GEOMETRIC STRUCTURE OF SPHERICALLY SYMMETRIC SPACETIME IN MASSIVE GRAVITY THEORY

Mr. Supakorn Luekullaphawong

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Physics
Department of Physics
Faculty of Science
Chulalongkorn University
Academic Year 2015

โครงสร้างเชิงเรขาคณิตของกาลอวกาศที่มีสมมาตรทรงกลมในทฤษฎีแมสสีฟแกรวิตี

นายศุภกร ลือกุลภาวงษ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาฟิสิกส์ ภาควิชาฟิสิกส์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2558

Thesis Title	GEOMETRIC STRUCTURE OF SPHERICALLY SYMMETRIC SPACETIME IN MASSIVE GRAVITY THEORY
By	Mr. Supakorn Luekullaphawong
Field of Study	Physics
Thesis Advisor	Parinya Karndumri, Ph.D.
Thesis Co-Advisor	Pitayuth Wongjun, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science
(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman
(Assistant Professor Rattachat Mongkolnavin, Ph.D.)

. . . . Thesis Advisor
(Parinya Karndumri, Ph.D.)

. . . . Thesis Co-Advisor
(Pitayuth Wongjun, Ph.D.)

. . . . Examiner
(Varagorn Hengpunya, Ph.D.)

. . . . External Examiner
(Khumphee Karwan, Ph.D.)

ศุภกร ลือกุลภาวงษ์: โครงสร้างเชิงเรขาคณิตของกาลอวกาศที่มีสมมาตรทรงกลมในทฤษฎีแมสสีฟแกรวิตี. (GEOMETRIC STRUCTURE OF SPHERICALLY SYMMETRIC SPACETIME IN MASSIVE GRAVITY THEORY) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : อ.ดร.ปริญญา การดำริห์, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : อ.ดร.พิทยุทธ วงศ์จันทร์, 67 หน้า.

ข้อมูลจากการสังเกตการณ์ในปัจจุบัน ชี้ให้เห็นว่า เอกภพของเรากำลังขยายตัวด้วยความเร่ง เราคาดหวังว่าทฤษฎีสัมพัทธภาพทั่วไปจะสามารถอธิบายปรากฏการณ์นี้ได้ อย่างไรก็ตาม หากเอกภพไร้ซึ่งสสารนอกเหนือจากที่เราเห็นได้ เช่น พลังงานมืด เราก็จะไม่สามารถอธิบายปรากฏการณ์นี้ได้เช่นกัน หนึ่งในความน่าจะเป็นที่เราอาจอธิบายสิ่งเหล่านี้ได้คือการนำทฤษฎีสัมพัทธภาพทั่วไปมาปรับแต่ง ซึ่งเราเรียกทฤษฎนี้ว่าทฤษฎีโมดิฟายด์แกรวิตี สำหรับทฤษฎีแมสสีฟแกรวิตี เป็นหนึ่งในทฤษฎีที่แตกแขนงออกมาจากทฤษฎีข้างต้น ที่ยอมให้แกรวิตอนที่มีสปินสองมีมวล ซึ่งต่างจากทฤษฎีสัมพัทธภาพทั่วไป ที่กล่าวว่า แกรวิตอนไม่มีมวล แบบจำลองที่สามารถอธิบายการขยายตัวของเอกภพด้วยความเร่งถูกเสนอโดย เดอ ราม กาบาแดดซ์ และโทลเล่ย์ เรียกว่า ดีอาร์จีที แมสสีฟแกรวิตี ถึงแม้ว่าทฤษฎีนี้จะสามารถอธิบายการขยายตัวของเอกภพด้วยความเร่งได้ แต่ในแง่หนึ่งทฤษฎีก็ต้องสามารถอธิบายในระดับความโน้มถ่วงเฉพาะที่ลงมาได้ เช่นเดียวกับการที่ทฤษฎีสัมพัทธภาพทั่วไปอธิบายได้ดีอยู่แล้ว เช่น ในระบบสุริยะ ดังนั้นเพื่อศึกษาผลจากแมสสีฟแกรวิตีในระดับความโน้มถ่วงเฉพาะที่ เป้าหมายในการวิจัยครั้งนี้คือ การศึกษาเรขาคณิตของกาลอวกาศโดยใช้ผลเฉลยสมมาตรทรงกลมในทฤษฎีดีอาร์จีทีแมสสีฟแกรวิตี จากผลเฉลยที่ได้นี้ สามารถนำมาหาวงโคจรของอนุภาคได้โดยวิเคราะห์จากศักย์ยังผล จากข้อมูลของดาวพุธ เราพบว่าวงโคจรของดาวพุธที่ได้จากทฤษฎีแมสสีฟแกรวิตี เหมือนกับผลที่ได้จากทฤษฎีสัมพัทธภาพทั่วไป ซึ่งหมายความว่า ทฤษฎี ดีอาร์จีที แมสสีฟแกรวิตี สามารถลดรูปไปเป็นทฤษฎีสัมพัทธภาพทั่วไปได้ในระดับความโน้มถ่วงเฉพาะที่

ภาควิชา ฟิสิกส์	ลายมือชื่อนิสิต ....
สาขาวิชา ฟิสิกส์	ลายมือชื่อ อ.ที่ปรึกษาหลัก ....
ปีการศึกษา 2558	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

5572132923 : MAJOR PHYSICS
KEYWORDS : BLACKHOLE / MODIFIED GRAVITY / STATIC AND SPHERICALLY SYMMETRIC / ACCELERATING EXPANSION OF THE UNIVERSE / FIERZ AND PUALI LENEARIZED GRAVITY

SUPAKORN LUEKULLAPHAWONG : GEOMETRIC STRUCTURE OF SPHERICALLY SYMMETRIC SPACETIME IN MASSIVE GRAVITY THEORY. ADVISOR : PARINYA KARNDUMRI, Ph.D., CO-ADVISOR : PITAYUTH WONGJUN, Ph.D., 67 pp.

Recent observations suggest that the universe is expanding with acceleration. General relativity theory is supposed to be a theory to describe this phenomenon. However, without introducing exotic matter such as dark energy, it cannot explain this phenomenon. One possibility to explain this phenomenon is a modification of general relativity which is usually called modified gravity theory. Massive gravity theory is one of the modifications in which the massless spin-2 graviton acquires masses in contrast to usual general relativity corresponding to massless graviton. A model that can explain acceleration of the expanding universe is presented by de Rham, Gabadadze, and Tolley and is called dRGT massive gravity theory. Even though this massive gravity theory can explain the expanding universe with acceleration, it must reduce to the usual explanation of local gravity scale such as the solar system. In order to study consequences of massive gravity at local gravity scale, the aim of this research is therefore to study spacetime geometry by using the spherically symmetric solutions in this theory. By using these solutions, one can find particle trajectories by analyzing the effective potential. From the data of Mercury's orbit, we found that the trajectory of Mercury obtained by massive gravity theory is same as the result predicted from GR. It implies that the dRGT massive gravity theory can reduce to GR at the local gravity scale.

Department : Physics .... Student's Signature ....

Field of Study : Physics .... Advisor's Signature ....

Academic Year : 2015 .... Co-Advisor's Signature ....

Acknowledgements

I am very delighted to express my gratitude and appreciation to their help in leading my thesis to complete, especially my advisor, Dr. Parinya Karndumri and co-advisor, Dr. Pitayuth Wongjun. They always give me insightful suggestions and discussions. Their encouragements and supports also made me pass through any obstrucles. This thesis cannot be complete without them. I would like to thank Dr. Khumphee Karwan for his discussions and useful recommendations and thanks to my partner laboratory, Mr. Lunchakorn Tannukij. My life as a master student could never be this pleasant without them.

Above all, this thesis would not be finished without an encouragement, help and financial support from my family. There is no word as to thank my father, my mother, my sisters, my aunts, and my relatives for all the sacrifices they made on my behalf. They keep supporting me even at hardship. Thus, I would like to dedicate this work to them. I also want to express my thanks to all of my friends and my special one, including the colleagues at Prompt International Institute and every my partners, for making enjoyable moments and giving energy to my life. Nevertheless, I have no regret being friend with them.

Contents

page

Chapter **page**

List of Figures

Figure **page**

CHAPTER I

Introduction

It is well-known that general relativity (GR) is one of pillars of modern physics. It is also expected that general relativity could be used to explain the dynamics of the universe. The recent observations suggest that the universe is expanding with acceleration [1, 2]. The unknown object which drives the cosmic acceleration is named as dark energy. The simplest way to describe this phenomenon is to add cosmological constant Λ into Einstein action. This model is an alternative model of dark energy. However, there is a problem in this model in case of the energy to drive the cosmic acceleration which is addressed in terms of vacuum energy. Since the cosmological constant model spreads throughout the universe, it can be interpreted as vacuum energy. The vacuum energy density evaluated by particle physics theory with Planck scale cutoff is about 10^{74} $(\mathrm{GeV})^4$ which is around 10^{121} times larger than the observed energy density of cosmological constant [3]. In order to solve this problem, scientists attempt to propose dynamical models of dark energy such as scalar field namely quintessence which denotes canonical scalar fields [3]. For more complicated models, there are various efforts to construct dark energy model for examples, k-essence [4, 5], three-form model [6, 7], and vector field model [8].

Instead of adding an exotic matter into the theory, one may obtain the cosmic acceleration by modifying gravity. This is known as modified gravity theory. The simplest model to modify gravity theory is $f(R)$ gravity theory which is the higher order invariants to the gravitational action coming from high-energy physics and an attempt to generalize GR in the sense of cosmology and astrophysics [9, 10]. For other complicated models, there are many models studied by theorists such as $f(G)$ model [11], braneworld [12], DGP model [13], and massive gravity theory [14,15,17–19]. In this research, we are interested in massive gravity theory. From the particle physics point of view, general relativity corresponds to a theory of massless spin-2 particle (graviton). Therefore, one of modifications to general relativity is to add mass to the graviton resulting in massive gravity theory.

In 1939, the first massive gravity theory was constructed by Fierz and Pauli. They introduced the mass terms to the linearized perturbation of Einstein-Hilbert action [14]. The linearized Einstein-Hilbert action known as linearized GR is invariant under guage transformation and corresponds to massless spin-2. The mass terms added into the action break the gauge invariance. As a result, there are five degrees of freedom in the theory, instead of two found in general relativity. The Fierz-Pauli massive gravity theory successfully describes massive spin-2. However, in 1970 van Dam, Veltman, and Zakharov studied the Fierz-Pauli action by adding a symmetric source into the action [15, 16]. They discovered that there is a discontinuity in the theory called vDVZ discontinuity when taking graviton mass to be zero. Light bending angle resulting from massless limit of massive gravity theory is different to one from the general relativity.

In 1972, Vainshtein [18] proposed the idea to explain this discontinuity. The idea is that the non-linear perturbation is dominated over the linear perturbation when mass of graviton is set to be zero. This suggests us that the non-linear perturbation should be considered in the massless limit. Just about the same time, Boulware and Deser also found a ghost mode (called BD ghost) when they considered the non-linear massive gravity theory [17]. The massive gravity theory has been continuously developed in order to alleviate this problem in the theory. Until 2010, de Rham, Gabadadze, and Tolley proposed a viable model of massive gravity theory without BD-ghost [19–21]. This is one of the main topics in this thesis and we will discuss on this massive gravity theory in details later.

Most of modifications of general relativity are constructed in order to explain gravity at the cosmological scale since general relativity with ordinary matter cannot correctly describe. However, at small scale such as our solar system scale, some modified gravity theories cannot be reduced to general relativity. This means that the theories cannot be used to explain local gravity or gravity at small scale [23–28]. In a simple modified gravity model, there is a parameter region of the model at which the universe can expand with acceleration. However, this region is not compatible with local gravity constraints [29–31]. It is accordingly worthwhile to investigate the compatibility of massive gravity theory with local gravity constraints. As we have mentioned, one of viable model of massive gravity theory without BD ghost is the de Rham-Gabadadze-Tolley (dRGT) massive gravity theory. It was found that graviton mass in dRGT massive gravity can play a role of cosmological constant driving the universe to expand with acceleration. As the same strategy with usual modified gravity theory, dRGT massive gravity theory should explain the local gravity such as the solar system.

Since most of astronomical objects in the universe are approximately spherical object, it is able to assume that our system has the spherical symmetry. This symmetry represents a general and simple form of solution. Therefore, the spherically symmetric solution in gravity theory is a powerful tool to investigate the particle's trajectory around spherical object. In dRGT massive gravity theory, spherically symmetric solutions are very complicated and not easy to compare the result with one from general relativity [22]. This is due to the complicated non-linear mass term. However, this can be simplified by choosing proper the fiducial metric. We will use this solution to analyze the trajectory of a particle. We will also investigate the compatibility of dRGT massive gravity theory with local gravity constraints within small radius from a mass source with respect to Vainshtein radius. At local gravity scale, it is well-known that many phenomena can be predicted by Einstein's gravity. It implies that the massive gravity theory should predict these phenomena as the prediction in GR. Most of local gravity constraints come from the observations of the solar system. One of the most stringent constraints is the variation of semi-major axis of the planetary motion [19]. This variation comes from the deviation of Newtonian potential. Generally, modified gravity model will provide some corrections to Newtonian potential which may be tested by observations. There are ten parameters for characterizing the Newtonian corrections from modified gravity models called "parametized post Newtonian (PPN) parameters" [32]. Some of these parameters can be obtained by analyzing spherically symmetric solutions.

By using the spherically symmetric solutions, we can find the effective potential and then analyze Mercury's trajectories. As a result, there are the correction terms of the effective potential corresponding to the graviton mass in the dRGT massive gravity theory. At local gravity scale, the correction terms are suppressed due to Vainshtein mechanism. We can check the result numerically by using the Mercury's information. We found that the trajectory of Mercury obtained by the dRGT massive gravity theory is same as the result predicted by GR. It implies that the dRGT massive gravity theory can reduce to GR at the local gravity scale.

This thesis is organized as follows. In Chapter II, we firstly review linearized general relativity by using perturbation theory in order to obtain the second order perturbed action and the linearized equation of motion. In Chapter III, we review the Fierz-Pauli theory in which the mass term is added to the linaerized Einstein-Hilbert action. The dVDZ discontinuity which is the problem of the Fierz-Pauli massive gravity theory is also presented in this chapter. In Chapter IV, we will review an introduction to the dRGT massive gravity theory which is one of a viable model of massive gravity theory. The Vainshtein mechanism and the spherically symmetric solution for this theory are also presented in this chapter. In Chapter V, we will find the effective potential for a particle and analyze the particle's trajectory. Finally, the results are summarized in Chapter VI.

CHAPTER II

Linearized Perturbations

Massive gravity theory has been started by studying a generalization of the linearized gravity theory. For the case of weak gravitational field, we can derive solutions of general relativity (GR) by using the perturbation theory. The Einstein field equation, which is the equation of motion in GR, is coupled non-linear differential equations. It is very complicated to solve the equation by using analytical method. As we have mentioned in the previous chapter, one of possible way to solve this equation is to impose some symmetry to the system, for example a spherical symmetry. However, the object in nature does not perfectly obey spherical symmetry, it slightly deviates from one in spherical symmetry. This allows us to use the perturbation theory to describe nature of gravity for such object. Note that the background solution is not necessary to obey the spherical symmetry. It can be any simple solutions but it should be associated with physical situations, for example Minkowski metric is a solution for flat spacetime. By using linearized gravity theory, one can describe the phenomena which cannot be explained by using Newtonian theory such as the light deflection, gravitational radiation, and the solar system. The linearized gravity is one of powerful tool to study gravity at local gravity scale. We will briefly review some important ideas and calculations in order to provide the basic concept for massive gravity theory.

Since the linearized gravity theory is based on perturbation theory of general relativity, we will briefly review concepts of general relativity. In the case of a very massive object which gives very strong gravitational field, results of the general relativity differ from one in Newtonian gravity. The general relativity is also consistent with Lorentz symmetry involving boots and space rotation in flat Minkowski space. The main equation in GR is the Einstein field equation which describes relation between the geometry of spacetime and energy-momentum tensor representing the mass and energy of gravitational source. The Einstein field equation can be written as,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.0.1}$$

where G is the gravitational constant. $g_{\mu\nu}$ is the metric tensor describing the geometry of spacetime. $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, both of them contain the second derivatives of the metric tensor. We explicitly show their definitions later. Therefore, these quantities associate with curvature of spacetime. $T_{\mu\nu}$ is the energy-momentum tensor. $G_{\mu\nu}$ is the Einstein tensor corresponding to spacetime curvature since it contains $R_{\mu\nu}$ and R. Therefore, this equation shows that the geometry of spacetime is curved by matter and vice versa the matter is a effected by the geometry of spacetime. Both sides of this equation satisfy the same identity that is the covariant derivatives of $G_{\mu\nu}$ and $T_{\mu\nu}$ is zero. The vanishing of covariant derivatives of $T_{\mu\nu}$ corresponds to conservations of energy and momentum in the system.

Most of fundamental modern physics are based on field theory in which the equations of motion are obtained by using variational principle. Since in GR equation of motion is Einstein equation, one may have to find the action corresponding to this equation through the variational principle. This action can be written as

$$S = \int d^4x\sqrt{-g}\big[\frac{1}{2\kappa}R + \mathcal{L}_M\big] \tag{2.0.2}$$

where $\kappa = 8\pi G$ and $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$. The first part is known as Einstein-Hilbert action. By varying this part with respect to the metric tensor, $g_{\mu\nu}$, one obtains Einstein tensor, $G_{\mu\nu}/2\kappa$. The second part is the matter Lagrangian, $\mathcal{L}_M$, describing matter field. By varying this term with respect to the metric tensor, we will get the energy-momentum tensor. From field theory, a symmetry of the action will provide a conserved quantity of the system. This action is invariant under general coordinate transformation which naturally provides the covariant conservations of energy-momentum tensor. This is one of advantage points in field theory approach of general relativity. It also provides us a reasonable way to modify the general relativity, for example replacing R by a function of R which is known as $f(R)$ gravity theory.

2.1 The Second Order Perturbed Action

According to general relativity, the metric tensor, $g_{\mu\nu}$, is a dynamical field in the theory. Linearized gravity theory is a perturbation theory based on general relativity. Therefore, one can decompose the metric tensor into background and perturbed part as follows

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{2.1.1}$$

where $h_{\mu\nu}$ is a small perturbation on the background metric, $g_{\mu\nu}$. As we have mentioned, the background metric can be any solution of Einstein solution. For the simple one, it may be the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$, which is the solution for $R = 0$ corresponding to flat spacetime. Here we will consider the general solution in which the metric, in principle, corresponds to curved spacetime. The quantities with tilde, $\tilde{X}$, is represented the full quantity including the perturbation and background part while the quantities without tilde is the background part. With this notation, Einstein-Hilbert action can be written as (the first term in Eq.(2.0.2))

$$S = \int d^4x \sqrt{-\tilde{g}}\tilde{R} = \int d^4x \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}, \tag{2.1.2}$$

Note that we omit the constant, κ, in the action. However, we can add it when we consider the source term. In this case, we will mention later. We will now consider linearized gravity perturbations on curved spacetime. The second order perturbed action must be considered in order to obtain the linearized equation of motion when we use variational principle. From this action, there are three parts including $\sqrt{-\tilde{g}}$, $\tilde{g}^{\mu\nu}$, and $\tilde{R}_{\mu\nu}$. Each part should be expanded up to the second order in order to obtain the second order perturbation of the action. We can firstly find the inverse metric tensor which can be expresses as (see Appendix A.1 for details),

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho}h^{\nu}_{\rho}. \tag{2.1.3}$$

The square root of determinant of the metric tensor can be expanded as follows

$$\begin{aligned}\sqrt{-\tilde{g}} &= \sqrt{-g} + \frac{1}{2}h\sqrt{-g} + \frac{1}{8}h^2\sqrt{-g} - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\sqrt{-g}, \\ &= \sqrt{-g} + \sqrt{-\tilde{g}}^{(1)} + \sqrt{-\tilde{g}}^{(2)}\end{aligned} \tag{2.1.4}$$

where

$$\begin{aligned}\sqrt{-\tilde{g}}^{(1)} &= \frac{1}{2}h\sqrt{-g}, \\ \sqrt{-\tilde{g}}^{(2)} &= \frac{1}{8}h^2\sqrt{-g} - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\sqrt{-g}.\end{aligned} \tag{2.1.5}$$

The superscripts (1) and (2) represent the first and the second order of perturbation, respectively. Now we will expand the Ricci tensor up to the second order of perturbation. In order to define the Ricci tensor, one has to introduce an important structure called Christoffel connection, sometimes called Christoffel symbol. It is the structure which can be formed in the Ricci tensor, and it is defined as,

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{2.1.6}$$

The Christoffel symbol, $\Gamma^{\rho}_{\mu\nu}$, is the important structure to define spacetime curvature. It is helpful to simply define covariant derivatives. In curved spacetime, two vectors from different spacetime points cannot be directly used to compare since they are in different tangent space. The Christoffel connection is a connection in the sense that it takes the vectors to the same tangent space through parallel transport. The parallel transportation is transportation of a vector along a curve in manifold by which its magnitude and angle between the vector and a tangent space is preserved. The Christoffel symbol can be also written in background and perturbation part up to the second order as (see more details in Appendix A.2).

