamma\left(\lfloor \frac{k+2}{2} \rfloor + \alpha + 1\right)}{\Gamma(\alpha+1)}.$$

If k is even, then $k = 2m$ for some $m \in \mathbb{N}$. Thus,

$$\begin{aligned}
d_{k+1}(\alpha) &= \frac{2^{2m+1}\lfloor \frac{2m+1}{2} \rfloor! \, \Gamma\left(\lfloor \frac{2m+2}{2} \rfloor + \alpha + 1\right)}{\Gamma(\alpha+1)} \\
&= \frac{2^{2m+1} m! \, \Gamma(m+\alpha+2)}{\Gamma(\alpha+1)} \\
&= \frac{2^{2m+1} m! \, (m+\alpha+1)\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \\
&= 2(m+\alpha+1) d_{2m}(\alpha) \\
&\geq 2(m+\alpha+1)(2m)! \\
&\geq (2m+1)(2m)! = (2m+1)! = (k+1)!.
\end{aligned}$$

If k is odd, then $n = 2m+1$ for some $m \in \mathbb{N}$. Thus,

$$\begin{aligned}
d_{k+1}(\alpha) &= \frac{2^{2m+2}\lfloor \frac{2m+2}{2} \rfloor! \, \Gamma\left(\lfloor \frac{2m+3}{2} \rfloor + \alpha + 1\right)}{\Gamma(\alpha+1)} \\
&= \frac{2^{2m+2}(m+1)! \, \Gamma(m+\alpha+2)}{\Gamma(\alpha+1)} \\
&= 2(m+1) d_{2m+1}(\alpha) \\
&\geq 2(m+1)(2m+1)! = (2m+2)! = (k+1)!.
\end{aligned}$$

By mathematical induction, $d_n(\alpha) \geq n!$ for all $n \in \mathbb{N} \cup \{0\}$. $\square$

Proposition 3.3. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |a_n|^2 d_n(\alpha) < \infty$. Then the power series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for* *all* $z \in \mathbb{C}$ *and hence the function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is an entire function. Moreover,* $\sum_{n=0}^{\infty} a_n z^n$ *converges to* f *in* $\mathcal{H}L^2_\alpha(\mathbb{C})$. *Hence the set of complex polynomials is dense in* $\mathcal{H}L^2_\alpha(\mathbb{C})$.

Proof. Let $z \in \mathbb{C}$. By Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n z^n| &= \sum_{n=0}^{\infty} |a_n| \sqrt{d_n(\alpha)} \frac{|z|^n}{\sqrt{d_n(\alpha)}} \\ &\le \left(\sum_{n=0}^{\infty} |a_n|^2 d_n(\alpha) \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{d_n(\alpha)} \right)^{1/2}.\end{aligned}$$

By Lemma 3.2, we have $n! \le d_n(\alpha)$ and so $\dfrac{1}{d_n(\alpha)} \le \dfrac{1}{n!}$. Then

$$\sum_{n=0}^{\infty} \frac{|z|^{2n}}{d_n(\alpha)} \le \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} = e^{|z|^2}.$$

Thus, for each $z \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} |a_n z^n| \le \left(\sum_{n=0}^{\infty} |a_n|^2 d_n(\alpha) \right)^{1/2} e^{|z|^2/2} < \infty.$$

Therefore, $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for all $z \in \mathbb{C}$.

Next, we define a sequence (F_N) by

$$F_N(z) := \sum_{n=0}^{N} a_n z^n$$

for any $z \in \mathbb{C}$. Then, by Proposition 3.1, we have

$$\|F_N - F_M\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=M+1}^{N} |a_n|^2 d_n(\alpha) \to 0 \quad \text{as } M, N \to \infty.$$

That is, (F_N) is a Cauchy sequence in $\mathcal{H}L^2_\alpha(\mathbb{C})$. Thus, (F_N) converges in $\mathcal{H}L^2_\alpha(\mathbb{C})$ to some function, say g. Then there is a subsequence of (F_N) converging pointwise almost everywhere to g. Since (F_N) converges pointwise to f, $g = f$ almost everywhere. This implies that $g = f$ in $\mathcal{H}L^2_\alpha(\mathbb{C})$ and so (F_N) converges to f in $\mathcal{H}L^2_\alpha(\mathbb{C})$. $\square$

Proposition 3.4. *Let $f \in \mathcal{H}(\mathbb{C})$. Then $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$ if and only if there exists a sequence $(a_n)_{n=0}^\infty$ of complex numbers such that $\sum_{n=0}^\infty |a_n|^2 d_n(\alpha) < \infty$ and $f(z) = \sum_{n=0}^\infty a_n z^n$ for all $z \in \mathbb{C}$.*

Proof. Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. Then f is an entire function and $\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Thus, there exists a sequence $(a_n)_{n=0}^\infty$ of complex numbers such that $f(z) = \sum_{n=0}^\infty a_n z^n$ for all $z \in \mathbb{C}$. By Proposition 3.1, we have $\sum_{n=0}^\infty |a_n|^2 d_n(\alpha) = \|f\|^2_{\mathcal{H}L^2(\mathbb{C})} < \infty$.

On the other hand, suppose that $f(z) = \sum_{n=0}^\infty a_n z^n$ and $\sum_{n=0}^\infty |a_n|^2 d_n(\alpha) < \infty$. It follows from Proposition 3.3 that f is an entire function and so, by Proposition 3.1, we have $\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=0}^\infty |a_n|^2 d_n(\alpha) < \infty$. Hence, $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. □

By direct computation, we obtain a pointwise bound for a function in the Dunkl-type Segal-Bargmann space. This pointwise bound is of the same form as the pointwise bound of the Segal-Bargmann space.

Proposition 3.5. *For any $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$ and for any $z \in \mathbb{C}$,*

$$|f(z)| \leq e^{|z|^2/2} \|f\|_{\mathcal{H}L^2_\alpha(\mathbb{C})}.$$

Proof. Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$ and $z \in \mathbb{C}$. From the proof of Propositions 3.3, we have

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^\infty |a_n z^n| \\ &\leq \left(\sum_{n=0}^\infty |a_n|^2 d_n(\alpha) \right)^{1/2} e^{|z|^2/2} \\ &= e^{|z|^2/2} \|f\|_{\mathcal{H}L^2_\alpha(\mathbb{C})}. \end{aligned}$$

This finishes the proof. □

Proposition 3.6. *Let $n \in \mathbb{N} \cup \{0\}$. Define*

$$e_n(z) = \frac{z^n}{\sqrt{d_n(\alpha)}}$$

for all $z \in \mathbb{C}$. Then $\{e_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathcal{H}L^2_\alpha(\mathbb{C})$.

Proof. We note that

$$\langle z^n, z^m \rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} = d_n(\alpha)\delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker delta function. This show that $\{z^n\}_{n=0}^\infty$ is an orthogonal set in $\mathcal{H}L^2_\alpha(\mathbb{C})$ and so $\{e_n\}_{n=0}^\infty$ is an orthonormal set in $\mathcal{H}L^2_\alpha(\mathbb{C})$. To prove that it is complete, let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$ be such that

$$\langle f, e_m \rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} = 0,$$

for all $m \in \mathbb{N} \cup \{0\}$. Write $f(z) = \sum_{n=0}^\infty a_n z^n$ where $\sum_{n=0}^\infty |a_n|^2 d_n(\alpha) < \infty$. It follows from Proposition 3.3 that $\sum_{n=0}^\infty a_n z^n$ converges in $\mathcal{H}L^2_\alpha(\mathbb{C})$. By the continuity of an inner product, we have

$$\begin{aligned} 0 = \left\langle f, \frac{z^m}{\sqrt{d_m(\alpha)}} \right\rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} &= \left\langle \sum_{n=0}^\infty a_n z^n, \frac{z^m}{\sqrt{d_m(\alpha)}} \right\rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} \\ &= \sum_{n=0}^\infty \frac{a_n}{\sqrt{d_m(\alpha)}} \langle z^n, z^m \rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} \\ &= \frac{a_m}{\sqrt{d_m(\alpha)}} \langle z^m, z^m \rangle_{\mathcal{H}L^2_\alpha(\mathbb{C})} \\ &= \frac{a_m}{\sqrt{d_m(\alpha)}} d_m(\alpha). \end{aligned}$$

Hence, $a_m\sqrt{d_m(\alpha)} = 0$ for all $m \in \mathbb{N} \cup \{0\}$. Since $d_m(\alpha) > 0$, $a_m = 0$ for all $m \in \mathbb{N} \cup \{0\}$ and so $f = 0$. □

Proposition 3.7. *The reproducing kernel of $\mathcal{H}L^2_\alpha(\mathbb{C})$ is given by*

$$\kappa(w, z) = \sum_{n=0}^\infty \frac{w^n \overline{z}^n}{d_n(\alpha)}$$

for all $w, z \in \mathbb{C}$.

