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RATE OF CONVERGENCE OF BINOMIAL FORMULA FOR OPTION PRICING

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สูตรทวินามและสูตรแบล็ค-โชลส์เป็นเครื่องมือสำหรับการประเมินราคาของคอลออปชัน ณ เวลาที่กำหนด และเรายังทราบว่าสูตรทวินามจะลู่เข้าสู่สูตรแบล็ค-โชลส์ เมื่อจำนวนของคาบ (n) ลู่เข้าสู่อนันต์ ในงานวิจัยนี้เราได้ให้อัตราการลู่เข้านี้ อันดับการลู่เข้าของเราคือ $\frac{1}{n\sqrt{n}}$ ซึ่งดีกว่า Cox, Ross and Robistein (1979), Leisen and Reimer (1996), Heston and Zhon (2000), Francine and Marc Diener (2004) and Chang and Palmer (2007) นอกจากนี้เรายังหาค่าคงที่ของขอบเขตการลู่เข้าได้อย่างชัดเจน

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The binomial and Black–Scholes formulas are tools for valuating a call option at any specified time. We have already known that the binomial formula converges to the Black–Scholes formula as the number of periods (n) converges to infinity. In this research, we give the rate of this convergence. Our order is $\frac{1}{n\sqrt{n}}$ which is better than Cox, Ross and Robistein (1979), Leisen and Reimer (1996), Heston and Zhon (2000), Francine and Marc Diener (2004) and Chang and Palmer (2007). We also provide the explicit constant of the bound.

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CONTENTS

CHAPTER I
INTRODUCTION

An option is a financial derivative that represents a contract sold by one party (the option writer) to another party (the option holder). The contract offers the buyer the right, but not the obligation, to buy (call) or sell (put) a security or other financial asset at an agreed-upon price (the strike price) during a certain period of time or on a specific date (exercise date).

A call option is an agreement that gives an investor the right, but not the obligation, to buy a stock, bond, commodity, or other instrument at a specified price within a specific time period.

A European option is an option that can only be exercised at the end of its life, at its maturity.

Conventionally, we denote

$$\begin{aligned} S_0 &\quad \text{as the current stock price,} \\ K &\quad \text{as the strike price,} \\ r &\quad \text{as the risk–free rate,} \\ T &\quad \text{as time to maturity} \quad \text{and} \\ \delta &\quad \text{as the volatility.} \end{aligned}$$

The Black–Scholes formula which was introduced by three economists, Fischer Black, Myron Scholes and Robert Merton, in 1973 have been used to calculate the theoretical price of European call option. This formula is widely used in many stock markets. The formula for call price is given by

$$C_{BS} = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \frac{\delta^2}{2})T}{\delta\sqrt{T}},$$
$$d_2 = d_1 - \delta\sqrt{T},$$

and Φ is the standard normal distribution function (see Fischer Black and Myron Scholes [4] and Robert Merton [17] for more details).

The binomial model is a tool for valuating a call option price at each point in the specified time. We assume the current stock price S_0 either rises to S_0u with probability p or falls to S_0d with probability $1-p$ at the end of next period, where $0 < d \leq 1 \leq u$.

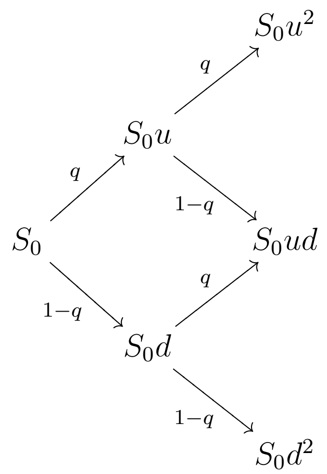

Fig 1.1 : Example for $n = 2$, 2–step binomial.

Let S_n be the stock price at period n. Then for $k = 0, 1, \ldots, n$ the probability that

$$S_n = S_0u^kd^{n-k} \quad \text{is} \quad \binom{n}{k} q^k(1-q)^{n-k}.$$

Let C_n be the value of the call option at period n. Then

$$C_n = \max\{0, S_n - K\}.$$

We use $C(n)$ as the n-period binomial model call option price. Then

$$C(n) = e^{-rT} E(C_n) = e^{-rT} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \max\{0, S_0 u^k d^{n-k} - K\}.$$

Let $a = \min\{j \in \{0, 1, \ldots, n\} \mid S_0 u^j d^{n-j} \geq K\}$. Then

$$C(n) = e^{-rT} \sum_{k=a}^{n} \binom{n}{k} q^k (1-q)^{n-k} (S_0 u^k d^{n-k} - K).$$

If we assume that $E[S_{k+1} \mid S_k] = S_k e^{r\Delta t}$, where $\Delta t = \dfrac{T}{n}$ (see [1] p.330), then

$$S_k e^{r\Delta t} = p S_k u + (1-p) S_k d$$

and

$$C(n) = S_0 \sum_{k=a}^{n} \binom{n}{k} p^k (1-p)^{n-k} - K e^{-rT} \sum_{k=a}^{n} \binom{n}{k} q^k (1-q)^{n-k}$$

where

$$q = \frac{e^{r\Delta t} - d}{u - d} \quad \text{and} \quad p = q u e^{-r\Delta t}$$

(see [1]).

In 1979, Cox, Ross and Rubinstein ([5]) obtained the CRR model by taking

$$u = e^{\delta\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\delta\sqrt{\Delta t}}$$

in the binomial model, and proved that for European call options the binomial model converges to the Black–Scholes formula as n tends to infinity. See Cox et al. [5] and Rendleman and Bartter [21] for more details.

After that, Heston and Zhou ([9], 2000) obtained the rate $\dfrac{1}{\sqrt{n}}$ of convergence, is

$$C(n) = C_{BS} + O\left(\frac{1}{\sqrt{n}}\right). \tag{1.1}$$

Francine and Marc Diener ([8], 2004) improved Heston and Zhou work by adding additional term in (1.1). Their result is stated in theorem 1.1.

Theorem 1.1. *If* $S_0 = 1$ *and* $T = 1$, *then*

$$C(n) = C_{BS} + \frac{e^{-\frac{d_1^2}{2}}}{24\delta\sqrt{2\pi}} \frac{A - 12\delta^2\left(\Delta_n^2 - 1\right)}{n} + O\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$\Delta_n = 1 - 2\,\mathrm{frac}\left[\frac{\log(1/K) + n\log d}{\log(u/d)}\right] \quad \textit{and}$$
$$A = -\delta^2(6 + d_1^2 + d_2^2) + 4(d_1^2 - d_2^2)r - 12r^2$$

with $\mathrm{frac}[x]$ *is the fractional part of a real number* x.

Besides the CRR model, there are Jarrow and Rudd model ([12], 1983) and Tian model ([24], 1993). Jarrow and Rudd considered the binomial model by taking

$$u = e^{\delta\sqrt{\Delta t} + (r - \frac{1}{2}\delta^2)\Delta t}, \quad d = e^{-\delta\sqrt{\Delta t} + (r - \frac{1}{2}\delta^2)\Delta t},$$

while Tian let

$$u = \frac{MV}{2}\left(V + 1 + \sqrt{V^2 + 2V - 3}\right), \quad d = \frac{MV}{2}\left(V + 1 - \sqrt{V^2 + 2V - 3}\right),$$

with $M = e^{r\Delta t}$, $V = e^{\delta^2\Delta t}$.

Leisen and Reimer ([13], 1996) proved that $C(n)$ in CRR model, Jarrow and Rudd model and Tian model converge to the Black–Scholes formula with a rate

$\frac{1}{n}$. Their result is

$$C(n) = C_{BS} + O\left(\frac{1}{n}\right).$$

In 2007, Chang and Palmer ([14]) considered general class of binomial models by taking

$$u = e^{\delta\sqrt{\Delta t}+\lambda_n\delta^2\Delta t} \quad \text{and} \quad d = e^{-\delta\sqrt{\Delta t}+\lambda_n\delta^2\Delta t},$$

where λ_n is a general bounded sequence.

In specific case, if we take

$$\lambda_n = 0 \quad \text{and} \quad \lambda_n = \frac{r}{\delta^2} - \frac{1}{2},$$

we will obtain CRR model and Jarrow and Rudd model, respectively. They also showed that $C(n)$ converges to C_{BS} at rate $\frac{1}{n}$. Their result is stated in theorem 1.2.

Theorem 1.2.

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\delta\sqrt{2\pi T}} \frac{A - 12\delta^2 T\left(\Delta_n^2 - 1\right)}{n} + O\left(\frac{1}{n}\right),$$

where

$$\Delta_n = 1 - 2\text{frac}\left[\frac{\log(S_0/K) + n\log d}{\log(u/d)}\right] \quad \textit{and}$$
$$A = -\delta^2 T\left(6 + d_1^2 + d_2^2\right) + 4T\left(d_1^2 - d_2^2\right)\left(r - \lambda_n\delta^2\right) - 12T^2\left(r - \lambda_n\delta^2\right)^2.$$

In this thesis, we improve the rate of convergence of the Binomial formula converging to the Black–Scholes formula in Theorem 1.2. Our result is stated in Theorem 1.3

Theorem 1.3. *Let* $n \geq \max\{100T, \frac{60}{\delta^4}, \frac{1.2657\max\{d_1^2, d_2^2\}}{\delta^4}, 30\max\{d_1^2, d_2^2\}\}$. *Then* *for the n-period binomail model, the price of a European call option satisfies*

$$C(n) = C_{BS} + S_0 F(d_1, \alpha, \beta) - Ke^{-rT} F(d_2, \hat{\alpha}, \hat{\beta}) + r,$$

where

$$F(d, s, t) = \frac{e^{\frac{-d^2}{2}}}{\sqrt{2\pi}} \left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} - \frac{2T(s^2 d + \sqrt{T} t)}{n} - \frac{d\,(1 - 2\,\mathrm{frac}(-b))^2}{2n} \right) + \frac{2(d^2 - 1)s\sqrt{T}}{3n\sqrt{2\pi}} e^{-d^2/2} + \frac{d(1 - d^2)}{12\sqrt{2\pi}} e^{-d^2/2},$$

$$b = \frac{\log(K/S_0) - n \log d}{\log(u/d)},\ \mathrm{frac}(-b) = -b - \lfloor -b \rfloor,\ \lfloor -b \rfloor = \max\{m \in \mathbb{N} \mid m \leq -b\},$$

$$\alpha = \frac{2(r - \lambda_n \delta^2) + \delta^2}{4\delta}, \quad \beta = -\frac{\delta^4 + 4\delta^2(r - \lambda_n \delta^2) + 12(r - \lambda_n \delta^2)^2}{48\delta},$$

$$\hat{\alpha} = \frac{2(r - \lambda_n \delta^2) - \delta^2}{4\delta}, \quad \hat{\beta} = \frac{\delta^4 - 4\delta^2(r - \lambda_n \delta^2) + 12(r - \lambda_n \delta^2)^2}{48\delta},$$

$$|r| \leq S_0 r(d_1) + K r(d_2)$$

$$\text{and} \quad r(d) = \frac{1.7185|d|^3 + 19.4642}{\delta^4 n\sqrt{n}} + \frac{47.1537}{n\sqrt{n}}.$$

CHAPTER II
REFINEMENT ON NORMAL APPROXIMATION OF POISSON BINOMIAL

In this chapter, we use Taylor's formula to improve the approximation of the Poisson binomial distribution by the standard normal distribution Φ, by adding some correction terms.

Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variable such that $P(X_j = 1) = p_j$ and $P(X_j = 0) = q_j$, where $0 < p_j < 1$ and $q_j = 1 - p_j$ for all $j = 1, 2, \ldots, n$ and

$$S_n = \sum_{j=1}^{n} X_j.$$

We call S_n a Poisson binomial random variable. Let

$$\mu := E(S_n) = \sum_{j=1}^{n} p_j \quad \text{and} \quad \sigma^2 := Var(S_n) = \sum_{j=1}^{n} p_j q_j.$$

It is well-known from central limit theorem ([2]) that

$$\lim_{n\to\infty} P\left(\frac{S_n - \mu}{\sigma} \leq z\right) = \Phi(z) \quad \text{for all } z \in \mathbb{R}.$$

Berry–Esseen theorem given by Shevtsova ([23]) showed that

$$\sup_{z\in\mathbb{R}} \left| P\left(\frac{S_n - \mu}{\sigma} \leq z\right) - \Phi(z) \right| \leq 0.56 \sum_{j=1}^{n} E\left|\frac{X_j - p_j}{p_j q_j}\right|^3.$$

In this chapter, we investigate the approximation of S_n by adding some correction terms. Makabe ([16]) gave the correction term $C_1(x)$ defined by

$$C_1(x) = \frac{1}{6\sqrt{2\pi}\sigma^3}\sum_{j=1}^{n} p_j q_j (p_j - q_j)(1-x^2)e^{-x^2/2},$$

to approximate the probability of S_n by $G(x) = \Phi(x) + C_1(x)$. Let

$$\delta_n := \left| P(a \le S_n \le b) - (G(x_2) - G(x_1)) \right|,$$

where

$$x_1 = \frac{1}{\sigma}\left(a - \mu - \frac{1}{2}\right) \quad \text{and} \quad x_2 = \frac{1}{\sigma}\left(b - \mu + \frac{1}{2}\right).$$

He showed that there exists a positive constant C such that

$$\delta_n \le \frac{C}{\sigma^2}, \tag{2.1}$$

for $\sigma^2 \ge 25$ and $p_j \le \frac{1}{2}$ for $j = 1, \dots, n$.

Later on, in 1993, Mikhailov ([18]) calculated the constant C in (2.1) and found that

$$\delta_n \le \frac{2(\sigma+3)}{\sigma^3}$$

under the condition $\sigma^2 \ge 100$. In 1995, Volkova ([26]) showed that if $\sigma^2 \ge 100$, then

$$\begin{aligned}\delta_n \le \frac{2}{\sigma^2}\left(0.05\beta_4 + 0.1\beta_3^4 + 0.08\right) + \frac{2}{\sigma^3}\left(0.05\beta_3 + 0.17\beta_3\beta_4 + 0.056\right) \\ + \frac{2}{\sigma^4}\left(0.06\beta_4^2 + 0.27\beta_4 + 0.002\right),\end{aligned}$$

where β_3 and β_4 are the third and fourth semi-invariants of S_n (see more detail in

[6] and [11]), respectively. Neammanee ([19]) showed in 2005 that

$$\delta_n \leq \frac{0.1618}{\sigma^2} \quad \text{for } \sigma^2 \geq 100.$$

In case of $p_1 = p_2 = \cdots = p_n =: p$, S_n is a binomial random variable. Shevtsova ([22]) showed that

$$\sup_{z\in\mathbb{R}} \left| P\left(\frac{S_n - \mu}{\sigma} \leq z\right) - \Phi(z) \right| \leq 0.3328 \left(\frac{(2p^2 - 3p + 1)/(q\sqrt{pq}) + 0.415}{\sqrt{n}} \right), \tag{2.2}$$

where $q = 1 - p$.

From (2.2), we see that the rate of convergence is $1/\sqrt{n}$. Many authors improved this rate of convergence. For example, Uspensky ([25]) showed that

$$\delta_n \leq \frac{0.26 + 0.36|p - q|}{npq} + 2e^{-(3/2)\sigma}, \quad \text{for } npq \geq 25.$$

Later, in 1955 Makabe ([15]) improved the result of Uspensky by showing that

$$\delta_n \leq \frac{0.106 + 0.054(q - p) + 0.108(q - p)^2}{npq} + 2e^{-(3/2)\sigma}$$

under the condition that $p < \frac{1}{2}$, $npq \geq 25$ and $n \geq 100$.

Observe that the rate of convergence of Uspensky ([25]) and Makabe ([15]) is of order $O(1/n)$ which is sharper than (2.2).

In this work, we improve the bound of Neammanee ([19]) by adding additional terms. Our result is in Theorem 1.1 and Corollary 1.2. The techniques are Taylor's formula and the idea from Uspensky ([25]) and Neammanee ([19]).

Theorem 2.1. *Assume that* $\sigma^2 \geq 25$. *Then*

$$P(a \leq S_n \leq b) = G(x_2) - G(x_1) + \varepsilon_n,$$

where

$$G(x) = \Phi(x) + C_2(x)e^{-x^2/2},$$

$$\begin{aligned}
C_2(x) &= \frac{1-x^2}{6\sqrt{2\pi}\sigma^3}\sum_{j=1}^{n} p_j q_j (p_j - q_j) + \frac{x(3-x^2)}{24\sqrt{2\pi}\sigma^4}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_j \\
&\quad - \frac{x\left(15-10x^2+x^4\right)}{72\sqrt{2\pi}\sigma^6}\left(\sum_{j=1}^{n} p_j q_j (p_j - q_j)\right)^2 + \frac{x}{24\sqrt{2\pi}\sigma^2}, \\
|\varepsilon_n| &\le \frac{4.9132}{\sigma^3} + 0.948e^{-(3/2)\sigma}, \\
x_1 &= \frac{1}{\sigma}\left(a - \mu - \frac{1}{2}\right), \quad \textit{and} \quad x_2 = \frac{1}{\sigma}\left(b - \mu + \frac{1}{2}\right).
\end{aligned}$$

In case of binomial random variable, we have the following corollary.