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho(0)}_{\mu\nu} + \Gamma^{\rho(1)}_{\mu\nu} + \Gamma^{\rho(2)}_{\mu\nu}, \tag{2.1.7}$$

where

$$\begin{aligned}
\Gamma^{\rho(0)}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \\
\Gamma^{\rho(1)}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu}), \\
\Gamma^{\rho(2)}_{\mu\nu} &= -\frac{1}{2}h^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu}).
\end{aligned} \tag{2.1.8}$$

The covariant derivative of a rank-2 tensor, $T_{\mu\nu}$, is defined by

$$\nabla_\rho T_{\mu\nu} = \partial_\rho T_{\mu\nu} - \Gamma^\lambda_{\rho\mu}T_{\lambda\nu} - \Gamma^\lambda_{\rho\nu}T_{\lambda\mu}. \tag{2.1.9}$$

In order to obtain quantity representing in the spacetime curvature, one has to introduce a curvature tensor which is known as the Riemann tensor. The Riemann tensor is constructed from difference of a vector resulting from parallel transport in different paths. If there is no curvature, the Riemann tensor is vanished, the transported vector will be the same. If the spacetime is curved, the Riemann tensor is generally not zero. The Riemann tensor can be defined as

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\sigma\nu} - \partial_\nu\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\sigma\nu} - \Gamma^\rho_{\alpha\nu}\Gamma^\alpha_{\sigma\mu}. \tag{2.1.10}$$

From the Riemann tensor defined above, we can see that there are the second derivatives of the metric with respect to the spacetime coordinates. Therefore, it represents a curvature of spacetime by meaning of differential geometry. We can write the Riemann tensor in background and perturbation parts up to the second order as follows (see Appendix A.3 for details)

$$\tilde{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} + \nabla_\mu\Gamma^{\rho(1)}_{\sigma\nu} - \nabla_\nu\Gamma^{\rho(1)}_{\sigma\mu} + \nabla_\mu\Gamma^{\rho(2)}_{\sigma\nu} - \nabla_\nu\Gamma^{\rho(2)}_{\sigma\mu} + \Gamma^{\rho(1)}_{\alpha\mu}\Gamma^{\alpha(1)}_{\sigma\nu} - \Gamma^{\rho(1)}_{\alpha\mu}\Gamma^{\alpha(1)}_{\sigma\mu}. \tag{2.1.11}$$

The Ricci tensor relates to the Riemann tensor as follows

$$R_{\sigma\nu} = R^\rho_{\sigma\rho\nu} \tag{2.1.12}$$

We can write the Ricci tensor in background and perturbation parts up to the second order as follows (see Appendix A.4 for details)

$$\tilde{R}_{\sigma\nu} = R^{(0)}_{\sigma\nu} + R^{(1)}_{\sigma\nu} + R^{(2)}_{\sigma\nu}, \tag{2.1.13}$$

where

$$
\begin{aligned}
R^{(0)}_{\sigma\nu} &= R_{\sigma\nu}, \\
R^{(1)}_{\sigma\nu} &= \nabla_\rho \Gamma^{\rho(1)}_{\sigma\nu} - \nabla_\nu \Gamma^{\rho(2)}_{\sigma\rho}. \\
R^{(2)}_{\sigma\nu} &= \nabla_\rho \Gamma^{\rho(1)}_{\sigma\nu} - \nabla_\nu \Gamma^{\rho(2)}_{\sigma\rho} + \Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\rho}.
\end{aligned}
\tag{2.1.14}
$$

We have already expanded all three parts in Eq.(2.1.2) up to the second order. The Ricci scalar can be written in terms of the Ricci tensor as follows

$$
R = g^{\sigma\nu} R_{\sigma\nu}. \tag{2.1.15}
$$

From this definition, the Ricci scalar can be expanded up to the second order as (see Appendix A.5 for details)

$$
\tilde{R} = R^{(0)} + R^{(1)} + R^{(2)}, \tag{2.1.16}
$$

where

$$
\begin{aligned}
R^{(0)} &= R, \\
R^{(1)} &= g^{\sigma\nu} R^{(1)}_{\sigma\nu} - h^{\sigma\nu} R_{\sigma\nu}, \\
&= g^{\sigma\nu} \nabla_\rho \Gamma^{\rho(1)}_{\sigma\nu} - g^{\sigma\nu} \nabla_\nu \Gamma^{\rho(1)}_{\sigma\rho} - h^{\sigma\nu} R_{\sigma\nu} \\
R^{(2)} &= g^{\sigma\nu} R^{(2)}_{\sigma\nu} - h^{\sigma\nu} R^{(1)}_{\sigma\nu} + h^{\sigma\lambda} h^{\nu}_{\lambda} R_{\sigma\nu} \\
&= g^{\sigma\nu} \nabla_\rho \Gamma^{\rho(2)}_{\sigma\nu} - g^{\sigma\nu} \nabla_\nu \Gamma^{\rho(2)}_{\sigma\rho} + g^{\sigma\nu} \Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu} \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\rho} \\
&\quad - h^{\sigma\nu} \nabla_\rho \Gamma^{\rho(1)}_{\sigma\nu} + h^{\sigma\nu} \nabla_\nu \Gamma^{\rho(1)}_{\sigma\rho} + h^{\sigma\lambda} h^{\nu}_{\lambda} R_{\sigma\nu}.
\end{aligned}
\tag{2.1.17}
$$

We now multiply the result, Eq.(2.1.16), to the first part, $\sqrt{-\tilde{g}}$, to expand up to the second order of perturbation of $\sqrt{-\tilde{g}}\tilde{R}$ and then keep only the second order terms. As a result, we obtain

$$
(\sqrt{-\tilde{g}}\tilde{R})^{(2)} = \sqrt{-g} R^{(2)} + \sqrt{-g}^{(1)} R^{(1)} + \sqrt{-g}^{(2)} R. \tag{2.1.18}
$$

Therefore, the second order perturbed action in Eq.(2.1.2) can be rewritten by substituting the results from Eq.(2.1.5) and Eq.(2.1.18) as follows

$$
S^{(2)} = \int \sqrt{-g} d^4x \Big[R^{(2)} + \frac{1}{2} h R^{(1)} + \frac{1}{8} h^2 R - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} R \Big]. \tag{2.1.19}
$$

We can substitute the Ricci scalar from Eq.(2.1.17) into Eq.(2.1.19) and rewrite the action in terms of covariant derivative as,

$$S^{(2)} = \int \sqrt{-g}d^4x \Big[g^{\sigma\nu}\nabla_\rho \Gamma^{\rho(2)}_{\sigma\nu} - g^{\sigma\nu}\nabla_\nu \Gamma^{\rho(2)}_{\sigma\rho} + g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha\rho}$$
$$- h^{\sigma\nu}\nabla_\rho \Gamma^{\rho(1)}_{\sigma\nu} + h^{\sigma\nu}\nabla_\nu \Gamma^{\rho(1)}_{\sigma\rho} + h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu} + \frac{1}{2}hg^{\sigma\nu}\nabla_\rho \Gamma^{\rho(1)}_{\sigma\nu} - \frac{1}{2}hg^{\sigma\nu}\nabla_\nu \Gamma^{\rho(1)}_{\sigma\rho}$$
$$- \frac{1}{2}hh^{\sigma\nu}R_{\sigma\nu} + \frac{1}{8}h^2R - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}R \Big]. \tag{2.1.20}$$

From Eq.(2.1.20), we can use the equation from the background, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, and then integrate by part of some terms in the above action. There are the surface terms resulting from integrating by part which we can set to be zero by demanding that the variation vanished at the surface (see more details in Appendix A.6). As a result, the action becomes

$$S^{(2)} = \int \sqrt{-g}d^4x \Big[(\nabla_\rho h^{\sigma\nu} - \frac{1}{2}g^{\sigma\nu}\nabla_\rho h)\Gamma^{\rho(1)}_{\sigma\nu} + (\frac{1}{2}g^{\sigma\nu}\nabla_\nu h - \nabla_\nu h^{\sigma\nu})\Gamma^{\rho(1)}_{\sigma\rho}$$
$$+ g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\rho} + \frac{1}{4}R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2) \Big]. \tag{2.1.21}$$

By substituting $\Gamma^{\alpha(1)}_{\mu\nu}$ from Eq.(2.1.8) into Eq.(2.1.21) and then simplifying the above equation, we have

$$S^{(2)} = \int \sqrt{-g}d^4x \Big[\frac{1}{4}\nabla^\mu h\nabla_\mu h - \frac{1}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu} + \frac{1}{2}\nabla^\mu h^{\lambda\nu}\nabla_\lambda h_{\mu\nu} - \frac{1}{4}\nabla^\lambda h^{\mu\nu}\nabla_\lambda h_{\mu\nu}$$
$$+ \frac{1}{4}R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2) \Big]. \tag{2.1.22}$$

The second order perturbed action in Eq.(2.1.22) contains the kinetic terms which are written in terms of covariant derivatives and the term looks like a mass term given by the combination of all possible contractions of two power of $h_{\mu\nu}$. Even though there are mass terms, the theory still has guage symmetry, $h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. The gauge symmetry provides 2 propagating degrees of freedom in the theory so that the theory corresponds to the massless theory. We will show explicitly how to count the propagating degrees of freedom in next section.

If we consider the action in flat Minkowski spacetime, we have $g_{\mu\nu} = \eta_{\mu\nu}$. It provides that the Christoffel symbols from Eq.(2.1.7) becomes zero, $\Gamma^\rho_{\mu\nu} = 0$. From Eq.(2.1.9), we see that the covariant derivatives can be reduced to partial derivatives because the geometry of spacetime will be no longer curved when we consider in the flat Minkowski spacetime, $\nabla_\mu = \partial_\mu$. For flat Minskowki spacetime, it also implies that $R_{\mu\nu} = 0$ and then $R = 0$. As a result, we will get the second order perturbed action in flat Minkowski spacetime as

$$S^{(2)} = \int d^4x \Big[\frac{1}{4}\partial^\mu h \partial_\mu h - \frac{1}{2}\partial^\mu h \partial^\nu h_{\mu\nu} + \frac{1}{2}\partial^\mu h^{\lambda\nu}\partial_\lambda h_{\mu\nu} - \frac{1}{4}\partial^\lambda h^{\mu\nu}\partial_\lambda h_{\mu\nu}\Big]. \quad (2.1.23)$$

From this action, there is no mass term explicitly in contrast to one in the curved spacetime. One can check that this action is invariant under gauge transformation [14],

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (2.1.24)$$

where $\xi_\mu(x)$ is a spacetime dependent gauge parameter. Therefore, this is a theory of massless spin-2 or massless graviton.

2.2 The Equations of Motion

As we have mentioned before, the equation of motion can be obtained by varying the second order perturbed action with respect to the dynamical field which, in this case, is $h_{\mu\nu}$. From the action in curved spacetime in Eq.(2.1.22), applying variational method, we have

$$\int \sqrt{-g} d^4x \Big[\frac{1}{2}\delta(\nabla^\mu h \nabla_\mu h) - \frac{1}{2}\delta(\nabla^\mu h \nabla^\nu h_{\mu\nu}) + \frac{1}{2}\delta(\nabla^\mu h^{\lambda\nu}\nabla_\lambda h_{\mu\nu}) \\ - \frac{1}{4}\delta(\nabla^\lambda h^{\mu\nu}\nabla_\lambda h_{\mu\nu}) + \frac{1}{4}\delta R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2)\Big] = 0.$$

From $\delta S = 0$, the variation of second order perturbed action provides the linearized equation of motion which can be written as

$$\Box h^{\mu\nu} - \nabla^\alpha \nabla^\nu h^\mu_\alpha - \nabla^\alpha \nabla^\mu h^\nu_\alpha + g^{\mu\nu}\nabla^\alpha\nabla^\beta h_{\alpha\beta} + \nabla^\mu\nabla^\nu h - g^{\mu\nu}\Box h + \frac{R}{2}(h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}h) = 0 \quad (2.2.2)$$

where $\Box \equiv \nabla_\mu \nabla^\mu$. One can reduce the equation of motion to one in flat Minkowski spacetime by using the same manner as we have done in the action before. From the equation of motion in curved spacetime in Eq.(2.2.2), we can set $g_{\mu\nu} = \eta_{\mu\nu}$ and then $R = 0$, $\nabla_\mu = \partial_\mu$. As a result, Eq.(2.2.2) becomes

$$\partial_\alpha \partial^\alpha h^{\mu\nu} - \partial^\alpha \partial^\nu h^\mu_\alpha - \partial^\alpha \partial^\mu h^\nu_\alpha + \eta^{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} + \partial^\mu \partial^\nu h - \eta^{\mu\nu} \partial_\alpha \partial^\alpha h = 0. \quad (2.2.3)$$

One can check that this equation of motion is still invariant under gauge transformation, $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. For convenience, we can choose the gauge choice as Lorentz gauge defined as

$$\partial_\mu h^{\mu\nu} = 0. \quad (2.2.4)$$

From these gauge conditions, the equation of motion in Eq.(2.2.3) can be reduced as

$$\partial_\alpha \partial^\alpha h^{\mu\nu} - \eta^{\mu\nu} \partial_\alpha \partial^\alpha h = 0. \quad (2.2.5)$$

By taking the trace to Eq.(2.2.5), we obtain

$$\partial_\alpha h = 0. \quad (2.2.6)$$

By substituting Eq.(2.2.4) and Eq.(2.2.6) into the equation of motion in Eq.(2.2.3), we obtain

$$\partial_\alpha \partial^\alpha h^{\mu\nu} = 0. \quad (2.2.7)$$

In order to preserve the Lorentz gauge, there give more conditions obtained by using Eq.(2.2.4), Eq.(2.2.6), and the gauge transformation; namely $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. These conditions can be written as

$$\partial^\mu \partial_\mu \xi_\nu = 0. \quad (2.2.8)$$

From Eq.(2.2.7), we can see that it is a waves equation of the symmetric rank-2 tensor field which has 10 propagating degrees of freedom. However, we have two constraint equations; from Eq.(2.2.4) which provides 4 conditions and Eq.(2.2.8) which provides more 4 conditions. Therefore, the propagating degrees of freedom are $10 - 4 - 4 = 2$. It implies that there are 2 degrees of freedom in the case of massless spin-2 field. It also corresponds to 2 polarizations of gravitational waves.

Note that we are freedom to choose other gauge choices which they can provide the same result. We choose the Lorentz gauge because it is simply to consider. If we consider the theory in curved spacetime, the propagating degrees of freedom are also 2. This is due to the gauge symmetry in the theory.

In this chapter, we found the second order perturbed action by expanding the Einstein-Hilbert action. From this action, we obtained the linearized equation of motion which invariant under the gauge transformation. We found that there are 2 propagating degrees of freedom which corresponds the massless spin-2 theory. Next chapter, we will study the massive gravity theory obtained by adding mass terms into the linearized Einstein-Hilbert action. We will also calculate light bending angle and show that the light bending angle in massive theory is different from one in GR.

CHAPTER III

vDVZ Discontinuity

From the previous chapter, we now have the second order perturbed action for massless theory, Eq.(2.1.23) in flat spacetime. The Fierz-Pauli (FP) theory is a linear massive gravity theory. Therefore, the linear mass term must be added into the action in Eq.(2.1.23) in order to obtain FP theory. In order to consider the general form of Fierze and Pauli action, we consider the action for a massive spin-2 particle in D-dimensional flat spacetime which can be written as

$$S = \int d^D x \Big[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2)\Big]. \tag{3.0.1}$$

Since the FP theory is a linear massive gravity theory, the additional mass terms in the action can be formed with two possible contractions of $h_{\mu\nu}$ as seen in the above action. The relative coefficient of -1 between h^2 and $h_{\mu\nu}h^{\mu\nu}$ is known as the Fierz-Pauli tuning. This Fierz-Pauli tuning is approached to eliminate a scalar ghost term. We will discuss this issue later in details. In the previous chapter, the massless action is invariant under gauge transformation. However, for the massive action considering in this chapter, the mass term breaks gauge symmetry. This massive theory has 5 degrees of freedom while the massless theory has 2 degrees of freedom. We will show how can we obtain 5 degrees of freedom for the FP massive theory in next section. From this action, we will add an external sourced term and then find the equation of motion by varying this action. Then we will show that the FP massive gravity theory has the vDVZ discontinuity. As we have noticed before, the light bending angle calculated from the FP massive gravity theory is not equal to one calculated from GR.

3.1 General Solution to the Source Equation

In the previous chapter, one of the results of the GR theory is the gravitational waves propagating in vacuum. Now we will consider how the gravitational waves can be generated. Therefore, we have to add a source term into the action of the FP massive theory. For convenience, we add the fixed symmetric external source $T^{\mu\nu}(x)$ to the action Eq.(3.0.1) as follows

$$S = \int d^D x \Big[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu}\Big] \tag{3.1.1}$$

where κ is introduced into the action again comparing to the action in Eq.(2.0.2), but now it is put in front of the source term for convenience. By varying the above action with respect to the dynamical field, $h_{\mu\nu}$, the equation of motion is obtained as

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\lambda\partial_\nu h^\lambda_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = -\kappa T_{\mu\nu}. \tag{3.1.2}$$

For the case $m = 0$, taking ∂^μ on the left hand side of Eq.(3.1.2) gives identically zero, so we then get the conservation condition $\partial^\mu T_{\mu\nu} = 0$. For the case $m \neq 0$, we can take partial derivative, ∂^μ, on Eq.(3.1.2), and then we have

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu}. \tag{3.1.3}$$

By substituting this equation into Eq.(3.1.2), we obtain the equation as

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = -\kappa T_{\mu\nu} + \frac{\kappa}{m^2}\left[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \eta_{\mu\nu}\partial\partial T\right], \tag{3.1.4}$$

where $\partial\partial T$ is defined for $\partial_\mu\partial_\nu T^{\mu\nu}$. Taking the trace of Eq.(3.1.4), we obtain

$$h = -\frac{\kappa}{m^2(D-1)}T - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T. \tag{3.1.5}$$

By plugging Eq.(3.1.5) into Eq.(3.1.3), we obtain

$$\partial^\mu h_{\mu\nu} = -\frac{\kappa}{m^2(D-1)}\partial_\nu T + \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial_\nu\partial\partial T. \tag{3.1.6}$$

Now we substitute Eq.(3.1.5) and Eq.(3.1.6) into the equation of motion, Eq. (3.1.2). As a result, the Eq.(3.1.2) can be rewritten as

$$\begin{aligned}(\partial^2 - m^2)h_{\mu\nu} =& -\kappa\Big[T_{\mu\nu} - \frac{1}{D-1}\Big(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\Big)T\Big] \\ &+ \frac{\kappa}{m^2}\Big[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \frac{1}{D-1}\Big(\eta_{\mu\nu} + (D-2)\frac{\partial_\mu\partial_\nu}{m^2}\Big)\partial\partial T\Big].\end{aligned} \tag{3.1.7}$$

In the case of conserved source, $\partial_\mu T^{\mu\nu} = 0$, we can reduce Eq.(3.1.7) to

$$(\partial^2 - m^2)h_{\mu\nu} = -\kappa\Big[T_{\mu\nu} - \frac{1}{D-1}\Big(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\Big)T\Big]. \tag{3.1.8}$$

Therefore, we can see that the equation of motion in Eq.(3.1.2) implies the following three equations

$$\begin{aligned}(\partial^2 - m^2)h_{\mu\nu} =& -\kappa\Big[T_{\mu\nu} - \frac{1}{D-1}\Big(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\Big)T\Big], \\ \partial^\mu h_{\mu\nu} =& -\frac{\kappa}{m^2(D-1)}\partial_\nu T, \\ h =& -\frac{\kappa}{m^2(D-1)}T.\end{aligned} \tag{3.1.9}$$

The first equation in Eq.(3.1.9) is the equation of motion while the second and third equation are the constraint equations for $h_{\mu\nu}$. By removing the source term in Eq.(3.1.9), we can rewrite these equations as

$$(\partial^2 - m^2)h_{\mu\nu} = 0, \tag{3.1.10a}$$

$$\partial^\mu h_{\mu\nu} = 0, \tag{3.1.10b}$$

$$h = 0. \tag{3.1.10c}$$

Similarly, to the previous chapter, these equations can provide us how many the propagating degrees of freedom in the theory are. The first equation, Eq.(3.1.10a), represents as the waves equation. In D-dimensional spacetime, we know that the symmetric rank-2 tensor field in the waves equation, Eq.(3.1.10a), has $D(D+1)/2$ degrees of freedom. The second one, Eq.(3.1.10b) which provides D constraints, and third one, Eq.(3.1.10c) which provides 1 constraint, give totally $D+1$ constraint equations. There are now only $D(D-1)/2-1$ degrees of freedom which correspond to number of degrees of freedom of the massive spin-2. In 4-dimensional spacetime, there are therefore 5 degrees of freedom as we have mentioned at the beginning of this chapter.