Proof. The proof can be done by Theorem 2.4 and Proposition 3.6. □

Proposition 3.8. *Let $n \in \mathbb{N} \cup \{0\}$. Define*

$$u_n(z) = \frac{z^{2n}}{\sqrt{d_{2n}(\alpha)}}$$

for all $z \in \mathbb{C}$. Then $\{u_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$.

Proof. The proof is similar to that of Proposition 3.6, so we omit the details of the proof. □

Proposition 3.9. *The reproducing kernel of* $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ *is given by*

$$\kappa_e(w, z) = \sum_{n=0}^{\infty} \frac{w^{2n}\overline{z}^{2n}}{d_{2n}(\alpha)}$$

for all $w, z \in \mathbb{C}$. *In particular,*

$$\kappa_e(z, z) = \frac{2^\alpha \Gamma(\alpha + 1) I_\alpha(|z|^2)}{|z|^{2\alpha}}$$

for all $z \in \mathbb{C}$.

Proof. From Theorem 2.4 and Proposition 3.8 we obtain that the reproducing kernel of $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ is given by

$$\kappa_e(w, z) = \sum_{n=0}^{\infty} \frac{w^{2n}\overline{z}^{2n}}{d_{2n}(\alpha)}$$

for all $w, z \in \mathbb{C}$. Hence, by Proposition 2.1 (1), we have

$$\begin{aligned} \kappa_e(z, z) &= \sum_{n=0}^{\infty} \frac{|z|^{4n}}{d_{2n}(\alpha)} \\ &= \sum_{n=0}^{\infty} \frac{|z|^{4n}\Gamma(\alpha + 1)}{2^{2n} n!\, \Gamma(n + \alpha + 1)} \\ &= \frac{2^\alpha \Gamma(\alpha + 1) I_\alpha(|z|^2)}{|z|^{2\alpha}} \end{aligned}$$

for all $z \in \mathbb{C}$. □

Corollary 3.10. *For any* $f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ *and for any* $z \in \mathbb{C}$,

$$|f(z)|^2 \leq \frac{2^\alpha \Gamma(\alpha + 1) I_\alpha(|z|^2)}{|z|^{2\alpha}} \|f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})}.$$

Proof. The proof follows from Theorem 2.3 (3) and Proposition 3.9. □

By estimating the asymptotic behavior of the modified Bessel function, we obtain in Corollary 3.10 in a more familiar form.

Theorem 3.11. *Let $f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$. Then there is a constant $C > 0$ such that*

$$|f(z)| \leq \frac{Ce^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}}$$

for any $z \in \mathbb{C}$.

Proof. Let $f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$. It follows from Corollary 3.10 that

$$|f(z)|^2 \leq \frac{2^\alpha \Gamma(\alpha+1) I_\alpha(|z|^2)}{|z|^{2\alpha}} \|f\|^2_{L^2(\mathbb{C},m_{e,\alpha})}$$

for any $z \in \mathbb{C}$. We note that the asymptotic behavior of the modified Bessel function of the first kind $I_\alpha(|z|^2)$ as $|z| \to \infty$ is given by

$$I_\alpha(|z|^2) \approx \frac{e^{|z|^2}}{\sqrt{2\pi|z|^2}}.$$

Thus, there exists $R > 0$ such that for all $z \in \mathbb{C}$ with $|z| > R$

$$\left| \frac{I_\alpha(|z|^2)}{\frac{e^{|z|^2}}{\sqrt{2\pi|z|^2}}} - 1 \right| < 1,$$

which implies that

$$0 < \frac{I_\alpha(|z|^2)}{\frac{e^{|z|^2}}{\sqrt{2\pi|z|^2}}} < 2$$

and thus

$$I_\alpha(|z|^2) < \frac{2e^{|z|^2}}{\sqrt{2\pi|z|^2}}.$$

By Corollary 3.10, we obtain

$$\begin{aligned} |f(z)|^2 &\leq 2^\alpha \Gamma(\alpha+1) I_\alpha(|z|^2)|z|^{-2\alpha}\|f\|^2_{L^2(\mathbb{C},m_{e,\alpha})} \\ &< 2^\alpha \Gamma(\alpha+1) \left(\frac{2e^{|z|^2}}{\sqrt{2\pi|z|^2}} \right) |z|^{-2\alpha}\|f\|^2_{L^2(\mathbb{C},m_{e,\alpha})} \\ &= \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}} \left(\frac{e^{|z|^2}}{|z|^{2\alpha+1}} \right) \|f\|^2_{L^2(\mathbb{C},m_{e,\alpha})} \end{aligned}$$

for all $z \in \mathbb{C}$ and $|z| > R$. This implies that

$$|f(z)|^2 \leq \frac{Ae^{|z|^2}}{|z|^{2\alpha+1}}$$

for all $z \in \mathbb{C}$ and $|z| > R$, where $A = \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}}\|f\|^2_{L^2(\mathbb{C},m_{e,\alpha})}$. Note that

$$\frac{1}{x} \le \frac{2}{1+x}$$

for any $x > 1$. We choose $S = \max\{R, 1\}$. Hence, we obtain

$$|f(z)|^2 \le \frac{2Ae^{|z|^2}}{1+|z|^{2\alpha+1}} \tag{3.1}$$

for any $z \in \mathbb{C}$ and $|z| > S$. Since f is a continuous function and $\{z \in \mathbb{C} : |z| \le S\}$ is a compact set, f is bounded on this set. That is, there exists $M > 0$ such that

$$|f(z)| \le M$$

for any $z \in \mathbb{C}$ and $|z| \le S$. The function $z \longmapsto \frac{e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}}$ is continuous on the set $\{z \in \mathbb{C} : |z| \le S\}$, so it has the minimum value, say L. Thus, we have

$$|f(z)| \le M = \frac{M}{L} \cdot L \le \frac{M}{L}\left(\frac{e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}}\right) \tag{3.2}$$

for all $z \in \mathbb{C}$ and $|z| \le S$. From inequalities (3.1) and (3.2), there exists a constant $C > 0$ such that

$$|f(z)| \le \frac{Ce^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}}$$

for all $z \in \mathbb{C}$. □

Next, we turn to the odd subspace of the Dunkl-type Segal-Bargmann space. We will state its analogous results to the even counterparts.

Proposition 3.12. *Let $n \in \mathbb{N} \cup \{0\}$. Define*

$$v_n(z) = \frac{z^{2n+1}}{\sqrt{d_{2n+1}(\alpha)}}$$

for all $z \in \mathbb{C}$. Then $\{v_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$.

Proof. The proof is similar to that of Proposition 3.6. □

Proposition 3.13. *The reproducing kernel of* $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ *is given by*

$$\kappa_o(w,z) = \sum_{n=0}^{\infty} \frac{w^{2n+1}\overline{z}^{2n+1}}{d_{2n+1}(\alpha)}$$

for all $w, z \in \mathbb{C}$*. In particular,*

$$\kappa_o(z,z) = \frac{2^\alpha \Gamma(\alpha+1) I_{\alpha+1}(|z|^2)}{|z|^{2\alpha}}$$

for all $z \in \mathbb{C}$*.*

Proof. By Theorem 2.4 and Proposition 3.12, we obtain that the reproducing kernel of $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ is given by

$$\kappa_o(w,z) = \sum_{n=0}^{\infty} \frac{w^{2n+1}\overline{z}^{2n+1}}{d_{2n+1}(\alpha)}$$

for all $w, z \in \mathbb{C}$. By Proposition 2.1 (1), we obtain that

$$\begin{aligned} \kappa_o(z,z) &= \sum_{n=0}^{\infty} \frac{|z|^{4n+2}}{d_{2n+1}(\alpha)} \\ &= \sum_{n=0}^{\infty} \frac{|z|^{4n+2}\Gamma(\alpha+1)}{2^{2n+1} n!\, \Gamma(n+\alpha+2)} \\ &= \frac{2^\alpha \Gamma(\alpha+1) I_{\alpha+1}(|z|^2)}{|z|^{2\alpha}} \end{aligned}$$

for all $z \in \mathbb{C}$. □

Corollary 3.14. *For any* $f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ *and for any* $z \in \mathbb{C}$*,*

$$|f(z)|^2 \leq \frac{2^\alpha \Gamma(\alpha+1) I_{\alpha+1}(|z|^2)}{|z|^{2\alpha}} \|f\|^2_{L^2(\mathbb{C}, m_{o,\alpha})}.$$

Proof. The proof follows from Theorem 2.3 (3) and Proposition 3.13. □

From Corollary 3.14, we obtain a pointwise bound for a function in the space $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$. This pointwise bound is given in terms of the modified Bessel function of the first kind of order $\alpha+1$ which differs from the pointwise bound for a function in $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$. However, the pointwise bound estimation for functions in $\mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ and $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ are the same because the asymptotic behavior of the modified Bessel function for large $|z|$ does not depend on the order.