Corollary 2.2. *Let S_n be a binomial random variable. For $\sigma^2 \ge 25$, we have*

$$P(a \le S_n \le b) = G(x_2) - G(x_1) + \varepsilon_n,$$

where

$$G(x) = \Phi(x) + \frac{Q_1(x)e^{-x^2/2}}{\sqrt{n}} + \frac{Q_2(x)e^{-x^2/2}}{n},$$

$$\begin{aligned}
Q_1(x) &= \frac{(p-q)\left(1-x^2\right)}{6\sqrt{2\pi pq}}, \\
Q_2(x) &= \frac{(1-6pq)(3-x^2)x}{24\sqrt{2\pi}pq} - \frac{(p-q)^2\left(15-10x^2+x^4\right)x}{72\sqrt{2\pi}pq} + \frac{x}{24pq\sqrt{2\pi}}, \\
|\varepsilon_n| &\le \frac{4.9132}{npq\sqrt{npq}} + 0.948e^{-(3/2)\sqrt{npq}}, \\
x_1 &= \frac{1}{\sqrt{npq}}\left(a - np - \frac{1}{2}\right) \quad \textit{and} \quad x_2 = \frac{1}{\sqrt{npq}}\left(b - np + \frac{1}{2}\right).
\end{aligned}$$

Note that, our order is of $O(\frac{1}{n\sqrt{n}})$ which is sharper than Neammanee ([19]) and Mikhailov ([18]).

In this chapter, we organize as the follows. In Section 2.1, we prove auxiliary results. A proof of main result is in Section 2.2.

2.1 Auxiliary Results

Let $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ and $\varphi(t)$ be the characteristic functions of $X_1, X_2, \ldots, X_n$ and S_n, respectively. Then

$$\varphi_j(t) = q_j + p_j e^{it},$$

and

$$\varphi(t) = \prod_{j=1}^{n} \varphi_j(t) = \prod_{j=1}^{n} (q_j + p_j e^{it}) \quad \text{where} \quad i = \sqrt{-1}.$$

For each $j = 1, 2, \ldots, n$, we note that the complex number $\varphi_j(t)$ can be represented in the form

$$\varphi_j(t) = \rho_j(t) e^{i\theta_j(t)},$$

where

$$\rho_j(t) := |\varphi_j(t)| = \left(1 - 4p_j q_j \sin^2 \frac{t}{2}\right)^{\frac{1}{2}} \tag{2.3}$$

and

$$\theta_j(t) := \arctan\left(\frac{p_j \sin t}{q_j + p_j \cos t}\right)$$

([19] pp.719–720).

Hence

$$\varphi(t) = \rho(t) e^{i\theta(t)}$$

where

$$\rho(t) := |\varphi(t)| = \prod_{j=1}^{n} \rho_j(t) = \prod_{j=1}^{n} \left(1 - 4p_j q_j \sin^2 \frac{t}{2}\right)^{\frac{1}{2}}$$

and

$$\theta(t) := \sum_{j=1}^{n} \theta_j(t) = \sum_{j=1}^{n} \arctan\left(\frac{p_j \sin t}{q_j + p_j \cos t}\right) \pmod{2\pi}.$$

Neammanee ([19], p.720) showed that

$$P(a \leq S_n \leq b) = R(x_2) - R(x_1),$$

where

$$x_1 = \frac{1}{\sigma}\left(a - \mu - \frac{1}{2}\right), \quad x_2 = \frac{1}{\sigma}\left(b - \mu + \frac{1}{2}\right).$$

and

$$R(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{\sin(t/2)}dt. \tag{2.4}$$

with $\alpha(t) = \theta(t) - \mu t$.

Next step, we approximate $R(x)$ by using the techniques of Neammanee ([19]) and Uspensky ([25]).

Proposition 2.3. *For* $\sigma^2 \geq 25$ *and* $t \in [0, \sqrt{\frac{3}{\sigma}}]$, *we have*

$$\left|\rho(t) - e^{-(1/2)\sigma^2 t^2} - \frac{1}{24}\sum_{j=1}^{n}(1 - 6p_j q_j)p_j q_j t^4 e^{-(1/2)\sigma^2 t^2}\right|$$
$$\leq \left(0.08056\sigma^2 t^6 + 0.0578\sigma^4 t^8\right) e^{-(1/2)\sigma^2 t^2}.$$

Proof. For each $j = 1, 2, \ldots, n$, we observe that

$$0 \leq 4p_j q_j \sin^2\left(\frac{t}{2}\right) < 1 \quad \text{for all} \quad t \in \left[0, \sqrt{\frac{3}{\sigma}}\right].$$

From this fact and (2.3), we have that

$$\begin{aligned}
&\ln \rho_j(t) \\
&= \ln\left(1 - 4p_j q_j \sin^2\frac{t}{2}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\ln\left(1 - 4p_j q_j \sin^2\frac{t}{2}\right)
\end{aligned}$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k$$
$$= -\frac{1}{2}\left[4p_jq_j\sin^2\frac{t}{2}+\frac{1}{2}\left(4p_jq_j\sin^2\frac{t}{2}\right)^2\right]-\frac{1}{2}\sum_{k=3}^{\infty}\frac{1}{k}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k. \qquad (2.5)$$

Since

$$\left|\sin\frac{t}{2}\right| \le \left|\frac{t}{2}\right|$$

for all $t \in \mathbb{R}$ ([19], p.721), we have

$$4p_jq_j\sin^2\frac{t}{2} \le p_jq_jt^2 < 1.$$

Hence

$$\begin{aligned}\sum_{k=3}^{\infty}\frac{1}{k}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k &\le \frac{1}{3}\sum_{k=3}^{\infty}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k\\ &= \frac{1}{3}\left(\frac{\left(4p_jq_j\sin^2(t/2)\right)^3}{1-4p_jq_j\sin^2(t/2)}\right)\\ &\le \frac{1}{3}\left(\frac{p_j^3q_j^3t^6}{1-p_jq_jt^2}\right)\\ &\le \frac{1}{40}p_jq_jt^6, \qquad (2.6)\end{aligned}$$

where we have used the fact that $p_jq_j \le \frac{1}{4}$ and $t^2 \le \frac{3}{5}$ in the last inequality. Using Taylor's formula, we can show that

$$\frac{t^2}{4}-\frac{t^4}{48}-\frac{t^6}{1440} \le \sin^2\left(\frac{t}{2}\right) \le \frac{t^2}{4}-\frac{t^4}{48}+\frac{t^6}{1440} \qquad (2.7)$$

and

$$\frac{t^4}{16}-\frac{t^6}{15} \le \sin^4(\frac{t}{2}) \le \frac{t^4}{16}+\frac{t^6}{15}. \qquad (2.8)$$

From (2.5)–(2.8) and the fact that $p_jq_j \le \frac{1}{4}$, we obtain

$$\begin{aligned}
\ln \rho_j(t) &\geq -2p_jq_j \sin^2 \frac{t}{2} - 4p_j^2q_j^2 \sin^4 \frac{t}{2} - \frac{1}{80}p_jq_jt^6 \\
&\geq -\frac{1}{2}p_jq_jt^2 + \frac{1}{24}p_jq_jt^4 - \frac{1}{720}p_jq_jt^6 - \frac{1}{4}p_j^2q_j^2t^4 - \frac{4}{15}p_j^2q_j^2t^6 - \frac{1}{80}p_jq_jt^6 \\
&\geq -\frac{1}{2}p_jq_jt^2 + \frac{1}{24}(1-6p_jq_j)p_jq_jt^4 - \frac{1}{720}p_jq_jt^6 - \frac{1}{15}p_jq_jt^6 - \frac{1}{80}p_jq_jt^6 \\
&= -\frac{1}{2}p_jq_jt^2 + \frac{1}{24}(1-6p_jq_j)p_jq_jt^4 - \frac{29}{360}p_jq_jt^6
\end{aligned}$$

and

$$\begin{aligned}
\ln \rho_j(t) &\leq -2p_jq_j \sin^2 \frac{t}{2} - 4p_j^2q_j^2 \sin^4 \frac{t}{2} \\
&\leq -\frac{1}{2}p_jq_jt^2 + \frac{1}{24}p_jq_jt^4 + \frac{1}{720}p_jq_jt^6 - \frac{1}{4}p_j^2q_j^2t^4 + \frac{4}{15}p_j^2q_j^2t^6 \\
&\leq -\frac{1}{2}p_jq_jt^2 + \frac{1}{24}(1-6p_jq_j)p_jq_jt^4 + \frac{49}{720}p_jq_jt^6.
\end{aligned}$$

Hence

$$\rho_j(t) \geq e^{-(1/2)p_jq_jt^2+(1/24)(1-6p_jq_j)p_jq_jt^4-(29/360)p_jq_jt^6} \tag{2.9}$$

and

$$\rho_j(t) \leq e^{-(1/2)p_jq_jt^2+(1/24)(1-6p_jq_j)p_jq_jt^4+(49/720)p_jq_jt^6}. \tag{2.10}$$

Since $e^x - 1 \geq x$ for $x \in \mathbb{R}$, by (2.9),

$$\begin{aligned}
\rho(t) - e^{-(1/2)\sigma^2t^2} &\geq \left(e^{(1/24)\sum\limits_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4-(29/360)\sigma^2t^6} - 1 \right) e^{-(1/2)\sigma^2t^2} \\
&\geq \left(\frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4 - \frac{29}{360}\sigma^2t^6 \right) e^{-(1/2)\sigma^2t^2}. \qquad (2.11)
\end{aligned}$$

By the fact that $t^2 \leq \dfrac{3}{\sigma}$, $\sigma^2 \geq 25$ and $p_jq_j \leq \dfrac{1}{4}$,

$$\left|\frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+(49/720)\sigma^2t^6\right| \le \frac{1}{24}\sigma^2t^4+\frac{49}{720}\sigma^2t^6$$
$$\le \frac{1}{24}\sigma^2\left(\frac{9}{\sigma^2}\right)+\frac{49}{720}\sigma^2\left(\frac{27}{\sigma^3}\right)$$
$$\le \frac{3}{8}+\frac{147}{400}$$
$$= 0.7425.$$

From this fact and the fact that

$$e^x = 1 + x + \frac{x^2}{2}e^{x_0} \qquad \text{for some } x_0 \text{ between } 0 \text{ and } x,$$

we have

$$e^{(1/24)\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+(49/720)\sigma^2t^6}-1$$
$$\le \frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+\frac{49}{720}\sigma^2t^6+\left[\frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+\frac{49}{720}\sigma^2t^6\right]^2 e^{0.7425}$$
$$\le \frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+\frac{49}{720}\sigma^2t^6+\left[\frac{1}{24}\sigma^2t^4+\frac{49}{1200}\sigma^2t^4\right]^2 e^{0.7425}$$
$$\le \frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+\frac{49}{720}\sigma^2t^6+0.0578\sigma^4t^8. \qquad (2.12)$$

By (2.10) and (2.12), we have

$$\rho(t)-e^{-(1/2)\sigma^2t^2}$$
$$\le e^{-(1/2)\sigma^2t^2}\left(e^{(1/24)\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4+(49/720)\sigma^2t^6}-1\right)$$
$$\le \frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4e^{-(1/2)\sigma^2t^2}+\frac{49}{720}\sigma^2t^6e^{-(1/2)\sigma^2t^2}+0.0578\sigma^4t^8e^{-(1/2)\sigma^2t^2}. \qquad (2.13)$$

By (2.11) and (2.13), we obtain

$$\left|\rho(t) - e^{-(1/2)\sigma^2 t^2} - \frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jt^4e^{-(1/2)\sigma^2t^2}\right|$$
$$\leq \left(0.08056\sigma^2t^6 + 0.0578\sigma^4t^8\right)e^{-(1/2)\sigma^2t^2}.$$

□

Proposition 2.4. *Assume that* $\sigma^2 \geq 25$. *Then*

1. $\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t)}{t}dt$
$=\sqrt{\frac{\pi}{2}}\int\limits_0^x e^{-(1/2)t^2}dt + \frac{\sqrt{2\pi}}{48\sigma^4}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_jx(3-x^2)e^{-(x^2/2)} + \Delta_1,$
where $|\Delta_1| \leq \frac{0.6838}{\sigma^3} + 0.0495e^{(-3/2)\sigma}$.

2. $\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t)tdt = \frac{x\sqrt{2\pi}}{2\sigma^2}e^{-x^2/2} + \Delta_2,$ *where* $|\Delta_2| \leq \frac{1}{2\sigma^6} + \frac{e^{-(3/2)\sigma}}{\sigma^2}$.

3. $\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t)t^5dt = \frac{\sqrt{2\pi}x\left(15-10x^2+x^4\right)e^{-x^2/2}}{2\sigma^6} + \Delta_3,$ *where*
$|\Delta_3| \leq \frac{24}{\sigma^8} + \frac{11.72}{\sigma^4}e^{-(3/2)\sigma}$.

4. $\int_0^{\sqrt{3/\sigma}} \rho(t)\cos(\sigma x t)t^2dt = -\frac{\sqrt{2\pi}\left(1-x^2\right)e^{-x^2/2}}{2\sigma^3} + \Delta_4,$ *where*
$|\Delta_4| \leq \frac{5.8904}{\sigma^5} + \frac{1.963}{\sigma^2\sqrt{\sigma}}e^{-(3/2)\sigma}$.

Proof. 1. Note that, for $k \in \mathbb{N}$,

$$\begin{aligned}\int\limits_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2t^2}t^{2k+1}dt &\leq \int\limits_0^{\infty} e^{-(1/2)\sigma^2t^2}t^{2k+1}dt \\ &= \frac{2^k}{\sigma^{2k+2}}\int\limits_0^{\infty} e^{-u}u^kdt \\ &= \frac{2^k\Gamma(k+1)}{\sigma^{2k+2}} \\ &= \frac{2^kk!}{\sigma^{2k+2}},\end{aligned} \tag{2.14}$$

where $\Gamma(\alpha) = \int\limits_0^\infty t^{\alpha-1}e^{-t}dt$ and $\Gamma(k+1) = k!$ for $k \in \mathbb{N}$ ([7], p. 855).
From Proposition 2.3 and (2.14),

$$\begin{aligned} &\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t)}{t}dt \\ &= \int_0^{\sqrt{3/\sigma}} \frac{e^{-(1/2)\sigma^2 t^2}\sin(\sigma x t)}{t}dt \\ &\quad + \frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_j \int_0^{\sqrt{3/\sigma}} t^3 e^{-(1/2)\sigma^2 t^2}\sin(\sigma x t)dt + \Delta_{11}, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} |\Delta_{11}| &\leq 0.08056\sigma^2 \int\limits_0^{\sqrt{3/\sigma}} t^5 e^{-(1/2)\sigma^2 t^2}dt + 0.0578\sigma^4 \int\limits_0^{\sqrt{3/\sigma}} t^7 e^{-(1/2)\sigma^2 t^2}dt \\ &\leq 0.08056\sigma^2\left(\frac{8}{\sigma^6}\right) + 0.0578\sigma^4\left(\frac{48}{\sigma^8}\right) \\ &\leq \frac{0.6838}{\sigma^3}, \end{aligned} \tag{2.16}$$

where we have used the fact that $\sigma^2 \geq 25$ in the last inequality.
Note from Neammanee ([19], p.725) that

$$\int_0^{\sqrt{3/\sigma}} \frac{e^{-(1/2)\sigma^2 t^2}\sin(\sigma x t)}{t}dt = \sqrt{\frac{\pi}{2}}\int\limits_0^x e^{-(1/2)t^2}dt + \Delta_{12}, \tag{2.17}$$

where

$$|\Delta_{12}| \leq \frac{0.1061}{\sigma}e^{(-3/2)\sigma} \leq 0.02122e^{(-3/2)\sigma}. \tag{2.18}$$

We see that

$$\frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_j\int_0^{\sqrt{3/\sigma}} t^3e^{-(1/2)\sigma^2t^2}\sin(\sigma xt)dt$$
$$=\frac{1}{24}\sum_{j=1}^{n}(1-6p_jq_j)p_jq_j\int_0^{\infty} t^3e^{-(1/2)\sigma^2t^2}\sin(\sigma xt)dt+\Delta_{13} \tag{2.19}$$

where

$$\begin{aligned}|\Delta_{13}| &\le \frac{1}{24}\sum_{j=1}^{n}|1-6p_jq_j|p_jq_j\int_{\sqrt{3/\sigma}}^{\infty} t^3e^{-(1/2)\sigma^2t^2}dt\\ &\le \frac{\sigma^2}{24}\int_{\sqrt{3/\sigma}}^{\infty} t^3e^{-(1/2)\sigma^2t^2}dt\\ &= \frac{1}{24}\left(\frac{2}{\sigma}\right)\left(\frac{3}{2}+\frac{1}{\sigma}\right)e^{-(3/2)\sigma}\\ &\le 0.0283e^{(-3/2)\sigma}.\end{aligned} \tag{2.20}$$

From the well–known integral

$$\int_0^{\infty} e^{-at^2}\cos(bt)dt=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-b^2/4a} \qquad \text{for} \quad a>0 \tag{2.21}$$

([19], p.725), if we differentiate (2.21) under the integral sign three times with respect to b, we have

$$\int_0^{\infty} t^3e^{-at^2}\sin(bt)dt=\frac{b}{8a^2}\sqrt{\frac{\pi}{a}}\left(3-\frac{b^2}{2a}\right)e^{-b^2/4a}.$$