For the first equation in Eq.(3.1.9), it is an inhomogeneous differential equation. One possible way to obtain the solution for this equation is that we transform the equation into the momentum space. Then we can algebraically solve for Fourier transformation of $h_{\mu\nu}$. The solution is just the integral over the momentum space of the Fourier transformation which can be written as

$$h_{\mu\nu}(x) = \kappa \int \frac{d^D p}{(2\pi)^D} e^{ipx} \frac{1}{p^2+m^2} \Big[T_{\mu\nu}(p) - \frac{1}{D-1}\Big(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\Big)T(p)\Big], \quad (3.1.11)$$

where $T^{\mu\nu}(p)$ is the Fourier transformation of the source in Fourier space, $T^{\mu\nu}(p) = \int d^D x e^{-ixp} T^{\mu\nu}(x)$. However, we cannot solve the exact solution for $h_{\mu\nu}$ since the exact form of the source is not given yet. In the next section, we will introduce the specific form of source by which the above integral will be evaluated to obtain the solution for $h_{\mu\nu}$.

3.2 Solution for a Point Source

In order to obtain the result compatible to real physical situations, we will consider the theory in 4-dimensional spacetime. Most of objects observed from far away can be considered as a point. It is also easy in mathematic point of view. Therefore, we can consider the gravitational source in Eq.(3.1.11) as a point source. The energy-momentum tensor of a point mass M at rest at the origin can be written as

$$T^{\mu\nu}(x) = M\delta_0^\mu \delta_0^\nu \delta^3(\mathbf{x}), \quad T_p^{\mu\nu} = 2\pi M \delta_0^\mu \delta_0^\nu \delta(p^0). \quad (3.2.1)$$

This source is conserved. By substituting Eq.(3.2.1) into Eq.(3.1.11) and then integrating p_0 part, we obtain the particular components of $h_{\mu\nu}$ as follows

$$\begin{aligned} h_{00}(x) =& \frac{2M}{3M_p} \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2+m^2}, \\ h_{0i}(x) =& 0, \\ h_{ij}(x) =& \frac{M}{3M_p} \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2+m^2} \Big(\delta_{ij} + \frac{p_i p_j}{m^2}\Big). \end{aligned} \quad (3.2.2)$$

In the weak field limit, h_{00} corresponds to the Newtonian potential. h_{ij} also relates to gravitational potential through the diagonal parts. By using contour integral

$$\begin{aligned}\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2+m^2} =& \frac{1}{4\pi}\frac{e^{-mr}}{r},\\ \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{p_i p_j}{\mathbf{p}^2+m^2} =& -\partial_i\partial_j \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} \frac{1}{\mathbf{p}^2+m^2}\\ =& \frac{1}{4\pi}\frac{e^{-mr}}{r}\Big[\frac{1}{r^2}(1+mr)\delta_{ij} - \frac{1}{r^4}(3+3mr+m^2r^2)x_i x_j,\Big]\end{aligned} \tag{3.2.3}$$

where $r \equiv \sqrt{x_i x_j}$, we now have

$$\begin{aligned}h_{00}(x) =& \frac{2M}{3M_p}\frac{1}{4\pi}\frac{e^{-mr}}{r},\\ h_{0i}(x) =& 0,\\ h_{ij}(x) =& \frac{M}{3M_p}\frac{1}{4\pi}\frac{e^{-mr}}{r}\Big[\frac{1+mr+m^2r^2}{m^2r^2}\delta_{ij} - \frac{1}{m^2r^4}(3+3mr+m^2r^2)x_i x_j,\Big].\end{aligned} \tag{3.2.4}$$

For the solutions above, they are inconvenient to read off the Newtonian potential. In order to simplify these solutions, we will discuss some essential details. We see that the term $p_i p_j$ in the third equation of Eq.(3.2.2) will diverge when we consider in the massless limit. Because there is gauge symmetry in massless gravity theory, it implies that we can remove this diverging term by using gauge transformation in order to avoid this problem. In the massive gravity theory, we expect that this term can be eliminated by using gauge transformation which will not effect to the physical properties of the system. Therefore, we can simplify Eq.(3.2.4) as follows

$$\begin{aligned}h_{00}(x) =& \frac{2M}{3M_p}\frac{1}{4\pi}\frac{e^{-mr}}{r},\\ h_{0i}(x) =& 0,\\ h_{ij}(x) =& \frac{M}{3M_p}\frac{1}{4\pi}\frac{e^{-mr}}{r}\delta_{ij}.\end{aligned} \tag{3.2.5}$$

The e^{-mr} term is called Yukawa suppression factor which is characteristic of a massive field.

3.3 Solution for the Massless Graviton

We have obtained the solution of $h_{\mu\nu}$ for massive gravity theory. As we have mentioned in the first chapter, the result in the massive gravity theory should be reduced to one in the massless gravity theory in order to explain the well-known phenomena which is can be explained by GR. Therefore, we will also calculate the point source solution for the massless case in order to compare the results obtained from both theories. For massless gravity theory, we can choose the Lorentz gauge, $\partial_\mu h^{\mu\nu} = 0$, and then take mass in Eq.(3.1.2) to be zero. It provides the equations of motion as

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu}. \tag{3.3.1}$$

Taking the trace of the above equation, we obtain $\Box h = \frac{2}{D-2}\kappa T$. By substituting this result to Eq.(3.3.1), we obtain

$$\Box h_{\mu\nu} = -\kappa\Big[T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T\Big] \tag{3.3.2}$$

As the same method we used to evaluate Eq.(3.1.11), we transform Eq.(3.3.2) into the momentum space. Then we can algebraically solve for Fourier transformation of $h_{\mu\nu}$. The solution will be in form of the integral over the momentum space of the Fourier transformation. As a result, we obtain

$$h_{\mu\nu}(x) = \kappa\int\frac{d^D p}{(2\pi)^D}e^{ipx}\frac{1}{p^2}\Big[T_{\mu\nu}(p) - \frac{1}{D-1}\eta_{\mu\nu}T(p)\Big] \tag{3.3.3}$$

where $T_{\mu\nu}(p) = \int d^D x e^{-ip.x}T_{\mu\nu}(x)$, is the Fourier transform of the source.

Now we are considering the system in 4-dimension ($D = 4$) and the point source of mass M at the origin referred from Eq.(3.2.1). By substituting $T_{\mu\nu}(p)$ from Eq. (3.2.1) into Eq.(3.3.3) and integrating over p_0, we obtain the components of $h_{\mu\nu}$ in the general integral form as

$$\begin{aligned} h_{00}(x) =& \frac{M}{2M_p}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2} = \frac{M}{2M_p}\frac{1}{4\pi r}, \\ h_{0i}(x) =& 0, \\ h_{ij}(x) =& \frac{M}{2M_p}\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{px}}\frac{1}{\mathbf{p}^2}\delta_{ij} = \frac{M}{2M_p}\frac{1}{4\pi r}\delta_{ij}. \end{aligned} \tag{3.3.4}$$

We obtain the general solutions of $h_{\mu\nu}$ in curved spacetime for both massive and massless gravity theory. Next section, we will calculate the Newtonian potential by using the results from this section, and then calculate a light bending angle in each theory. We will see that light ray is bent by the curvature of spacetime determined from $h_{\mu\nu}$. We also see how the value of light bending angle in each case are different.

3.4 The vDVZ Discontinuity

Now we are considering system of a test particle moving in the field $h_{\mu\nu}$ to predict some physical results. In general relativity, we know that a test particle responds to the metric deviation as $\delta g_{\mu\nu} = \frac{2}{M_p} h_{\mu\nu}$ [46].

$$\delta g_{00} = \frac{2}{M_p} h_{00} = -2\phi, \quad \delta g_{0i} = \frac{2}{M_p} h_{0i} = 0, \quad \delta g_{ij} = \frac{2}{M_p} h_{ij} = -2\psi \delta_{ij}. \tag{3.4.1}$$

where $\phi = -\frac{GM}{r}$ is the Newtonian potentian, $\psi = \gamma\phi$, and γ is parametrized post-Newtonial parameters. As we have mentioned in Chapter I, modified gravity theory will provide some corrections to Newtonian potential which may be tested by observations. There are ten parameters for characterizing the Newtonian corrections from modified gravity models called parametized post Newtonian (PPN) parameters. Some of these parameters can be obtained by analyzing spherically symmetric solutions. In other words, it is an approximation in the Newtonian theory to GR because GR can very well explain many phenomena at solar system or local gravity scale. It implies that any modified gravity theory has to reduce to GR at local gravity scale. As this discussion, we can find the parameters to characterize the deviation from GR by parameterizing the metric as corrections of Newtonian theory. Therefore, γ is one of ten PPN parameters which can be used to calculate the light bending angle. Then the angle for the light bending at impact parameter b from the source is given by

$$\alpha = 2(1+\gamma)\frac{GM}{b}. \tag{3.4.2}$$

For massless case, by comparing Eq.(3.3.4) and Eq.(3.4.1) in the components h_{00} and h_{ij}, we obtain

$$\phi(r) = -\frac{GM}{r}, \quad \psi(r) = -\frac{GM}{r}. \tag{3.4.3}$$

where $G = 1/8\pi M_p^2$. In this case, we have the PPN parameter $\gamma = 1$. The magnitude of the light bending angle is,

$$\alpha = \frac{4GM}{b}, \tag{3.4.4}$$

For the massive graviton case, we can see in Eq.(3.2.5) which we have evaluated before as follows

$$\begin{aligned} h_{00}(x) =& \frac{2M}{3M_p}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \\ h_{0i}(x) =& 0, \\ h_{ij}(x) =& \frac{M}{3M_p}\frac{1}{4\pi}\frac{e^{-mr}}{r}\delta_{ij}. \end{aligned} \tag{3.4.5}$$

As we have noticed before, any modified gravity theory should be reduce to GR at the local gravity scale. Since the massive gravity theory is a modified gravity theory by which the mass of graviton is given to GR, it should be reduced to GR by taking mass of graviton to be zero. From Eq.(3.4.5), we can find the massless limit of massive gravity theory by taking mass to be zero. By taking mass to be zero and then substituting Eq.(3.4.5) into Eq.(3.4.1), we obtain the solution for massive gravity theory as follows

$$\phi = -\frac{4}{3}\frac{GM}{r} \quad \psi = -\frac{2}{3}\frac{GM}{r}\delta_{ij}. \tag{3.4.6}$$

In the massive graviton case, we obtain the PPN parameter $\gamma = \frac{1}{2}$ by using relation $\psi(r) = \gamma\phi$ and $\phi(r) = -\frac{GM}{r}$. This provides some inconsistent signals from massive gravity theory in massless limit which is different from GR. By substituting γ into Eq.(3.4.2), The magnitude of the light bending angle is obtained as

$$\alpha = \frac{3GM}{b}. \tag{3.4.7}$$

We see that the light bending angle of both cases are different. In order to compare these results obtained by both theories, we will make a reference value for the light bending angle to agree with GR by rescaling $G \to \frac{4}{3}G$. The light bending angle in the case of massive graviton will change to $\alpha = \frac{4GM}{b}$. It implies that if we take the same value of light bending angle, we will get the different value of G from both theories. The difference of value of G will give a different value of ϕ so that it should not occur because the value of Newtonian potential and light bending angle which can be measured in laboratory must be the same for both theories.

This means that the results obtained from massless limit of the linearized massive gravity theory are different from ones obtained by the linearized general relativity. In fact, if we predict the same thing, it has to get the same result. When we consider the result predicted by the massive gravity theory in massless limit, it should give the same result predicted by GR. However, it breaks our intuition by which the massive theory provides the result different from one obtained in GR. This is known as vDVZ (van Dam, Veltman, Zakharov) discontinuity [15, 16].

We have firstly studied the massive gravity theory by adding mass terms into the linearized Einstein-Hilbert action which we have mentioned in Chapter II. By varying this massive action, it provides the linearized equations of motion. We obtained the solution of $h_{\mu\nu}$ by solving the equation of motion. As a result, we calculated the light bending angle for massive gravity theory and then compared the result to one obtained from GR. When we take graviton mass to be zero, the result from massive gravity theory differs from the one obtained by GR. Therefore, we have seen the discontinuity in this linear Fierz-Pauli theory called vDVZ discontinuity. This vDVZ discontinuity can be explained by using idea of Vainshtein in which the non-linearized action should be considered. For this idea, it is found that the theory suffers from ghost instability found by Boulware and Deser. As we have noticed in Chapter I, one of the interesting model of non-linear massive gravity theory without instabilities is dRGT massive gravity theory. The Vainshtein mechanism and the dRGT massive gravity theory will be considered in next chapter. We then find the analytical solutions in dRGT massive gravity theory in order to find the particle's trajectory in the Chapter V.

CHAPTER IV

The dRGT Massive Gravity

Is the graviton possible to obtain a mass? It is possible, it can be evaluated by theoretical and experimental bounds [47]. By adding mass to the graviton, it is a choice to explain the accelerating expansion of the universe. It is found that the mass of graviton can play a role of the cosmological constant. Therefore, at large scales, the massive gravity theory can explain cosmological acceleration. However, we have shown that the massive gravity theory encounters the vDVZ discontinuity. As we have mentioned in the previous chapter, this vDVZ discontinuity arises by adding FP mass term to the linearized gravity theory. By considering non-linear massive gravity theory, this discontinuity can be explained. This idea is proposed by Vainshtein in 1972.

Just about the same time, Boulware and Deser (BD) found that the non-linear massive gravity theory is suffered from instability. This instability is occurred due to an additional scalar ghost mode in the non-linear FP massive gravity theory, called BD-ghost. It is the sixth degree of freedom additionally to 5 degrees of freedom for massive gravity theory mentioned in Chapter III. Therefore, there are some attempts trying to explain these problems. They can explain the origin of this ghost by using the effective field approach and introducing the Stükelberg fields. The Stükelberg fields play a role of the additional scalar and vector graviton polarization [48] and make the action to be covariant. In order to solve this problem, the non-linear action has to contain well-constructed mass terms so that the BD-ghost will be not appeared. This well-constructed mass terms will be represented in this chapter.

In this chapter, we will firstly introduce the covariant non-linear massive gravity without BD-ghost known as dRham, Gabadadze, and Tolley (dRGT) massive gravity theory. By using spherical symmetry, we will show that Vainshtein mechanism can work in dRGT massive gravity theory. In other words, the result from dRGT massive gravity theory in the massless limit can be reduced to one from GR. By using the simple form of the fiducial metric, we then find an analytical solution in this theory. We will use this solution in order to cosider particle's trajectory in Chapter V.

4.1 Action and Equations of Motion

In this section, we will consider the dRGT massive gravity theory in details. The action which is proposed by de Rham, Gabadadze, and Tolley will be defined by specifying the form of mass terms. We begin this section by introducing Einstein-Hilbert action with the covariant Fierz-Pauli mass term. Therefore, the covariant FP action can be written as

$$S = \int \sqrt{-g} dx^4 \Big[R - m_g^2 \mathcal{U}_2 \Big], \tag{4.1.1}$$

m_g is a parameter in unit of mass representing mass of graviton. $\mathcal{U}_2 = (H_{\mu\nu}H^{\mu\nu} - H^2)$ is the potential function for the graviton mass term. $\mathcal{U}_n$ denotes the interaction terms at the n^{th} order in terms of $H_{\mu\nu}$. The tensor $H_{\mu\nu}$ is a covariantization of the metric perturbations, namely [19]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \equiv H_{\mu\nu} + \partial_\mu \phi^\alpha \partial_\nu \phi^\beta \eta_{\alpha\beta} \tag{4.1.2}$$

where ϕ^α are four scalar Stückelberg fields [36]. The action is invariant under transformations [48]

$$x'^\mu = x^\mu + \xi^\mu \tag{4.1.3}$$

$$\phi^\alpha(x') = \phi^\alpha(x) + \xi^\nu \partial_\nu \phi^\alpha(x). \tag{4.1.4}$$

We can see that one important role of Stückelberg field is to restore generalized coordinate transformation. Now we see explicitly that this action has the symmetry under general coordinate transformation. Therefore we can choose a gauge corresponding to this symmetry as $\phi^\alpha = x^\alpha$ namely unitary gauge. We can show that this unitary gauge can be reduced to the FP massive theory. By substituting the unitary gauge condition, $\phi^\alpha = x^\alpha$, into Eq.(4.1.2), we obtain $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ as the perturbed metric in FP massive theory reviewed in Chapter II.

As we mentioned before, Vainshtein proposed that the non-linear terms should be added into the action in order to eliminate discontinuity in the linear FP massive gravity theory. From this idea, the non-linear terms are dominated when mass goes to zero. A broad class of non-linear theories of massive gravity is plagued by BD ghost [17]. These non-linear terms should be specific. Otherwise, the BD-ghost terms will appear in the action. The same as FP-tuning idea, de Rham, Gabadadze, and Tolley found that BD-ghost can be systematically removed in the decoupling limit to all orders in the perturbation theory known as dRGT massive gravity theory [19–21]. The dRGT non-linear massive gravity action can be written as

$$S = \int \sqrt{-g} dx^4 \Big[M_p^2 R - m_g^2 \Big(\alpha_2 \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4 \Big) \Big]. \tag{4.1.5}$$

The terms of potential can be constructed [19] by tuning at each powers of $H_{\mu\nu}$ which is written in terms of other tensor, $\mathcal{K}_{\mu\nu}$ as

$$\begin{aligned} \mathcal{U}_2 &= \big[\mathcal{K}\big]^2 - \big[\mathcal{K}^2\big], \\ \mathcal{U}_3 &= \big[\mathcal{K}\big]^3 - 3\big[\mathcal{K}\big]\big[\mathcal{K}^2\big] + 2\big[\mathcal{K}^2\big], \\ \mathcal{U}_4 &= \big[\mathcal{K}\big]^4 - 6\big[\mathcal{K}\big]^2\big[\mathcal{K}^2\big] + 8\big[\mathcal{K}\big]\big[\mathcal{K}^3\big] + 3\big[\mathcal{K}^2\big]^2 - 6\big[\mathcal{K}^4\big] \end{aligned} \tag{4.1.6}$$

where

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \Big(\sqrt{\mathbb{I} - g^{-1}H}\Big)^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\sigma} f_{\alpha\beta} \partial_\sigma \phi^\alpha \partial_\nu \phi^\beta}. \tag{4.1.7}$$

$\big[\mathcal{K}\big]$ denotes the trace of $\mathcal{K}^\mu_\nu$, namely $\big[\mathcal{K}\big] = \mathcal{K}^\mu_\mu$. The interaction terms are symmetrical polynomials of $\mathcal{K}$. The coefficients in the combinations are chosen to make the equation of motion have no higher derivative terms. The metric tensor $g_{\mu\nu}$ is the observable metric describing the five degrees of freedom of the massive graviton. $f_{\alpha\beta}$ is the fiducial metric or the reference metric determining the form of the solution to the theory. It is the non-dynamical metric then plays the role of Lagrange multiplier.