Theorem 3.15. *Let $f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$. Then there is a constant $C > 0$ such that*

$$|f(z)| \leq \frac{Ce^{|z|^2/2}}{(1 + |z|^{2\alpha+1})^{1/2}}$$

for any $z \in \mathbb{C}$.

Proof. Since the proof is similar to Theorem 3.11, we omit the proof. □

Now we already obtained pointwise bounds for even part and odd part of function in the Dunkl-type Segal-Bargmann space. The pointwise bound is better than the one in Proposition 3.5. We then use this to obtain a pointwise bound for a function in the Dunkl-type Segal-Bargmann space.

Theorem 3.16. *Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. Then there is a constant $C > 0$ such that*

$$|f(z)| \leq \frac{Ce^{|z|^2/2}}{(1 + |z|^{2\alpha+1})^{1/2}}$$

for any $z \in \mathbb{C}$.

Proof. Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. Then we can write $f(z) = f_e(z) + f_o(z)$ for any $z \in \mathbb{C}$. By Theorems 3.11 and 3.15, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} |f(z)| &\leq |f_e(z)| + |f_o(z)| \\ &\leq \frac{C_1 e^{|z|^2/2}}{(1 + |z|^{2\alpha+1})^{1/2}} + \frac{C_2 e^{|z|^2/2}}{(1 + |z|^{2\alpha+1})^{1/2}} \end{aligned}$$

for any $z \in \mathbb{C}$. □

CHAPTER IV

CHARACTERIZATION THEOREMS

In this chapter, we will prove the characterization of smooth functions in the Dunkl-type Segal-Bargmann space and its even and odd subspaces in terms of the pointwise bound. Before proving this, we first study relations of the domain of the Dunkl operator, the multiplication operator and the differential operator.

4.1 Even subspaces

We consider the Schrödinger radial kinetic energy operator $D_{e,\alpha}$:

$$(D_{e,\alpha}f)(z) = \frac{d^2 f}{dz^2}(z) + \left(\frac{2\alpha+1}{z}\right)\frac{df}{dz}(z)$$

and the multiplication operator by z^2

$$(M^2 f)(z) = z^2 f(z).$$

Definition 4.1. For each $n \in \mathbb{N}$, we denote the domains of $D^n_{e,\alpha}$ and M^{2n} by

$$\mathcal{D}(D^n_{e,\alpha}) = \{f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}) \mid D^n_{e,\alpha} f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})\}$$

and

$$\mathcal{D}(M^{2n}) = \{f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}) \mid M^{2n} f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})\}.$$

Remark. Cholewinski [5] proved that the operators $D_{e,\alpha}$ and M^2 are densely defined, for the set of even polynomials is contained in each of these domains, and that they are adjoints of each other.

Proposition 4.2. $\mathcal{D}(D_{e,\alpha}) = \mathcal{D}(M^2)$.

Proof. Let $f \in \mathcal{H}L_e^2(\mathbb{C}, m_{e,\alpha})$ with $f(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$ for all $z \in \mathbb{C}$. Then

$$\begin{aligned}
(D_{e,\alpha} f)(z) &= \sum_{n=0}^{\infty} [(2n)(2n-1) + (2\alpha+1)(2n)]\, a_{2n} z^{2n-2} \\
&= \sum_{n=0}^{\infty} \left[4n^2 - 2n + 4\alpha n + 2n\right] a_{2n} z^{2n-2} \\
&= \sum_{n=0}^{\infty} [4n(n+\alpha)]\, a_{2n} z^{2n-2}
\end{aligned}$$

and $(M^2 f)(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n+2}$. We note that

$$d_{2n}(\alpha) = 4n(n+\alpha) d_{2n-2}(\alpha).$$

By Propositon 3.1 and the above equality, we have

$$\begin{aligned}
\|D_{e,\alpha} f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} &= \sum_{n=0}^{\infty} |4n(n+\alpha) a_{2n}|^2\, d_{2n-2}(\alpha) \\
&= \sum_{n=0}^{\infty} 4n\,(n+\alpha)\, |a_{2n}|^2 d_{2n}(\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\|M^2 f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} &= \sum_{n=0}^{\infty} |a_{2n}|^2 d_{2n+2}(\alpha) \\
&= \sum_{n=0}^{\infty} 4(n+1)(n+\alpha+1) |a_{2n}|^2 d_{2n}(\alpha) \\
&= \sum_{n=0}^{\infty} 4(n^2 + n\alpha + n + n + \alpha + 1) |a_{2n}|^2 d_{2n}(\alpha) \\
&= \sum_{n=0}^{\infty} [4n(n+\alpha) + 4(2n+\alpha+1)]\, |a_{2n}|^2 d_{2n}(\alpha).
\end{aligned}$$

Since $4(\alpha+1) = d_2(\alpha)$,

$$\|M^2 f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} = \|D_{e,\alpha} f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} + \sum_{n=0}^{\infty} 8n |a_{2n}|^2 d_{2n}(\alpha) + d_2(\alpha) \|f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})}.$$

It is clear that $\|M^2 f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} < \infty$ implies that the right-hand side of the above equality is finite and so $\|D_{e,\alpha} f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} < \infty$. Conversely, we suppose that $\|D_{e,\alpha}f\|^2_{L^2(\mathbb{C},m_{e,\alpha})} < \infty$. Then $\sum_{n=0}^{\infty} 4n(n+\alpha)|a_{2n}|^2 d_{2n}(\alpha) < \infty$ and so $\sum_{n=0}^{\infty} 8n|a_{2n}|^2 d_{2n}(\alpha) < \infty$. It follows that $\|M^2 f\|^2_{L^2(\mathbb{C},m_{e,\alpha})} < \infty$. Hence, $\mathcal{D}(D_{e,\alpha}) = \mathcal{D}(M^2)$. □

Corollary 4.3. *For each $n \in \mathbb{N}$, $\mathcal{D}(D^n_{e,\alpha}) = \mathcal{D}(M^{2n})$.*

Proof. By Proposition 4.2, we have $\mathcal{D}(D_{e,\alpha}) = \mathcal{D}(M^2)$. Let $k \in \mathbb{N}$. We assume that $\mathcal{D}(D^k_{e,\alpha}) = \mathcal{D}(M^{2k})$. Then

$$\begin{aligned} f \in \mathcal{D}(D^{k+1}_{e,\alpha}) &\iff D^{k+1}_{e,\alpha} f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}) \\ &\iff D_{e,\alpha} f \in \mathcal{D}(D^k_{e,\alpha}) = \mathcal{D}(M^{2k}) \\ &\iff M^{2k}(D_{e,\alpha} f) \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}) \\ &\iff z^{2k}(D_{e,\alpha} f) \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}). \end{aligned}$$

We claim $z^{2k}(D_{e,\alpha}f) \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$ if and only if $D_{e,\alpha}(z^{2k}f) \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$. Since $f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$, we can write $f(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$ for all $z \in \mathbb{C}$. From the proof of the previous Proposition, we have

$$(D_{e,\alpha}f)(z) = \sum_{n=0}^{\infty} 4n\,(n+\alpha)\,a_{2n} z^{2n-2}.$$

Hence,

$$z^{2k}(D_{e,\alpha}f)(z) = \sum_{n=0}^{\infty} 4n\,(n+\alpha)\,a_{2n} z^{2n+2k-2}$$

and

$$D_{e,\alpha}(z^{2k}f)(z) = \sum_{n=0}^{\infty} 4(n+k)\,(n+k+\alpha)\,a_{2n} z^{2n+2k-2}.$$

By Proposition 3.1,

$$\|z^{2k}(D_{e,\alpha}f)\|^2_{L^2(\mathbb{C},m_{e,\alpha})} = \sum_{n=0}^{\infty} 16n^2\,(n+\alpha)^2\,|a_{2n}|^2 d_{2n+2k-2}(\alpha)$$

and

$$\|D_{e,\alpha}(z^{2k}f)\|^2_{L^2(\mathbb{C},m_{e,\alpha})} = \sum_{n=0}^{\infty} 16(n+k)^2\,(n+k+\alpha)^2\,|a_{2n}|^2 d_{2n+2k-2}(\alpha).$$