Put $a=\dfrac{\sigma^2}{2}$ and $b=\sigma x$,

$$\int_0^{\infty} t^3e^{-(1/2)\sigma^2t^2}\sin(\sigma xt)dt=\frac{x(3-x^2)\sqrt{2\pi}}{2\sigma^4}e^{-x^2/2}. \tag{2.22}$$

From (2.15)–(2.20) and (2.22),

$$\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t)}{t} dt = \sqrt{\frac{\pi}{2}} \int\limits_0^x e^{-(1/2)t^2} dt + \frac{\sqrt{2\pi}}{48\sigma^4} \sum_{j=1}^n (1-6p_j q_j) p_j q_j x(3-x^2) e^{-x^2/2} + \Delta_1$$

where

$$|\Delta_1| \leq |\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}| \leq \frac{0.6838}{\sigma^3} + 0.0495 e^{(-3/2)\sigma}.$$

2. From the fact that

$$\left|\rho(t) - e^{-(1/2)\sigma^2 t^2}\right| \leq \frac{1}{16}\sigma^2 t^4 e^{-(1/2)\sigma^2 t^2} \tag{2.23}$$

([19], p.721), we have

$$\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t) t dt = \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t dt + \Delta_{21}, \tag{2.24}$$

where

$$|\Delta_{21}| \leq \frac{1}{16} \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} t^5 dt \leq \frac{1}{16}\left(\frac{8}{\sigma^6}\right) = \frac{1}{2\sigma^6}. \tag{2.25}$$

Using the same techniques of (2.17), we have

$$\int_0^{\infty} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t dt = \frac{x\sqrt{2\pi}}{2\sigma^2} e^{-x^2/2}. \tag{2.26}$$

This implies that

$$\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t dt = \int_0^{\infty} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t dt + \Delta_{22} \tag{2.27}$$

$$= \frac{x\sqrt{2\pi}}{2\sigma^2} e^{-x^2/2} + \Delta_{22}, \tag{2.28}$$

where

$$|\Delta_{22}| \le \int_{\sqrt{3/\sigma}}^{\infty} e^{-(1/2)\sigma^2 t^2} t dt = \frac{e^{-(3/2)\sigma}}{\sigma^2}. \tag{2.29}$$

Hence, by (2.24)–(2.29),

$$\int_0^{\sqrt{3/\sigma}} \rho(t) \sin(\sigma x t) t dt = \frac{x\sqrt{2\pi}}{2\sigma^2} e^{-x^2/2} + \Delta_2,$$

where

$$|\Delta_2| \le |\Delta_{21}| + |\Delta_{22}| \le \frac{1}{2\sigma^6} + \frac{e^{-(3/2)\sigma}}{\sigma^2}.$$

3. By (2.14) and (2.23), we have

$$\int_0^{\sqrt{3/\sigma}} \rho(t) \sin(\sigma x t) t^5 dt = \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t^5 dt + \Delta_{31}, \tag{2.30}$$

where

$$|\Delta_{31}| \le \frac{\sigma^2}{16} \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} t^9 dt \le \frac{\sigma^2}{16} \left(\frac{384}{\sigma^{10}} \right) = \frac{24}{\sigma^8}. \tag{2.31}$$

Similar to (2.22), we can show that

$$\int_0^{\infty} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t^5 dt = \frac{\sqrt{2\pi} x \left(15 - 10x^2 + x^4\right) e^{-x^2/2}}{2\sigma^6}. \tag{2.32}$$

Hence

$$\begin{aligned} \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t^5 dt &= \int_0^{\infty} e^{-(1/2)\sigma^2 t^2} \sin(\sigma x t) t^5 dt + \Delta_{32} \\ &= \frac{\sqrt{2\pi} x \left(15 - 10x^2 + x^4\right) e^{-x^2/2}}{2\sigma^6} + \Delta_{32}, \end{aligned} \tag{2.33}$$

where

$$|\Delta_{32}| \le \int_{\sqrt{3/\sigma}}^{\infty} e^{-(1/2)\sigma^2 t^2} t^5 dt = \left(\frac{4}{\sigma^6}\right)\left(\frac{9\sigma^2}{4} + 3\sigma + 2\right) e^{-(3/2)\sigma} \le \frac{11.72}{\sigma^4} e^{-(3/2)\sigma}. \tag{2.34}$$

By (2.30)–(2.34), the proof is completed.

4. Let $X \sim N\left(0, \dfrac{1}{\sigma}\right)$ and $M(t)$ be the moment generating function of X. It is well–known that

$$M(t) = e^{t^2/(2\sigma^2)} \quad \text{and} \quad M^{(k)}(0) = E(X^k).$$

Hence

$$\frac{1}{\sigma^{2k}} \prod_{j=1}^{k} (2j-1) = M^{(2k)}(0) = E(X^{2k}) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2k} e^{-(1/2)\sigma^2 x^2} dx \quad \text{for } k \in \mathbb{N}.$$

which implies that

$$\int_0^{\sqrt{3/\sigma}} x^{2k} e^{-(1/2)\sigma^2 x^2} dx \le \int_0^{\infty} x^{2k} e^{-(1/2)\sigma^2 x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} x^{2k} e^{-(1/2)\sigma^2 x^2} dx = \frac{\sqrt{2\pi}}{2\sigma^{2k+1}} \prod_{j=1}^{k} (2j-1). \tag{2.35}$$

From this fact and (2.23), we have

$$\int_0^{\sqrt{3/\sigma}} \rho(t) \cos(\sigma x t) t^2 dt = \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} \cos(\sigma x t) t^2 dt + \Delta_{41}, \tag{2.36}$$

where

$$|\Delta_{41}| \le \frac{\sigma^2}{16} \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} t^6 dt \le \frac{\sigma^2}{16} \left(\frac{15\sqrt{2\pi}}{\sigma^7}\right) = \frac{5.8904}{\sigma^5}. \tag{2.37}$$

By the same techniques of (2.22), we can show that

$$\int_0^\infty e^{-(1/2)\sigma^2 t^2}\cos(\sigma x t)t^2 dt = -\frac{\sqrt{2\pi}\,(1-x^2)\,e^{-x^2/2}}{2\sigma^3}.$$

Hence

$$\begin{aligned}\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2}\cos(\sigma x t)t^2 dt &= \int_0^\infty e^{-(1/2)\sigma^2 t^2}\cos(\sigma x t)t^2 dt + \Delta_{42}\\ &= -\frac{\sqrt{2\pi}\,(1-x^2)\,e^{-x^2/2}}{2\sigma^3} + \Delta_{42},\end{aligned} \tag{2.38}$$

where

$$|\Delta_{42}| \le \int_{\sqrt{3/\sigma}}^\infty e^{-(1/2)\sigma^2 t^2} t^2 dt \le \frac{\sqrt{\sigma}}{\sqrt{3}}\int_{\sqrt{3/\sigma}}^\infty e^{-(1/2)\sigma^2 t^2} t^3 dt \le \frac{1.963}{\sigma^2\sqrt{\sigma}} e^{-(3/2)\sigma}. \tag{2.39}$$

From (2.36)–(2.39),

$$\int_0^{\sqrt{3/\sigma}} \rho(t)\cos(\sigma x t)t^2 dt = -\frac{\sqrt{2\pi}\,(1-x^2)\,e^{-x^2/2}}{2\sigma^3} + \Delta_4,$$

where

$$|\Delta_4| \le |\Delta_{41}| + |\Delta_{42}| \le \frac{5.8904}{\sigma^5} + \frac{1.963}{\sigma^2\sqrt{\sigma}} e^{-(3/2)\sigma}.$$

□

Proposition 2.5. *Assume that* $\sigma^2 \ge 25$. *Then*

$$\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{t} dt = \sqrt{\frac{\pi}{2}} \int_0^x e^{-(1/2)t^2} dt + \pi C_2(x) e^{-x^2/2} - \frac{x\sqrt{\pi} e^{-x^2/2}}{24\sqrt{2}\sigma^2} + \Delta_5,$$

where $|\Delta_5| \le \dfrac{7.4364}{\sigma^3} + 0.3721 e^{-(3/2)\sigma}$.

Proof. From Uspensky ([25], p.124), for $t^2 \le \dfrac{3}{5}$ we have

$$\alpha(t) = \frac{1}{6}\sum_{j=1}^{n} p_j q_j (p_j - q_j) t^3 + M_1 t^5 \quad \text{and} \quad \alpha(t) = M_2 t^3, \tag{2.40}$$

where $|M_1| \le 0.1596\sigma^2$ and $|M_2| \le 0.3053\sigma^2$. Using Taylor's formula, we obtain that

$$\sin(\alpha(t)) = \alpha(t) - \frac{\alpha^3(t)}{6}\cos(t_0) \quad \text{and} \quad \cos(\alpha(t)) = 1 - \frac{\alpha^2(t)}{2} + \frac{\alpha^4(t)}{24}\cos(t_1)$$

for some t_0 and t_1 between 0 and $\alpha(t)$. Hence

$$\begin{aligned}
\sin(\sigma x t - \alpha(t)) &= \sin(\sigma x t)\cos(\alpha(t)) - \cos(\sigma x t)\sin(\alpha(t)) \\
&= \sin(\sigma x t) - \frac{1}{72}\left(\sum_{j=1}^{n} p_j q_j (p_j - q_j)\right)^2 t^6 \sin(\sigma x t) \\
&\quad - \frac{1}{6}\sum_{j=1}^{n} p_j q_j (p_j - q_j) t^3 \cos(\sigma x t) + \Delta_{51},
\end{aligned} \tag{2.41}$$

where

$$\begin{aligned}
|\Delta_{51}| &= \Bigg| \sin(\sigma x t)\left(-\frac{1}{3}\sum_{j=1}^{n} p_j q_j (p_j - q_j) M_1 t^8 - M_1^2 t^{10} + \frac{\alpha^4(t)}{24}\cos(t_1)\right) \\
&\quad - \cos(\sigma x t)\left(M_1 t^5 - \frac{\alpha^3(t)}{6}\cos(t_0)\right)\Bigg| \\
&\le |M_1| t^5 + \frac{\sigma^2 |M_1| t^8}{3} + \frac{|M_2^3| t^9}{6} + M_1^2 t^{10} + \frac{M_2^4 t^{12}}{24} \\
&\le 0.1596\sigma^2 t^5 + 0.0532\sigma^4 t^8 + 0.0047\sigma^6 t^9 + 0.0254\sigma^4 t^{10} + 0.0004\sigma^8 t^{12}.
\end{aligned} \tag{2.42}$$

By (2.41) and (2.42), we have

$$\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{t} dt$$

$$= \int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t)}{t} dt - \frac{1}{72}\left(\sum_{j=1}^n p_j q_j (p_j - q_j)\right)^2 \int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t) t^5 dt$$

$$- \frac{1}{6}\sum_{j=1}^n p_j q_j (p_j - q_j) \int_0^{\sqrt{3/\sigma}} \rho(t)\cos(\sigma x t) t^2 dt + \Delta_{52}, \tag{2.43}$$

where

$$|\Delta_{52}|$$
$$\leq \int_0^{\sqrt{3/\sigma}} \rho(t)\left(0.1596\sigma^2 t^4 + 0.0532\sigma^4 t^7 + 0.0047\sigma^6 t^8 + 0.0254\sigma^4 t^9 + 0.0004\sigma^8 t^{11}\right) dt.$$

Since $\rho(t) \leq e^{-(1/2)\sigma^2 t^2 + (1/24)\sigma^2 t^4}$ for $t \in [0, \pi]$ ([19], p.720),

$$\rho(t) \leq 1.5 e^{-(1/2)\sigma^2 t^2} \quad \text{for} \quad t \in [0, \sqrt{\frac{3}{\sigma}}]. \tag{2.44}$$

From this fact, (2.14) and (2.35), we obtain

$$|\Delta_{52}| \leq \frac{6.6598}{\sigma^3}. \tag{2.45}$$

From Proposition 2.4 (1), (3) and (4), (2.43) and (2.45), we have

$$\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{t} dt$$
$$= \sqrt{\frac{\pi}{2}} \int_0^x e^{-(1/2)t^2} dt + \pi C_2(x) e^{-x^2/2} - \frac{x\sqrt{\pi} e^{-x^2/2}}{24\sqrt{2}\sigma^2} + \Delta_1$$
$$- \frac{\Delta_2}{72}\left(\sum_{j=1}^n p_j q_j (p_j - q_j)\right)^2 - \frac{\Delta_4}{6}\sum_{j=1}^n p_j q_j (p_j - q_j) + \Delta_{52}$$
$$= \sqrt{\frac{\pi}{2}} \int_0^x e^{-(1/2)t^2} dt + \pi C_2(x) e^{-x^2/2} - \frac{x\sqrt{\pi} e^{-x^2/2}}{24\sqrt{2}\sigma^2} + \Delta_5,$$

where $|\Delta_5| \leq |\Delta_1| + \frac{\sigma^4}{72}|\Delta_3| + \frac{\sigma^2}{6}|\Delta_4| + |\Delta_{52}| \leq \frac{7.4364}{\sigma^3} + 0.3721 e^{-(3/2)\sigma}$. □

Proposition 2.6. *Assume that* $\sigma^2 \geq 25$. *Then*

$$\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t - \alpha(t))tdt = \frac{x\sqrt{2\pi}}{2\sigma^2}e^{-x^2/2} + \Delta_6,$$

where

$$|\Delta_6| \leq \frac{4.1231}{\sigma^3} + 0.04e^{-(3/2)\sigma}.$$

Proof. By Taylor's formula, we obtain $\cos(\alpha(t)) = 1 - \frac{\alpha^2(t)}{2}\cos(t_2)$ for some t_2 between 0 and $\alpha(t)$. From the fact that $|\sin(x)| \leq |x|$ for $x \in \mathbb{R}$ and (2.40), we have

$$\begin{aligned}\sin(\sigma x t - \alpha(t)) &= \sin(\sigma x t)\cos(\alpha(t)) - \cos(\sigma x t)\sin(\alpha(t)) \\ &= \sin(\sigma x t) - \frac{\alpha^2(t)}{2}\cos(t_2)\sin(\sigma x t) - \cos(\sigma x t)\sin(\alpha(t)) \\ &= \sin(\sigma x t) + \Delta_{61},\end{aligned}$$

where

$$|\Delta_{61}| \leq \frac{\alpha^2(t)}{2} + |\alpha(t)| \leq 0.0466\sigma^4 t^6 + 0.3053\sigma^2 t^3.$$

Then

$$\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t - \alpha(t))tdt = \int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t)tdt + \Delta_{62},$$

where

$$\begin{aligned}|\Delta_{62}| &\leq 0.0466\sigma^4 \int_0^{\sqrt{3/\sigma}} \rho(t)t^7 dt + 0.3053\sigma^2 \int_0^{\sqrt{3/\sigma}} \rho(t)t^4 dt \\ &\leq \frac{4.1151}{\sigma^3},\end{aligned}$$

where we have used (2.14), (2.35) and (2.44) in the last inequality.