In order to obtain the equation of motion, the action in Eq.(4.1.5) is varied with respect to metric $g_{\mu\nu}$, and then we have

$$G_{\mu\nu} + m_g^2 X_{\mu\nu} = 0, \tag{4.1.8}$$

where $X_{\mu\nu}$ is the effective energy-momentum tensor obtained by varying the potential terms Eq.(4.1.5) with respect to $g_{\mu\nu}$,

$$\begin{aligned} X_{\mu\nu} = \mathcal{K}_{\mu\nu} - \mathcal{K} g_{\mu\nu} - \alpha \Big(\mathcal{K}^2_{\mu\nu} - \mathcal{K}\mathcal{K}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \mathcal{U}_2 \Big) \\ + 3\beta \Big(\mathcal{K}^3_{\mu\nu} - \mathcal{K}\mathcal{K}^2_{\mu\nu} + \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{U}_2 - \frac{1}{6} g_{\mu\nu}\mathcal{U}_3 \Big). \end{aligned} \tag{4.1.9}$$

We have chosen to rescale the parameters by making $\alpha_2 = 1$ and redefine the two remaining parameters α_3 and α_4 of the graviton potential in Eq.(4.1.5) by introducing two new parameter α and β, given as

$$\alpha_3 = \frac{\alpha - 1}{3}, \quad \alpha_4 = \frac{\beta}{4} + \frac{1 - \alpha}{12}. \tag{4.1.10}$$

Then, we use the Bianchi identities to obtain the constraint equation as

$$\nabla^\mu X_{\mu\nu} = 0. \tag{4.1.11}$$

Most of astronomical objects can be approximately sphere. Therefore, we then try to explain an astronomical phenomena by finding the spherically symmetric solutions from Eq.(4.1.8) and Eq.(4.1.11). The form of the solutions can be divided into two parts depending on the form of the fiducial metric. For first part, the fiducial metric is the flat Minkowski metric. We use this solution to show how Vainshtein mechanism can work in dRGT massive gravity theory. The solution in the second part represents an analytic solution. We will use this solution to calculate the particle's trajectory in next chapter.

4.2 Spherically Symmetric Solutions and Vainshtein Mechanism

Since most objects in the universe are seem likely as spherical objects, we can assume that the system has the spherical symmetry. In this section, we will obtain the the simple and useful solution by imposing spherical symmetry. The most general form of the physical metric corresponding to static and spherically symmetric conditions can be written as

$$ds^2 = -b(r)dt^2 + 2d(r)dtdr + a(r)dr^2 + c(r)^2 d\Omega^2. \tag{4.2.1}$$

The fiducial metric is chosen in flat Minkowski form with spherical coordinate as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{4.2.2}$$

where $d\Omega^2$ denotes the solid angle as $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. By using this ansatz, the solutions for Eq. (4.1.8) and Eq. (4.1.11) are found and classified into two branches: $d(r) = 0$ or $c(r) = c_0 r$ where c_0 is a constant depending on the parameters α and β [33–35]. The most interesting branch is $d(r) = 0$ since it is simpler to analyze. Therefore, we choose to analyze the solution in this branch, $d(r) = 0$, for this thesis.

We firstly review the linearized metric in order to show the discontinuity in the linear dRGT massive gravity theory and then consider the non-linear metric to show how the Vainshtein mechanism can work. It is convenient to redefine the functions in the physical metric in Eq.(4.2.1) to satisfy perturbation form as follows

$$b(r) = (1 + N(r)), \quad a(r) = (1 + F(r))^{-1/2}, \quad c(r) = (1 + H(r))^{-1}. \tag{4.2.3}$$

In order to simplify the metric, we will change the radial coordinate as $\rho = \frac{r}{1+H(r)}$. Therefore, the linearized metric is expressed as

$$ds^2 = -(1 + n)dt^2 + (1 - f)d\rho^2 + \rho^2 d\Omega^2 \tag{4.2.4}$$

where $f(\rho) = F(r(\rho)) - 2h(\rho) - 2\rho h'(\rho)$, $n(\rho) = 2N(r(\rho))$, $h(\rho) = H(r(\rho))$ and the prime denotes derivative with respect to ρ. At the linear order of the functions $n(\rho)$, $f(\rho)$, and $h(\rho)$, by using Eq.(4.1.8) and Eq.(4.1.11), we obtain the equations for the functions, $n(\rho)$, $f(\rho)$, and $h(\rho)$ in terms of radial coordinate ρ as follow

[22, 33],

$$0 = (m_g^2\rho^2 + 2)f + 2\rho(f' + m_g^2\rho^2 h' + 3m_g^2\rho h), \tag{4.2.5}$$

$$0 = \frac{1}{2}m_g^2\rho^2(n - 4h) - \rho n' - f, \tag{4.2.6}$$

$$0 = f + \frac{1}{2}\rho n'. \tag{4.2.7}$$

From these equations, we rearrange each function in form of homogeneous differential equation as follows

$$\rho^2 n'' + 2\rho n' - m_g^2\rho^2 n = 0. \tag{4.2.8}$$

In order to solve this equation, we can change the variable such that $n = \frac{\tilde{n}}{\rho}$, where $\tilde{n}$ is a new variable. By substituting this relation into Eq.(4.2.8), we obtain

$$\tilde{n}'' - m_g^2\tilde{n} = 0. \tag{4.2.9}$$

From above equation, it is found that the solution for $\tilde{n}$ is in form of $ke^{-m_g\rho}$, where k is an integration constant. Therefore, we obtain the solutions for n and f as follows

$$n = -\frac{8GM}{3\rho}e^{-m_g\rho}, \tag{4.2.10}$$

$$f = -\frac{4GM}{3\rho}(1 + m\rho)e^{-m_g\rho}. \tag{4.2.11}$$

In order to obtain the solution which can be reduced to Newtonian theory, we can fix the integration constant so that M is a mass of a point source and G is the Newtonian constant. As a result, it is found that the post-Newtonian parameter $\gamma = 1/2$ obtained by using $\gamma = f/n$ is $\gamma = 1/2(1 + m_g\rho)$ as we have mentioned in Chapter III. These solutions disagree with solutions from GR as well as the Solar system observations ($\gamma = 1$ in GR and observations give $1 - \gamma \backsimeq 10^{-5}$ [39]) so that the theory encounters vDVZ discontinuity. By analyzing the massless limit approximations, the equation of motion can be truncated to linear order in f and n, but not in h. These suggest us that we have to consider higher order of h in the equation of motion [33].

Therefore, we have to consider the non-linear behaviour of h as $m_g \to 0$. By keeping non-linear terms in h, Eq.(4.1.8) and Bianchi identities in Eq.(4.1.11) can be written as

$$f = -2\frac{GM}{\rho} - (m_g\rho)^2\big(h - \alpha h^2 + \beta h^3\big), \tag{4.2.12}$$

$$n' = 2\frac{GM}{\rho^2} - m_g^2\rho\big(h - \beta h^3\big), \tag{4.2.13}$$

$$0 = \frac{3}{2}\beta^2 h^5(\rho) - \big(\alpha^2 + 2\beta\big)h^3(\rho) + 3\big(\alpha + \beta A(\rho)\big)h^2(\rho) - \frac{3}{2}h(\rho) - A(\rho), \tag{4.2.14}$$

where

$$A(\rho) \equiv (\rho_v/\rho),$$
$$\rho_v \equiv (GM/m_g^2)^{1/3}. \tag{4.2.15}$$

We call ρ_v as the Vainshtein radius. From Eq.(4.2.12), Eq.(4.2.13), and Eq. (4.2.14), there are the terms depending on h^3 and h^5. These terms make the equation of motion complicated to solve. Therefore, in order to simply solve these equations for f and n, we can take $\beta = 0$ to eliminate the terms proportional to h^5. In this case with assumption $\rho \ll \rho_v$, the solutions are obtained as

$$n = -\frac{2GM}{\rho}\Big(1 + \frac{1}{4\alpha}\big(\frac{\rho}{\rho_v}\big)^2\Big),$$
$$f = -\frac{2GM}{\rho}\Big(1 - \frac{1}{2\alpha}\big(\frac{\rho}{\rho_v}\big)\Big). \tag{4.2.16}$$

As a result, it is found that there is no discontinuity appearing in the solution when we consider the non-linear of h within $\rho \ll \rho_v$. This is the Vainshtein mechanism for dRGT massive gravity theory at which the theory can reduce to general relativity at local gravity scale (within Vainshtein radius). For more general solution which $\beta \neq 0$, the solution for h is more complicated. We must solve the fifth order equation of h in Eq.(4.2.14). The solution of h will be substituted into the others, Eq.(4.2.12) and Eq.(4.2.13), in order to solve differential equation for functions of n and f. It implies that it is hardly to solve for the exact solution or analytical solution. However, we can choose other form of the fiducial metric in order to solve for analytical solution. We will find this solution in the next section.

4.3 Analytical Solutions

In the previous section, the spherically symmetric solution, obtained by setting $\beta = 0$, is not general but it is easy to show how Vainshtein mechanism can work. However, we would like to find a general solution, in order to calculate particle's trajectory in next chapter. Therefore, we will try to find the analytical solutions in this theory. We still restrict our consideration in the diagonal branch of the physical metric, $d(r) = 0$, static condition, and spherical symmetry. For convenience, we will choose the function $c(r)$ of the physical metric in Eq.(4.2.1) as $c(r) = r$ and then the physical metric can be written as

$$ds^2 = -n(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{4.3.1}$$

From the previous section, we used the fiducial metric as the flat Minkowski metric, $f_{\mu\nu} = \eta_{\mu\nu}$. We found that we cannot find the analytical solution by using this form of the fiducial metric. Therefore, we can choose a suitable fiducial metric in order to solve for analytical solution. This fiducial metric can be written as [36–38]

$$f_{\alpha\beta} = \mathrm{diag}(0, 0, c^2, c^2 \sin^2\theta) \tag{4.3.2}$$

From the ansatz in Eq.(4.3.1) with the fiducial metric in Eq.(4.3.2), we can find the components of Einstein tensor, Eq.(4.1.8). These components of Einstein tensor can be obtained as

$$G^t_t = \frac{f'}{r} + \frac{f}{r^2} - \frac{1}{r^2}, \tag{4.3.3}$$

$$G^r_r = \frac{f(rn' + n)}{nr^2} - \frac{1}{r^2}, \tag{4.3.4}$$

$$G^\theta_\theta = G^\phi_\phi = f'\Big(\frac{n'}{4n} + \frac{1}{2r}\Big) + f\Big(\frac{n''}{2n} + \frac{n'}{2nr} - \frac{(n')^2}{4n^2}\Big). \tag{4.3.5}$$

The components of effective energy-momentum tensor, $X_{\mu\nu}$, in Eq.(4.1.9) with this ansatz can be written as

$$X^r_r = X^t_t = -\Big(\frac{\alpha(3r-c)(r-c)}{r^2} + \frac{3\beta(r-c)^2}{r^2} + \frac{3r-2c}{r}\Big), \tag{4.3.6}$$

$$X^\theta_\theta = X^\phi_\phi = \frac{\alpha(2c-3r)}{r} + \frac{3\beta(r-c)^2}{r} + \frac{c-3r}{r}. \tag{4.3.7}$$

There are specific values of the parameter c in which this effective energy-momentum tensor behaves like a cosmological constant. An interesting case is $c = 0$. This simplifies the components of the effective energy-momentum tensor to depend only on the parameters of the theory or, in other words to be constant. For $c = 0$, the tensor X_μ^ν in Eq.(4.3.6) will be proportional to the identity matrix. This means that the mass terms are all constants at the Lagrangian level and correspond to the cosmological constant term. Now, we will substitute all components of Einstein tensor and effective energy-momentum tensor into Eq.(4.1.8). The modified Einstein equations can be written as

$$\frac{f'}{r} + \frac{f}{r^2} - \frac{1}{r^2} = m_g^2\Big(\frac{\alpha(3r-c)(r-c)}{r^2} + \frac{3\beta(r-c)^2}{r^2} + \frac{3r-2c}{r}\Big), \tag{4.3.8}$$

$$\frac{f(rn'+n)}{nr^2} - \frac{1}{r^2} = m_g^2\Big(\frac{\alpha(3r-c)(r-c)}{r^2} + \frac{3\beta(r-c)^2}{r^2} + \frac{3r-2c}{r}\Big), \tag{4.3.9}$$

$$f'\Big(\frac{n'}{4n} + \frac{1}{2r}\Big) + f\Big(\frac{n''}{2n} + \frac{n'}{2nr} - \frac{(n')^2}{4n^2}\Big) = -m_g^2\Big(\frac{\alpha(2c-3r)}{r} + \frac{3\beta(r-c)^2}{r} + \frac{c-3r}{r}\Big). \tag{4.3.10}$$

From Eq.(4.3.8), it is a differential equation of only function f. Therefore, we can solve this differential equation for $f(r)$ to obtain solution as

$$f(r) = 1 - \frac{2MG}{r} + \frac{\Lambda}{3}r^2 + \gamma r + \zeta, \tag{4.3.11}$$

where M is the mass of the gravitational source and we redefine the parameters for convenience as

$$\Lambda = 3m_g^2(1+\alpha+\beta), \tag{4.3.12}$$

$$\gamma = -cm_g^2(1+2\alpha+3\beta), \tag{4.3.13}$$

$$\zeta = c^2m_g^2(\alpha+3\beta), \tag{4.3.14}$$

Λ plays a role of the cosmological constant depending on the graviton mass m_g which is not surprisingly. The mass of graviton also serves as the cosmological constant in the self-expanding cosmological solution in dRGT massive gravity theory [42, 43]. This solution can give rise to known solutions in GR as follows. In the case of $m_g = 0$, we have the Schwarzschild solution. For $c = 0$ which makes $\gamma = \zeta = 0$, The solution can be determined according to the value of α and β. If

$(1+\alpha+\beta) < 0$, the solution is in the form of Schwarzschild-de-Sitter while the case $(1+\alpha+\beta) > 0$, on the other hand, provides the Schwarzschild-anti-de-Sitter solution. By evaluating Eq.(4.3.8) minus by Eq.(4.3.9), we also obtain

$$n'f = f'n. \tag{4.3.15}$$

This equation shows that the functions f and n differ by a constant. Actually, we can choose the constant to obtain a solution such that

$$n(r) = f(r) = 1 - \frac{2MG}{r} + \frac{\Lambda}{3}r^2 + \gamma r + \zeta. \tag{4.3.16}$$

We can see that the solution obtained in this section is exact and analytic. Therefore, it is useful to use this form of the spherically symmetric solution for analyze the particle's trajectory. We will consider this issue in next chapter.

By comparing the form of solutions in this section and one in the previous section, we found that the solution in dRGT massive gravity theory hardly depends on the choice of the fiducial (reference) metric; changing to other forms of the fiducial metric will significantly affect the solution. This dependency is one of the important properties of massive gravity. For example, one cannot have a nontrivial flat cosmological solution with a Minkowski fiducial metric in a cosmological point of view [41], only the open FLRW solution is allowed [42] where the FLRW solution with arbitrary geometry exists when the FLRW fiducial metric is considered [43]. By generalizing the form of the fiducial metric, a nontrivial cosmological solution can be found [44].

We have shown that the non-linear dRGT massive gravity can explain the vDVZ discontinuity by adding the higher order mass terms into the covariant FP massive gravity action. This shows how Vainshtein mechanism can work in dRGT massive gravity theory. A broad class of non-linear massive gravity theory provided the BD-ghost making the theory unstable. In order to eliminate the BD-ghost, we can introduce well-constructed mass terms into the action. This specific mass terms can be constructed by tuning at each power of $H_{\mu\nu}$ including Stückelberg field. The dRGT massive gravity theory is such a theory without

BD-ghost. By imposing static condition and spherical symmetry, we obtain the analytical solution for dRGT massive gravity theory. The dRGT solution differs from GR solution such that it has additional terms parametrized by the graviton mass and other parameters of the theory. In next chapter, we will use this solution to find the effective potential and consider the particle's trajectory around the gravitational point source especially in the our solar system. If this theory can provide prediction of the Mercury's trajectory as GR provides, it means that this theory is possible to reduce to GR at the local gravity scale.

CHAPTER V

Particle's Trajectory

As we have learned in the previous chapter, there is the discontinuity called vDVZ discontinuity in the linear massive gravity theory. By using Vainshtein mechanism, it was found that the non-linear massive gravity theory must be considered in order to explain the vDVZ discontinuity [18]. This mechanism shows that the higher order in $h_{\mu\nu}$ is dominated inside a characteristic radius called the Vainshtein radius. However, non-linearity of the massive gravity theory provides the BD-ghost found by Boulware and Deser [17]. These BD-ghost terms lead the theory to be unstable. dRGT massive gravity theory is introduced by adding the well-constructed non-linear mass terms into the action in order to eliminate the BD-ghost. From the dRGT massive gravity theory, we found that the analytical solution is different from the solution obtained in GR. At the local gravity scale, the solution obtained in dRGT massive gravity theory must be reduced to one obtained in GR in order to explain the well-known phenomena. Therefore, we can consider the trajectory of an object around the gravitational source such as Mercury's trajectory in our solar system in order to show that the dRGT massive gravity theory can explain the phenomena at the local gravity scale as GR can.

In this chapter, we will analyze the particle's trajectory by using the effective potential. We firstly introduce the effective potential which can be determined by using the Killing vector theorem and by considering together with the dRGT massive gravity solution (4.3.11). As a result, we then categorize the effective potential into three interesting cases.

5.1 Effective Potential

If we consider a particle moving under influence a conservative force, the total energy per unit mass can be written in terms of kinetic and potential terms. Some of kinetic terms can be rearranged to be potential terms by imposing the constraint obtained from the symmetry of the system. The effective potential is an expression combining these terms with the original potential terms. For example, the effective potential of a particle for Newtonian theory can be written as

$$V(r) = \frac{L^2}{2r^2} - \frac{GM}{r}, \tag{5.1.1}$$

where L is the angular momentum per unit mass and M is the mass of a gravitational source. L in the first term of Eq.(5.1.1) is obtained by the constraint of the system, $d\phi/d\tau = L/r^2$. The second term in Eq.(5.1.1) is the original potential term. From this effective potential, it can be plotted as a function of r as shown in Fig.5.1. By using this effective potential, we can analyze how a particle moves around the gravitational source. By considering the effective potential in Fig.5.1, if the total energy per unit mass is equal to the minimum of the effective potential, the particle will move around the source as a circular orbit. If the total energy per unit mass is more than the minimum of effective potential and less than zero, there will be maximum and minimum radius at which the particle moves as an elliptical orbit. If the total energy per unit mass is more than or equal to zero, the orbit of the particle is unbound which means that the particle will not orbit around the source.