Thus, $\|D_{e,\alpha}(z^{2k}f)\|^2_{L^2(\mathbb{C},m_{e,\alpha})} < \infty$ implies $\|z^{2k}(D_{e,\alpha}f)\|^2_{L^2(\mathbb{C},m_{e,\alpha})} < \infty$. On the other hand, we assume that $\|z^{2k}(D_{e,\alpha}f)\|^2_{L^2(\mathbb{C},m_{e,\alpha})} < \infty$. We consider

$$\begin{aligned}(n+k)^2(n+k+\alpha)^2 &= (n^2+2nk+k^2)\left[(n+\alpha)^2+2(n+\alpha)k+k^2\right]\\ &= n^2(n+\alpha)^2+2n^2(n+\alpha)k+n^2k^2+2n(n+\alpha)^2k\\ &\quad +4n(n+\alpha)k^2+2nk^3+(n+\alpha)^2k^2+2(n+\alpha)k^3+k^4\\ &\leq Kn^2(n+\alpha)^2\end{aligned}$$

where K is a constant depending on k. It follows that $\|D_{e,\alpha}(z^{2k}f)\|^2_{L^2(\mathbb{C},m_{e,\alpha})} < \infty$. Hence,

$$\begin{aligned}f \in \mathcal{D}(D^{k+1}_{e,\alpha}) &\iff D_{e,\alpha}(z^{2k}f) \in \mathcal{H}L^2_e(\mathbb{C},m_{e,\alpha})\\ &\iff z^{2k}f \in \mathcal{D}(D_{e,\alpha}) = \mathcal{D}(M^2)\\ &\iff z^{2k+2}f \in \mathcal{H}L^2_e(\mathbb{C},m_{e,\alpha})\\ &\iff M^{2(k+1)}f \in \mathcal{H}L^2_e(\mathbb{C},m_{e,\alpha})\\ &\iff f \in \mathcal{D}(M^{2(k+1)}).\end{aligned}$$

By mathematical induction, we have that $\mathcal{D}(D^n_{e,\alpha}) = \mathcal{D}(M^{2n})$ for all $n \in \mathbb{N}$. □

Theorem 4.4. *Let $n \in \mathbb{N}$ and $f \in \mathcal{H}L^2_e(\mathbb{C},m_{e,\alpha})$.*

1. *If $f \in \mathcal{D}(D^n_{e,\alpha})$, then there is $C > 0$ such that*

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2n})^2}$$

for any $z \in \mathbb{C}$.

2. *If there is $C > 0$ such that*

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2(n+1)})^2} \tag{4.1}$$

for any $z \in \mathbb{C}$, then $f \in \mathcal{D}(D^n_{e,\alpha})$.

3. $f \in \bigcap_{n=1}^{\infty} \mathcal{D}(D^n_{e,\alpha})$ *if and only if for each* $n \in \mathbb{N}$*, there is* $C_n > 0$ *such that*

$$|f(z)|^2 \leq \frac{C_n e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2n})^2}$$

for any $z \in \mathbb{C}$*.*

Proof. Let $n \in \mathbb{N}$. By Corollary 4.3, we have

$$f \in \mathcal{D}(D^n_{e,\alpha}) \iff f \in \mathcal{D}(M^{2n}) \iff z^{2n} f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha}).$$

1. Let $f \in \mathcal{D}(D^n_{e,\alpha})$. Then $z^{2n} f \in \mathcal{H}L^2_e(\mathbb{C}, m_{e,\alpha})$. By Theorem 3.11, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}(1+|z|^{2n})|f(z)| &= |f(z)| + |z^{2n} f(z)| \\ &\leq \frac{C_1 e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}} + \frac{C_2 e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}}\end{aligned}$$

for any $z \in \mathbb{C}$. This implies that there is $C > 0$ such that

$$|f(z)| \leq \frac{C e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}(1+|z|^{2n})}$$

for any $z \in \mathbb{C}$.

2. Assume that the inequality (4.1) holds. We will show that $\|z^{2n} f\|_{L^2(\mathbb{C}, m_{e,\alpha})}$ is finite. Then we consider

$$\begin{aligned}\|z^{2n} f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} &= \int_{\mathbb{C}} |z^{2n} f(z)|^2 m_{e,\alpha}(z)\, dz \\ &\leq \int_{\mathbb{C}} \frac{C|z|^{4n} e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2(n+1)})^2} m_{e,\alpha}(z)\, dz \\ &= C \int_0^{2\pi} \int_0^{\infty} \frac{r^{4n} e^{r^2}}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} m_{e,\alpha}(r) r\, dr d\theta \\ &= 2\pi C \int_0^{\infty} \frac{r^{4n} e^{r^2}}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} m_{e,\alpha}(r) r\, dr.\end{aligned}$$

First, we consider the behavior of integrand near ∞. For any $\alpha > -\frac{1}{2}$, the asymptotic behavior of $m_{e,\alpha}$ as $r \to \infty$ is

$$m_{e,\alpha}(z) \approx \frac{1}{\pi^{\frac{1}{2}} 2^{\alpha+\frac{1}{2}} \Gamma(\alpha+1)} e^{-r^2} r^{2\alpha+2}.$$

Thus, there exists a constant $A > 0$ such that for all $r \in (S, \infty)$, when S is large enough, we obtain

$$\begin{aligned} \int_S^\infty \frac{r^{4n} e^{r^2} m_{e,\alpha}(r)}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} r\,dr &\leq \int_S^\infty \frac{A r^{4n+2\alpha+2}}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2}\,dr \\ &\leq A \int_S^\infty \frac{1}{r^3}\,dr \\ &< \infty. \end{aligned}$$

Next, we observe that

$$\int_0^S \frac{r^{4n} e^{r^2} m_{e,\alpha}(r)}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} r\,dr < \infty$$

because the integrand is continuous function on compact set $[0, S]$. Hence, $\|z^{2n} f\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} < \infty$ and thus, $f \in \mathcal{D}(D^n_{e,\alpha})$.

3. The proof follows from 1 and 2. □

4.2 Odd subspaces

We define the operators $D_{o,\alpha}$ and M^2 by

$$(D_{o,\alpha} f)(z) = \frac{d^2 f}{dz^2}(z) + (2\alpha+1)\frac{d}{dz}\left(\frac{f(z)}{z}\right)$$

and

$$(M^2 f)(z) = z^2 f(z).$$

Definition 4.5. For each $n \in \mathbb{N}$, the domains of operators $D^n_{o,\alpha}$ and M^{2n} are defined as follow

$$\mathcal{D}(D^n_{o,\alpha}) = \{f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}) \mid D^n_{o,\alpha} f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})\}$$

and

$$\mathcal{D}(M^{2n}) = \{f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}) \mid M^{2n} f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})\}.$$

Remark. The operators $D_{o,\alpha}$ and M^2 are densely defined on $\mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$, for the set of odd polynomials is contained in each of these domains, and that they are adjoints of each other.

Proposition 4.6. $\mathcal{D}(D_{o,\alpha}) = \mathcal{D}(M^2)$.

Proof. Let $f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ and write $f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$ for all $z \in \mathbb{C}$. Then

$$\begin{aligned}
(D_{o,\alpha} f)(z) &= \sum_{n=0}^{\infty} [(2n+1)(2n) + (2\alpha+1)(2n)]\, a_{2n+1} z^{2n-1} \\
&= \sum_{n=0}^{\infty} \left[4n^2 + 2n + 4\alpha n + 2n\right] a_{2n+1} z^{2n-1} \\
&= \sum_{n=0}^{\infty} \left[4n^2 + 4\alpha n + 4n\right] a_{2n+1} z^{2n-1} \\
&= \sum_{n=0}^{\infty} [4n(n+\alpha+1)]\, a_{2n+1} z^{2n-1}
\end{aligned}$$

and $(M^2 f)(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+3}$. Note that

$$d_{2n+1}(\alpha) = 4n\,(n+\alpha+1)\, d_{2n-1}(\alpha).$$

By Proposition 3.1 and the above equality, we obtain that

$$\begin{aligned}
\|D_{o,\alpha} f\|^2_{L^2(\mathbb{C}, m_{o,\alpha})} &= \sum_{n=0}^{\infty} |4n(n+\alpha+1) a_{2n+1}|^2\, d_{2n-1}(\alpha) \\
&= \sum_{n=0}^{\infty} 4n(n+\alpha+1) |a_{2n+1}|^2 d_{2n+1}(\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\|M^2 f\|^2_{L^2(\mathbb{C}, m_{o,\alpha})} &= \sum_{n=0}^{\infty} |a_{2n+1}|^2 d_{2n+3}(\alpha) \\
&= \sum_{n=0}^{\infty} 4(n+1)(n+\alpha+2) |a_{2n+1}|^2 d_{2n+1}(\alpha) \\
&= \sum_{n=0}^{\infty} 4(n^2 + n\alpha + 2n + n + \alpha + 2) |a_{2n+1}|^2 d_{2n+1}(\alpha) \\
&= \sum_{n=0}^{\infty} [4n(n+\alpha+1) + 4(2n) + 4(\alpha+2)]\, |a_{2n+1}|^2 d_{2n+1}(\alpha).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|M^2 f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} &= \|D_{o,\alpha} f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} + \sum_{n=0}^{\infty} 8n|a_{2n+1}|^2 d_{2n+1}(\alpha) \\ &\quad + 4(\alpha+2)\|f\|^2_{L^2(\mathbb{C},m_{o,\alpha})}.\end{aligned}$$