From this fact and Proposition 2.4 (2), we have

$$\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t - \alpha(t))t dt = \frac{x\sqrt{2\pi}}{2\sigma^2} e^{-x^2/2} + \Delta_6,$$

where

$$|\Delta_6| \le |\Delta_{62}| + |\Delta_2| \le \frac{4.1151}{\sigma^3} + \frac{1}{2\sigma^6} + \frac{e^{-(3/2)\sigma}}{\sigma^2} \le \frac{4.1231}{\sigma^3} + 0.04e^{-(3/2)\sigma}.$$

□

2.2 Proof of Theorem 1.1

Proof. By (2.4), we have

$$\begin{aligned} R(x) &= \frac{1}{2\pi}\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{\sin(t/2)} dt + \frac{1}{2\pi}\int_{\sqrt{3/\sigma}}^{\sqrt{(3/4\sigma)\pi}} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{\sin(t/2)} dt \\ &\quad + \frac{1}{2\pi}\int_{\sqrt{(3/4\sigma)\pi}}^{\pi} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{\sin(t/2)} dt \\ &=: R_1 + R_2 + R_3, \end{aligned} \tag{2.46}$$

From ([19] p.723),

$$|R_2| \le 0.3383e^{-(3/2)\sigma} \qquad \text{and} \qquad |R_3| \le 0.0167e^{-(3/2)\sigma}.$$

Using Taylor's formula, we can show that for $x \in \mathbb{R}$ such that $0 < x \le 1$,

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120}\cos(x_0)$$

for some $x_0 \in [0, x]$. Then

$$
\begin{aligned}
\frac{1}{\sin(x)} &= \frac{1}{x}\left(\frac{1}{1-\left(\frac{x^2}{6}-\frac{x^4}{120}\cos(x_0)\right)}\right)\\
&= \frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{x^2}{6}-\frac{x^4}{120}\cos(x_0)\right)^k\\
&= \frac{1}{x}\left(1+\frac{x^2}{6}-\frac{x^4}{120}\cos(x_0)+\sum_{k=2}^{\infty}\left(\frac{x^2}{6}-\frac{x^4}{120}\cos(x_0)\right)^k\right)\\
&= \frac{1}{x}+\frac{x}{6}+\Delta_{71},
\end{aligned}
$$

where

$$
\begin{aligned}
|\Delta_{71}| &\le \frac{x^3}{120}+\frac{1}{x}\sum_{k=2}^{\infty}\left(\frac{x^2}{6}+\frac{x^4}{120}\right)^k\\
&= \frac{x^3}{120}+\frac{1}{x}\left(\frac{x^2}{6}+\frac{x^4}{120}\right)^2\left(\frac{1}{1-\left(\frac{x^2}{6}+\frac{x^4}{120}\right)}\right)\\
&\le \frac{x^3}{120}+\frac{1}{x}\left(\frac{x^2}{6}+\frac{x^4}{120}\right)^2\left(\frac{1}{1-\frac{21}{120}}\right)\\
&= \frac{x^3}{120}+\frac{120}{99x}\left(\frac{x^2}{6}+\frac{x^4}{120}\right)^2\\
&\le \frac{x^3}{120}+\frac{120}{99x}\left(\frac{21x^2}{120}\right)^2\\
&= 0.0455x^3.
\end{aligned}
$$

So $\dfrac{1}{\sin(t/2)} = \dfrac{2}{t}+\dfrac{t}{12}+\Delta_{72}$, where $|\Delta_{72}| \le 0.0057t^3$ and $0 < t < 2$. Thus

$$
R_1 = \frac{1}{2\pi}\int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t-\alpha(t))\left(\frac{2}{t}+\frac{t}{12}\right)dt+\Delta_{73}, \tag{2.47}
$$

where

$$
|\Delta_{73}| \le \frac{0.0057}{2\pi}\int_0^{\sqrt{3/\sigma}} \rho(t)t^3dt.
$$

By (2.14) and (2.44), we obtain

$$|\Delta_{73}| \le \frac{0.0086}{2\pi} \int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} t^3 dt \le \frac{0.0086}{2\pi} \left(\frac{2}{\sigma^4}\right) \le \frac{0.0005}{\sigma^3}. \tag{2.48}$$

By (2.46)–(2.48), proposition 2.5 and proposition 2.6, we have

$$\begin{aligned}
R(x) &= \frac{1}{2\pi} \int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin(\sigma x t - \alpha(t))}{\sin(t/2)} dt + R_2 + R_3 \\
&= \frac{1}{2\pi} \int_0^{\sqrt{3/\sigma}} \rho(t)\sin(\sigma x t - \alpha(t)) \left(\frac{2}{t} + \frac{t}{12}\right) dt + \Delta_{73} + R_2 + R_3 \\
&= \frac{1}{\pi}\left[\sqrt{\frac{\pi}{2}} \int\limits_0^x e^{-(1/2)t^2} dt + \pi C_2(x) e^{-x^2/2} - \frac{x\sqrt{\pi} e^{-x^2/2}}{24\sqrt{2}\sigma^2} + \Delta_5\right] \\
&\quad + \frac{1}{24\pi}\left[\frac{x\sqrt{2\pi}}{2\sigma^2} e^{-x^2/2} + \Delta_6\right] \\
&= \Phi(x) + C_2(x) e^{-x^2/2} + \Delta_{74} \\
&= G(x) + \Delta_{74},
\end{aligned}$$

where

$$|\Delta_{74}| \le \frac{|\Delta_5|}{\pi} + \frac{|\Delta_6|}{24\pi} + |\Delta_{73}| + |R_2| + |R_3| \le \frac{2.4566}{\sigma^3} + 0.474 e^{-(3/2)\sigma}.$$

Therefore

$$P(a \le S_n \le b) = R(x_2) - R(x_1) = G(x_2) - G(x_1) + \varepsilon_n,$$

where

$$x_1 = \frac{1}{\sigma}\left(a - \mu - \frac{1}{2}\right), \quad x_2 = \frac{1}{\sigma}\left(b - \mu + \frac{1}{2}\right)$$

and

$$|\varepsilon_n| \le 2|\Delta_{74}| \le \frac{4.9132}{\sigma^3} + 0.948 e^{-(3/2)\sigma}.$$

□

CHAPTER III

RATE OF CONVERGENCE OF BINOMIAL FORMULA FOR OPTION PRICING

Let

$$
\begin{aligned}
&r && \text{be the risk–free rate,}\\
&T && \text{be time to maturity,}\\
&\delta && \text{be the volatility,}\\
&n && \text{be a number of period.}
\end{aligned}
$$

In real situation, we can assume that $0 < T \leq 1$, $0 < \delta \leq 1$ and $0 < r \leq 1$.
To give Binomial formula, let

$$u = e^{\delta\sqrt{\frac{T}{n}} + \frac{\lambda_n \delta^2 T}{n}}, \qquad d = e^{-\delta\sqrt{\frac{T}{n}} + \frac{\lambda_n \delta^2 T}{n}}, \tag{3.1}$$

$$q = \frac{e^{\frac{rT}{n}} - d}{u - d}, \qquad p = que^{-\frac{rT}{n}}, \tag{3.2}$$

where (λ_n) is a bounded sequence of real numbers. We define Binomial formula by

$$C(n) = S_0 \sum_{k=a}^{n} \binom{n}{k} p^k (1-p)^{n-k} - Ke^{-rT} \sum_{k=a}^{n} \binom{n}{k} q^k (1-q)^{n-k}$$

where S_0 is the initial stock price, K is the strike price,

$$a = \min\{m \in \mathbb{N} \mid m \geq b\}, \quad b = \frac{\log(K/S_0) - n\log d}{\log(u/d)} \tag{3.3}$$

and $\log(x)$ is understood as logarithm with natural base.

Observe that

$$a = \lceil b \rceil \quad \text{and} \quad a = b + \mathrm{frac}(-b), \tag{3.4}$$

where $\mathrm{frac}(x) = x - \lfloor x \rfloor$, $\lfloor x \rfloor = \max\{m \in \mathbb{N} \mid m \leq x\}$ and $\lceil x \rceil = \min\{m \in \mathbb{N} \mid m \geq x\}$.

The formula of Black–Scholes model is given by

$$C_{BS} = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \frac{\delta^2}{2})T}{\delta\sqrt{T}}, \quad d_2 = d_1 - \delta\sqrt{T} \tag{3.5}$$

and Φ is the standard normal distribution function.

It is well–known that $\lim\limits_{n\to\infty} C(n) = C_{BS}$ ([5]). In this chapter, we give the rate of convergence. Our rate is of order $\dfrac{1}{n\sqrt{n}}$ which is better than that before. We organize the contents in this chapter as the follows. In Section 3.1, we prove auxiliary results which is used to prove our main result is in Section 3.2.

3.1 Auxiliary Results

In this section, we give auxiliary results to prove our main result in Section 3.2.

Lemma 3.1. *Let $s, t \in \mathbb{R}$ be such that $|s| \leq 1$, $0 < |t| \leq 1$ and $f : \left(0, \frac{1}{10}\right) \to \mathbb{R}$ be defined by*

$$f(x) = \frac{e^{sx^2+tx} - 1}{e^{2tx} - 1}.$$

Then the followings hold.

1. $f(x) = \dfrac{1}{2} + \dfrac{2s - t^2}{4t}x + r_1(x)$, *where* $|r_1(x)| \leq \dfrac{1.7872x^3}{|t|}$.

2. $f(x)=\frac{1}{2}+\frac{2s-t^2}{4t}x+\frac{t^4-4st^2+12s^2}{48t}x^3+r_2(x)$, *where* $|r_2(x)|\le\frac{1.5021x^4}{|t|}$.

3. For $x>0$ *such that* $\frac{2.3585x^2}{t^2}\le\frac{1}{4}$, *we have*

$$\frac{1}{2\sqrt{(1-f(x))\,f(x)}}=1+\frac{\left(2s-t^2\right)^2x^2}{8t^2}+r_3(x),$$

where $|r_3(x)|\le\frac{8.2067x^4}{t^4}$.

Proof. 1. Let $g(x)=e^{sx^2+tx}-1$ and $h(x)=e^{2tx}-1$. By Taylor expansion, there exist x_0 and x_1 between 0 and x such that

$$h(x)=2tx+\frac{(2tx)^2}{2!}+\frac{(2tx)^3}{3!}+\frac{(2tx)^4e^{2tx_0}}{4!} \tag{3.6}$$

and

$$\begin{aligned} g(x)&=g'(0)x+\frac{g''(0)}{2!}x^2+\frac{g'''(0)}{3!}x^3+\frac{g^{(4)}(x_1)}{4!}x^4\\ &=tx+\frac{t^2+2s}{2}x^2+\frac{t^3+6st}{6}x^3+\frac{g^{(4)}(x_1)}{24}x^4, \end{aligned} \tag{3.7}$$

where

$$g^{(4)}(x_1)=\left[12s^2+12s(t+2sx_1)^2+(t+2sx_1)^4\right]e^{sx_1^2+tx_1}. \tag{3.8}$$

Hence,

$$\begin{aligned} f(x)=\frac{g(x)}{h(x)}&=\frac{tx+\frac{t^2+2s}{2}x^2+\frac{t^3+6st}{6}x^3+\frac{g^{(4)}(x_1)}{24}x^4}{2tx+2t^2x^2+\frac{4t^3x^3}{3}+\frac{2t^4x^4e^{2tx_0}}{3}}\\ &=\frac{t+\frac{t^2+2s}{2}x+\frac{t^3+6st}{6}x^2+\frac{g^{(4)}(x_1)}{24}x^3}{2t+2t^2x+\frac{4t^3}{3}x^2+\frac{2t^4e^{2tx_0}}{3}x^3}\\ &=\frac{1}{2}+\frac{2s-t^2}{4t}x+r_1(x), \end{aligned} \tag{3.9}$$

where

$$r_1(x) = \frac{x}{h(x)}\left[\left(\frac{g^{(4)}(x_1)}{24} - \frac{t^4e^{2tx_0}}{3} - \frac{t^2\,(2s-t^2)}{3}\right)x^3 - \frac{(2s-t^2)\,t^3e^{2tx_0}}{6}x^4\right]$$
$$= \frac{x^4}{h(x)}\left[\frac{g^{(4)}(x_1)}{24} - \frac{t^4e^{2tx_0}}{3} - \frac{t^2\,(2s-t^2)}{3} - \frac{(2s-t^2)\,t^3e^{2tx_0}}{6}x\right].$$

Since $|s| \le 1$, $0 < |t| \le 1$, $0 \le x_0 \le x < 0.1$ and $0 \le x_1 \le x < 0.1$,

$$|r_1(x)| \le \frac{x^4}{|h(x)|}\left[\frac{(12+12(1+0.2)^2+(1+0.2)^4)\,e^{0.11}}{24} + \frac{e^{0.2}}{3} + 1 + \frac{e^{0.2}}{2}(0.1)\right]$$
$$= \frac{x^4}{|h(x)|}\left[\frac{34.9993}{24} + 0.4071 + 1 + 0.0611\right]$$
$$= \frac{2.9265x^4}{|h(x)|}. \tag{3.10}$$

By Taylor formula, we know that $h(x) = e^{2tx} - 1 = 2txe^{u_0}$, for some u_0 between 0 and $2tx$.

If $0 < t \le 1$, then $0 \le u_0 \le 2x$. This implies that $|h(x)| \ge 2|t|xe^0 = 2|t|x$.

In case of $-1 \le t < 0$, we have $-2x < u_0 \le 0$ which implies that

$$|h(x)| \ge 2|t|xe^{-2x} \ge 2|t|xe^{-0.2} = 1.6375|t|x.$$

Hence for $x \in \left(0, \frac{1}{10}\right)$,

$$\frac{1}{|h(x)|} \le \frac{0.6107}{|t|x}. \tag{3.11}$$

From this fact, (3.9) and (3.10),

$$f(x) = \frac{1}{2} + \frac{2s-t^2}{4t}x + r_1(x),$$

where

$$|r_1(x)| \le \frac{1.7872x^3}{|t|}.$$

2. By the same techniques of (3.6) and (3.7), there exist x_2 and x_3 between 0 and x such that

$$h(x) = 2tx + \frac{(2tx)^2}{2!} + \frac{(2tx)^3}{3!} + \frac{(2tx)^4}{4!} + \frac{(2tx)^5 e^{2tx_2}}{5!} \tag{3.12}$$

and

$$\begin{aligned} g(x) &= g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 + \frac{g^{(5)}(x_3)}{5!}x^4 \\ &= tx + \frac{t^2+2s}{2}x^2 + \frac{t^3+6st}{6}x^3 + \frac{t^4+12st^2+12s^2}{24}x^4 + \frac{g^{(5)}(x_3)}{120}x^5, \end{aligned} \tag{3.13}$$

where

$$g^{(5)}(x_3) = \left[60s^2(t+2sx_3) + 20s(t+2sx_3)^3 + (t+2sx_3)^5\right] e^{sx_3^2+tx_3}. \tag{3.14}$$

Hence

$$\begin{aligned} f(x) = \frac{g(x)}{h(x)} &= \frac{tx + \frac{t^2+2s}{2}x^2 + \frac{t^3+6st}{6}x^3 + \frac{t^4+12st^2+12s^2}{24}x^4 + \frac{g^{(5)}(x_3)}{120}x^5}{2tx + 2t^2x^2 + \frac{4t^3x^3}{3} + \frac{2t^4x^4}{3} + \frac{4t^5x^5e^{2tx_2}}{15}} \\ &= \frac{t + \frac{t^2+2s}{2}x + \frac{t^3+6st}{6}x^2 + \frac{t^4+12st^2+12s^2}{24}x^3 + \frac{g^{(5)}(x_3)}{120}x^4}{2t + 2t^2x + \frac{4t^3}{3}x^2 + \frac{2t^4}{3}x^3 + \frac{4t^5e^{2tx_2}}{15}x^4} \\ &= \frac{1}{2} + \frac{2s-t^2}{4t}x + + \frac{t^4-4st^2+12s^2}{48t}x^3 + r_2(x), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &r_2(x) \\ &= \frac{x}{h(x)}\Bigg[\left(\frac{g^{(5)}(x_3)}{120} - \frac{2t^5e^{2tx_2}}{15} - \frac{2t^3s-t^5}{6} - \frac{t^5-4t^3s+12ts^2}{24}\right)x^4 \\ &\quad - \frac{t^6-4t^4s+12t^2s^2}{36}x^5 - \frac{(2st^4-t^6)e^{2tx_2}}{15}x^5 - \frac{t^7-4t^5s+12t^3s^2}{72}x^6 \\ &\quad - \frac{\left(t^8-4t^6s+12t^4s^2\right)e^{2tx_2}}{180}x^7\Bigg] \end{aligned}$$

$$= \frac{x^5}{h(x)}\left[\frac{g^{(5)}(x_3)}{120} - \frac{2t^5e^{2tx_2}}{15} - \frac{2t^3s-t^5}{6} - \frac{t^5-4t^3s+12ts^2}{24}\right.$$
$$- \frac{t^6-4t^4s+12t^2s^2}{36}x - \frac{(2st^4-t^6)e^{2tx_2}}{15}x - \frac{t^7-4t^5s+12t^3s^2}{72}x^2$$
$$\left. - \frac{\left(t^8-4t^6s+12t^4s^2\right)e^{2tx_2}}{180}x^3\right].$$

From $|s| \le 1$, $0 < |t| \le 1$, $0 \le x_2 \le x < 0.1$, $0 \le x_3 \le x < 0.1$ and (3.11),

$$r_2(x) \le \frac{0.6107x^4}{|t|}\left[\frac{\left(60(1+0.2)+20(1+0.2)^3+(1+0.2)^5\right)e^{0.11}}{120} + \frac{2e^{0.2}}{15} + \frac{1}{2} + \frac{17}{24}\right.$$
$$\left. + \frac{e^{0.2}}{5}(0.1) + \frac{17}{36}(0.1) + \frac{17}{72}(0.1)^2 + \frac{17e^{0.2}}{180}(0.1)^3\right]$$
$$= \frac{0.6107x^4}{|t|}\left[1.0144 + 0.1629 + 0.5 + 0.7083 + 0.0244 + 0.0472 + 0.0024 + 0.0001\right]$$
$$= \frac{1.5021x^4}{|t|}. \tag{3.16}$$

Hence, by (3.15) and (3.16), we have 2.

3. By 1., we have

$$f(x) = \frac{1}{2} + \frac{2s-t^2}{4t}x + r_1(x)$$

and

$$1 - f(x) = \frac{1}{2} - \frac{2s-t^2}{4t}x - r_1(x),$$

which implies that

$$(1-f(x))\,f(x) = \frac{1}{4} - \left(\frac{2s-t^2}{4t}x + r_1(x)\right)^2 = \frac{1}{4} - \frac{\left(2s-t^2\right)^2}{16t^2}x^2 + r_{31}(x), \tag{3.17}$$

where

$$|r_{31}(x)| \le \left|\frac{2s-t^2}{2t}xr_1(x)\right| + \left(r_1(x)\right)^2$$

$$\begin{aligned}
&\le \frac{3}{2|t|}\left(\frac{1.7872x^3}{|t|}\right)x + \left(\frac{1.7872x^3}{|t|}\right)^2 \\
&= \frac{2.6808x^4}{t^2} + \frac{3.1941x^6}{t^2} \\
&\le \frac{2.6808x^4}{t^2} + \frac{0.0319x^4}{t^2} \\
&= \frac{2.7127x^4}{t^2}.
\end{aligned}$$

Note that, for $\gamma, y \in \mathbb{R}$,

$$(1+y)^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} y^k, \tag{3.18}$$

where

$$\binom{\gamma}{k} = \begin{cases} \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!} & if \quad k \in \mathbb{N} \\ 1 & if \quad k = 0 \end{cases}$$

([3], p. 356).