We firstly find the constants of the particle's motion. One way to find these constants is to use the Killing vector theorem. The Killing vector theorem provides the concept of conservation of energy and angular momentum corresponding to the symmetry of the system. By considering the physical metric tensor corresponding to spherically symmetric solution of dRGT massive gravity theory as we have mentioned before, it can be written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{5.1.2}$$

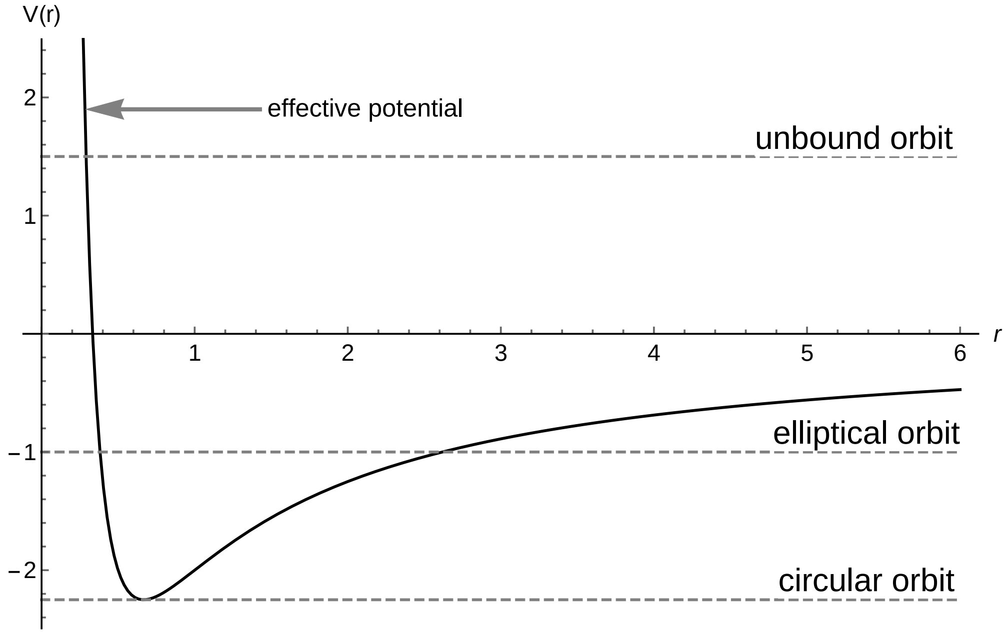

Figure 5.1: Newtonian effective potential is plotted from Eq.(5.1.1). The solid line shows the effective potential and dash line shows the total energy per unit mass of the particle. We have set the Eq.(5.1.1) as $V(r) = 1/r^2 - 3/r$.

where $f(r)$ was found in the previous chapter in Eq.(4.3.11) as follows

$$f(r) = 1 - \frac{2MG}{r} + \frac{\Lambda}{3}r^2 + \gamma r + \zeta$$

As we noticed before, a simple way to find the constants of motion is obtained by using the Killing vector theorem. We will use the four Killing vectors, one for time translation and three for the spherical symmetry. These will provide the constant of the motion for a free particle; if K^μ is a Killing vector, the theorem provides

$$-K^\nu \frac{dx^\mu}{d\tau} g_{\mu\nu} = \text{constant}. \tag{5.1.3}$$

Because of the symmetries, we can reduce some of the terms involving these symmetries to the familiar form. Invariance under time translations provides the conservation of energy, while invariance under spatial rotations gives the conservation of the three components of angular momentum. We can think that the angular momentum is a three-vector with a component of magnitude and two components of direction. The conservation of directions indicates that the particle will move in a plane. We can choose the plane in which the particle moves to easily solve the problem by rotating the coordinates. It implies that we can choose the plane $\theta = \frac{\pi}{2}$ which make $\sin\theta = 1$. Thus, two remaining Killing vectors correspond to the energy and magnitude of angular momentum can be expressed as

$$E = -K^{\mu}\frac{dx^{\nu}}{d\tau}g_{\mu\nu},$$
$$E = f(r)\frac{dt}{d\tau} \tag{5.1.4}$$

and

$$L = -R_{\mu}\frac{dx^{\mu}}{d\tau},$$
$$L = r^2\frac{d\phi}{d\tau}, \tag{5.1.5}$$

where $K^{\mu} = (1,0,0,0)$ and $R_{\mu} = (0,0,0,r^2\sin\theta) = (0,0,0,r^2)$. These conserved quantities which are the energy (E) and angular momentum per unit mass of the particle (L) give a convenient way to understand the orbits of a particle. In order to obtain the effective potential, we have to use the constraint of geodesic equation which can be written as

$$\varepsilon = -g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}, \tag{5.1.6}$$

where $\varepsilon = 0$ for massless particle and $\varepsilon = -1$ for massive particle. By considering the metric tensor Eq.(5.1.2) and then expanding Eq.(5.1.6) for massive particle, $\epsilon = -1$, we obtain

$$-f(r)\Big(\frac{dt}{d\tau}\Big)^2 + \frac{1}{f(r)}\Big(\frac{dr}{d\tau}\Big)^2 + r^2\Big(\frac{d\theta}{d\tau}\Big)^2 + r^2\Big(\frac{d\phi}{d\tau}\Big)^2 = -1,$$
$$-f(r)\Big(\frac{E}{f(r)}\Big)^2 + \frac{1}{f(r)}\Big(\frac{dr}{d\tau}\Big)^2 + r^2\Big(\frac{L}{r^2}\Big)^2 = -1,$$
$$-E^2 + \Big(\frac{dr}{d\tau}\Big)^2 + \frac{L^2}{r^2}f(r) = -f(r),$$
$$\Big(\frac{dr}{d\tau}\Big)^2 + \frac{L^2}{r^2}f(r) + f(r) - E^2 = 0,$$
$$\Big(\frac{dr}{d\tau}\Big)^2 + \Big(\frac{L^2}{r^2} + 1\Big)f(r) = E^2. \tag{5.1.7}$$

By substituting Eq.(4.3.11) into Eq.(5.1.7), we obtain

$$\Big(\frac{dr}{d\tau}\Big)^2 + 1 - \frac{2MG}{r} + \frac{\Lambda r^2}{3} + \gamma r + \zeta + \frac{L^2}{r^2} - \frac{2MGL^2}{r^3} + \frac{\Lambda L^2}{3} + \frac{\gamma L^2}{r} + \frac{\zeta L^2}{r^2} = E^2,$$
$$\frac{1}{2}\Big(\frac{dr}{d\tau}\Big)^2 + \frac{\Lambda r^3}{6} + \gamma r + (\gamma L^2 - 2MG)\frac{1}{2r} + (1+\zeta)\frac{L^2}{2r^2} - \frac{MGL^2}{r^3} = \frac{E^2}{2} - \frac{\Lambda L^2}{2} - \frac{\zeta}{2}. \tag{5.1.8}$$

In order to find the effective potential, we can group the terms in Eq.(5.1.8) into three parts. The first part represents the kinetic term. The second part is the effective potential which includes the terms depending on r. The last part contains the constant terms corresponding to the total energy per unit mass.

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) = K \text{ (constant)}, \tag{5.1.9}$$

where

$$V_{eff} = \frac{\Lambda r^2}{6} + \frac{\gamma r}{2} + (\gamma L^2 - 2MG)\frac{1}{2r} + (\zeta + 1)\frac{L^2}{2r^2} - \frac{MGL^2}{r^3}. \tag{5.1.10}$$

We now obtain the effective potential per unit mass of a particle. In order to analyze the particle's trajectory in the dRGT massive gravity theory, the effective potential must be written in terms of the original parameters in dRGT massive gravity theory such as α, β, and m_g. By substituting Λ, γ, and ζ from Eq. (4.3.12), Eq.(4.3.13), and Eq.(4.3.14) respectively into Eq.(5.1.10), we will obtain the desired effective potential for the particle's trajectory. We will consider this effective potential containing these parameters in the next section.

5.2 Equation of Particle's Motion in the dRGT Massive Gravity Theory

From the effective potential, we can consider the contribution of graviton mass to the particle's trajectory. In order to compare the strength of the effective potential due to dRGT massive gravity theory with that due to GR, we can ignore the effect of angular momentum or, in order words, we can set angular momentum to be zero. The effective potential for vanishing angular momentum can be written as

$$V_{eff} = \frac{\Lambda r^2}{6} + \frac{\gamma r}{2} - \frac{MG}{r}. \tag{5.2.1}$$

When the first and the third terms in Eq.(5.2.1) is in the same order, it provides the characteristic radius for the massive gravity theory. This characterisic radius is

$r^3 = r_s/m_g^2 \equiv r_v^3$. r_v is known as the Vainshtein radius and r_s is the Schwarzschild radius, ($r_s = 2GM$). When r is greater than the Vainshtein radius, the first term in Eq.(5.2.1) dominates. It implies that we are considering the system at the cosmological scale. When r is less than the Vainshtein radius, the third term in Eq.(5.2.1) dominates. It implies that we are considering the system at the local gravity scale. By considering the second and the third terms in Eq.(5.2.1) to be in the same order, we found that c and r_v is also in the same order. Therefore, we will rewrite the effective potential in Eq.(5.1.10) in terms of α, β, r_s, and r_v. By substituting Λ, γ, and ζ from Eq.(4.3.12), Eq.(4.3.13), and Eq.(4.3.14) respectively into Eq.(5.1.10) and rearranging Eq.(5.1.10), we obtain

$$\begin{aligned} V_{eff} = \frac{r_s}{2r_v}\Big[&(1+\alpha+\beta)\Big(\frac{r}{r_v}\Big)^2 - (1+2\alpha+3\beta)\Big(\frac{r}{r_v}\Big) \\ &- \big[1+(1+2\alpha+3\beta)\frac{L^2\epsilon}{r_v r_s}\big]\Big(\frac{r_v}{r}\Big) + \big[\epsilon(\alpha+3\beta)+1\big]\frac{L^2 r_v}{r_s r^2} - \frac{L^2 r_v}{r^3}\Big], \end{aligned} \quad (5.2.2)$$

where $\epsilon = c^2 m_g^2$. If we consider the system at the cosmological scale ($r \gg r_v$), the first two terms will dominate. By considering this effective potential at the cosmological scale, we can see that force, $F = -\nabla V_{eff}$, will be negative when $(1 + \alpha + \beta) > 0$. This shows that particles are attracted. This attraction corresponds to a collapsing universe whose spacetime is the anti-de-Sitter (AdS) spacetime. The force will be positive when $(1 + \alpha + \beta) < 0$. It can imply that the universe is expanding which corresponds to the de-Sitter (dS) spacetime. Normally, if the effective potential has only the first term, we cannot find the stable orbit at the cosmological scale. However, the second term can provide the stable orbital radius at the cosmological scale when $(1 + \alpha + \beta) > 0$ and $(1 + 2\alpha + 3\beta) > 0$.

In order to consider the system at the local gravity scale such as the scale of solar system, we will consider the orbital radius of a particle which is very smaller than the Vainshtein radius ($r \ll r_v$). Therefore, the first two terms in Eq.(5.2.2) will be negligible. By this condition, we obtain

$$V_{eff} = \frac{r_s}{2r_v}\Big[-\big[1+(1+2\alpha+3\beta)\frac{L^2\epsilon}{r_v r_s}\big]\Big(\frac{r_v}{r}\Big) + \big[\epsilon(\alpha+3\beta)+1\big]\frac{L^2 r_v}{r_s r^2} - \frac{L^2 r_v}{r^3}\Big]. \quad (5.2.3)$$

From this effective potential, we will consider in three cases. For the first case, angular momentum goes to zero. The second case, angular momentum goes to infinity but $L^2\epsilon/r_v r_s$ approaches zero. This case represents the particle orbiting around the source but it is not fast enough to make the term $L^2\epsilon/r_v r_s$ dominates. The last one, also angular momentum goes to infinity but $L^2\epsilon/r_v r_s$ is close to one. It implies that the graviton mass will affect the effective potential.

For the first case, we obtain the effective potential as

$$V_{eff} = -\frac{r_s}{2r}. \tag{5.2.4}$$

This case is interesting because this effective potential is the same as one obtained in GR. When we consider the vanishing angular momentum particle, it is found that the effective potential and then the motion of a particle are not affected by graviton mass. Therefore, the effective potential obtained in dRGT massive gravity theory can be reduced to one in GR. In other words, we cannot distinguish the dRGT massive gravity theory from GR via a gravitational experiment on a test particle with zero angular momentum.

For the second case, $L \to \infty$ and $L^2\epsilon/r_v r_s \sim 0$, we obtain

$$V_{eff} = -\frac{r_s}{2r} + \frac{L^2}{2r^2} - \frac{L^2 r_s}{2r^3}. \tag{5.2.5}$$

This case, angular momentum goes to infinity but is not enough to make the term $L^2\epsilon/r_v r_s$ contributes significantly to the effective potential. It is found that the effective potential is the same as one obtained in GR. The radius of the circular orbit of this particle can be solved by using $dV_{eff}/dr = 0$. We can easily obtain the possible radii as

$$r = \frac{L^2 \pm L\sqrt{L^2 - 3r_s^2}}{r_s}. \tag{5.2.6}$$

Since we are considering the case of high angular momentum, we can estimate the solutions as

$$\begin{aligned} r &= \frac{L^2 \pm L^2(1 - 3r_s^2/2L^2)}{r_s} \\ r &= \frac{2L^2}{r_s}, \frac{3}{2}r_s \end{aligned} \tag{5.2.7}$$

or,

$$r = \frac{L^2}{GM}, 3GM.$$

For $r = 3GM$, it provides the unstable circular orbit corresponding to the maximum point of the effective potential as shown in Fig.5.2 for the case of the Mercury. For $r = L^2/GM$, it provides the stable circular orbit corresponding to the minimum point of the effective potential as shown in Fig.5.3 for the case of the Mercury.

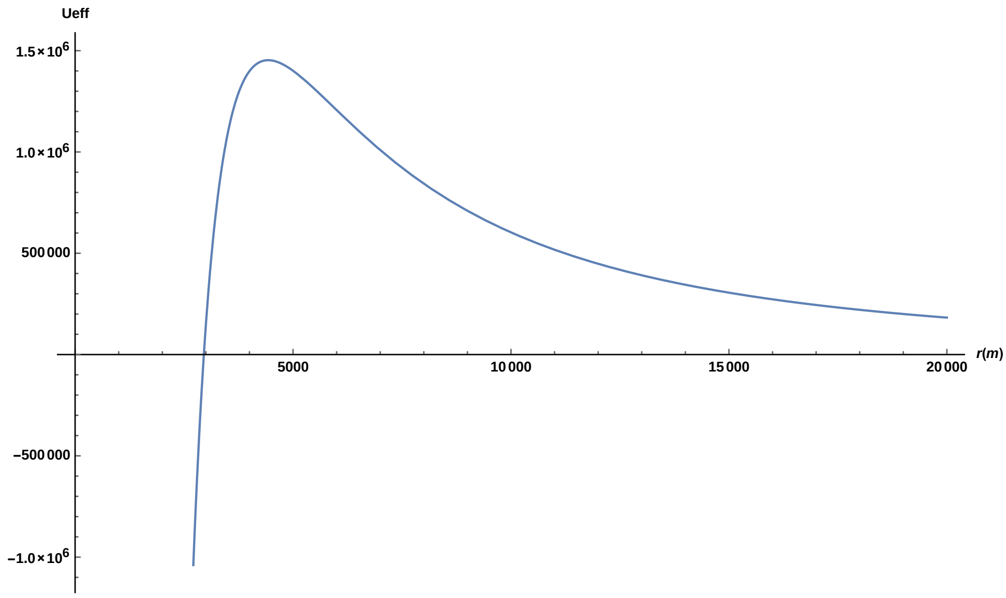

Figure 5.2: Plotting of unstable orbit of Mercury which is not the present radius of Mercury. However, It represents the possibility to have this radius which is 4430 m.

Moreover, we can find a minimum value of angular momentum which can still provide the circular orbit. By taking the value under discriminant in Eq. (5.2.6) to be zero, we obtain

$$L = \sqrt{12}GM. \tag{5.2.8}$$

The radius of the circular orbit corresponding to this value of angular momentum is

$$r = 6GM. \tag{5.2.9}$$

In particular, a circular orbit is not possible for $L < \sqrt{12}GM$. For $r = 6GM$, it is also the smallest possible radius of stable circular orbit in Schwarzschild solution.

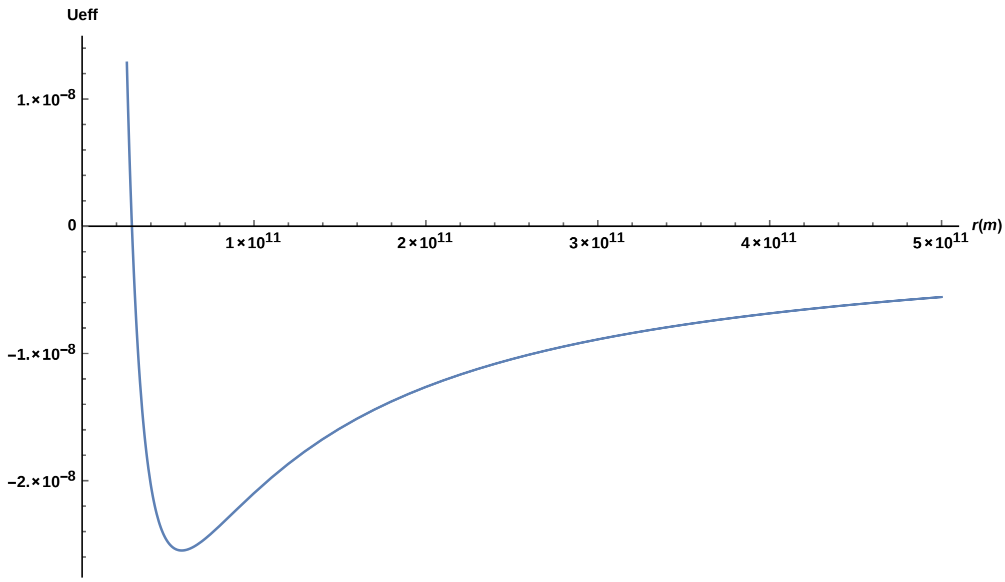

Figure 5.3: Plotting of stable orbit of Mercury which is quite close to the observations. It is about 5.79×10^{10} m.

For the third case, $L \to \infty$ and $L^2\epsilon/r_v r_s \sim 1$. The effective potential for this case can be written as

$$V_{eff} = \frac{r_s}{2r_v}\Big[-\big[1+(1+2\alpha+3\beta)\frac{L^2\epsilon}{r_v r_s}\big]\Big(\frac{r_v}{r}\Big)+\big[\epsilon(\alpha+3\beta)+1\big]\frac{L^2 r_v}{r_s r^2}-\frac{L^2 r_v}{r^3}\Big]. \quad (5.2.10)$$

This case, angular momentum goes to infinity and is enough to make the term $L^2\epsilon/r_v r_s$ contributes significantly to the effective potential. It is found that this effective potential contains the correction terms which invloves the parameters of the dRGT massive gravity theory; namely α, β, and graviton mass in term of ϵ. These parameters can be tuned to fit with observational data, as well as we can reduce this case to that of the GR when mass of graviton goes to zero. The circular radius for this case can be written as

$$r = \frac{XL^2 \pm XL^2\sqrt{1-3r_s^2 Y/X^2L^2}}{r_s Y}, \quad (5.2.11)$$

where $X = 1 + \alpha\epsilon + 3\beta\epsilon$ and $Y = 1 + \frac{\epsilon L^2}{r_v r_s^2} + \frac{2\alpha\epsilon L^2}{r_v r_s^2} + \frac{3\beta\epsilon L^2}{r_v r_s^2}$. We found that the circular radius not only depend on angular momentum and but also depend on the parameters of the dRGT massive gravity theory. We also found stable and unstable orbits as in GR case. The circular radii can be estimated as

$$r = \frac{XL^2 \pm XL^2(1 - 3r_s^2 Y/2X^2L^2)}{r_s Y},$$
$$r = \frac{2XL^2}{r_s Y}, \frac{3}{2}\frac{r_s}{X}. \tag{5.2.12}$$

When graviton mass goes to zero, these circular radii obtained in dRGT massive gravity theory can be reduced to those obtained in GR. Moreover, we can interpret the condition on the angular momentum as in the case of the GR by considering discriminant of Eq.(5.2.11). The minimum value of angular momentum in this case can be found as

$$L = \frac{\sqrt{3Y}}{X} r_s. \tag{5.2.13}$$

The radius of the circular orbit corresponding to this value of angular momentum can be written as

$$r = \frac{3r_s}{X} = \frac{6GM}{X}. \tag{5.2.14}$$

If $L < \sqrt{3Y} r_s/X$, a circular orbit will not occur. Therefore, $r = 6GM/X$ is smallest possible radius of stable circular orbit in dRGT massive gravity theory.

We can calculate the angular momentum of the particle which provides the result in dRGT massive gravity theory significantly different to the result in GR. That is

$$\frac{L^3}{r_s} \sim \frac{1}{m_g^2} \sim 10^{44}\ (eV)^{-2}. \tag{5.2.15}$$

Particularly, if L^3/r_s is considerably more than $10^{44}\ (eV)^{-2}$, we can use the corresponding particle to observe clear-cut effects due to the dRGT massive gravity theory.