Clearly, if $\|M^2 f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$, then the right-hand side of the above equality is finite and so $\|D_{o,\alpha} f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$. On the other hand, we suppose that $\|D_{o,\alpha} f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$. Then $\sum_{n=0}^{\infty} 4n\,(n+\alpha+1)\,|a_{2n+1}|^2 d_{2n+1}(\alpha) < \infty$. Thus, $\sum_{n=0}^{\infty} 8n|a_{2n+1}|^2 d_{2n+1}(\alpha) < \infty$ and so $\|M^2 f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$. Hence, $\mathcal{D}(D_{o,\alpha}) = \mathcal{D}(M^2)$. □

Corollary 4.7. *For each $n \in \mathbb{N}$, $\mathcal{D}(D^n_{o,\alpha}) = \mathcal{D}(M^{2n})$.*

Proof. By Proposition 4.6, we have $\mathcal{D}(D_{o,\alpha}) = \mathcal{D}(M^2)$. Let $k \in \mathbb{N}$. Now, we assume that $\mathcal{D}(D^k_{o,\alpha}) = \mathcal{D}(M^{2k})$. Then

$$\begin{aligned} f \in \mathcal{D}(D^{k+1}_{o,\alpha}) &\iff D^{k+1}_{o,\alpha} f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}) \\ &\iff D_{o,\alpha} f \in \mathcal{D}(D^k_{o,\alpha}) = \mathcal{D}(M^{2k}) \\ &\iff M^{2k}(D_{o,\alpha} f) \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}) \\ &\iff z^{2k}(D_{o,\alpha} f) \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}). \end{aligned}$$

We claim that $z^{2k}(D_{o,\alpha} f) \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$ if and only if $D_{o,\alpha}(z^{2k} f) \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$. From the proof of Proposition 4.6, we have

$$(D_{o,\alpha} f)(z) = \sum_{n=0}^{\infty} 4n\,(n+\alpha+1)\,a_{2n+1} z^{2n-1}.$$

Thus,

$$z^{2k}(D_{o,\alpha} f)(z) = \sum_{n=0}^{\infty} 4n\,(n+\alpha+1)\,a_{2n+1} z^{2n+2k-1}$$

and

$$D_{o,\alpha}(z^{2k} f)(z) = \sum_{n=0}^{\infty} 4(n+k)\,(n+k+\alpha+1)\,a_{2n+1} z^{2n+2k-1}.$$

By Proposition 3.1,

$$\|z^{2k}(D_{o,\alpha}f)\|^2_{L^2(\mathbb{C},m_{o,\alpha})} = \sum_{n=0}^{\infty} 16n^2\left(n+\alpha+1\right)^2|a_{2n+1}|^2 d_{2n+2k-1}(\alpha)$$

and

$$\|D_{o,\alpha}(z^{2k}f)\|^2_{L^2(\mathbb{C},m_{o,\alpha})} = \sum_{n=0}^{\infty} 16(n+k)^2\left(n+k+\alpha+1\right)^2|a_{2n+1}|^2 d_{2n+2k-1}(\alpha).$$

Thus, if $\|D_{o,\alpha}(z^{2k}f)\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$, we have $\|(z^{2k}(D_{o,\alpha}f)\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$. Next, consider

$$\begin{aligned}(n+k)^2(n+k+\alpha+1)^2 &= (n^2+2nk+k^2)\left[(n+\alpha+1)^2+2(n+\alpha+1)k+k^2\right]\\ &= n^2(n+\alpha+1)^2+2n^2(n+\alpha+1)k+n^2k^2\\ &\quad+\ 2n(n+\alpha+1)^2k+4n(n+\alpha+1)k^2+2nk^3\\ &\quad+\ (n+\alpha+1)^2k^2+2(n+\alpha+1)k^3+k^4\\ &\leq Kn^2(n+\alpha+1)^2\end{aligned}$$

where K is a constant depending on k. Suppose $\|z^{2k}(D_{o,\alpha}f)\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$. Using the above inequality, we obtain $\|D_{o,\alpha}(z^{2k}f)\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$. Hence,

$$\begin{aligned} f\in\mathcal{D}(D^{k+1}_{o,\alpha}) &\iff D_{o,\alpha}(z^{2k}f)\in\mathcal{H}L^2_o(\mathbb{C},m_{o,\alpha})\\ &\iff z^{2k}f\in\mathcal{D}(D_{o,\alpha})=\mathcal{D}(M^2)\\ &\iff z^{2k+2}f\in\mathcal{H}L^2_o(\mathbb{C},m_{o,\alpha})\\ &\iff M^{2(k+1)}f\in\mathcal{H}L^2_o(\mathbb{C},m_{o,\alpha})\\ &\iff f\in\mathcal{D}(M^{2(k+1)}).\end{aligned}$$

Thus, $\mathcal{D}(D^{k+1}_{o,\alpha}) = \mathcal{D}(M^{2(k+1)})$. By mathematical induction, we conclude that $\mathcal{D}(D^{n}_{o,\alpha}) = \mathcal{D}(M^{2n})$ for all $n\in\mathbb{N}$. □

From Corollary 4.7, we obtain that the domains of the operators $D^n_{o,\alpha}$ and M^{2n} are equal. We use this to characterize functions in the domain of $D^n_{o,\alpha}$.

Theorem 4.8. *Let $n \in \mathbb{N}$ and $f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$.*

1. *If $f \in \mathcal{D}(D^n_{o,\alpha})$, then there is $C > 0$ such that*

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2n})^2}$$

for any $z \in \mathbb{C}$.

2. *If there is $C > 0$ such that*

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2(n+1)})^2} \tag{4.2}$$

for any $z \in \mathbb{C}$, then $f \in \mathcal{D}(D^n_{o,\alpha})$.

3. *$f \in \bigcap_{n=1}^{\infty} \mathcal{D}(D^n_{o,\alpha})$ if and only if for each $n \in \mathbb{N}$, there is $C_n > 0$ such that*

$$|f(z)|^2 \leq \frac{C_n e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2n})^2}$$

for any $z \in \mathbb{C}$.

Proof. Let $n \in \mathbb{N}$. By Corollary 4.7, we have

$$f \in \mathcal{D}(D^n_{o,\alpha}) \iff f \in \mathcal{D}(M^{2n}) \iff z^{2n} f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha}).$$

1. Let $f \in \mathcal{D}(D^n_{o,\alpha})$. Then $z^{2n} f \in \mathcal{H}L^2_o(\mathbb{C}, m_{o,\alpha})$. By Theorem 3.15, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} (1+|z|^{2n})|f(z)| &= |f(z)| + |z^{2n} f(z)| \\ &\leq \frac{C_1 e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}} + \frac{C_2 e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}} \end{aligned}$$

for any $z \in \mathbb{C}$. It follows that there is $C > 0$ such that

$$|f(z)| \leq \frac{Ce^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}(1+|z|^{2n})}$$

for any $z \in \mathbb{C}$.