We note that for $k \in \mathbb{N}$,

$$\begin{aligned}
\left|\binom{-\frac{1}{2}}{k+1}\right| &= \left|\frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-k+1)(-\frac{1}{2}-k)}{(k+1)!}\right| \\
&= \left|\frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-k+1)}{k!}\right| \frac{\frac{1}{2}+k}{k+1} \\
&\le \left|\binom{-\frac{1}{2}}{k}\right|.
\end{aligned}$$

Hence

$$\left|\binom{-\frac{1}{2}}{k}\right| \le \left|\binom{-\frac{1}{2}}{2}\right| = \frac{3}{8} \quad \text{for} \quad k \ge 2.$$

From this fact, (3.17) and (3.18), we obtain

$$\frac{1}{2\sqrt{(1-f(x))\,f(x)}} = \frac{1}{2}\left(\frac{1}{4} - \frac{\left(2s-t^2\right)^2 x^2}{16t^2} + r_{31}(x)\right)^{-1/2}$$

$$
\begin{aligned}
&= \left(1 - \frac{\left(2s - t^2\right)^2 x^2}{4t^2} + 4r_{31}(x)\right)^{-1/2} \\
&= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(-\frac{\left(2s - t^2\right)^2 x^2}{4t^2} + 4r_{31}(x)\right)^k \\
&= 1 + \frac{\left(2s - t^2\right)^2 x^2}{8t^2} + r_3(x),
\end{aligned}
$$

where

$$
\begin{aligned}
|r_3(x)| &\le 2|r_{31}(x)| + \frac{3}{8} \sum_{k=2}^{\infty} \left| -\frac{\left(2s - t^2\right)^2 x^2}{4t^2} + 4r_{31}(x) \right|^k \\
&\le \frac{5.4254x^4}{t^2} + \frac{3}{8} \sum_{k=2}^{\infty} \left| \frac{\left(2s - t^2\right)^2 x^2}{4t^2} + \frac{10.8508x^4}{t^2} \right|^k .
\end{aligned}
$$

Since $0 < x \le 0.1$, $|s| \le 1$, $0 < |t| \le 1$ and $\dfrac{2.3585x^2}{t^2} \le \dfrac{1}{4}$,

$$
\begin{aligned}
|r_3(x)| &\le \frac{5.4254x^4}{t^2} + \frac{3}{8} \sum_{k=2}^{\infty} \left(\frac{9x^2}{4t^2} + \frac{0.1085x^2}{t^2} \right)^k \\
&= \frac{5.4254x^4}{t^2} + \frac{3}{8} \sum_{k=2}^{\infty} \left(\frac{2.3585x^2}{t^2} \right)^k \\
&\le \frac{5.4254x^4}{t^2} + \frac{3}{8} \left(\frac{2.3585x^2}{t^2} \right)^2 \left(\frac{4}{3} \right) \\
&= \frac{5.4254x^4}{t^2} + \frac{2.7813x^4}{t^4} \\
&\le \frac{8.2067x^4}{t^4}.
\end{aligned}
$$

□

Lemma 3.2. *Let $n_1 \in \mathbb{N}$ such that $n_1 \ge \max\left\{100T, \frac{60}{\delta^4}, \frac{1.2657d_1^2}{\delta^4}\right\}$. Let $n \ge n_1$ and $T_1 = \frac{a - np - \frac{1}{2}}{\sqrt{np(1-p)}}$. If $|r - \lambda_n\delta^2| \le 1$, then we have*

1. $T_1 = -d_1 - \dfrac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} - \dfrac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} + r_1,$

where $\alpha = \dfrac{2(r - \lambda_n\delta^2) + \delta^2}{4\delta}, \quad \beta = -\dfrac{\delta^4 + 4\delta^2(r - \lambda_n\delta^2) + 12(r - \lambda_n\delta^2)^2}{48\delta}$ *and*

$|r_1| \le \dfrac{19.4739T^2 + 1.2279}{\delta^4 n\sqrt{n}}$,

2. $T_1 = -d_1 - \dfrac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} + r_2,$ *where* $|r_2| \le \dfrac{1.125|d_1| + 2.514T^2 + 0.8668}{\delta^2 n}$,

3. $T_1 = -d_1 + r_3,$ *where* $|r_3| \le \dfrac{2.436}{\sqrt{n}}$.

Proof. 1. By (3.1) and (3.2), we have

$$\begin{aligned} p &= que^{r(\frac{T}{n})} \\ &= \frac{u - ude^{-r(\frac{T}{n})}}{u-d} \\ &= \frac{e^{\delta\sqrt{\frac{T}{n}} + \lambda_n\delta^2(\frac{T}{n})} - e^{2\lambda_n\delta^2(\frac{T}{n})}e^{-r(\frac{T}{n})}}{e^{\delta\sqrt{\frac{T}{n}} + \lambda_n\delta^2(\frac{T}{n})} - e^{-\delta\sqrt{\frac{T}{n}} + \lambda_n\delta^2(\frac{T}{n})}} \\ &= \frac{e^{-\delta\sqrt{\frac{T}{n}} - (r-\lambda_n\delta^2)(\frac{T}{n})} - 1}{e^{-2\delta\sqrt{\frac{T}{n}}} - 1} \\ &= \frac{e^{tx+sx^2} - 1}{e^{2tx} - 1}, \end{aligned} \tag{3.19}$$

where

$$t = -\delta,\ s = -(r - \lambda_n\delta^2) \text{ and } x = \sqrt{\frac{T}{n}}. \tag{3.20}$$

Note that

$$0 \le x = \sqrt{\frac{T}{n}} \le \sqrt{\frac{T}{100T}} = \frac{1}{10}. \tag{3.21}$$

From Lemma 3.1(2.), we obtain

$$p = \frac{1}{2} + \alpha\sqrt{\frac{T}{n}} + \beta\left(\sqrt{\frac{T}{n}}\right)^3 + r_{11},$$

where $\alpha = \dfrac{2(r - \lambda_n\delta^2) + \delta^2}{4\delta}$, $\beta = -\dfrac{\delta^4 + 4\delta^2(r-\lambda_n\delta^2) + 12(r-\lambda_n\delta^2)^2}{48\delta}$ and

$$|r_{11}| \le \frac{1.5021T^2}{\delta n^2}. \tag{3.22}$$

So

$$2a - 2np - 1 = 2(b + \mathrm{frac}(-b)) - 2n\left(\frac{1}{2} + \alpha\sqrt{\frac{T}{n}} + \beta\left(\sqrt{\frac{T}{n}}\right)^3 + r_{11}\right) - 1$$
$$= 2b - n - 2\alpha\sqrt{Tn} - 2T\beta\sqrt{\frac{T}{n}} - 1 + 2\mathrm{frac}(-b) - 2nr_{11}.$$

Chang and Palmer ([5], p. 98) showed that $2b - n - 2\alpha\sqrt{Tn} = -\sqrt{n}d_1$. Hence

$$2a - 2np - 1 = -\sqrt{n}d_1 - 1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11}. \tag{3.23}$$

Since $n \geq \frac{60}{\delta^4}$ and $T \leq 1$,

$$\frac{2.3585T}{\delta^2 n} \leq \frac{2.3585}{\delta^2 n} < \frac{1}{4}.$$

From this fact, (3.19), (3.20) and (3.21), we can apply Lemma 3.1(3.) to obtain that

$$\frac{1}{2\sqrt{p\,(1-p)}} = 1 + \frac{2\alpha^2 T}{n} + r_{12}, \tag{3.24}$$

where

$$|r_{12}| \leq \frac{8.2067T^2}{\delta^4 n^2}. \tag{3.25}$$

By (3.23) and (3.24), we have

$$T_1 = (2a - 2np - 1)\,\frac{1}{2\sqrt{np(1-p)}}$$
$$= \frac{1}{\sqrt{n}}\left(\frac{1}{2\sqrt{p(1-p)}}\right)(2a - 2np - 1)$$
$$= \frac{1}{\sqrt{n}}\left(1 + \frac{2\alpha^2 T}{n} + r_{12}\right)\left(-\sqrt{n}d_1 - 1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11}\right)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n}}\Bigg[-\sqrt{n}d_1 - 1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11} - \frac{2\alpha^2 T d_1}{\sqrt{n}} \\
&\quad + \frac{2\alpha^2 T}{n}\left(-1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11}\right) \\
&\quad + r_{12}\left(-\sqrt{n}d_1 - 1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11}\right)\Bigg] \\
&= -d_1 - \frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} - \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} + r_1,
\end{aligned}
$$

where

$$
\begin{aligned}
&r_1 \\
&= \frac{1}{\sqrt{n}}\Bigg[- 2nr_{11} + \frac{2\alpha^2 T}{n}\left(-1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11}\right) \\
&\quad + r_{12}\left(-\sqrt{n}d_1 - 1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}} - 2nr_{11}\right)\Bigg] \\
&= \frac{1}{\sqrt{n}}\Bigg[- 2nr_{11}\left(1 + \frac{2\alpha^2 T}{n}\right) + \frac{2\alpha^2 T}{n}\left(-1 + 2\,\mathrm{frac}(-b) - 2T\beta\sqrt{\frac{T}{n}}\right) \\
&\quad + r_{12}\left(2a - 2np - 1\right)\Bigg].
\end{aligned}
$$

Note that

$$
|\alpha| \leq \frac{3}{4\delta} \tag{3.26}
$$

and

$$
|\beta| \leq \frac{17}{48\delta}. \tag{3.27}
$$

From (3.19)–(3.21) and Lemma 3.1(1.),

$$
p = \frac{1}{2} + \alpha\sqrt{\frac{T}{n}} + r_{13}, \tag{3.28}
$$

where

$$|r_{13}| \le \frac{1.7872}{\delta n\sqrt{n}}. \tag{3.29}$$

By (3.26), (3.28), (3.29), $n \ge \dfrac{60}{\delta^4}$ and $0 < \delta \le 1$, we obtain

$$\begin{aligned}\left|p - \frac{1}{2}\right| &\le \frac{|\alpha|}{\sqrt{n}} + |r_{13}| \\ &\le \frac{0.75}{\delta\sqrt{n}} + \frac{1.7872}{\delta n\sqrt{n}} \\ &\le 0.0968\delta + 0.0038\delta^5 \\ &\le 0.1006. \end{aligned} \tag{3.30}$$

This implies that

$$0.3994 \le p \le 0.6006. \tag{3.31}$$

Hence

$$0 < 2np + 1 \le 1.2012n + 1 \le 1.2012n + 0.0167n = 1.2179n.$$

From this fact and the fact that $0 \le a \le n$, we have

$$|2a - 2np - 1| \le 2n.$$

From this fact, (3.22), (3.25), (3.26), (3.27), $n \ge \dfrac{60}{\delta^4}$ and $0 < \delta \le 1$,

$$\begin{aligned}|r_1| &\le \frac{1}{\sqrt{n}}\left[2n|r_{11}|\left(1 + \frac{9}{8\delta^2 n}\right) + \frac{9}{8\delta^2 n}\left(1 + \frac{17}{24\delta\sqrt{n}}\right) + |r_{12}|\,|2a - 2np - 1|\right] \\ &\le \frac{1}{\sqrt{n}}\left[\frac{3.0042T^2}{\delta^4 n}\left(1 + \frac{9}{8\delta^2 n}\right) + \frac{9}{8\delta^2 n} + \frac{153}{192\delta^3 n\sqrt{n}} + \frac{8.2067T^2}{\delta^4 n^2}(2n)\right] \\ &= \frac{1}{\sqrt{n}}\left[\frac{3.0042T^2}{\delta^4 n} + \frac{3.3797T^2}{\delta^6 n^2} + \frac{1.125}{\delta^2 n} + \frac{0.7969}{\delta^3 n\sqrt{n}} + \frac{16.4134T^2}{\delta^4 n}\right]\end{aligned}$$

$$\leq \frac{1}{\sqrt{n}}\left[\frac{3.0042T^2}{\delta^4 n}+\frac{0.0563T^2}{\delta^2 n}+\frac{1.125}{\delta^2 n}+\frac{0.1029}{\delta n}+\frac{16.4134T^2}{\delta^4 n}\right]$$
$$\leq \frac{19.4739T^2+1.2279}{\delta^4 n\sqrt{n}}.$$

2. By 1., we have

$$T_1 = -d_1 - \frac{1-2\,\mathrm{frac}(-b)}{\sqrt{n}} + r_2,$$

where

$$r_2 = -\frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} + r_1.$$

From (3.26), (3.27) and $n \geq \frac{60}{\delta^4}$,

$$\begin{aligned}
|r_2| &\leq \frac{2(\alpha^2|d_1|+|\beta|)}{n} + |r_1| \\
&\leq \frac{1.125|d_1|}{\delta^2 n} + \frac{0.7083}{\delta n} + \frac{19.4739T^2+1.2279}{\delta^4 n\sqrt{n}} \\
&\leq \frac{1.125|d_1|}{\delta^2 n} + \frac{0.7083}{\delta^2 n} + \frac{2.514T^2+0.1585}{\delta^2 n} \\
&= \frac{1.125|d_1|+2.514T^2+0.8668}{\delta^2 n}.
\end{aligned}$$

3. By 2., we have

$$T_1 = -d_1 + r_3,$$

where

$$r_3 = -\frac{1-2\,\mathrm{frac}(-b)}{\sqrt{n}} + r_2.$$

Since $n \geq \frac{60}{\delta^4}$ and $n \geq \frac{1.2657d_1^2}{\delta^4}$,

$$
\begin{aligned}
|r_3| &\le \frac{|1-2\,\mathrm{frac}(-b)|}{\sqrt{n}} + |r_2| \\
&\le \frac{1}{\sqrt{n}} + \frac{1.125|d_1| + 2.514T^2 + 0.8668}{\delta^2 n} \\
&\le \frac{1}{\sqrt{n}} + \frac{1.125|d_1|}{\delta^2 n} + \frac{3.3808}{\delta^2 n} \\
&\le \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{0.436}{\sqrt{n}} \\
&= \frac{2.436}{\sqrt{n}}. \qquad (3.32)
\end{aligned}
$$

□

Lemma 3.3. *Let $Q_1(x)$, $Q_2(x)$ be defined in the Corollary 2.2 and $n_2 \in \mathbb{N}$ such that*

$$n_2 \ge 30d_1^2. \qquad (3.33)$$

Let $n \ge \max\{n_1, n_2\}$ and $T_1 = \dfrac{a - np - \frac{1}{2}}{\sqrt{np(1-p)}}$, then we have

1. $Q_1(T_1) = \dfrac{2(1-d_1^2)\alpha\sqrt{T}e^{-d_1^2/2}}{3\sqrt{2n\pi}} + r_4$, *where* $|r_4| \le \dfrac{0.775d_1^2 + 2.55}{\delta n}$.

2. $Q_2(T_1) = \dfrac{d_1(1-d_1^2)e^{-d_1^2/2}}{12\sqrt{2\pi}} + r_5$,

where $|r_5| \le \dfrac{0.129|d_1|^3 + 0.0406d_1^2 + 0.7739|d_1| + 3.852}{\sqrt{n}}$.