We have now analyzed the dRGT massive gravity theory by considering an effective potential of a particle in order to compare the particle's trajectory obtained in dRGT massive gravity theory with one obtained in GR. In the first case, we consider the effective potential for the particle which has no angular momentum at the local gravity scale. This effective potential can be reduced to one obtained in GR. This case shows that in the massive gravity theory the particle which has no angular momentum will be attracted to the gravitational source similarly to what will occur in GR. The second case corresponding to high angular momentum situation while ϵ is very small so that $L^2\epsilon/r_v r_s \sim 0$, it is found that the dRGT massive gravity theory can explain the Mercury's trajectory as GR can. It implies that in this sense the dRGT massive gravity theory can recover the result obtained in GR in such a limit. The third case corresponds to very high angular momentum situation while ϵ is still very small so that $L^2\epsilon/r_v r_s \sim 1$. The effective potential has the correction terms involving the dRGT massive gravity parameters which can be tuned to fit with the observational data. We finally find the condition for the particle's angular momentum; namely $L^3/r_s > 10^{44}\ (eV)^{-2}$, by which the dRGT massive gravity theory prediction is significantly different from one predicted in GR.

CHAPTER VI

Conclusion

As we have reviewed and discussed, the linear massive gravity theory has been started by Fierz and Pauli in 1939. They introduced the mass terms to the the linearized Einstein-Hilbert action. The additional mass terms in massive gravity theory make the theory has five degrees of freedom in the Fierz-Pauli (FP) theory, instead of two degrees of freedom found in general relativity (GR). The FP massive gravity theory successfully describes massive spin-2 theory. In 1970 van Dam, Veltman, and Zakharov studied FP massive gravity theory by adding a symmetric source into the action. In Chapter III, we have derived the linearized equation of motion by varying the massive gravity action. We then obtained the solution of $h_{\mu\nu}$ by solving the linearized equation of motion. As a result, we calculated the light bending angle in the massive gravity theory and then compared the result to one obtained in GR. The result from linear massive gravity theory in massless limit differs from the one obtained in GR. Therefore, we have seen the appearance of discontinuity in this linear massive gravity theory called vDVZ discontinuity. The vDVZ discontinuity can be explained by considering non-linear massive gravity theory. This idea was proposed by Vainshtein in 1972. However, just about the same time, Boulware and Deser (BD) found that a broad class of non-linear massive gravity theories suffers from an instability. This instability is occured due to an additional scalar ghost mode in the non-linear FP massive gravity theory, called BD-ghost. It is the sixth degree of freedom in addition to 5 degrees of freedom for massive gravity theory. In Chapter IV, we presented some attempts trying to explain these problems. The explanation of the origin of this ghost can be found by using the effective field theory approach and introducing the

Stükelberg fields. In order to solve this problem, the action has to contain well-constructed non-linear mass terms so that the BD-ghost will not appear. Such an action was proposed by de Rham, Gabadadze, and Tolley in 2010. They found such mass terms of massive gravity theory which is known as dRGT massive gravity theory. We know that the general relativity can very well explain phenomena at local gravity scale such as those in our solar system. In Chapter IV, we have seen how Vainshtein mechanism can work in dRGT massive gravity theory. We consider this theory by imposing a static condition and spherical symmetry so that we obtained an analytic solution for dRGT massive gravity theory. At the local gravity scale the solution obtained in dRGT massive gravity theory can be reduced to one obtained in GR. It is found that this solution is expressed in terms of the parameters α, β, and, ϵ. This solution, Eq.(4.3.1) can be written as

$$ds^2 = -n(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

where $n(r) = f(r)$, and the function $f(r)$ in this solution can be written as

$$f(r) = 1 - \frac{2MG}{r} + \frac{\Lambda}{3}r^2 + \gamma r + \zeta,$$

where

$$\begin{aligned}
\Lambda =& 3m_g^2(1+\alpha+\beta), \\
\gamma =& -cm_g^2(1+2\alpha+3\beta), \\
\zeta =& c^2 m_g^2(\alpha+3\beta),
\end{aligned}$$

The terms in the function $(f(r))$ containing mass of graviton can be thought of as correction terms. Λ which is the coefficient in front r^2-term plays a role of cosmological constanct. This solution can give rise to known solutions in GR as follows. In the case of $m_g = 0$, we have the Schwarzschild solution. For $c = 0$ which makes $\gamma = \zeta = 0$, the solution can be classified according to the value of α and β. If $(1+\alpha+\beta) < 0$, the solution is in the form of Schwarzschild-de-Sitter while the case that $(1+\alpha+\beta) > 0$, on the other hand, provides the Schwarzschild-anti-de-Sitter solution. From the Vainshtein mechanism, a chacteristic radius can be found which is known as Vainshtein radius (r_v) where $r_v^3 = r_s/m_g^2$. This radius can distinguish the non-linear regime ($r \ll r_v$) from the linear regime ($r \gg r_v$) where the non-linear terms will be suppressed inside this radius. We then discussed this solution in order to consider the particle's trajectory and possible further studies in Chapter V. At the local gravity scale, we expected that the solution obtained in dRGT massive gravity theory must be reducible to one obtained in GR in order to explain the well-known phenomena. Therefore, we have considered the trajectory of an object around a gravitational source such as Mercury's trajectory in our solar system in order to show that the dRGT massive gravity theory can explain the phenomena at the local gravity scale as GR can. We have analyzed the particle's trajectory by using the effective potential. This effective potential can be determined from the corresponding geodesic equation with the help of the Killing vector theorem. The effective potential can be written as

$$V_{eff} = \frac{r_s}{2r_v}\Big[(1+\alpha+\beta)\Big(\frac{r}{r_v}\Big)^2 - (1+2\alpha+3\beta)\Big(\frac{r}{r_v}\Big) \\ - \big[1+(1+2\alpha+3\beta)\frac{L^2\epsilon}{r_v r_s}\big]\Big(\frac{r_v}{r}\Big) + \big[\epsilon(\alpha+3\beta)+1\big]\frac{L^2 r_v}{r_s r^2} - \frac{L^2 r_v}{r^3}\Big].$$

As a result, at the local gravity scale we then categorized the particle's motion into three interesting cases. In the first case, we considered the effective potential for the particle which has no angular momentum. This effective potential can be reduced to one obtained in GR. This case shows that in the massive gravity theory the particle which has no angular momentum will be attracted to the gravitational source similarly to what will occur in GR. The second case corresponds to high angular momentum situation while ϵ is very small so that $L^2\epsilon/r_v r_s \sim 0$. It is found that the dRGT massive gravity theory can explain the Mercury's trajectory as GR can. It implies that in this sense the dRGT massive gravity theory can recover the result obtained in GR in such a situation. The third case corresponds to very high angular momentum situation while ϵ is still very small so that $L^2\epsilon/r_v r_s \sim 1$. We can show this effective potential for this case again as follows

$$V_{eff} = \frac{r_s}{2r_v}\Big[-\big[1+(1+2\alpha+3\beta)\frac{L^2\epsilon}{r_v r_s}\big]\Big(\frac{r_v}{r}\Big) + \big[\epsilon(\alpha+3\beta)+1\big]\frac{L^2 r_v}{r_s r^2} - \frac{L^2 r_v}{r^3}\Big].$$

We can see that this effective potential has the correction terms involving the dRGT massive gravity parameters which can be tuned to fit with the observational data. We finally find the condition for the particle's angular momentum; namely $L^3/r_s > 10^{44}\ (eV)^{-2}$, by which the prediction of dRGT massive gravity theory is significantly different from one predicted in GR.

The further studies can possibly involve these correction terms such as finding constraint of the parameters in the theory from the observations. One can also perform the same analyzes on a more complicated system such as a system with a spinning or charged source. Though, this theory is not the theory of everything, it provides a plenty of insights on the nature of the universe. The massive gravity theory may be more useful than Einstein's general relativity theory in some situations. We now are waiting for the further evidences and knowledges to develop this modified gravity theory to be more accurate.

References

[1] A. G. Riess *et al.* [Supernova Search Team Collaboration], "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant," Astron. J. **116**, 1009 (1998) [astro-ph/9805201].

[2] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], "Measurements of Omega and Lambda from 42 high redshift supernovae," Astrophys. J. **517**, 565 (1999) [astro-ph/9812133].

[3] L. Amendola and S. Tsujikawa, *Dark Energy Theory and Observations,* United Kingdom: The Cambridge University Press, (2010)

[4] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, "Essentials of k essence," Phys. Rev. D **63**, 103510 (2001) [astro-ph/0006373].

[5] L. Tannukij and P. Wongjun, "Mass-Varying Massive Gravity with k-essence," Eur. Phys. J. C **76**, no. 1, 17 (2016) [arXiv:1511.02164 [gr-qc]].

[6] P. Wongjun, "A Perfect Fluid in Lagrangian Formulation due to Generalized Three-Form Field," (2016) [arXiv:1602.00682v1 [gr-qc]].

[7] T. Ngampitipan and P. Wongjun, "Dynamics of three-form dark energy with dark matter couplings," JCAP **1111**, 036 (2011) [arXiv:1108.0140 [hep-ph]].

[8] T. Koivisto and D. F. Mota, "Vector Field Models of Inflation and Dark Energy," JCAP **0808**, 021 (2008) [arXiv:0805.4229 [astro-ph]].

[9] T. P. Sotiriou and V. Faraoni, "f(R) theories of gravity," Rev. Mod. Phys. **82**, 451 (2010) [arXiv:0805.1726 [gr-qc]].

[10] A. De Felice and S. Tsujikawa, "f(R) Theories," Living Rev. Rel. **13**, 3 (2010) [arXiv:1002.4928 [gr-qc]].

[11] G. Abbas, D. Momeni, M. Aamir Ali, R. Myrzakulov, S. Qaisar, "Anisotropic Compact Stars in f(G) Gravity," Astrophys. Space Sci. **357**, no. 2, 158 (2015) [arXiv:1501.00427 [gr-qc]].

[12] R. Maartens and K. Koyama, "Brane-World Gravity," Living Rev. Rel. **13**, 5 (2010) [arXiv:1004.3962 [hep-th]].

[13] K. Atazadeh and H. R. Sepangi, "Modified gravity inspired DGP brane cosmology," Phys. Lett. B **643**, 76 (2006) [gr-qc/0610107].

[14] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[15] H. van Dam and M. J. G. Veltman, "Massive and massless Yang Mills and gravitational field," Nucl. Phys. B **22**, 397 (1970).

[16] V. I. Zakharov, "Linearized gravitation theory and the graviton mass," JETP Lett. **12**, 312 (1970) [Pisma Zh. Eksp. Teor. Fiz. 12, 447 (1970)].

[17] D. G. Boulware and S. Deser, "Can gravitation have a finite range?," Phys. Rev. D **6**, 3368 (1972).

[18] A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," Phys. Lett. B **39**, 393 (1972).

[19] C. de Rham and G. Gabadadze, "Generalization of the Fierz-Pauli Action," Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].

[20] C. de Rham, G. Gabadadze and A. J. Tolley, "Resummation of Massive Gravity," Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].

[21] C. de Rham and G. Gabadadze, Phys. Lett. B693, **334** (2010) [arXiv: 1006.4367 [hep-th]].

[22] K. Koyama, G. Niz and G. Tasinato, "Exact Solutions in Massive Gravity," Class. Quant. Grav. **30** 184002 (2013) [arXiv:1304.0601[hep-th]].

[23] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, "Is Cosmic Speed-Up Due to New Gravitational Physics?," Phys. Rev. D **70** 043528 (2004). [arXiv:0306438v2 [hep-th]].

[24] G. J. Olmo, "Post-Newtonian constraints on f(R) cosmologies in metric formalism," Phys. Rev. D **72** 083505, (2005). [arXiv:0505135v1 [hep-th]].

[25] G. J. Olmo, "The gravity lagrangian according to solar system experiments," Phys. Rev. Lett. **95** 261102 (2005). [arXiv:0505101v2 [hep-th]].

[26] L. Amendola, D. Polarski, S. Tsujikawa, "Are f(R) dark energy models osmologially viable?," Phys. Rev. Lett. **98** 131302 (2007). [arXiv:0603703v2 [hep-th]].

[27] A. De Felice and M. Hindmarsh, J. Cosmol, "Solar system constraints on f(G) gravity models," Astropart. Phys. 0706, 028 (2007). [arXiv:0907.1830 [hep-th]].

[28] A. De Felice, J. M. Gerard and S. Tsujikawa, "Cosmological perturbation in f(R, G) theories with a perfect fluid," Phys. Rev. D **81** 023532 (2010). [arXiv:0810.5712v2 [hep-th]].

[29] A. De Felice and S. Tsujikawa, "Construction of cosmologically viable f(G) gravity models," Phys. Lett. B **675** 1 (2009). [arXiv:0810.5712v2 [hep-th]].

[30] E. V. Pitjeva, Solar System Research **39** 176 (2005).

[31] B. Bertotti, L. Iess and P. Tortora, Nature **425** 374 (2003).

[32] A. Salam and J. A. Strathdee, Phys. Rev. D **16** (1977) 2668.

[33] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. D **84** 064033 (2011) [arXiv: 1104.2143 [hep-th]].

[34] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. Lett. **107** 131101 (2011) [arXiv:1103.4708 [hep-th]].

[35] F. Sbisa, G. Niz, K. Koyama and G. Tasinato, Phys. Rev. D **86** 024033 (2012) [arXiv:1204.1193 [hep-th]].

[36] D. Vegh, "Holography without translational symmetry," (2013) [arXiv: 1301.0537 [hep-th]].

[37] A. Adams, D. A. Roberts and O. Saremi, "Hawking-Page transition in holographic massive gravity," Phys. Rev. D **91**, no. 4, 046003 (2015) [arXiv: 1408.6560 [hep-th]].

[38] R. G. Cai, Y. P. Hu, Q. Y. Pan and Y. L. Zhang, "Thermodynamics of Black Holes in Massive Gravity," Phys. Rev. D **91** 024032 (2015) [arXiv: 1409.2369 [hep-th]].

[39] C. M. Will, "The Confrontation between general relativity and experiment," Living Rev. Rel. **9**, 3 (2006) [gr-qc/0510072].

[40] H. Kodama and I. Arraut, "Stability of the Schwarzschild-de Sitter black hole in the dRGT massive gravity theory," PTEP **2014**, 023E02 (2014) [arXiv: 1312.0370 [hep-th]]

[41] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalalva and A. J. Tolley, "Massive Cosmologies," Phys. Rev. D **84**, 124046 (2011) [arXiv:1108.5231 [hep-th]].

[42] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, "Open FRW universes and self-acceleration from nonlinear massive gravity," JCAP **1111**, 030 (2011) [arXiv:1109.3845 [hep-th]]

[43] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, "Cosmological perturbations of self-accelerating universe in nonlinear massive gravity," JCAP **1203**, 006 (2012) [arXiv:1111.4107 [hep-th]]

[44] T. Chollaphan, L. Tannukij and P. Wongjun, "Extended DBI massive gravity with generalized fiducial metric," JHEP **06**, 038 (2015) [arXiv:1502.08018 [gr-qc]]

[45] L. Berezhiani, G. Chakareneuli and G. Gabadadze, “Mixed Galileons and Spherically Symmetric Solutions,” Class. Quant. Grav. **30**, 184003 (2013) [arXiv:1305.0271 [hep-th]].

[46] S. M. Carroll, *Spacetime and geometry: An introduction to general relativity,* San Francisco, USA: Addison-Wesley (2004).

[47] A. F. Ali, S. Das, “Stringent theoretical and experimental bounds on graviton mass,” (2016) [arXiv:1605.05928v1 [gr-qc]]

[48] K. Hinterbichler “Theoretical Aspects of Massive Gravity,” Rev. Mod. Phys. **84**, 671 (2012) [arXiv:1105.3735 [hep-th]].

APPENDIX

APPENDIX A

Einstein's Linearized Perturbation

In this appendix, we discuss how to derive a linearized Einstein's action on curved space we refer to in chapter II.

A.1 The Metric Tensor

Starting to collect the second order perturbation from metric tensor. The metric tensor $\tilde{g}_{\mu\nu}$ obeys the relation

$$\tilde{g}^{\mu\nu}\tilde{g}_{\rho\nu} = \delta^{\mu}_{\nu}, \tag{A.1.1}$$

we have the metric tensor can be written as background and perturbed part as $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ and we assume the inverse metric tensor as

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + f^{\mu\nu}, \tag{A.1.2}$$

where $f^{\mu\nu}$ is any second order metric that need to find. we can rewrite Eq.(A.1.1) as

$$\begin{aligned} \left(g^{\mu\nu} - h^{\mu\nu} + f^{\mu\nu}\right)\left(g_{\rho\nu} + h_{\rho\nu}\right) &= \delta^{\mu}_{\nu}, \\ \delta^{\mu}_{\nu} - h^{\mu\rho}h_{\rho\nu} + f^{\mu}_{\nu} + f^{\mu\rho}h_{\rho\nu} &= \delta^{\mu}_{\nu}. \end{aligned}$$

We keep only second order term

$$\begin{aligned} f^{\mu}_{\nu} &= h^{\mu\rho}h_{\rho\nu}, \\ f^{\mu\nu} &= h^{\mu\rho}h^{\nu}_{\rho}. \end{aligned}$$

so, we obtain

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho}h^{\nu}_{\rho}. \tag{A.1.5}$$

Moreoever, we can evaluate the second order of $\sqrt{-\tilde{g}}$ by using Taylor expansion of $\sqrt{-\tilde{g}} = \sqrt{-\det(\tilde{g_{\mu\nu}})} = \sqrt{-\det(g_{\mu\nu} + h_{\mu\nu})} \equiv F(g_{\mu\nu} + h_{\mu\nu})$, that is

$$F(g_{\mu\nu} + h_{\mu\nu}) = F(g_{\mu\nu}) + \frac{1}{2}\frac{\partial F}{\partial \tilde{g}_{\mu\nu}}|_{\tilde{g}_{\mu\nu}=g_{\mu\nu}}.h_{\mu\nu} + \frac{1}{2}\frac{\partial^2 F}{\partial \tilde{g}_{\mu\nu}\partial \tilde{g}_{\alpha\beta}}|_{\tilde{g}_{\mu\nu}=g_{\mu\nu}}.h^{\mu\nu}h_{\alpha\beta} + ... \tag{A.1.6}$$

We will interprete the solution of each term, the first on is

$$\frac{\partial F}{\partial \tilde{g}_{\mu\nu}} = \frac{\partial\sqrt{-\tilde{g}}}{\partial \tilde{g}_{\mu\nu}}. \tag{A.1.7}$$

In order to evaluate Eq.(A.1.7), we have to introduce the formula as follows

$$\ln(\det(M)) = \mathrm{Tr}(\ln M), \tag{A.1.8}$$

where M is any metric tensor. When we take differentiate on this correlation, and M is g, we obtain

$$\begin{aligned}
\frac{1}{g}\delta g &= g^{\mu\nu}\delta g_{\mu\nu}, \\
\frac{\partial g}{\partial g_{\mu\nu}} &= g g^{\mu\nu}, \\
\frac{\partial\sqrt{-\tilde{g}}}{\partial \tilde{g}_{\mu\nu}} &= \frac{-1}{2\sqrt{-g}}\frac{\partial g}{\partial g_{\mu\nu}}, \\
&= \frac{-\sqrt{-g}}{2(-g)} g g^{\mu\nu}.
\end{aligned}$$

$$\therefore \frac{\partial\sqrt{-\tilde{g}}}{\partial \tilde{g}_{\mu\nu}} = \frac{1}{2}g^{\mu\nu}\sqrt{-g}, \tag{A.1.10}$$

By substituting Eq(A.1.10) into Eq(A.1.7) which firstly has been differentiated with respect to $\tilde{g}_{\alpha\beta}$, we obtain

$$\begin{aligned}
\frac{\partial^2 F}{\partial \tilde{g}_{\mu\nu}\partial \tilde{g}_{\alpha\beta}} &= \frac{1}{2}g^{\mu\nu}\frac{\sqrt{-g}}{\partial g_{\alpha\beta}} + \frac{1}{2}\sqrt{-g}\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}}, \\
&= \frac{1}{2}g^{\mu\nu}(\frac{1}{2}g^{\alpha\beta}\sqrt{-g}) + \frac{1}{2}\sqrt{-g}\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}}.
\end{aligned} \tag{A.1.11}$$