2. We assume that the inequality (4.2) holds. By assumption, we have

$$\begin{aligned}
\|z^{2n}f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} &= \int_{\mathbb{C}} |z^{2n}f(z)|^2 m_{o,\alpha}(z)\,dz \\
&\leq \int_{\mathbb{C}} \frac{C|z|^{4n}e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{2(n+1)})^2} m_{o,\alpha}(z)\,dz \\
&= C\int_0^{2\pi}\int_0^{\infty} \frac{r^{4n}e^{r^2}}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} m_{o,\alpha}(r) r\,dr d\theta \\
&= 2\pi C\int_0^{\infty} \frac{r^{4n}e^{r^2}}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} m_{o,\alpha}(r) r\,dr.
\end{aligned}$$

Since the asymptotic behavior of $m_{o,\alpha}$ as $r \to \infty$ is given by

$$m_{o,\alpha}(z) \approx \frac{1}{\pi^{\frac{1}{2}}2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} e^{-r^2} r^{2\alpha+2},$$

there is a constant $A > 0$ such that for all $r \in (S, \infty)$, when S is large enough, we have

$$\begin{aligned}
\int_S^{\infty} \frac{r^{4n}e^{r^2}m_{o,\alpha}(r)}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} r\,dr &\leq \int_S^{\infty} \frac{Ar^{4n+2\alpha+2}}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2}\,dr \\
&\leq A\int_S^{\infty} \frac{1}{r^3}\,dr \\
&< \infty.
\end{aligned}$$

Next, consider

$$\int_0^{S} \frac{r^{4n}e^{r^2}m_{o,\alpha}(r)}{(1+r^{2\alpha+1})(1+r^{2(n+1)})^2} r\,dr < \infty,$$

since the integrand is continuous on $[0, S]$. Hence, $\|z^{2n}f\|^2_{L^2(\mathbb{C},m_{o,\alpha})} < \infty$ and so $f \in \mathcal{D}(D^n_{o,\alpha})$.

3. The proof follows from 1 and 2. □

4.3 The Dunkl-type Segal-Bargmann space

The Dunkl operator Λ_α and the multiplication operator M are defined by

$$\Lambda_\alpha f(z) = \frac{df}{dz}(z) + \frac{2\alpha+1}{z}\left(\frac{f(z)-f(-z)}{2}\right)$$

and

$$(Mf)(z) = zf(z).$$

Proposition 4.9. *If f is an entire function, then*

$$(\Lambda_\alpha^2 f)(z) = \frac{d^2 f}{dz^2}(z) + \frac{2\alpha+1}{z}\left(\frac{df}{dz}(z)\right) - \frac{2\alpha+1}{z^2}\left(\frac{f(z)-f(-z)}{2}\right).$$

Proof. We calculate

$$\begin{aligned}
(\Lambda_\alpha^2 f)(z) &= \frac{d^2 f}{dz^2}(z) + \frac{2\alpha+1}{2}\left(\frac{z\frac{df}{dz}(z) + z\frac{df}{dz}(-z) - (f(z)-f(-z))}{z^2}\right) \\
&\quad + \frac{2\alpha+1}{2z}\left(\frac{df}{dz}(z) + \frac{2\alpha+1}{z}\left(\frac{f(z)-f(-z)}{2}\right)\right) \\
&\quad - \frac{2\alpha+1}{2z}\left(\frac{df}{dz}(-z) - \frac{2\alpha+1}{z}\left(\frac{f(-z)-f(z)}{2}\right)\right) \\
&= \frac{d^2 f}{dz^2}(z) + \frac{2\alpha+1}{2}\left(2\frac{\frac{df}{dz}(z)}{z} - \frac{f(z)}{z^2} + \frac{f(-z)}{z^2}\right) \\
&= \frac{d^2 f}{dz^2}(z) + \frac{2\alpha+1}{z}\left(\frac{df}{dz}(z)\right) - \frac{2\alpha+1}{z^2}\left(\frac{f(z)-f(-z)}{2}\right).
\end{aligned}$$

□

Corollary 4.10. *If f is an even entire function, then*

$$(\Lambda_\alpha^2 f)(z) = \frac{d^2 f}{dz^2}(z) + \left(\frac{2\alpha+1}{z}\right)\frac{df}{dz}(z).$$

Proof. Since f is an even function, this statement follows immediately from Proposition 4.9. □

From the above corollary, we see that the Schrödinger radial kinetic energy operator $D_{e,\alpha}$, mentioned in Section 4.1, is the Dunkl operator composed with itself and restricted on the even subspace $\mathcal{H}L_e^2(\mathbb{C}, m_{e,\alpha})$. Moreover, if we restrict to the odd subspace $\mathcal{H}L_o^2(\mathbb{C}, m_{o,\alpha})$, the composition of two Dunkl operators is the operator $D_{o,\alpha}$, mentioned in Section 4.2.

Corollary 4.11. *If f is an odd entire function, then*

$$(\Lambda_\alpha^2 f)(z) = \frac{d^2 f}{dz^2}(z) + (2\alpha+1)\frac{d}{dz}\left(\frac{f(z)}{z}\right).$$

Proof. Let f be an odd entire function. Then $f(z) = -f(-z)$ for all $z \in \mathbb{C}$. Thus, by Proposition 4.9, we have

$$\begin{aligned}(\Lambda_\alpha^2 f)(z) &= \frac{d^2 f}{dz^2}(z) + \left(\frac{2\alpha+1}{z}\right)\frac{df}{dz}(z) - \frac{2\alpha+1}{z^2}\left(\frac{f(z)-f(-z)}{2}\right) \\ &= \frac{d^2 f}{dz^2}(z) + \left(\frac{2\alpha+1}{z}\right)\frac{df}{dz}(z) - \frac{2\alpha+1}{z^2}f(z) \\ &= \frac{d^2 f}{dz^2}(z) + (2\alpha+1)\frac{d}{dz}\left(\frac{f(z)}{z}\right). \end{aligned}$$

□

Definition 4.12. For each $n \in \mathbb{N}$, we denote the domains of Λ_α and M^2 by

$$\mathcal{D}(\Lambda_\alpha^n) = \{f \in \mathcal{H}L_\alpha^2(\mathbb{C}) \mid \Lambda_\alpha^n f \in \mathcal{H}L_\alpha^2(\mathbb{C})\}$$

and

$$\mathcal{D}(M^n) = \{f \in \mathcal{H}L_\alpha^2(\mathbb{C}) \mid M^n f \in \mathcal{H}L_\alpha^2(\mathbb{C})\}.$$

Remark. According to Sifi and Soltani [9], the Dunkl operator Λ_α and the multiplication operator M are densely defined operators on $\mathcal{H}L^2(\mathbb{C})$ and are adjoints of each other.

Proposition 4.13. *$\mathcal{D}(\Lambda_\alpha) = \mathcal{D}(M)$.*

Proof. Let $f \in \mathcal{H}L_\alpha^2(\mathbb{C})$ and write $f(z) = \sum_{n=0}^\infty a_n z^n$ for all $z \in \mathbb{C}$. Then

$$\begin{aligned}(\Lambda_\alpha f)(z) &= \sum_{n=1}^\infty n a_n z^{n-1} + (2\alpha+1)\sum_{n=1}^\infty \left(\frac{1-(-1)^n}{2}\right) a_n z^{n-1} \\ &= \sum_{n=1}^\infty \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] a_n z^{n-1}\end{aligned}$$

and $(Mf)(z) = \sum_{n=0}^\infty a_n z^{n+1}$. We note that

$$d_{n+1}(\alpha) = \left[n + 1 + (2\alpha+1)\left(\frac{1+(-1)^n}{2}\right)\right] d_n(\alpha). \tag{4.3}$$

Using Propositon 3.1 and the above equality, we obtain

$$\begin{aligned}\|\Lambda_\alpha f\|^2_{\mathcal{H}L_\alpha^2(\mathbb{C})} &= \sum_{n=1}^\infty \left|n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right|^2 |a_n|^2 d_{n-1}(\alpha) \\ &= \sum_{n=0}^\infty \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] |a_n|^2 d_n(\alpha)\end{aligned}$$

and

$$
\begin{aligned}
\|Mf\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} &= \sum_{n=0}^{\infty} |a_n|^2 d_{n+1}(\alpha) \\
&= \sum_{n=0}^{\infty} \left[n + 1 + (2\alpha+1)\left(\frac{1+(-1)^n}{2}\right)\right] |a_n|^2 d_n(\alpha) \\
&= \sum_{n=0}^{\infty} \left[n + 1 + (2\alpha+1)\left(\frac{1-(-1)^n+2(-1)^n}{2}\right)\right] |a_n|^2 d_n(\alpha) \\
&= \sum_{n=0}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right) + 1 + (2\alpha+1)(-1)^n\right] |a_n|^2 d_n(\alpha).
\end{aligned}
$$

Hence,

$$
\|Mf\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} + \|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} + (2\alpha+1)\sum_{n=0}^{\infty}(-1)^n |a_n|^2 d_n(\alpha).
$$

Thus, if $\|Mf\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$, then $\|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Conversely, suppose that $\|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Because of the absolute convergence, $\sum_{n=0}^{\infty}(-1)^n|a_n|^2 d_n(\alpha) < \infty$. Hence, $\|Mf\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. This shows that $\mathcal{D}(\Lambda_\alpha) = \mathcal{D}(M)$. □