Proof. 1. By Lemma 3.2(3.) and $n \ge n_1 \ge 60$, we have

$$T_1^2 = (-d_1 + r_3)^2 = d_1^2 + r_{41}, \qquad (3.34)$$

where

$$|r_{41}| \le |2d_1r_3| + |r_3|^2 \le \frac{4.872|d_1|}{\sqrt{n}} + \frac{5.9341}{n} \le \frac{4.872|d_1| + 0.7661}{\sqrt{n}}. \qquad (3.35)$$

For $n \in \mathbb{N}$ such that $n \geq \max\{60, 30d_1^2\}$, we have

$$|r_{41}| \leq 1. \tag{3.36}$$

From this fact and the fact that $|e^x - 1| \leq 2|x|$ for $|x| \leq 1$, we obtain

$$\begin{aligned} e^{-T_1^2/2} &= e^{-d_1^2/2} e^{-r_{41}/2} \\ &= e^{-d_1^2/2} + \left(e^{-r_{41}/2} - 1\right) e^{-d_1^2/2} \\ &= e^{-d_1^2/2} + r_{42}, \end{aligned} \tag{3.37}$$

where

$$|r_{42}| \leq \left|e^{-r_{41}/2} - 1\right| e^{-d_1^2/2} \leq |r_{41}| e^{-d_1^2/2} \leq \frac{4.872|d_1|e^{-d_1^2/2} + 0.7661}{\sqrt{n}}. \tag{3.38}$$

Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \dfrac{x^m}{e^{x^2/2}}$, where $m \in \mathbb{N}$. We can show that f has maximum at $x = \sqrt{m}$. This implies that

$$\frac{x^m}{e^{x^2/2}} \leq \frac{(\sqrt{m})^m}{e^{m/2}} \quad \text{for } x \geq 0 \text{ and } m \in \mathbb{N}. \tag{3.39}$$

From this fact and (3.38),

$$|r_{42}| \leq \frac{4.872(0.6065) + 0.7661}{\sqrt{n}} = \frac{3.721}{\sqrt{n}}. \tag{3.40}$$

By (3.24), (3.28) and (3.34),

$$\begin{aligned} &Q_1(T_1) \\ &= \frac{(2p-1)\left(1 - T_1^2\right)}{6\sqrt{2p(1-p)\pi}} \\ &= \frac{2p-1}{3\sqrt{2\pi}} \left(\frac{1}{2\sqrt{p(1-p)}}\right) \left(1 - T_1^2\right) \\ &= \frac{1}{3\sqrt{2\pi}} \left(2\alpha\sqrt{\frac{T}{n}} + 2r_{13}\right) \left(1 + \frac{2\alpha^2 T}{n} + r_{12}\right) \left(1 - d_1^2 - r_{41}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\alpha\sqrt{T}}{3\sqrt{2n\pi}}\left(1+\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)+\frac{2r_{13}}{3\sqrt{2\pi}}\left(1+\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)\\
&= \frac{2\alpha\sqrt{T}}{3\sqrt{2n\pi}}\left(1-d_1^2-r_{41}+\left(\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)\right)\\
&\quad+\frac{2r_{13}}{3\sqrt{2\pi}}\left(1+\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)\\
&= \frac{2(1-d_1^2)\alpha\sqrt{T}}{3\sqrt{2n\pi}}+\frac{2\alpha\sqrt{T}}{3\sqrt{2n\pi}}\left(\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)+\frac{2\alpha r_{41}\sqrt{T}}{3\sqrt{2n\pi}}\\
&\quad+\frac{2r_{13}}{3\sqrt{2\pi}}\left(1+\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)\\
&= \frac{2(1-d_1^2)\alpha\sqrt{T}}{3\sqrt{2n\pi}}+r_{43}, \qquad (3.41)
\end{aligned}
$$

where

$$
\begin{aligned}
r_{43} &= \frac{2\alpha\sqrt{T}}{3\sqrt{2n\pi}}\left(\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)+\frac{2\alpha r_{41}\sqrt{T}}{3\sqrt{2n\pi}}\\
&\quad+\frac{2r_{13}}{3\sqrt{2\pi}}\left(1+\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right).\\
&=: a_1+a_2+a_3.
\end{aligned}
$$

From (3.37) and (3.41), we obtain

$$
\begin{aligned}
Q_1(T_1)e^{-T_1^2/2} &= Q_1(T_1)\left(e^{-d_1^2/2}+r_{42}\right)\\
&= Q_1(T_1)e^{-d_1^2/2}+Q_1(T_1)r_{42}\\
&= \left(\frac{2(1-d_1^2)\alpha\sqrt{T}}{3\sqrt{2n\pi}}+r_{43}\right)e^{-d_1^2/2}+Q_1(T_1)r_{42}\\
&= \frac{2(1-d_1^2)\alpha\sqrt{T}e^{-d_1^2/2}}{3\sqrt{2n\pi}}+r_4, \qquad (3.42)
\end{aligned}
$$

where

$$
r_4 = Q_1(T_1)r_{42}+a_1e^{-d_1^2/2}+a_2e^{-d_1^2/2}+a_3e^{-d_1^2/2}. \qquad (3.43)
$$

By (3.24)–(3.26), $n \geq \frac{60}{\delta^4}$ and $0 < \delta \leq 1$, we have

$$\begin{aligned}
\frac{1}{2\sqrt{p\,(1-p)}} &\leq 1 + \frac{2\alpha^2 T}{n} + \frac{8.2067T^2}{\delta^4 n^2} \\
&\leq 1 + \frac{0.1125}{\delta^2 n} + + \frac{8.0267}{\delta^4 n^2} \\
&\leq 1 + 0.0019\delta^2 + 0.0022\delta^4 \\
&\leq 1.0041. \qquad (3.44)
\end{aligned}$$

From this fact, (3.26), (3.28), (3.29), (3.34), (3.36), (3.40) and $n \geq 60$, we have

$$\begin{aligned}
|Q_1(T_1)r_{42}| &\leq \left| \frac{(2p-1)\,(1-T_1^2)}{6\sqrt{2p(1-p)\pi}} \right| |r_{42}| \\
&\leq \frac{|2p-1|}{3\sqrt{2\pi}} \left(\frac{1}{2\sqrt{p\,(1-p)}} \right) \left(1+T_1^2\right) \left(\frac{3.721}{\sqrt{n}} \right) \\
&\leq \frac{1.0041}{3\sqrt{2\pi}} \left(\frac{2|\alpha|}{\sqrt{n}} + 2|r_{13}| \right) \left(1 + d_1^2 + |r_{41}|\right) \left(\frac{3.721}{\sqrt{n}} \right) \\
&= \frac{0.4969}{\sqrt{n}} \left(\frac{2|\alpha|}{\sqrt{n}} + 2|r_{13}| \right) \left(1 + d_1^2 + |r_{41}|\right) \\
&\leq \frac{0.4969}{\sqrt{n}} \left(\frac{1.5}{\delta\sqrt{n}} + 2\left(\frac{1.7872}{\delta n\sqrt{n}} \right) \right) \left(d_1^2 + 2\right) \\
&= \left(\frac{0.7454}{\delta n} + \frac{1.7761}{\delta n^2} \right) \left(d_1^2 + 2\right) \\
&\leq \left(\frac{0.7454}{\delta n} + \frac{0.0296}{\delta n} \right) \left(d_1^2 + 2\right) \\
&= \frac{0.775d_1^2 + 1.55}{\delta n}. \qquad (3.45)
\end{aligned}$$

By (3.25), (3.26), (3.36), (3.39), $n \geq \frac{60}{\delta^4}$ and $0 < \delta \leq 1$,

$$\begin{aligned}
|a_1|e^{-d_1^2/2} &\leq \left| \frac{2\alpha\sqrt{T}}{3\sqrt{2n\pi}} \left(\frac{2\alpha^2 T}{n} + r_{12} \right) \left(1 - d_1^2 - r_{41}\right) \right| e^{-d_1^2/2} \\
&\leq \frac{2|\alpha|}{3\sqrt{2n\pi}} \left(\frac{2|\alpha|^2}{n} + |r_{12}| \right) \left(2 + d_1^2\right) e^{-d_1^2/2} \\
&\leq \frac{0.1995}{\delta\sqrt{n}} \left(\frac{1.125}{\delta^2 n} + \frac{8.2067}{\delta^4 n^2} \right) \left(2 + d_1^2\right) e^{-d_1^2/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{0.1995}{\delta\sqrt{n}}\left(\frac{0.1454}{\sqrt{n}}+\frac{0.0177}{\sqrt{n}}\right)\left(2+d_1^2\right)e^{-d_1^2/2}\\
&= \frac{0.0325}{\delta n}\left(2e^{-d_1^2/2}+d_1^2e^{-d_1^2/2}\right)\\
&\leq \frac{0.0325}{\delta n}(2+0.7358)\\
&= \frac{0.0889}{\delta n}. \qquad (3.46)
\end{aligned}
$$

From (3.26), (3.35) and (3.39),

$$
\begin{aligned}
|a_2|e^{-d_1^2/2} &= \left|\frac{2\alpha r_{41}\sqrt{T}}{3\sqrt{2n\pi}}\right|e^{-d_1^2/2}\\
&\leq \frac{2|\alpha|}{3\sqrt{2n\pi}}|r_{41}|e^{-d_1^2/2}\\
&\leq \frac{0.1995}{\delta\sqrt{n}}\left(\frac{4.872|d_1|+0.7661}{\sqrt{n}}\right)e^{-d_1^2/2}\\
&= \frac{0.972|d_1|e^{-d_1^2/2}+0.1529e^{-d_1^2/2}}{\delta n}\\
&\leq \frac{0.972(0.6065)+0.1529}{\delta n}\\
&= \frac{0.7424}{\delta n}. \qquad (3.47)
\end{aligned}
$$

By (3.24), (3.29), (3.36), (3.39), (3.44) and $n \geq 60$,

$$
\begin{aligned}
|a_3|e^{-d_1^2/2} &= \left|\frac{2r_{13}}{3\sqrt{2\pi}}\left(1+\frac{2\alpha^2T}{n}+r_{12}\right)\left(1-d_1^2-r_{41}\right)\right|e^{-d_1^2/2}\\
&\leq \frac{2|r_{13}|}{3\sqrt{2\pi}}\left(\frac{1}{2\sqrt{p\,(1-p)}}\right)\left(1+d_1^2+|r_{41}|\right)e^{-d_1^2/2}\\
&\leq \frac{0.4753}{\delta n\sqrt{n}}(1.0041)\left(d_1^2+2\right)e^{-d_1^2/2}\\
&= \frac{0.4772d_1^2e^{-d_1^2/2}+0.9545e^{-d_1^2/2}}{\delta n\sqrt{n}}\\
&\leq \frac{0.4772(0.7358)+0.9545}{\delta n}\left(\frac{1}{\sqrt{60}}\right)\\
&= \frac{0.1687}{\delta n}. \qquad (3.48)
\end{aligned}
$$

Hence, by (3.43), (3.45)–(3.48)

$$\begin{aligned} |r_4| &\le \frac{0.775d_1^2+1.55}{\delta n}+\frac{0.0889}{\delta n}+\frac{0.7424}{\delta n}+\frac{0.1687}{\delta n} \\ &= \frac{0.775d_1^2+2.55}{\delta n}. \end{aligned}$$

From this fact and (3.42), the proof is completed. 2. Let

$$\begin{aligned} &Q_2(T_1) \\ &= \frac{(1-6p(1-p))(3-T_1^2)T_1}{24\sqrt{2\pi}p(1-p)}+\frac{T_1}{24\sqrt{2\pi}p(1-p)}-\frac{(2p-1)^2\,(15-10T_1^2+T_1^4)\,T_1}{72\sqrt{2\pi}p(1-p)} \\ &=: A_1+A_2+A_3. \end{aligned}$$

Then

$$Q_2(T_1)e^{-T_1^2/2} = A_1e^{-T_1^2/2}+A_2e^{-T_1^2/2}+A_3e^{-T_1^2/2}. \tag{3.49}$$

From (3.24), (3.26), (3.28), $n \ge \dfrac{60}{\delta^4}$ and $0<\delta\le 1$, we have

$$\begin{aligned} \frac{1}{4p\,(1-p)} &= \left(1+\frac{2\alpha^2T}{n}+r_{12}\right)^2 \\ &= 1+r_{51}, \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} |r_{51}| &= \left|2\left(\frac{2\alpha^2T}{n}+r_{12}\right)+\left(\frac{2\alpha^2T}{n}+r_{12}\right)^2\right| \\ &\le 2\left(\frac{2|\alpha|^2T}{n}+|r_{12}|\right)+\left(\frac{2|\alpha|^2T}{n}+|r_{12}|\right)^2 \\ &\le 2\left(\frac{1.125}{\delta^2 n}+\frac{8.2067}{\delta^4n^2}\right)+\left(\frac{1.125}{\delta^2 n}+\frac{8.2067}{\delta^4n^2}\right)^2 \\ &\le 2\left(\frac{0.1452}{\sqrt{n}}+\frac{0.0177}{\sqrt{n}}\right)+\left(\frac{0.1452}{\sqrt{n}}+\frac{0.0177}{\sqrt{n}}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{0.3258}{\sqrt{n}} + \frac{0.0265}{n} \\
&\leq \frac{0.3292}{\sqrt{n}} \qquad (3.51)
\end{aligned}
$$

and

$$
\begin{aligned}
p(1-p) &= \left(\frac{1}{2} + \alpha\sqrt{\frac{T}{n}} + r_{13}\right)\left(\frac{1}{2} - \alpha\sqrt{\frac{T}{n}} - r_{13}\right) \\
&= \frac{1}{4} + r_{52}, \qquad (3.52)
\end{aligned}
$$

where

$$
\begin{aligned}
|r_{52}| &= \left(\alpha\sqrt{\frac{T}{n}} + r_{13}\right)^2 \\
&\leq \left(\frac{0.75}{\delta\sqrt{n}} + \frac{1.7872}{\delta n\sqrt{n}}\right)^2 \\
&= \frac{0.5625}{\delta^2 n} + \frac{2.6808}{\delta^2 n^2} + \frac{3.1941}{\delta^2 n^3} \\
&\leq \frac{0.0726}{\sqrt{n}} + \frac{0.0058}{\sqrt{n}} + \frac{0.0001}{\sqrt{n}} \\
&= \frac{0.0785}{\sqrt{n}}. \qquad (3.53)
\end{aligned}
$$

By Lemma 3.2(3.), (3.34), (3.37), (3.50), (3.52), we have

$$
\begin{aligned}
A_1 &= \frac{(1-6p(1-p))(3-T_1^2)T_1}{24p(1-p)\sqrt{2\pi}} \\
&= \frac{1-6p(1-p)}{6\sqrt{2\pi}}\left(\frac{1}{4p(1-p)}\right)(3-T_1^2)T_1 \\
&= \frac{1-6(0.25+r_{52})}{6\sqrt{2\pi}}(1+r_{51})(3-d_1^2-r_{41})(-d_1+r_3) \\
&= \frac{-0.5-6r_{52}}{6\sqrt{2\pi}}(1+r_{51})\big(-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\big) \\
&= \frac{-0.5-0.5r_{51}-6r_{52}(1+r_{51})}{6\sqrt{2\pi}}\big(-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\big) \\
&= \frac{0.5d_1(3-d_1^2)}{6\sqrt{2\pi}} - \frac{0.5}{6\sqrt{2\pi}}\big((3-d_1^2)r_3+r_{41}(d_1-r_3)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-0.5r_{51}-6r_{52}(1+r_{51})}{6\sqrt{2\pi}}\left(-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\right)\\
&=\frac{d_1(3-d_1^2)}{12\sqrt{2\pi}}-\frac{1}{12\sqrt{2\pi}}\left((3-d_1^2)r_3+r_{41}(d_1-r_3)\right)\\
&-\frac{r_{51}+12r_{52}(1+r_{51})}{12\sqrt{2\pi}}\left(-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\right).
\end{aligned}
$$

and

$$
\begin{aligned}
A_1e^{-T_1^2/2}&=A_1\left(e^{-d_1^2/2}+r_{42}\right)\\
&=A_1e^{-d_1^2/2}+A_1r_{42}\\
&=\frac{d_1(3-d_1^2)e^{-d_1^2/2}}{12\sqrt{2\pi}}+r_{53},
\end{aligned}
\tag{3.54}
$$

where

$$
\begin{aligned}
r_{53}&=A_1r_{42}-\frac{1}{12\sqrt{2\pi}}\left((3-d_1^2)r_3+r_{41}(d_1-r_3)\right)e^{-d_1^2/2}\\
&-\frac{r_{51}+12r_{52}(1+r_{51})}{12\sqrt{2\pi}}\left(-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\right)e^{-d_1^2/2}\\
&=: b_1+b_2+b_3.
\end{aligned}
$$

By (3.31), we have $p(1-p)\geq 0.1595$ and $0\leq p\leq 1$. It's well–known that $p(1-p)\leq 0.25$. This implies that $0.1595\leq p(1-p)\leq 0.25$.
Hence

$$
|1-6p(1-p)|\leq 0.5. \tag{3.55}
$$

From this fact, Lemma 3.2(3.), (3.34), (3.40), (3.50), (3.51) and $n\geq 60$,

$$
\begin{aligned}
|b_1|&=|A_1r_{42}|\\
&=\left|\frac{(1-6p(1-p))(3-T_1^2)T_1}{24p(1-p)\sqrt{2\pi}}\right||r_{42}|\\
&\leq\frac{|1-6p(1-p)|}{6\sqrt{2\pi}}\left(\frac{1}{4p(1-p)}\right)(3+T_1^2)|T_1||r_{42}|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{0.5}{6\sqrt{2\pi}}\left(1+\frac{0.3292}{\sqrt{n}}\right)(3+d_1^2+|r_{41}|)\left(|d_1|+\frac{2.436}{\sqrt{n}}\right)\frac{3.721}{\sqrt{n}} \\
&\le \frac{0.5}{6\sqrt{2\pi}}(1+0.0425)(4+d_1^2)(|d_1|+0.3145)\frac{3.721}{\sqrt{n}} \\
&= \frac{0.129(|d_1|^3+0.3145d_1^2+4|d_1|+1.258)}{\sqrt{n}} \\
&= \frac{0.129|d_1|^3+0.0406d_1^2+0.516|d_1|+0.1623}{\sqrt{n}}.
\end{aligned}
$$

By (3.32), (3.35) and $n \ge 60$, we have

$$
\begin{aligned}
\left|(3-d_1^2)r_3+r_{41}(d_1-r_3)\right| &\le (3+d_1^2)|r_3|+|r_{41}|(|d_1|+|r_3|) \\
&\le (3+d_1^2)\frac{2.436}{\sqrt{n}}+\frac{4.872|d_1|+0.7661}{\sqrt{n}}(|d_1|+\frac{2.436}{\sqrt{n}}) \\
&\le (3+d_1^2)\frac{2.436}{\sqrt{n}}+\frac{4.872|d_1|+0.7661}{\sqrt{n}}(|d_1|+0.3145) \\
&= \frac{7.308d_1^2+2.2983|d_1|+7.5489}{\sqrt{n}} \qquad (3.56)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|b_2| &= \left|\frac{1}{12\sqrt{2\pi}}\left((3-d_1^2)r_3+r_{41}(d_1-r_3)\right)\right|e^{-d_1^2/2} \\
&\le \frac{1}{12\sqrt{2\pi}}\left(\frac{7.308d_1^2+2.2983|d_1|+7.5489}{\sqrt{n}}\right)e^{-d_1^2/2} \\
&= \frac{0.2426d_1^2e^{-d_1^2/2}+0.0763|d_1|e^{-d_1^2/2}+0.2506e^{-d_1^2/2}}{\sqrt{n}} \\
&\le \frac{0.2426(0.7358)+0.0763(0.6065)+0.2506}{\sqrt{n}} \\
&= \frac{0.4754}{\sqrt{n}},
\end{aligned}
$$

where we have used (3.39) in the second inequality.