From identity of metric tensor, we have

$$g^{\mu\nu}g_{\mu\sigma} = \delta^{\nu}_{\sigma},$$
$$\frac{\partial}{\partial g_{\alpha\beta}}(g^{\mu\nu}g_{\mu\sigma}) = 0,$$
$$\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}}.g_{\mu\sigma} + g^{\mu\nu}\frac{\partial g_{\mu\sigma}}{\partial g_{\alpha\beta}} = 0.$$

Taking $g^{\gamma\sigma}$ on both sides of an above eqaution

$$\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}}\delta^{\gamma}_{\mu} = -g^{\mu\nu}g^{\gamma\sigma}\delta^{\mu}_{\alpha}\delta^{\sigma}_{\beta},$$
$$\frac{\partial g^{\gamma\nu}}{\partial g_{\alpha\beta}} = -g^{\alpha\nu}g^{\gamma\beta},$$
$$\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}} = -g^{\alpha\nu}g^{\mu\beta}. \tag{A.1.13}$$

By substituting Eq.(A.1.13) into Eq.(A.1.11), we obtain

$$\therefore \frac{\partial^2 F}{\partial\tilde{g}_{\mu\nu}\partial\tilde{g}_{\alpha\beta}} = \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\sqrt{-g} - \frac{1}{2}g^{\alpha\nu}g^{\mu\beta}\sqrt{-g}. \tag{A.1.14}$$

By substituting quation Eq.(A.1.7) and Eq.(A.1.14) into Eq.(A.1.6) to find out $\sqrt{-g}$,

$$\sqrt{-\tilde{g}} = \sqrt{-g} + \frac{1}{2}h\sqrt{-g} + \frac{1}{8}h^2\sqrt{-g} - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\sqrt{-g}. \tag{A.1.15}$$

A.2 The Christoffel Symbol

The Christoffel symbol, $\Gamma^{\rho}_{\mu\nu}$, which is the important structure to definespacetime curvature. It is defined as

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\lambda}(\partial_\mu\tilde{g}_{\lambda\nu} + \partial_\nu\tilde{g}_{\lambda\mu} - \partial_\lambda\tilde{g}_{\mu\nu}), \tag{A.2.1a}$$
$$= \Gamma^{\rho}_{\mu\nu} + \frac{1}{2}g^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) - \frac{1}{2}h^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}),$$
$$- \frac{1}{2}h^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu} + \frac{1}{2}h^{\rho\gamma}h^{\lambda}_{\gamma}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{A.2.1b}$$

We can use the covariant derivative of rank 2-tensor, $\nabla_\rho T_{\mu\nu} = \partial_\rho T_{\mu\nu} - \Gamma^\lambda_{\rho\mu} T_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} T_{\lambda\mu}$, to simplify the Christoffel symbol. We obtain

$$\nabla_\rho h_{\lambda\nu} = \partial_\sigma h_{\lambda\nu} - \Gamma^\rho_{\sigma\lambda} h_{\rho\nu} - \Gamma^\rho_{\sigma\nu} h_{\rho\lambda}, \tag{A.2.2a}$$

$$\nabla_\nu h_{\lambda\sigma} = \partial_\nu h_{\lambda\sigma} - \Gamma^\rho_{\nu\lambda} h_{\rho\sigma} - \Gamma^\rho_{\nu\sigma} h_{\rho\lambda}, \tag{A.2.2b}$$

$$\nabla_\lambda h_{\sigma\nu} = \partial_\lambda h_{\sigma\nu} - \Gamma^\rho_{\lambda\sigma} h_{\rho\nu} - \Gamma^\rho_{\lambda\nu} h_{\rho\sigma}. \tag{A.2.2c}$$

Eq.(A.2.2a)+Eq.(A.2.2b)-Eq.(A.2.2c), we get

$$\partial_\sigma h_{\lambda\nu} + \partial_\nu h_{\lambda\sigma} - \partial_\lambda h_{\sigma\nu} = \nabla_\sigma h_{\lambda\nu} + \nabla_\nu h_{\lambda\sigma} - \nabla_\lambda h_{\sigma\nu} - 2\Gamma^\rho_{\sigma\nu} h_{\rho\lambda}. \tag{A.2.3}$$

From Eq.(A.2.1b)

$$\begin{aligned}
\tilde{\Gamma}^\rho_{\mu\nu} =& \Gamma^\rho_{\mu\nu} + \frac{1}{2} g^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu} + 2\Gamma^{\rho\sigma\nu} h_{\rho\lambda}) - \frac{1}{2} h^{\rho\lambda}(2\Gamma^\rho_{\mu\nu} g_{\rho\lambda}), \\
&- \frac{1}{2} h^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu} + 2\Gamma^\rho_{\mu\nu} h_{\rho\lambda}) + \frac{1}{2} h^{\rho\gamma} h^\lambda_\gamma (2\Gamma^\rho_{\mu\nu} g_{\rho\lambda}), \\
=& \Gamma^\rho_{\mu\nu} + \frac{1}{2} g^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu}) - \frac{1}{2} h^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu}).
\end{aligned}$$

Therefore, we can rewrite the Christoffel symbol in terms of Christoffel symbol up to second order as follows

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^{\rho(0)}_{\mu\nu} + \Gamma^{\rho(1)}_{\mu\nu} + \Gamma^{\rho(2)}_{\mu\nu}, \tag{A.2.5}$$

where

$$\Gamma^{\rho(0)}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \tag{A.2.6a}$$

$$\Gamma^{\rho(1)}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu}), \tag{A.2.6b}$$

$$\Gamma^{\rho(2)}_{\mu\nu} = -\frac{1}{2} h^{\rho\lambda}(\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\lambda\mu} - \nabla_\lambda h_{\mu\nu}). \tag{A.2.6c}$$

A.3 The Riemann Curvature Tensor

The Riemann curvature tensor or Riemann-Christoffel tensor is the tensor which shows the curvature of Riemannian manifolds. We can calculate it by using definition as

$$\begin{aligned}
\tilde{R}^{\rho}_{\sigma\mu\nu} =&\partial_\mu \tilde{\Gamma}^{\rho}_{\sigma\nu} - \partial_\nu \tilde{\Gamma}^{\rho}_{\sigma\mu} + \tilde{\Gamma}^{\rho}_{\alpha\mu}\tilde{\Gamma}^{\alpha}_{\sigma\nu} - \tilde{\Gamma}^{\rho}_{\alpha\nu}\tilde{\Gamma}^{\alpha}_{\sigma\mu}, \qquad \text{(A.3.1)}\\
=&\partial_\mu \Gamma^{\rho(0)}_{\sigma\nu} + \partial_\mu \Gamma^{\rho(1)}_{\sigma\nu} + \partial_\mu \Gamma^{\rho(2)}_{\sigma\nu} - \partial_\nu \Gamma^{\rho(0)}_{\sigma\mu} - \partial_\nu \Gamma^{\rho(1)}_{\sigma\mu} - \partial_\nu \Gamma^{\rho(2)}_{\sigma\mu},\\
&+ \left(\Gamma^{\rho(0)}_{\alpha\mu} + \Gamma^{\rho(1)}_{\alpha\mu} + \Gamma^{\rho(2)}_{\alpha\mu}\right)\left(\Gamma^{\alpha(0)}_{\sigma\nu} + \Gamma^{\alpha(1)}_{\sigma\nu} + \Gamma^{\alpha(2)}_{\sigma\nu}\right),\\
&- \left(\Gamma^{\rho(0)}_{\alpha\nu} + \Gamma^{\rho(1)}_{\alpha\nu} + \Gamma^{\rho(2)}_{\alpha\nu}\right)\left(\Gamma^{\alpha(0)}_{\sigma\mu} + \Gamma^{\alpha(1)}_{\sigma\mu} + \Gamma^{\alpha(2)}_{\sigma\mu}\right),\\
=&R^{\rho}_{\sigma\mu\nu} + \partial_\mu \Gamma^{\rho(1)}_{\sigma\mu} - \partial_\nu \Gamma^{\rho(1)}_{\sigma\mu} + \partial_\mu \Gamma^{\rho(2)}_{\sigma\nu} - \partial_\nu \Gamma^{\rho(2)}_{\sigma\mu},\\
&+ \Gamma^{\rho(0)}_{\alpha\mu}\Gamma^{\alpha(1)}_{\sigma\nu} + \Gamma^{\rho(0)}_{\alpha\mu}\Gamma^{\alpha(2)}_{\sigma\nu} + \Gamma^{\rho(1)}_{\alpha\mu}\Gamma^{\alpha(0)}_{\sigma\nu} + \Gamma^{\rho(1)}_{\alpha\mu}\Gamma^{\alpha(1)}_{\sigma\nu} + \Gamma^{\rho(2)}_{\alpha\mu}\Gamma^{\alpha(0)}_{\sigma\nu},\\
&- \Gamma^{\rho(0)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\mu} - \Gamma^{\rho(0)}_{\alpha\nu}\Gamma^{\alpha(2)}_{\sigma\mu} - \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(0)}_{\sigma\mu} - \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\mu} - \Gamma^{\rho(2)}_{\alpha\nu}\Gamma^{\alpha(0)}_{\sigma\mu}. \qquad \text{(A.3.2)}
\end{aligned}$$

We can write the terms of partial derivatives in form of covariant derivatives by using definition of the covariant derivatives of a tensor shown as

$$\nabla_\mu T^{\mu_1}_{\nu_1\nu_2} = \partial_\mu T^{\mu_1}_{\nu_1\nu_2} + \Gamma^{\mu_1}_{\mu\lambda} T^{\lambda}_{\nu_1\nu_2} - \Gamma^{\lambda}_{\mu\nu_1} T^{\mu_1}_{\lambda\nu_2} - \Gamma^{\lambda}_{\mu\nu_1} T^{\mu_1}_{\lambda\nu_1}, \qquad \text{(A.3.3)}$$

By making the term of partial derivatives to be a subject, we obtain

$$\partial_\mu T^{\mu_1}_{\nu_1\nu_2} = \nabla_\mu T^{\mu_1}_{\nu_1\nu_2} - \Gamma^{\mu_1}_{\mu\lambda} T^{\lambda}_{\nu_1\nu_2} + \Gamma^{\lambda}_{\mu\nu_1} T^{\mu_1}_{\lambda\nu_2} + \Gamma^{\lambda}_{\mu\nu_1} T^{\mu_1}_{\lambda\nu_1}. \qquad \text{(A.3.4)}$$

Therefore, we obtain

$$\begin{aligned}
\partial_\mu \Gamma^{\rho(1)}_{\sigma\nu} =&\nabla_\mu \Gamma^{\rho(1)}_{\sigma\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda(1)}_{\sigma\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho(1)}_{\lambda\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho(1)}_{\lambda\sigma} \qquad \text{(A.3.5a)}\\
\partial_\nu \Gamma^{\rho(1)}_{\sigma\mu} =&\nabla_\nu \Gamma^{\rho(1)}_{\sigma\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda(1)}_{\sigma\mu} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho(1)}_{\lambda\mu} + \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho(1)}_{\lambda\sigma}. \qquad \text{(A.3.5b)}
\end{aligned}$$

Eq.(A.3.5a)-Eq.(A.3.5b), we have

$$\partial_\mu \Gamma^{\rho(1)}_{\sigma\nu} - \partial_\nu \Gamma^{\rho(1)}_{\sigma\mu} = \nabla_\mu \Gamma^{\rho(1)}_{\sigma\nu} - \nabla_\nu \Gamma^{\rho(1)}_{\sigma\mu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda(1)}_{\sigma\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho(1)}_{\lambda\nu} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda(1)}_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho(1)}_{\lambda\mu}. \qquad \text{(A.3.6)}$$

The second order terms are rearranged in the same way,

$$\begin{aligned}
\partial_\mu \Gamma^{\rho(2)}_{\sigma\nu} =&\nabla_\mu \Gamma^{\rho(2)}_{\sigma\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda(2)}_{\sigma\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho(2)}_{\lambda\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho(2)}_{\lambda\sigma}, \qquad \text{(A.3.7a)}\\
\partial_\nu \Gamma^{\rho(2)}_{\sigma\mu} =&\nabla_\nu \Gamma^{\rho(2)}_{\sigma\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda(2)}_{\sigma\mu} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho(2)}_{\lambda\mu} + \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho(2)}_{\lambda\sigma}. \qquad \text{(A.3.7b)}
\end{aligned}$$

Eq.(A.3.7a)-Eq.(A.3.7b), we have

$$\partial_\mu \Gamma^{\rho(2)}_{\sigma\nu} - \partial_\nu \Gamma^{\rho(2)}_{\sigma\mu} = \nabla_\mu \Gamma^{\rho(2)}_{\sigma\nu} - \nabla_\nu \Gamma^{\rho(2)}_{\sigma\mu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda(2)}_{\sigma\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho(2)}_{\lambda\nu} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda(2)}_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho(2)}_{\lambda\mu}. \qquad \text{(A.3.8)}$$

By substituting Eq.(A.3.6) and Eq.(A.3.8) into Eq.(A.3.2), we obtain

$$\tilde{R}^{\rho}_{\sigma\mu\nu} = R^{\rho}_{\sigma\mu\nu} + \nabla_{\mu}\Gamma^{\rho(1)}_{\sigma\nu} - \nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\mu} + \nabla_{\mu}\Gamma^{\rho(2)}_{\sigma\nu} - \nabla_{\nu}\Gamma^{\rho(2)}_{\sigma\mu} + \Gamma^{\rho(1)}_{\alpha\mu}\Gamma^{\alpha(1)}_{\sigma\nu} - \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\mu}. \tag{A.3.9}$$

This is the Riemann curvature tensor using to calculate the Ricci tensor which is represented in the next section.

A.4 The Ricci Curvature Tensor

The Ricci curvature tensor or Ricci tensor, it shows the way of measuring to which the geometry. The Ricci tensor is determined by a trace of Riemann tensor as follows

$$\tilde{R}_{\sigma\nu} = \tilde{R}^{\rho}_{\sigma\rho\nu}, \tag{A.4.1}$$

$$\tilde{R}_{\sigma\nu} = R_{\sigma\nu} + \nabla_{\rho}\Gamma^{rho(1)}_{\sigma\nu} - \nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho} + \nabla_{\rho}\Gamma^{\rho(2)}_{\sigma\nu} - \nabla_{\nu}\Gamma^{\rho(2)}_{\sigma\rho} + \Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}\sigma\rho.$$

we can rewrite the previous equation in background and pertuerbation part up to the second order as follows

$$\tilde{R}_{\sigma\nu} = R^{(0)}_{\sigma\nu} + R^{(1)}_{\sigma\nu} + R^{(2)}_{\sigma\nu}, \tag{A.4.2}$$

where

$$R^{(0)}_{\sigma\nu} = R_{\sigma\nu}, \tag{A.4.3a}$$

$$R^{(1)}_{\sigma\nu} = \nabla_{\rho}\Gamma^{rho(1)}_{\sigma\nu} - \nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho}, \tag{A.4.3b}$$

$$R^{(2)}_{\sigma\nu} = \nabla_{\rho}\Gamma^{\rho(2)}_{\sigma\nu} - \nabla_{\nu}\Gamma^{\rho(2)}_{\sigma\rho} + \Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - \Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}\sigma\rho. \tag{A.4.3c}$$

A.5 The Scalar Curvature

The scalar curvature or Ricci scalar, it is the Lagrangian density for Einstein-Hilbert action. It is defined as the trace of Ricci tensor. That is

$$\tilde{R} = \tilde{g}^{\sigma\nu}\tilde{R}_{\sigma\nu}, \tag{A.5.1}$$

and we know that

$$\tilde{g}^{\sigma\nu} = g^{\sigma\nu} - h^{\sigma\nu} + h^{\sigma\lambda}h^{\nu}_{\lambda}. \tag{A.5.2}$$

We now have

$$\begin{aligned}\tilde{R} =& R + g^{\sigma\nu}R^{(1)}_{\sigma\nu} + g^{\sigma\nu}R^{(2)}_{\sigma\nu} - h^{\sigma\nu}R_{\sigma\nu} - h^{\sigma\nu}R^{(1)}_{\sigma\nu} + h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu},\\ =& R + g^{\sigma\nu}R^{(1)}_{\sigma\nu} - h^{\sigma\nu}R_{\sigma\nu} + g^{\sigma\nu}R^{(2)}_{\sigma\nu} - h^{\sigma\nu}R_{\sigma\nu} + h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu}.\end{aligned}$$

We can rewrite it as

$$\tilde{R} = R^{(0)} + R^{(1)} + R^{(2)}, \tag{A.5.4}$$

where

$$R^{(0)} = R, \tag{A.5.5a}$$

$$\begin{aligned}R^{(1)} =& g^{\sigma\nu}R^{(1)}_{\sigma\nu} - h^{\sigma\nu}R_{\sigma\nu}\\ =& g^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(1)}_{\sigma\nu} - g^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho} - h^{\sigma\nu}R_{\sigma\nu},\end{aligned} \tag{A.5.5b}$$

$$\begin{aligned}R^{(2)} =& g^{\sigma\nu}R^{(2)}_{\sigma\nu} - h^{\sigma\nu}R_{\sigma\nu} + h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu},\\ =& g^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(2)}_{\sigma\nu} - g^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(2)}_{\sigma\rho} + g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\rho},\\ &- h^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(1)}_{\sigma\nu} + h^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho} + h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu}.\end{aligned} \tag{A.5.5c}$$

A.6 The Linearized Action

From Eq.(2.1.19), the second order perturbed action can be rewritten by substituting the results in Eq.(A.1.15), Eq.(A.3.9), and Eq.(A.5.4) as follows

$$\begin{aligned}S^{(2)} = \int \sqrt{-g}d^4x\Big[& g^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(2)}_{\sigma\nu} - g^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(2)}_{\sigma\rho} + g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha\rho},\\ &- h^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(1)}_{\sigma\nu} + h^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho} + h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu} + \frac{1}{2}hg^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(1)}_{\sigma\nu} - \frac{1}{2}hg^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho},\\ &- \frac{1}{2}hh^{\sigma\nu}R_{\sigma\nu} + \frac{1}{8}h^2R - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}R\Big].\end{aligned} \tag{A.6.1}$$

From Eq.(A.6.1), we can use the equation from background, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ or $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$, and then integrate by part of some terms in this above action. There are the surface terms resulting from integrating by part which we can set them to be zero by demanding that the variation vanished at the surface. As a serult, we can rewrite some terms as

$$
\begin{aligned}
h^{\sigma\lambda}h^{\nu}_{\lambda}R_{\sigma\nu} - \frac{1}{2}hh^{\sigma\nu}R_{\sigma\nu} + \frac{1}{8}h^2R - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}R =& \frac{1}{2}h^{\sigma\lambda}h^{\nu}_{\lambda}g_{\sigma\nu}R - \frac{1}{4}hh^{\sigma\nu}g_{\sigma\nu}R \\
&+ \frac{1}{8}h^2R - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}R, \\
=& \frac{1}{2}h^{\sigma\lambda}h_{\sigma\lambda}R - \frac{1}{4}hhR + \frac{1}{8}h^2R \\
&- \frac{1}{4}h^{\mu\nu}h_{\mu\nu}R, \\
=& \frac{1}{4}h^{\mu\nu}h_{\mu\nu}R - \frac{1}{8}h^2R, \\
=& \frac{1}{4}R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2). \qquad \text{(A.6.2)}
\end{aligned}
$$