Corollary 4.14. *For each $n \in \mathbb{N}$, $\mathcal{D}(\Lambda_\alpha^n) = \mathcal{D}(M^n)$.*

Proof. By Proposition 4.13, we have $\mathcal{D}(\Lambda_\alpha) = \mathcal{D}(M)$. To show that $\mathcal{D}(\Lambda_\alpha^n) = \mathcal{D}(M^n)$, let $k \in \mathbb{N}$ and we assume that $\mathcal{D}(\Lambda_\alpha^k) = \mathcal{D}(M^k)$. Then

$$
\begin{aligned}
f \in \mathcal{D}(\Lambda_\alpha^{k+1}) &\iff \Lambda_\alpha^{k+1} f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\
&\iff \Lambda_\alpha f \in \mathcal{D}(\Lambda_\alpha^k) = \mathcal{D}(M^k) \\
&\iff M^k(\Lambda_\alpha f) \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\
&\iff z^k(\Lambda_\alpha f) \in \mathcal{H}L^2_\alpha(\mathbb{C}).
\end{aligned}
$$

We claim that $z^k(\Lambda_\alpha f) \in \mathcal{H}L^2_\alpha(\mathbb{C})$ if and only if $\Lambda_\alpha(z^k f) \in \mathcal{H}L^2_\alpha(\mathbb{C})$. To show this claim, we write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$
(\Lambda_\alpha f)(z) = \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] a_n z^{n-1}.
$$

Hence,

$$z^k(\Lambda_\alpha f)(z) = \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] a_n z^{n+k-1}$$

and

$$\Lambda_\alpha(z^k f)(z) = \sum_{n=1}^{\infty} \left[n + k + (2\alpha+1)\left(\frac{1-(-1)^{n+k}}{2}\right)\right] a_n z^{n+k-1}.$$

By Theorem 3.1,

$$\|z^k(\Lambda_\alpha f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right]^2 |a_n|^2 d_{n+k-1}(\alpha)$$

and

$$\|\Lambda_\alpha(z^k f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=1}^{\infty} \left[n + k + (2\alpha+1)\left(\frac{1-(-1)^{n+k}}{2}\right)\right]^2 |a_n|^2 d_{n+k-1}(\alpha).$$

Clearly, $\|\Lambda_\alpha(z^k f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$ implies that $\|z^k(\Lambda_\alpha f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Conversely, we consider

$$\begin{aligned} n + k + (2\alpha+1)\left(\frac{1-(-1)^{n+k}}{2}\right) &\leq n + k + (2\alpha+1) \\ &\leq Kn \\ &\leq K\left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] \end{aligned}$$

where K is a constant depending on k. Suppose that $\|z^k(\Lambda_\alpha f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Using the above inequality, we obtain $\|\Lambda_\alpha(z^k f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Thus,

$$\begin{aligned} f \in \mathcal{D}(\Lambda_\alpha^{k+1}) &\iff \Lambda_\alpha(z^k f) \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\ &\iff z^k f \in \mathcal{D}(\Lambda_\alpha) = \mathcal{D}(M) \\ &\iff z^{k+1} f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\ &\iff M^{k+1} f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\ &\iff f \in \mathcal{D}(M^{k+1}). \end{aligned}$$

By mathematical induction, we conclude that $\mathcal{D}(\Lambda_\alpha^n) = \mathcal{D}(M^n)$ for all $n \in \mathbb{N}$. $\square$

Next, consider the differential operator $Df(z) = \frac{df}{dz}(z)$. For each $n \in \mathbb{N}$, define

$$\mathcal{D}(D^n) = \{f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \mid D^n f \in \mathcal{H}L^2_\alpha(\mathbb{C})\}$$

and

$$C^\infty_\alpha(\mathbb{C}) = \bigcap_{n=1}^{\infty} \mathcal{D}(D^n).$$

That is, $f \in C^\infty_\alpha(\mathbb{C})$ if and only if the derivatives of f of all orders are in $\mathcal{H}L^2_\alpha(\mathbb{C})$.

Proposition 4.15. $\mathcal{D}(D) = \mathcal{D}(\Lambda_\alpha)$

Proof. Let $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$ with $f(z) = \sum_{n=0}^\infty a_n z^n$. Then $Df(z) = \sum_{n=1}^\infty n a_n z^{n-1}$ and

$$\Lambda_\alpha f(z) = \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] a_n z^{n-1}.$$

By Proposition 3.1 and the equality (4.3), we obtain

$$\begin{aligned} \|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} &= \sum_{n=0}^{\infty} |a_n|^2 d_n(\alpha) \\ &= \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] |a_n|^2 d_{n-1}(\alpha), \end{aligned}$$

$$\|Df(z)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=1}^{\infty} n^2 |a_n|^2 d_{n-1}(\alpha)$$

and

$$\|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right]^2 |a_n|^2 d_{n-1}(\alpha).$$

We calculate

$$\begin{aligned} &\left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right]^2 \\ &= n^2 + n(2\alpha+1)(1-(-1)^n) + (2\alpha+1)^2 \frac{(1-(-1)^n)^2}{4} \\ &= n^2 + n(2\alpha+1)(1-(-1)^n) + (2\alpha+1)^2 \frac{(1-(-1)^n)}{2} \\ &= n^2 - n(2\alpha+1)(-1)^n + (2\alpha+1)\left(n + (2\alpha+1)\frac{(1-(-1)^n)}{2}\right). \end{aligned}$$

Thus,

$$\|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \|Df\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} - (2\alpha+1)\sum_{n=1}^{\infty} n(-1)^n|a_n|^2 d_{n-1}(\alpha)$$
$$+ (2\alpha+1)\|f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})}.$$

Clearly, $\|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$ implies that $\|Df\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. On the other hand, suppose that $\|Df\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Then

$$\sum_{n=1}^{\infty} |n(-1)^{n-1}||a_n|^2 d_{n-1}(\alpha) \le \|Df\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty.$$

This show that $\|\Lambda_\alpha f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Therefore, $\mathcal{D}(D) = \mathcal{D}(\Lambda_\alpha)$. □

Corollary 4.16. *For each $n \in \mathbb{N}$, $\mathcal{D}(\Lambda_\alpha^n) = \mathcal{D}(D^n)$.*

Proof. By Proposition 4.15, we have $\mathcal{D}(\Lambda_\alpha) = \mathcal{D}(D)$. Let $k \in \mathbb{N}$. Assume that $\mathcal{D}(\Lambda_\alpha^k) = \mathcal{D}(D^k)$. Then

$$\begin{aligned} f \in \mathcal{D}(\Lambda_\alpha^{k+1}) &\iff \Lambda_\alpha^{k+1} f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\ &\iff \Lambda_\alpha f \in \mathcal{D}(\Lambda_\alpha^k) = \mathcal{D}(D^k) \\ &\iff D^k(\Lambda_\alpha f) \in \mathcal{H}L^2_\alpha(\mathbb{C}). \end{aligned}$$

We claim that $D^k(\Lambda_\alpha f) \in \mathcal{H}L^2_\alpha(\mathbb{C})$ if and only if $\Lambda_\alpha(D^k f) \in \mathcal{H}L^2_\alpha(\mathbb{C})$. Note that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$D^k f(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}.$$

By direct calculation, we have

$$(\Lambda_\alpha f)(z) = \sum_{n=1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] a_n z^{n-1}.$$

Thus,

$$\begin{aligned} D^k(\Lambda_\alpha f)(z) &= \sum_{n=k+1}^{\infty} \left[n + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\right] \frac{(n-1)!}{(n-1-k)!} a_n z^{n-1-k} \\ &= \sum_{n=k+1}^{\infty} \left[n-k + (2\alpha+1)\left(\frac{1-(-1)^n}{2}\right)\frac{n-k}{n}\right] \frac{n!}{(n-k)!} a_n z^{n-k-1} \end{aligned}$$

and

$$\Lambda_\alpha(D^k f)(z) = \sum_{n=k+1}^{\infty} \left[n - k + (2\alpha+1)\left(\frac{1-(-1)^{n-k}}{2}\right)\right] \frac{n!}{(n-k)!} a_n z^{n-k-1}.$$