From (3.39), (3.51), (3.53), (3.56) and $n \ge 60$, we obtain

$$
\begin{aligned}
|r_{51}+12r_{52}(1+r_{51})| &\le |r_{51}|+12|r_{52}|(1+|r_{51}|) \\
&\le \frac{0.3292}{\sqrt{n}}+\frac{12(0.0785)}{\sqrt{n}}\left(1+\frac{0.3292}{\sqrt{n}}\right)
\end{aligned}
$$

$$\leq \frac{0.3292}{\sqrt{n}} + \frac{0.942}{\sqrt{n}}(1+0.0425)$$
$$= \frac{1.3112}{\sqrt{n}}$$

and

$$\begin{aligned}
&|-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)| \\
&\leq |d_1|(3+d_1^2) + \left|(3-d_1^2)r_3 + r_{41}(d_1-r_3)\right| \\
&\leq 3|d_1| + |d_1|^3 + \frac{7.308d_1^2 + 2.2983|d_1| + 7.5489}{\sqrt{n}} \\
&\leq 3|d_1| + |d_1|^3 + 0.9435d_1^2 + 0.2967|d_1| + 0.9746 \\
&= |d_1|^3 + 0.9435d_1^2 + 3.2967|d_1| + 0.9746
\end{aligned}$$

which implies that

$$\begin{aligned}
|b_3| &= \left|\frac{r_{51}+12r_{52}(1+r_{51})}{12\sqrt{2\pi}}\left(-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\right)\right| e^{-d_1^2/2} \\
&\leq \frac{|r_{51}+12r_{52}(1+r_{51})|}{12\sqrt{2\pi}}\left|-d_1(3-d_1^2)+(3-d_1^2)r_3+r_{41}(d_1-r_3)\right| e^{-d_1^2/2} \\
&\leq \frac{1.3112}{12\sqrt{2n\pi}}\left(|d_1|^3 + 0.9435d_1^2 + 3.2967|d_1| + 0.9746\right) e^{-d_1^2/2} \\
&= \frac{0.0436\left(|d_1|^3 e^{-d_1^2/2} + 0.9435d_1^2 e^{-d_1^2/2} + 3.2967|d_1| e^{-d_1^2/2} + 0.9746 e^{-d_1^2/2}\right)}{\sqrt{n}} \\
&= \frac{0.0436|d_1|^3 e^{-d_1^2/2} + 0.0411d_1^2 e^{-d_1^2/2} + 0.1438|d_1| e^{-d_1^2/2} + 0.0425 e^{-d_1^2/2}}{\sqrt{n}} \\
&\leq \frac{0.0436(1.1594) + 0.0411(0.7358) + 0.1438(0.6065) + 0.0425}{\sqrt{n}} \\
&= \frac{0.2105}{\sqrt{n}}.
\end{aligned}$$

Then

$$\begin{aligned}
|r_{53}| &\leq |b_1| + |b_2| + |b_3| \\
&\leq \frac{0.129|d_1|^3 + 0.0406d_1^2 + 0.516|d_1| + 0.1623}{\sqrt{n}} + \frac{0.4754}{\sqrt{n}} + \frac{0.2105}{\sqrt{n}}
\end{aligned}$$

$$= \frac{0.129|d_1|^3 + 0.0406d_1^2 + 0.516|d_1| + 0.8482}{\sqrt{n}}. \tag{3.57}$$

By Lemma 3.2(3.), (3.37), (3.50), we have

$$\begin{aligned}
A_2 e^{-T_1^2/2} &= A_2\left(e^{-d_1^2/2} + r_{42}\right) \\
&= A_2 e^{-d_1^2/2} + A_2 r_{42} \\
&= \left(\frac{T_1}{24\sqrt{2\pi}p(1-p)}\right) e^{-d_1^2/2} + A_2 r_{42} \\
&= \frac{1}{6\sqrt{2\pi}}\left(\frac{1}{4p(1-p)}\right) T_1 e^{-d_1^2/2} + A_2 r_{42} \\
&= \frac{1}{6\sqrt{2\pi}}\left(1 + r_{51}\right)(-d_1 + r_3) e^{-d_1^2/2} + A_2 r_{42} \\
&= \frac{1}{6\sqrt{2\pi}}\left(-d_1 + r_3 + r_{51}(-d_1 + r_3)\right) e^{-d_1^2/2} + A_2 r_{42} \\
&= \frac{-d_1 e^{-d_1^2/2}}{6\sqrt{2\pi}} + r_{54}, \qquad (3.58)
\end{aligned}$$

where

$$\begin{aligned}
r_{54} &= \frac{\left(r_3 + r_{51}(-d_1 + r_3)\right) e^{-d_1^2/2}}{6\sqrt{2\pi}} + A_2 r_{42} \\
&=: c_1 + c_2.
\end{aligned}$$

From (3.32), (3.39), (3.51) and $n \geq 60$,

$$\begin{aligned}
|c_1| &= \left|\frac{(r_3 + r_{51}(-d_1 + r_3))}{6\sqrt{2\pi}}\right| e^{-d_1^2/2} \\
&\leq \frac{e^{-d_1^2/2}}{6\sqrt{2\pi}}\left(|r_3| + |r_{51}|(|d_1| + |r_3|)\right) \\
&\leq \frac{e^{-d_1^2/2}}{6\sqrt{2\pi}}\left(\frac{2.436}{\sqrt{n}} + \frac{0.3292}{\sqrt{n}}\left(|d_1| + \frac{2.436}{\sqrt{n}}\right)\right) \\
&\leq \frac{e^{-d_1^2/2}}{6\sqrt{2\pi}}\left(\frac{2.436}{\sqrt{n}} + \frac{0.3292}{\sqrt{n}}\left(|d_1| + 0.3145\right)\right) \\
&= \frac{0.3292|d_1|e^{-d_1^2/2} + 2.5395e^{-d_1^2/2}}{\sqrt{n}}
\end{aligned}$$

$$\leq \frac{0.3292(0.6065) + 2.5395}{\sqrt{n}}$$
$$= \frac{2.7392}{\sqrt{n}}.$$

By Lemma 3.2(3.), (3.50), (3.51) and $n \geq 60$

$$\begin{aligned}
|c_2| &= |A_2 r_{42}| \\
&\leq \frac{|T_1|}{6\sqrt{2\pi}} \left(\frac{1}{4p(1-p)} \right) |r_{42}| \\
&\leq \frac{1}{6\sqrt{2\pi}} \left(|d_1| + |r_3| \right) \left(1 + |r_{51}| \right) \frac{3.721}{\sqrt{n}} \\
&\leq \frac{0.2474}{\sqrt{n}} \left(|d_1| + \frac{2.436}{\sqrt{n}} \right) \left(1 + \frac{0.3292}{\sqrt{n}} \right) \\
&\leq \frac{0.2474}{\sqrt{n}} \left(|d_1| + 0.3145 \right) \left(1 + 0.0425 \right) \\
&= \frac{0.2579|d_1| + 0.0811}{\sqrt{n}}.
\end{aligned}$$

Thus

$$\begin{aligned}
|r_{54}| &\leq |c_1| + |c_2| \\
&\leq \frac{2.7392}{\sqrt{n}} + \frac{0.2579|d_1| + 0.0811}{\sqrt{n}} \\
&= \frac{0.2579|d_1| + 2.8203}{\sqrt{n}}.
\end{aligned} \tag{3.59}$$

By (3.26), (3.28), (3.29), (3.39), (3.50) (3.51), $n \geq \frac{60}{\delta^4}$ and $0 < \delta \leq 1$, we obtain

$$\begin{aligned}
(2p-1)^2 &\leq \left(\frac{2|\alpha|}{\sqrt{n}} + 2|r_{13}| \right)^2 \\
&\leq \left(\frac{1.5}{\delta\sqrt{n}} + \frac{3.5744}{\delta n\sqrt{n}} \right)^2 \\
&= \frac{2.25}{\delta^2 n} + \frac{10.7232}{\delta^2 n^2} + \frac{12.7763}{\delta^2 n^3} \\
&\leq \frac{0.2905}{\sqrt{n}} + \frac{0.0231}{\sqrt{n}} + \frac{0.0005}{\sqrt{n}}
\end{aligned}$$

$$= \frac{0.3141}{\sqrt{n}}. \tag{3.60}$$

and

$$\begin{aligned}\left|15 - 10T_1^2 + T_1^4\right| e^{-T_1^2/2} &\leq 15|T_1|e^{-T_1^2/2} + 10|T_1|^3 e^{-T_1^2/2} + |T_1|^5 e^{-T_1^2/2} \\ &\leq 15(0.6065) + 10(1.1594) + 4.5887 \\ &= 25.2799\end{aligned}$$

which implies that

$$\begin{aligned}|A_3|e^{-T_1^2/2} &= \left|\frac{(2p-1)^2\,(15 - 10T_1^2 + T_1^4)\,T_1}{72p(1-p)\sqrt{2\pi}}\right| e^{-T_1^2/2} \\ &\leq \frac{(2p-1)^2}{18\sqrt{2\pi}}\left(\frac{1}{4p(1-p)}\right)\left|15 - 10T_1^2 + T_1^4\right| e^{-T_1^2/2} \\ &\leq \frac{0.3141}{18\sqrt{2n\pi}}\,(1 + |r_{51}|)\,(25.2799) \\ &\leq \frac{0.176}{\sqrt{n}}\left(1 + \frac{0.3292}{n}\right) \\ &\leq \frac{0.176}{\sqrt{n}}\,(1 + 0.0425) \\ &= \frac{0.1835}{\sqrt{n}}.\end{aligned} \tag{3.61}$$

From (3.49), (3.54), (3.57), (3.58), (3.59) and (3.61) , we have

$$\begin{aligned}Q_2(T_1)e^{-T_1^2/2} &= A_1 e^{-T_1^2/2} + A_2 e^{-T_1^2/2} + A_3 e^{-T_1^2/2} \\ &= \frac{d_1(3 - d_1^2)e^{-d_1^2/2}}{12\sqrt{2\pi}} + r_{53} - \frac{d_1 e^{-d_1^2/2}}{6\sqrt{2\pi}} + r_{54} + A_3 e^{-T_1^2/2} \\ &= \frac{d_1(1 - d_1^2)e^{-d_1^2/2}}{12\sqrt{2\pi}} + r_5,\end{aligned}$$

where

$$\begin{aligned}
|r_5| &\le |r_{53}| + |r_{54}| + |A_3| e^{-T_1^2/2} \\
&\le \frac{0.129|d_1|^3 + 0.0406 d_1^2 + 0.516|d_1| + 0.8482}{\sqrt{n}} + \frac{0.2579|d_1| + 2.8203}{\sqrt{n}} + \frac{0.1835}{\sqrt{n}} \\
&= \frac{0.129|d_1|^3 + 0.0406 d_1^2 + 0.7739|d_1| + 3.852}{\sqrt{n}}.
\end{aligned}$$

□

Lemma 3.4. *Let $Q_1(x)$, $Q_2(x)$ be defined in the Corollary 2.2. If $n \ge \max\{n_1, n_2\}$ and $T_2 = \dfrac{n(1-p)+\frac{1}{2}}{\sqrt{np(1-p)}}$, then*

1. $|Q_1(T_2)|\, e^{-T_2^2/2} \le \dfrac{0.0872}{n}$.
2. $|Q_2(T_2)|\, e^{-T_2^2/2} \le \dfrac{0.368}{\sqrt{n}}$.

Proof. By (3.39), we have

$$\begin{aligned}
\left|1 - T_2^2\right| e^{-T_2^2/2} &\le \max\left\{e^{-T_2^2/2}, T_2^2 e^{-T_2^2/2}\right\} \\
&= \max\left\{\frac{T_2^2 e^{-T_2^2/2}}{T_2^2}, \frac{T_2^4 e^{-T_2^2/2}}{T_2^2}\right\} \\
&\le \max\left\{\frac{0.7358}{T_2^2}, \frac{2.1654}{T_2^2}\right\} \\
&= \frac{2.1654}{T_2^2}.
\end{aligned}$$

From this fact, (3.44), (3.60) and $n \ge \dfrac{60}{\delta^4}$,

$$\begin{aligned}
|Q_1(T_2)|\, e^{-T_2^2/2} &= \left|\frac{(1-2p)\left(1-T_2^2\right)}{6\sqrt{2p(1-p)\pi}}\right| e^{-T_2^2/2} \\
&= \frac{|1-2p|}{3\sqrt{2\pi}} \left(\frac{1}{2\sqrt{p(1-p)}}\right) \left|1 - T_2^2\right| e^{-T_2^2/2} \\
&\le \frac{1}{3\sqrt{2\pi}} \sqrt{\frac{0.3141}{\sqrt{n}}}\, (1.0041) \left(\frac{2.1654}{T_2^2}\right)
\end{aligned}$$

$$\leq \frac{0.2014}{3\sqrt{2\pi}}\left(\frac{2.1654}{T_2^2}\right)$$
$$= \frac{0.058}{T_2^2}.$$

By (3.31), we obtain

$$\begin{aligned}
T_2 &= \frac{n(1-p)+\frac{1}{2}}{\sqrt{np(1-p)}} \\
&\geq \frac{\sqrt{n(1-p)}}{\sqrt{p}} \\
&= \left(\sqrt{\frac{1}{p}-1}\right)\sqrt{n} \\
&\geq \left(\sqrt{\frac{1}{0.6006}-1}\right)\sqrt{n} \\
&= 0.8155\sqrt{n}. \qquad (3.62)
\end{aligned}$$

Hence

$$|Q_1(T_2)|\, e^{-T_2^2/2} \leq \frac{0.058}{0.665n} = \frac{0.0872}{n}.$$

2. By (3.39), we have

$$T_2 e^{-T_2^2/2} = \frac{T_2^2 e^{-T_2^2/2}}{T_2} \leq \frac{0.7358}{T_2}, \qquad (3.63)$$

$$\begin{aligned}
\left|3-T_2^2\right| T_2 e^{-T_2^2/2} &\leq \max\left\{3T_2 e^{-T_2^2/2}, T_2^3 e^{-T_2^2/2}\right\} \\
&= \max\left\{\frac{3T_2^2 e^{-T_2^2/2}}{T_2}, \frac{T_2^4 e^{-T_2^2/2}}{T_2}\right\} \\
&\leq \max\left\{\frac{3(0.7358)}{T_2}, \frac{2.1654}{T_2}\right\} \\
&= \frac{2.2074}{T_2} \qquad (3.64)
\end{aligned}$$

and

$$
\begin{aligned}
\left|15-10T_2^2+T_2^4\right| T_2 e^{-T_2^2/2} &\leq \max\left\{15T_2 e^{-T_2^2/2}+T_2^5 e^{-T_2^2/2}, 10T_2^3 e^{-T_2^2/2}\right\} \\
&= \max\left\{\frac{15T_2^2 e^{-T_2^2/2}}{T_2}+\frac{T_2^6 e^{-T_2^2/2}}{T_2}, \frac{10T_2^4 e^{-T_2^2/2}}{T_2}\right\} \\
&\leq \max\left\{\frac{15(0.7358)}{T_2}+\frac{10.754}{T_2}, \frac{10(2.1654)}{T_2}\right\} \\
&= \frac{21.791}{T_2}. \qquad (3.65)
\end{aligned}
$$

From (3.44), (3.60), (3.55), (3.62)–(3.65) and $n \geq 60$,

$$
\begin{aligned}
&|Q_2(T_2)|\, e^{-T_2^2/2} \\
&\leq \frac{1}{6\sqrt{2\pi}}\left(\frac{1}{4p(1-p)}\right)\Bigg(|1-6p(1-p)|\left|3-T_2^2\right| T_2 e^{-T_2^2/2}+T_2 e^{-T_2^2/2} \\
&\quad +\frac{(2p-1)^2}{3}\left|\left(15-10T_2^2+T_2^4\right)T_2\right| e^{-T_2^2/2}\Bigg) \\
&\leq \frac{1.0082}{6\sqrt{2\pi}}\left(1.5612\left(\frac{2.2074}{T_2}\right)+\frac{0.7358}{T_2}+\frac{0.3141}{3n}\left(\frac{21.791}{T_2}\right)\right) \\
&\leq \frac{1.0082}{6\sqrt{2\pi}}\left(1.5612\left(\frac{2.2074}{T_2}\right)+\frac{0.7358}{T_2}+0.0135\left(\frac{21.791}{T_2}\right)\right) \\
&= \frac{0.3001}{T_2} \\
&\leq \frac{0.368}{\sqrt{n}}.
\end{aligned}
$$

□

Lemma 3.5. *Let* $n \geq \max\{n_1, n_2\}$, $T_1 = \dfrac{a - np - \frac{1}{2}}{\sqrt{np(1-p)}}$ *and* $T_2 = \dfrac{n(1-p) + \frac{1}{2}}{\sqrt{np(1-p)}}$. *Then*

$$\int_{T_1}^{T_2} e^{-\frac{t^2}{2}} dt$$

$$= \sqrt{2\pi}\Phi(d_1) + e^{\frac{-d_1^2}{2}} \left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} + \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - \frac{d_1\,(1 - 2\,\mathrm{frac}(-b))^2}{2n} \right)$$

$$+ r_6,$$

where $|r_6| \leq \dfrac{28.2158}{\delta^4 n\sqrt{n}}$.