We then integrate some terms in Eq.(A.6.1), we obtain

$$h^{\sigma\nu}\nabla_{\rho}\Gamma^{\rho(1)}_{\sigma\nu} = \nabla_{\rho}(h^{\sigma\nu}\Gamma^{\rho(1)}_{\sigma\nu}) - \nabla_{\rho}h^{\sigma\nu}\Gamma^{\rho(1)}_{\sigma\nu}, \qquad \text{(A.6.3)}$$

$$h^{\sigma\nu}\nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho} = \nabla_{\nu}(h^{\sigma\nu}\Gamma^{\rho(1)}_{\sigma\rho}) - \nabla_{\nu}h^{\sigma\nu}\Gamma^{\rho(1)}_{\sigma\rho}, \qquad \text{(A.6.4)}$$

$$\frac{1}{2}g^{\sigma\nu}h\nabla_{\rho}\Gamma^{\rho(1)}_{\sigma\nu} = \nabla_{\rho}(\frac{1}{2}g^{\sigma\nu}h\Gamma^{\rho(1)}_{\sigma\nu}) - \frac{1}{2}g^{\sigma\nu}\nabla_{\rho}h\Gamma^{\rho(1)}_{\sigma\nu}, \qquad \text{(A.6.5)}$$

$$\frac{1}{2}g^{\sigma\nu}h\nabla_{\nu}\Gamma^{\rho(1)}_{\sigma\rho} = \nabla_{\nu}(\frac{1}{2}g^{\sigma\nu}h\Gamma^{\rho(1)}_{\sigma\rho}) - \frac{1}{2}g^{\sigma\nu}\nabla_{\nu}h\Gamma^{\rho(1)}_{\sigma\rho}. \qquad \text{(A.6.6)}$$

By substituting the above equations into Eq.(A.6.1), we can rewrite Eq.(A.6.1) without the surface terms as follows

$$
\begin{aligned}
S = \int \sqrt{-g}d^4x \Big[&(\nabla_{\rho}h^{\sigma\nu} - \frac{1}{2}g^{\sigma\nu}\nabla_{\rho}h)\Gamma^{\rho(1)}_{\sigma\nu} + (\frac{1}{2}g^{\sigma\nu}\nabla_{\nu}h - \nabla_{\nu}h^{\sigma\nu})\Gamma^{\rho(1)}_{\sigma\rho} \\
&+ g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha(1)}_{\sigma\rho} + \frac{1}{4}R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2)\Big]. \qquad \text{(A.6.7)}
\end{aligned}
$$

In order to simplify this action, we can use definition of Christoffel symbol in Eq.(A.2.6a), Eq.(A.2.6b), and Eq.(A.2.6c) to substitute into Eq.(A.6.7). Then, the first, second, third, and forth terms in Eq.(A.6.7) can be rewritten as

$$\begin{aligned}(\nabla_\rho h^{\sigma\nu} - \frac{1}{2}g^{\sigma\nu}\nabla_\rho h)\Gamma^{\rho(1)}_{\sigma\nu} =&(\nabla_\rho h^{\sigma\nu} - \frac{1}{2}g^{\sigma\nu}\nabla_\rho h)\frac{1}{2}g^{\rho\lambda}(\nabla_\sigma h_{\lambda\nu} + \nabla_\nu h_{\lambda\sigma} - \nabla_\lambda h_{\sigma\rho}),\\ =&(\nabla^\lambda h^{\sigma\nu} - \frac{1}{2}g^{\sigma\nu}\nabla^\lambda h)(\nabla_\sigma h_{\lambda\nu} + \nabla_\nu h_{\lambda\sigma} - \nabla_\lambda h_{\sigma\rho}),\\ =&\frac{1}{2}\nabla^\lambda h^{\sigma\nu}\nabla_\sigma h_{\lambda\nu} + \frac{1}{2}\nabla^\lambda h^{\sigma\nu}\nabla_\nu h_{\lambda\sigma} - \frac{1}{2}\nabla^\lambda h^{\sigma\nu}\nabla_\lambda h_{\sigma\nu},\\ &-\frac{1}{4}\nabla^\lambda h\nabla^\nu h_{\lambda\nu} - \frac{1}{4}\nabla^\lambda h\nabla^\sigma h_{\lambda\sigma} + \frac{1}{4}\nabla^\lambda h\nabla_\lambda h. \quad \text{(A.6.8)}\end{aligned}$$

$$\begin{aligned}(\frac{1}{2}g^{\sigma\nu}\nabla_\nu h - \nabla_\nu h^{\sigma\nu})\Gamma^{\rho(1)}_{\sigma\rho} =&(\frac{1}{2}g^{\sigma\nu}\nabla_\nu h - \nabla_\nu h^{\sigma\nu})\frac{1}{2}g^{\rho\lambda}(\nabla_\sigma h_{\lambda\rho} + \nabla_\rho h_{\lambda\sigma} - \nabla_\lambda h_{\sigma\rho}),\\ =&(\frac{1}{2}g^{\sigma\nu}\nabla^\sigma h - \frac{1}{2}\nabla_\nu h^{\sigma\nu})(\nabla_\sigma h + \nabla^\lambda h_{\lambda\sigma} - \nabla^\rho h_{\sigma\rho}),\\ =&\frac{1}{4}\nabla^\sigma h\nabla_\sigma h + \frac{1}{4}\nabla^\sigma h\nabla_\lambda h_{\lambda\sigma} - \frac{1}{4}\nabla^\sigma h\nabla^\rho h_{\sigma\rho},\\ &-\frac{1}{2}\nabla_\nu h^{\sigma\nu}\nabla_\sigma h - \frac{1}{2}\nabla_\nu h^{\sigma\nu}\nabla^\lambda h_{\lambda\sigma} + \frac{1}{2}\nabla_\nu h^{\sigma\nu}\nabla^\rho h_{\sigma\rho}.\end{aligned} \quad \text{(A.6.9)}$$

By adding Eq.(A.6.8) with Eq.(A.6.9), we obtain the following equation,

$$\begin{aligned}(\nabla_\rho h^{\sigma\nu} - \frac{1}{2}g^{\sigma\nu}\nabla_\rho h)\Gamma^{\rho(1)}_{\sigma\nu} + (\frac{1}{2}g^{\sigma\nu}\nabla_\nu h - \nabla_\nu h^{\sigma\nu})\Gamma^{\rho(1)}_{\sigma\rho} = \frac{1}{2}\nabla^\sigma h\nabla_\sigma h - \nabla^\lambda h\nabla^\nu h_{\lambda\nu}\\ + \nabla^\lambda h^{\sigma\nu}\nabla_\sigma h_{\lambda\nu} - \frac{1}{2}\nabla^\lambda h^{\sigma\nu}\nabla_\lambda h_{\sigma\nu}.\end{aligned} \quad \text{(A.6.10)}$$

$$\begin{aligned}g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} =&\frac{1}{4}g^{\sigma\nu}g^{\rho\lambda}g^{\alpha\beta}(\nabla_\alpha h_{\lambda\rho} + \nabla_\rho h_{\lambda\alpha} - \nabla_\lambda h_{\alpha\rho})(\nabla_\sigma h_{\beta\nu} + \nabla_\nu h_{\beta\sigma} - \nabla_\beta h_{\sigma\nu}),\\ =&\frac{1}{4}(\nabla_\alpha h + \nabla^\lambda h_{\lambda\alpha} - \nabla^\rho h_{\alpha\rho})(\nabla_\sigma h^{\alpha\sigma} + \nabla_\nu h^{\alpha\nu} - \nabla^\alpha h),\\ =&\frac{1}{2}\nabla_\alpha h\nabla_\nu h^{\alpha\nu} - \frac{1}{4}\nabla_\alpha h\nabla^\alpha h. \quad \text{(A.6.11)}\end{aligned}$$

$$\begin{aligned}g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\rho} =&\frac{1}{4}g^{\sigma\nu}g^{\rho\lambda}g^{\alpha\beta}(\nabla_\alpha h_{\lambda\nu} + \nabla_\nu h_{\lambda\alpha} - \nabla_\lambda h_{\alpha\nu})(\nabla_\sigma h_{\beta\rho} + \nabla_\rho h_{\beta\sigma} - \nabla_\beta h_{\sigma\rho}),\\ =&\frac{1}{4}(\nabla_\alpha h^{\sigma\rho} + \nabla^\sigma h^\rho_\alpha - \nabla^\rho h^\sigma_\alpha)(\nabla_\sigma h_{\beta\rho} + \nabla_\rho h_{\beta\sigma} - \nabla_\beta h_{\sigma\rho}),\\ =&\frac{1}{4}(\nabla_\alpha h^{\sigma\rho}\nabla_\sigma h^\alpha_\rho + \nabla_\alpha h^{\sigma\rho}\nabla_\rho h^\alpha_\sigma - \nabla_\alpha h^{\sigma\rho}\nabla^\alpha h_{\sigma\rho} + \nabla^\sigma h^\rho_\alpha\nabla_\sigma h^\alpha_\rho\\ &+ \nabla^\sigma h^\rho_\alpha\nabla_\rho h^\alpha_\sigma - \nabla^\sigma h^\rho_\alpha\nabla^\alpha h_{\sigma\rho} - \nabla^\rho h^\sigma_\alpha\nabla_\sigma h^\alpha_\rho - \nabla^\rho h^\sigma_\alpha\nabla_\rho h^\alpha_\sigma\\ &+ \nabla^\rho h^\sigma_\alpha\nabla^\alpha h_{\sigma\rho}),\\ =&\frac{1}{2}\nabla^\alpha h^{\sigma\rho}\nabla_\sigma h_{\alpha\rho} - \frac{1}{4}\nabla^\rho h^{\sigma\alpha}\nabla_\rho h_{\alpha\sigma}. \quad \text{(A.6.12)}\end{aligned}$$

By substracting Eq.(A.6.11) with Eq.(A.6.12), we obtain the following equation,

$$\begin{aligned} g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\nu} - g^{\sigma\nu}\Gamma^{\rho(1)}_{\alpha\rho}\Gamma^{\alpha(1)}_{\sigma\rho} =& \frac{1}{2}\nabla_\alpha h\nabla_\nu h^{\alpha\nu} - \frac{1}{4}\nabla_\alpha h\nabla^\alpha h \\ & - \frac{1}{2}\nabla^\alpha h^{\sigma\rho}\nabla_\sigma h_{\alpha\rho}\frac{1}{4}\nabla^\rho h^{\sigma\alpha}\nabla_\rho h_{\alpha\sigma}. \end{aligned} \tag{A.6.13}$$

We can rewrite Eq.(A.6.7) by substituting Eq.(A.6.10) and Eq.(A.6.13) as follows

$$\begin{aligned} S =& \int \sqrt{-g}d^4x\big[\frac{1}{2}\nabla^\sigma h\nabla_\sigma h - \nabla^\lambda h\nabla^\nu h_{\lambda\nu} + \nabla^\lambda h^{\sigma\nu}\nabla_\sigma h_{\lambda\nu} \\ & - \frac{1}{2}\nabla^\lambda h^{\sigma\nu}\nabla_\lambda h_{\sigma\nu}\frac{1}{2}\nabla_\alpha h\nabla_\nu h^{\alpha\nu} - \frac{1}{4}\nabla_\alpha h\nabla^\alpha h - \frac{1}{2}\nabla^\alpha h^{\sigma\rho}\nabla_\sigma h_{\alpha\rho} + \frac{1}{4}\nabla^\rho h^{\sigma\alpha}\nabla_\rho h_{\alpha\sigma}\big], \\ =& \int \sqrt{-g}d^4x\big[\frac{1}{4}\nabla^\nu h\nabla_\nu h - \frac{1}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu} + \frac{1}{2}\nabla^\mu h^{\lambda\nu}\nabla_\lambda h_{\mu\nu} - \frac{1}{4}\nabla^\lambda h^{\mu\nu}\nabla_\lambda h_{\mu\nu} \\ & + \frac{1}{4}R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2)\big]. \end{aligned} \tag{A.6.14}$$

A.7 The Equations of Motion

From Eq.(A.6.14), we now write it again as follows

$$\begin{aligned} \int \sqrt{-g}d^4x\Big[&\frac{1}{2}\delta(\nabla^\mu h\nabla_\mu h) - \frac{1}{2}\delta(\nabla^\mu h\nabla^\nu h_{\mu\nu}) + \frac{1}{2}\delta(\nabla^\mu h^{\lambda\nu}\nabla_\lambda h_{\mu\nu}) \\ & -\frac{1}{4}\delta(\nabla^\lambda h^{\mu\nu}\nabla_\lambda h_{\mu\nu}) + \frac{1}{4}\delta R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2)\Big] = 0. \end{aligned} \tag{A.7.1}$$

We will expand the terms in Eq.(A.7.1). These terms can be expanded as

$$\begin{aligned} \delta(\frac{1}{4}\nabla^\mu h\nabla_\mu h) =& \frac{1}{4}\nabla^\mu \delta h.\nabla_\mu h + \frac{1}{4}\nabla^\mu h.\nabla_\mu \delta h, \\ =& \frac{1}{2}\nabla^\mu h.\nabla_\mu \delta h, \\ =& \nabla_\mu(\frac{1}{2}\nabla^\mu h.\delta h) - \frac{1}{2}\nabla_\mu\nabla^\mu h.\delta h, \\ =& \nabla_\mu(\frac{1}{2}\nabla^\mu h.\delta h) - \frac{1}{2}g^{\alpha\beta}\nabla_\mu\nabla^\mu h.\delta h_{\alpha\beta}, \end{aligned} \tag{A.7.2}$$

$$\begin{aligned} \delta(\frac{1}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu}) =& \frac{1}{2}\nabla^\mu \delta h.\nabla^\nu h_{\mu\nu} + \frac{1}{2}\nabla^\mu h.\nabla^\nu \delta h_{\mu\nu}, \\ =& \nabla^\mu(\frac{1}{2}\nabla^\nu h_{\mu\nu}\delta h) + \nabla^\nu(\frac{1}{2}\nabla^\mu h\delta h_{\mu\nu}), \\ & - \frac{1}{2}g^{\alpha\beta}\nabla^\mu\nabla^\nu h_{\mu\nu}\delta h_{\alpha\beta} - \frac{1}{2}\nabla^\alpha\nabla^\beta h\delta h_{\alpha\beta}, \end{aligned} \tag{A.7.3}$$

$$\begin{aligned}\delta(\frac{1}{2}\nabla^{\mu}h^{\lambda\nu}\nabla_{\lambda}h_{\mu\nu}) =& \frac{1}{2}\nabla^{\mu}\delta h^{\lambda\nu}.\nabla_{\lambda}h_{\mu\nu} + \frac{1}{2}\nabla^{\mu}h^{\lambda\nu}\nabla_{\lambda}\delta h_{\mu\nu},\\ =& \nabla^{\mu}(\frac{1}{2}\nabla_{\lambda}h_{\mu\nu}.\delta h^{\lambda\nu}) + \nabla_{\lambda}(\frac{1}{2}\nabla^{\mu}h^{\lambda\nu}.\delta h_{\mu\nu}),\\ &- \frac{1}{2}\nabla^{\mu}\nabla^{\lambda}h^{\nu}_{\mu}.\delta h_{\lambda\nu} - \frac{1}{2}\nabla_{\lambda}\nabla^{\mu}h^{\lambda\nu}.\delta h_{\mu\nu},\end{aligned} \tag{A.7.4}$$

$$\begin{aligned}\delta(\frac{1}{4}\nabla^{\lambda}h^{\mu\nu}\nabla_{\lambda}h_{\mu\nu}) =& \frac{1}{4}\nabla^{\lambda}\delta h^{\mu\nu}.\nabla_{\lambda}h_{\mu\nu} + \frac{1}{4}\nabla^{\lambda}h^{\mu\nu}.\nabla_{\lambda}\delta h_{\mu\nu},\\ =& \frac{1}{2}\nabla^{\lambda}h^{\mu\nu}\nabla_{\lambda}\delta h_{\mu\nu},\\ =& \nabla_{\lambda}(\frac{1}{2}\nabla^{\lambda}h^{\mu\nu}.\delta h_{\mu\nu}) - \frac{1}{2}\nabla_{\lambda}\nabla^{\lambda}h^{\lambda\nu}\delta h_{\mu\nu},\end{aligned} \tag{A.7.5}$$

$$\begin{aligned}\delta(\frac{1}{4}R(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^{2})) =& \frac{1}{4}\delta(Rh^{\mu\nu}h_{\mu\nu}) - \frac{1}{8}\delta(Rh^{2}),\\ =& \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\delta R + \frac{1}{4}Rh^{\mu\nu}\delta h_{\mu\nu} + \frac{1}{4}Rh_{\mu\nu}\delta h^{\mu\nu} - \frac{1}{8}\delta Rh^{2} - \frac{1}{8}R\delta h^{2},\\ =& \frac{R}{2}(h^{\mu\nu}\delta h_{\mu\nu} - g^{\alpha\beta}h\delta h_{\alpha\beta}).\end{aligned} \tag{A.7.6}$$

By substituting Eq.(A.7.2) to Eq.(A.7.6) into Eq.(A.7.1), we obtain

$$\begin{aligned}\int \sqrt{-g}d^{4}x\Big[&\frac{1}{2}g^{\alpha\beta}\Box h\delta h_{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}\nabla^{\mu}\nabla^{\nu}h_{\mu\nu}\delta h_{\alpha\beta} + \frac{1}{2}\nabla^{\alpha}\nabla^{\beta}h\delta h_{\alpha\beta} - \frac{1}{2}\nabla^{\mu}\nabla^{\lambda}h^{\nu}_{\mu}\delta h_{\lambda\nu}\\ &- \frac{1}{2}\nabla^{\lambda}\nabla^{\mu}h^{\nu}_{\lambda}.\delta h_{\mu\nu} + \frac{1}{2}\Box h^{\mu\nu}\delta h_{\mu\nu} + \frac{R}{2}(h^{\mu\nu}\delta h_{\mu\nu} - g^{\alpha\beta}h\delta h_{\alpha})\Big] = 0.\end{aligned}$$

$$\begin{aligned}\int \sqrt{-g}d^{4}x\Big[&-\frac{1}{2}g^{\mu\nu}\Box h + \frac{1}{2}g^{\mu\nu}\nabla^{\alpha}\nabla^{\beta}h_{\alpha\beta} + \frac{1}{2}\nabla^{\mu}\nabla^{\nu}h - \frac{1}{2}\nabla^{\alpha}\nabla^{\nu}h^{\mu}_{\alpha}\\ &- \frac{1}{2}\nabla^{\alpha}\nabla^{\mu}h^{\nu}_{\alpha} + \frac{1}{2}\Box h^{\mu\nu} + \frac{R}{2}(h^{\mu\nu} - g^{\mu\nu}h)\Big]\delta h_{\mu\nu} = 0.\end{aligned}$$

This integration is equal to zero so that the term in the bracket must be zero. We have

$$\begin{aligned}-\frac{1}{2}g^{\mu\nu}\Box h + \frac{1}{2}g^{\mu\nu}\nabla^{\alpha}\nabla^{\beta}h_{\alpha\beta} + \frac{1}{2}\nabla^{\mu}\nabla^{\nu}h - \frac{1}{2}\nabla^{\alpha}\nabla^{\nu}h^{\mu}_{\alpha} - \frac{1}{2}\nabla^{\alpha}\nabla^{\mu}h^{\nu}_{\alpha} + \frac{1}{2}\Box h^{\mu\nu}\\ +\frac{R}{2}(h^{\mu\nu} - g^{\mu\nu}h) = 0.\end{aligned} \tag{A.7.7}$$

By rearranging above equation, we obtain

$$\Box h^{\mu\nu} - \nabla^{\alpha}\nabla^{\nu}h^{\mu}_{\alpha} - \nabla^{\alpha}\nabla^{\mu}h^{\nu}_{\alpha} + g^{\mu\nu}\nabla^{\alpha}\nabla^{\beta}h_{\alpha\beta} + \nabla^{\mu}\nabla^{\nu}h - g^{\mu\nu}\Box h + \frac{R}{2}(h^{\mu\nu} - g^{\mu\nu}h) = 0. \tag{A.7.8}$$

Vitae

Mr. Supakorn Luekullaphawong was born in Songkla on 13 July 1989 and received his Bachelor's degree in physics from Mahidol University in 2011. He has studied astrophysics, field theory, and general relativity for his Master's degree. His research interests are in theoretical physics, particularly in the area of modified gravity for black hole solution.

Presentations

1. Oral presentation in Siam Physics Congress 2011, Phuket, Thailand, 23 - 26 March 2011
2. Oral presentation 5th Congress on Science and Technology for Youths, Bitec Bangna, Bangkok, 19 - 20 March 2010.
3. The Institute for Innovative Learning took part of Mahidol Wichakarn 2011, Mahidol University, Bangkok, 19 - 20 August 2011.

International Schools

1. The Siam GR+HEP+COSMO Symposium IV, Naresuan University, Thailand, 26 - 28 July 2009.
2. 4th Conference on Science and Technology for Youths, Bitec Bangna, Bangkok, 20 - 21 March 2009.