By Theorem 3.1, we obtain that

$$\|D^k(\Lambda_\alpha f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})}$$
$$= \sum_{n=k+1}^{\infty} \left[n - k + (2\alpha+1)\left(\frac{1-(-1)^{n}}{2}\right)\frac{n-k}{n}\right]^2 \left(\frac{n!}{(n-k)!}\right)^2 |a_n|^2 d_{n-k-1}(\alpha)$$

and

$$\|\Lambda_\alpha(D^k f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})}$$
$$= \sum_{n=k+1}^{\infty} \left[n - k + (2\alpha+1)\left(\frac{1-(-1)^{n-k}}{2}\right)\right]^2 \left(\frac{n!}{(n-k)!}\right)^2 |a_n|^2 d_{n-k-1}(\alpha).$$

Conversely, we consider

$$\begin{aligned} n - k + (2\alpha+1)\left(\frac{1-(-1)^{n}}{2}\right)\left(\frac{n-k}{n}\right) &\le n - k + (2\alpha+1)\left(\frac{n-k}{n}\right) \\ &\le n-k+(2\alpha+1) \\ &\le K(n-k) \\ &\le K\left[n-k+(2\alpha+1)\left(\frac{1-(-1)^{n-k}}{2}\right)\right] \end{aligned}$$

where K is a constant depending on k. Thus, $\|\Lambda_\alpha(D^k f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$ implies that $\|D^k(\Lambda_\alpha f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Similarly, we calculate

$$\begin{aligned} n - k + (2\alpha+1)\left(\frac{1-(-1)^{n-k}}{2}\right) &\le n-k+(2\alpha+1) \\ &\le K(n-k) \\ &\le K\left[n-k+(2\alpha+1)\left(\frac{1-(-1)^{n}}{2}\right)\left(\frac{n-k}{n}\right)\right] \end{aligned}$$

where K is a constant depending on k. Suppose $\|D^k(\Lambda_\alpha f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Using the above inequality, we obtain that $\|\Lambda_\alpha(D^k f)\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} < \infty$. Thus,

$$\begin{aligned} f \in \mathcal{D}(\Lambda_\alpha^{k+1}) &\iff \Lambda_\alpha(D^k f) \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\ &\iff D^k f \in \mathcal{D}(\Lambda_\alpha) = \mathcal{D}(D) \\ &\iff D^{k+1} f \in \mathcal{H}L^2_\alpha(\mathbb{C}) \\ &\iff f \in \mathcal{D}(D^{k+1}). \end{aligned}$$

This show that $\mathcal{D}(\Lambda_\alpha^{k+1}) = \mathcal{D}(D^{k+1})$. By mathematical induction, we obtain $\mathcal{D}(\Lambda_\alpha^n) = \mathcal{D}(D^n)$ for all $n \in \mathbb{N}$. □

The above results tell us that the domain of the Dunkl operator, multiplication operator and differential operator are equal. We characterize smooth functions in the Dunkl-type Segal-Bargmann space in the following theorem.

Theorem 4.17. *Let $n \in \mathbb{N}$ and $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$.*

1. *If $f \in \mathcal{D}(D^n)$, then there is $C > 0$ such that*

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^n)^2}$$

   for any $z \in \mathbb{C}$.

2. *If there is $C > 0$ such that*

$$|f(z)|^2 \leq \frac{Ce^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{n+2})^2} \tag{4.4}$$

   for any $z \in \mathbb{C}$, then $f \in \mathcal{D}(D^n)$.

3. *$f \in C_\alpha^\infty(\mathbb{C})$ if and only if for each $n \in \mathbb{N}$, there is $C_n > 0$ such that*

$$|f(z)|^2 \leq \frac{C_n e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^n)^2}$$

   for any $z \in \mathbb{C}$.

Proof. Let $n \in \mathbb{N}$ and $f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. We note that

$$f \in \mathcal{D}(D^n) \iff f \in \mathcal{D}(\Lambda^n_\alpha) \iff f \in \mathcal{D}(M^n) \iff z^n f \in \mathcal{H}L^2_\alpha(\mathbb{C}).$$

1. Let $f \in \mathcal{D}(D^n)$. Then $z^n f \in \mathcal{H}L^2_\alpha(\mathbb{C})$. By Theorem 3.16, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}(1+|z|^n)|f(z)| &= |f(z)| + |z^n f(z)| \\ &\leq \frac{C_1 e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}} + \frac{C_2 e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}}\end{aligned}$$

for any $z \in \mathbb{C}$. This implies that there is $C > 0$ such that

$$|f(z)| \leq \frac{C e^{|z|^2/2}}{(1+|z|^{2\alpha+1})^{1/2}(1+|z|^n)}$$

for any $z \in \mathbb{C}$.

2. We assume that (4.4) holds. Note that

$$\|z^n f\|^2_{\mathcal{H}L^2_\alpha(\mathbb{C})} = \|(z^n f)_e\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} + \|(z^n f)_o\|^2_{L^2(\mathbb{C}, m_{o,\alpha})}.$$

Using Minkowski's inequality, we see that

$$\begin{aligned}&\|(z^n f)_e\|_{L^2(\mathbb{C}, m_{e,\alpha})} \\ &= \left(\int_{\mathbb{C}} |(z^n f)_e(z)|^2 \, m_{e,\alpha}(z)\, dz \right)^{1/2} \\ &= \left(\int_{\mathbb{C}} \left| \frac{z^n f(z) + (-z)^n f(-z)}{2} \right|^2 m_{e,\alpha}(z)\, dz \right)^{1/2} \\ &\leq \left(\int_{\mathbb{C}} \left| \frac{z^n f(z)}{2} \right|^2 m_{e,\alpha}(z)\, dz \right)^{1/2} + \left(\int_{\mathbb{C}} \left| \frac{(-z)^n f(-z)}{2} \right|^2 m_{e,\alpha}(z)\, dz \right)^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{C}} \left| \frac{z^n f(z)}{2} \right|^2 m_{e,\alpha}(z)\, dz &= \frac{1}{4} \int_{\mathbb{C}} |z|^{2n} \, |f(z)|^2 \, m_{e,\alpha}(z)\, dz \\ &\leq \frac{1}{4} \int_{\mathbb{C}} \frac{C|z|^{2n} e^{|z|^2}}{(1+|z|^{2\alpha+1})(1+|z|^{n+2})^2} m_{e,\alpha}(z)\, dz \\ &= \frac{C}{4} \int_0^{2\pi} \int_0^\infty \frac{r^{2n} e^{r^2}}{(1+r^{2\alpha+1})(1+r^{n+2})^2} m_{e,\alpha}(r) r \, dr d\theta \\ &= \frac{2\pi C}{4} \int_0^\infty \frac{r^{2n} e^{r^2}}{(1+r^{2\alpha+1})(1+r^{n+2})^2} m_{e,\alpha}(r) r \, dr.\end{aligned}$$

The asymptotic behavior of $m_{e,\alpha}$ as $r \to \infty$ is

$$m_{e,\alpha}(z) \approx \frac{1}{\pi^{\frac{1}{2}} 2^{\alpha+\frac{1}{2}} \Gamma(\alpha+1)} e^{-r^2} r^{2\alpha+2}.$$

Thus, there is a constant $A > 0$ such that for all $r \in (S, \infty)$, when S is large enough, we have

$$\begin{aligned} \int_S^\infty \frac{r^{2n} e^{r^2} m_{e,\alpha}(r)}{(1+r^{2\alpha+1})(1+r^{n+2})^2} r\,dr &\leq \int_S^\infty \frac{A r^{2n+2\alpha+2}}{(1+r^{2\alpha+1})(1+r^{n+2})^2}\,dr \\ &\leq A \int_S^\infty \frac{1}{r^3}\,dr \\ &< \infty. \end{aligned}$$

Since the integrand is continuous function on $[0, S]$,

$$\int_0^S \frac{r^{2n} e^{r^2} m_{e,\alpha}(r)}{(1+r^{2\alpha+1})(1+r^{n+2})^2} r\,dr < \infty.$$

Thus, we obtain

$$\int_{\mathbb{C}} \left| \frac{z^n f(z)}{2} \right|^2 m_{e,\alpha}(z)\,dz < \infty$$

and similarly, we obtain

$$\int_{\mathbb{C}} \left| \frac{(-z)^n f(-z)}{2} \right|^2 m_{e,\alpha}(z)\,dz < \infty.$$

Therefore, $\|(z^n f)_e\|^2_{L^2(\mathbb{C}, m_{e,\alpha})} < \infty$. By the same argument as in the proof of $\|(z^n f)_e\|^2_{L^2(\mathbb{C}, m_{e,\alpha})}$, we obtain $\|(z^n f)_o\|^2_{L^2(\mathbb{C}, m_{o,\alpha})} < \infty$. This implies that $z^n f \in \mathcal{H}L^2_\alpha(\mathbb{C})$.

3. The proof follows from 1 and 2. $\square$

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