Proof. We note that

$$\int_{T_1}^{T_2} e^{-\frac{t^2}{2}} dt = \int_{T_1}^{\infty} e^{-\frac{t^2}{2}} dt - \int_{T_2}^{\infty} e^{-\frac{t^2}{2}} dt. \tag{3.66}$$

Then

$$0 \leq \int_{T_2}^{\infty} e^{-\frac{t^2}{2}} dt \leq \frac{1}{T_2} \int_{T_2}^{\infty} t e^{-\frac{t^2}{2}} dt = \frac{e^{-\frac{T_2^2}{2}}}{T_2} = \frac{T_2^2 e^{-\frac{T_2^2}{2}}}{T_2^3}.$$

By (3.39) and (3.62), we obtain

$$0 \leq \int_{T_2}^{\infty} e^{-\frac{T^2}{2}} dt \leq \frac{0.7358}{T_2^3} \leq \frac{1.3567}{n\sqrt{n}}. \tag{3.67}$$

We see that

$$\int_{T_1}^{\infty} e^{-\frac{t^2}{2}} dt = \int_{-\infty}^{-T_1} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi}\Phi(d_1) + \int_{d_1}^{-T_1} e^{-\frac{t^2}{2}} dt. \tag{3.68}$$

From Lo-Bin and Ken ([14], p. 98), we have

$$\int_{d_1}^{-T_1} e^{-\frac{t^2}{2}} dt = e^{\frac{-d_1^2}{2}}(-T_1 - d_1) - \frac{d_1 e^{\frac{-d_1^2}{2}}}{2}(-T_1 - d_1)^2 + r_{61}, \tag{3.69}$$

where

$$r_{61} = \frac{-e^{-x_0^2/2} + x_0^2 e^{-x_0^2/2}}{6}(-T_1 - d_1)^3 = \frac{(x_0^2 - 1)\, e^{-x_0^2/2}}{6}(-T_1 - d_1)^3,$$

for some x_0 between $-T_1$ and d_1.

By Lemma 3.2(3.) and (3.39) we obtain

$$\begin{aligned} |r_{61}| &\leq \max\left\{x_0^2 e^{-x_0^2/2}, e^{-x_0^2/2}\right\} \frac{|r_{31}|^3}{6} \\ &\leq \max\{0.7358, 1\} \frac{14.4555}{6n\sqrt{n}} \\ &= \frac{2.4093}{n\sqrt{n}}. \end{aligned} \tag{3.70}$$

From lemma 3.2(1.), (2.), (3.39) and $n \geq 60$,

$$\begin{aligned} & e^{\frac{-d_1^2}{2}}(-T_1 - d_1) - \frac{d_1 e^{\frac{-d_1^2}{2}}}{2}(-T_1 - d_1)^2 \\ &= e^{\frac{-d_1^2}{2}}\left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} + \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - r_1\right) \\ &\quad - \frac{d_1 e^{\frac{-d_1^2}{2}}}{2}\left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} - r_2\right)^2 \\ &= e^{\frac{-d_1^2}{2}}\left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} + \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - \frac{d_1\,(1 - 2\,\mathrm{frac}(-b))^2}{2n}\right) \\ &\quad - r_1 e^{\frac{-d_1^2}{2}} - \frac{d_1 e^{\frac{-d_1^2}{2}}}{2}\left(-\frac{2(1 - 2\,\mathrm{frac}(-b))r_2}{\sqrt{n}} + r_2^2\right) \\ &= e^{\frac{-d_1^2}{2}}\left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} + \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - \frac{d_1\,(1 - 2\,\mathrm{frac}(-b))^2}{2n}\right) + r_{62}, \end{aligned} \tag{3.71}$$

where

$$
\begin{aligned}
|r_{62}| &\le |r_1|e^{\frac{-d_1^2}{2}} + \frac{|d_1|e^{\frac{-d_1^2}{2}}}{2}\left(\frac{2|r_2|}{\sqrt{n}} + |r_2|^2\right) \\
&\le \frac{20.672}{\delta^4 n\sqrt{n}} + \frac{|d_1|e^{\frac{-d_1^2}{2}}}{2}\left(\frac{2(1.125|d_1|+3.377)}{\delta^2 n\sqrt{n}} + \left(\frac{1.125|d_1|+3.377}{\delta^2 n}\right)^2\right) \\
&= \frac{20.672}{\delta^4 n\sqrt{n}} + \frac{|d_1|e^{\frac{-d_1^2}{2}}}{2}\left(\frac{2.25|d_1|+6.754}{\delta^2 n\sqrt{n}} + \frac{1.2656d_1^2+7.5983|d_1|+11.4041}{\delta^4 n^2}\right) \\
&\le \frac{20.672}{\delta^4 n\sqrt{n}} + \frac{|d_1|e^{\frac{-d_1^2}{2}}}{2}\left(\frac{2.25|d_1|+6.754}{\delta^4 n\sqrt{n}} + \frac{0.1634d_1^2+0.9803|d_1|+1.4723}{\delta^4 n\sqrt{n}}\right) \\
&= \frac{20.672}{\delta^4 n\sqrt{n}} + \frac{0.0817|d_1|^3e^{\frac{-d_1^2}{2}} + 1.6152|d_1|^2e^{\frac{-d_1^2}{2}} + 4.1132|d_1|e^{\frac{-d_1^2}{2}}}{\delta^4 n\sqrt{n}} \\
&\le \frac{20.672}{\delta^4 n\sqrt{n}} + \frac{0.0817(1.1594)+1.6152(0.7358)+4.1132(0.6065)}{\delta^4 n\sqrt{n}} \\
&= \frac{24.4498}{\delta^4 n\sqrt{n}}. \qquad (3.72)
\end{aligned}
$$

By (3.66)–(3.72), we have

$$
\begin{aligned}
&\int_{T_1}^{T_2} e^{-\frac{t^2}{2}}dt \\
&= \sqrt{2\pi}\Phi(d_1) + e^{\frac{-d_1^2}{2}}\left(\frac{1-2\,\mathrm{frac}(-b)}{\sqrt{n}} + \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - \frac{d_1\left(1-2\,\mathrm{frac}(-b)\right)^2}{2n}\right) \\
&\quad - \int_{T_2}^{\infty} e^{-\frac{t^2}{2}}dt + r_{61} + r_{62}, \\
&= \sqrt{2\pi}\Phi(d_1) + e^{\frac{-d_1^2}{2}}\left(\frac{1-2\,\mathrm{frac}(-b)}{\sqrt{n}} + \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - \frac{d_1\left(1-2\,\mathrm{frac}(-b)\right)^2}{2n}\right) + r_6,
\end{aligned}
$$

where

$$
|r_6| \le \int_{T_2}^{\infty} e^{-\frac{t^2}{2}}dt + |r_{61}| + |r_{62}| \le \frac{1.3567}{n\sqrt{n}} + \frac{2.4093}{n\sqrt{n}} + \frac{24.4498}{\delta^4 n\sqrt{n}} = \frac{28.2158}{\delta^4 n\sqrt{n}}.
$$

□

Lemma 3.6. *For* $n \geq \max\{100T, \frac{60}{\delta^4}, \frac{1.2657d_1^2}{\delta^4}, 30d_1^2\}$, *we have*

$$\sum_{k=a}^{n} \binom{n}{k} p^k(1-p)^{n-k} = \Phi(d_1) + F(d_1, \alpha, \beta) + r_7,$$

where $|r_7| \leq r(d_1)$,

$$F(d,s,t) = \frac{e^{\frac{-d^2}{2}}}{\sqrt{2\pi}} \left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} - \frac{2T(s^2 d + \sqrt{T}t)}{n} - \frac{d\,(1-2\,\mathrm{frac}(-b))^2}{2n} \right) + \frac{2(d^2-1)s\sqrt{T}}{3n\sqrt{2\pi}} e^{-d^2/2} + \frac{d(d^2-1)}{12n\sqrt{2\pi}} e^{-d^2/2}$$

and

$$r(d) = \frac{1.7185|d|^3 + 19.4642}{\delta^4 n\sqrt{n}} + \frac{47.1537}{n\sqrt{n}}.$$

Proof. By Corollary 2.2, we have

$$\sum_{k=a}^{n} \binom{n}{k} p^m (1-p)^{n-m}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{T_1}^{T_2} e^{-(1/2)t^2} dt + \frac{Q_1(T_2)e^{-T_2^2/2}}{\sqrt{n}} - \frac{Q_1(T_1)e^{-T_1^2/2}}{\sqrt{n}} + \frac{Q_2(T_2)e^{-T_2^2/2}}{n} - \frac{Q_2(T_1)e^{-T_1^2/2}}{n} + \varepsilon_n,$$

where $|\varepsilon_n| \leq \frac{4.9132}{np(1-p)\sqrt{np(1-p)}} + 0.9358e^{-(3/2)\sqrt{np(1-p)}}$ and $Q_1(x)$, $Q_2(x)$ are defined in the Corollary 2.2.

Since $g(x) = x^3 e^{-1.5x}$ has maximum at $x = 2$ on $[0, \infty)$,

$$\left(\sqrt{np(1-p)}\right)^3 e^{-(3/2)\sqrt{np(1-p)}} \leq 0.959.$$

This implies that

$$
\begin{aligned}
|\varepsilon_n| &\le \frac{4.9132}{np(1-p)\sqrt{np(1-p)}} + 0.948\left(\frac{0.959}{np(1-p)\sqrt{np(1-p)}}\right) \\
&= \frac{5.8223}{n\sqrt{n}}\left(\frac{1}{\sqrt{p(1-p)}}\right)^3 \\
&\le \frac{5.8223}{n\sqrt{n}}(2.0082)^3 = \frac{47.1537}{n\sqrt{n}},
\end{aligned}
$$

where we used (3.44) in the last inequality.

From this fact and Lemma 3.3–Lemma 3.5, we have

$$
\begin{aligned}
&\sum_{k=a}^{n}\binom{n}{k}p^k(1-p)^{n-k} \\
&= \Phi(d_1) + \frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}}\left(\frac{1-2\,\mathrm{frac}(-b)}{\sqrt{n}} - \frac{2T(\alpha^2 d_1 + \sqrt{T}\beta)}{n} - \frac{d_1\,(1-2\,\mathrm{frac}(-b))^2}{2n}\right) \\
&\quad + \frac{r_6}{\sqrt{2\pi}} + \frac{2(d_1^2-1)\alpha\sqrt{T}}{3\sqrt{2\pi}n}e^{-d_1^2/2} + \frac{r_4}{\sqrt{n}} + \frac{d_1(d_1^2-1)}{12\sqrt{2\pi}n}e^{-d_1^2/2} + \frac{r_5}{n} + \frac{Q_2(T_2)e^{-T_2^2/2}}{n} \\
&\quad + \frac{Q_1(T_2)e^{-T_2^2/2}}{\sqrt{n}} + \varepsilon_n \\
&= \Phi(d_1) + F(d_1,\alpha,\beta) + r_7,
\end{aligned}
$$

where

$$
\begin{aligned}
|r_7| &\le \frac{|r_6|}{\sqrt{2\pi}} + \frac{1}{\sqrt{n}}\left(|r_4| + \left|Q_1(T_2)e^{-T_2^2/2}\right|\right) + \frac{1}{n}\left(|r_5| + \left|Q_2(T_2)e^{-T_2^2/2}\right|\right) + |\varepsilon_n| \\
&\le \frac{28.2158}{\delta^4 n\sqrt{2n\pi}} + \frac{1}{\sqrt{n}}\left(\frac{0.775d_1^2+2.55}{\delta n} + \frac{0.0872}{n}\right) \\
&\quad + \frac{1}{n}\left(\frac{0.129|d_1|^3 + 0.0406d_1^2 + 0.7739|d_1| + 3.852}{\sqrt{n}} + \frac{0.368}{\sqrt{n}}\right) + \frac{47.1537}{n\sqrt{n}} \\
&\le \frac{11.2565}{\delta^4 n\sqrt{n}} + \frac{0.775d_1^2+2.55}{\delta^4 n\sqrt{n}} + \frac{0.0872}{\delta^4 n\sqrt{n}} \\
&\quad + \frac{0.129|d_1|^3 + 0.0406d_1^2 + 0.7739|d_1| + 3.852}{\delta^4 n\sqrt{n}} + \frac{0.368}{\delta^4 n\sqrt{n}} + \frac{47.1537}{n\sqrt{n}} \\
&= \frac{0.129|d_1|^3 + 0.8156d_1^2 + 0.7739|d_1| + 17.7457}{\delta^4 n\sqrt{n}} + \frac{47.1537}{n\sqrt{n}}.
\end{aligned}
$$

If $|d_1| \le 1$, then $|r_7| \le \dfrac{19.4642}{\delta^4 n\sqrt{n}} + \dfrac{47.1537}{n\sqrt{n}}$.
If $|d_1| > 1$, then $|r_7| \le \dfrac{1.7185|d_1|^3 + 17.7457}{\delta^4 n\sqrt{n}} + \dfrac{47.1537}{n\sqrt{n}}$.
Hence

$$|r_7| \le \frac{1.7185|d_1|^3 + 19.4642}{\delta^4 n\sqrt{n}} + \frac{47.1537}{n\sqrt{n}} = r(d_1).$$

□

Lemma 3.7. *For* $n \ge \max\{100T, \frac{60}{\delta^4}, \frac{1.2657d_2^2}{\delta^4}, 30d_2^2\}$, *we have*

$$\sum_{k=a}^{n} \binom{n}{k} q^m (1-q)^{n-m} = \Phi(d_2) + F(d_2, \hat{\alpha}, \hat{\beta}) + r_8,$$

where $\hat{\alpha} = \dfrac{2(r-\lambda_n\delta^2) - \delta^2}{4\delta}$, $\hat{\beta} = \dfrac{\delta^4 - 4\delta^2(r-\lambda_n\delta^2) + 12(r-\lambda_n\delta^2)^2}{48\delta}$,
and $|r_8| \le r(d_2)$

Proof. Observe that $|\hat{\alpha}| \le \dfrac{3}{4\delta}$ and $|\hat{\beta}| \le \dfrac{17}{48\delta}$.
By (3.1) and (3.2), we have

$$\begin{aligned} q &= \frac{e^{\frac{rT}{n}} - d}{u - d} \\ &= \frac{e^{\frac{rT}{n}} - e^{-\delta\sqrt{\frac{T}{n}} + \lambda_n\delta^2(\frac{T}{n})}}{e^{\delta\sqrt{\frac{T}{n}} + \lambda_n\delta^2(\frac{T}{n})} - e^{-\delta\sqrt{\frac{T}{n}} + \lambda_n\delta^2(\frac{T}{n})}} \\ &= \frac{e^{\delta\sqrt{\frac{T}{n}} + (r-\lambda_n\delta^2)(\frac{T}{n})} - 1}{e^{2\delta\sqrt{\frac{T}{n}}} - 1} \\ &= \frac{e^{tx+sx^2} - 1}{e^{2tx} - 1}, \end{aligned}$$

where $t = \delta$, $s = (r - \lambda_n\delta^2)$ and $x = \sqrt{\dfrac{T}{n}}$.
From this fact, we can follow the arguments of Lemma 3.2–Lemma 3.6 to show that the conclusion of the Lemma holds. □

3.2 Proof of Main Theorem

Theorem 3.8. *Let* $n \geq \max\{100T, \frac{60}{\delta^4}, \frac{1.2657\max\{d_1^2, d_2^2\}}{\delta^4}, 30\max\{d_1^2, d_2^2\}\}$. *Then for the n-period binomail model, the price of a European call option satisfies*

$$C(n) = C_{BS} + S_0 F(d_1, \alpha, \beta) - Ke^{-rT} F(d_2, \hat{\alpha}, \hat{\beta}) + r,$$

where $|r| \leq S_0|r(d_1)| + K|r(d_2)|$,

$$\begin{aligned} F(d,s,t) = \frac{e^{\frac{-d^2}{2}}}{\sqrt{2\pi}} &\left(\frac{1 - 2\,\mathrm{frac}(-b)}{\sqrt{n}} - \frac{2T(s^2 d + \sqrt{T}t)}{n} - \frac{d\,(1 - 2\,\mathrm{frac}(-b))^2}{2n} \right) \\ &+ \frac{2(d^2-1)s\sqrt{T}}{3n\sqrt{2\pi}} e^{-d^2/2} + \frac{d(1-d^2)}{12\sqrt{2\pi}} e^{-d^2/2} \end{aligned}$$

and

$$r(d) = \frac{1.7185|d|^3 + 19.4642}{\delta^4 n\sqrt{n}} + \frac{47.1537}{n\sqrt{n}}.$$

Proof. We know that

$$C(n) = S_0 \sum_{k=a}^{n} \binom{n}{k} p^k (1-p)^{n-k} - Ke^{-rT} \sum_{k=a}^{n} \binom{n}{k} q^k (1-q)^{n-k}.$$

By Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned} C(n) &= S_0 \left[\Phi(d_1) + F(d_1, \alpha, \beta) + r(d_1) \right] - Ke^{-rT} \left[\Phi(d_2) + F(d_2, \hat{\alpha}, \hat{\beta}) + r(d_2) \right] \\ &= C_{BS} + S_0 F(d_1, \alpha, \beta) - Ke^{-rT} F(d_2, \hat{\alpha}, \hat{\beta}) + r, \end{aligned}$$

where

$$|r| \leq S_0 r(d_1) + K r(d_2).$$

□

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