# วิธีการคำนวณรังสีฮอว์คิงแบบต่างๆ และฮอโลกราฟี

นาย ธัมม์ วุฒิชาญ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาฟิสิกส์ ภาควิชาฟิสิกส์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2558

# VARIOUS METHODS OF CALCULATING HAWKING RADIATION AND HOLOGRAPHY

Mr. Taum Wuthicharn

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Physics
Department of Physics
Faculty of Science
Chulalongkorn University
Academic Year 2015

| | |
|---|---|
| Thesis Title | VARIOUS METHODS OF CALCULATING HAWKING RADIATION AND HOLOGRAPHY |
| By | Mr. Taum Wuthicharn |
| Field of Study | Physics |
| Thesis Advisor | Piyabut Burikham, Ph.D. |

---

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.......................... Dean of the Faculty of Science
(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.......................... Chairman
(Assistant Professor Rattachat Mongkolnavin, Ph.D.)

.......................... Thesis Advisor
(Associate Professor Piyabut Burikham, Ph.D.)

.......................... Examiner
(Varagorn Hengpunya, Ph.D.)

.......................... External Examiner
(Tosaporn Angsachon, Ph.D.)

ธัมม์ วุฒิชาญ : วิธีการคำนวณรังสีฮอว์คิงแบบต่างๆ และ ฮอโลกราฟี (VARIOUS METHODS OF CALCULATING HAWKING RADIATION) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ. ดร. ปิยบุตร บุรีคำ, 69 หน้า

รังสีและอุณหภูมิฮอว์คิงนั้นมีวิธีการคำนวณอยู่หลายวิธี โดยแต่ละวิธีนั้น จะมีคำอธิบายสำหรับการเกิดรังสีที่แตกต่างกันอยู่เล็กน้อย อย่างไรก็ตาม แต่ละวิธีนั้นให้ผลการคำนวณที่เหมือนกันภายในขอบเขตบางช่วง ($T = \hbar c^3/8\pi GMk_B$) ซึ่งในวิทยานิพนธ์นี้ เราจะศึกษาวิธีการคำนวณรังสีฮอว์คิงที่แตกต่างกันอยู่สี่วิธี โดยเราจะเริ่มด้วยวิธีการที่เก่าแก่ที่สุด ซึ่งอาศัยปรากฏการณ์อุนรุ (Unruh Effect) ในการแสดงว่าผู้สังเกตการณ์ไกลมากจะรับรู้สถานะพื้นของผู้สังเกตการณ์ตกอิสระว่ามีรังสีความร้อนที่อุณหภูมิคงที่ หลักจากนั้นเราจะศึกษาอนุภาคเสมือนที่ผุดขึ้นมาในบริเวรขอบฟ้าเหตุการณ์แล้วหลบหนีจากหลุมดำผ่านปรากฏการณ์ลอดอุโมงควอนตัมและการเกิดรังสีฮอว์คิงจากอนุภาคเหล่านี้ ต่อมาเราจะใช้หลักความไม่แน่นอนในรูปแบบต่างๆ เพื่อหาอุณหภูมิฮอว์คิง โดยความไม่แน่นอนทางโมเมนตัมของอนุภาคเสมือนที่ผุดขึ้นมาบริเวรขอบฟ้าเหตุการณ์จะถูกใช้คำนวณหาพลังงานของอนุภาคเหล่านี้ และอุณหภูมิฮอว์คิงจะสามารถคำนวณได้จากพลังงานของอนุภาคที่ประกอบขึ้นมาเป็นรังสีฮอว์คิง นอกจากนี้เราจะทำการศึกษาสเกลเชิงมวลต่างๆ รวมไปถึงผลของสเกลต่อกระบวนการระเหยตัวของหลุมดำ และความเป็นไปได้ที่หลุมดำจะหยุดแผ่รังสีแล้วเหลือเศษบางอย่างไว้ อีกทั้งเรายังค้นพบว่าเอนโทรปีของเศษเหลือนี้จะมีคุณสมบัติตามฮอโลกราฟี โดยผลลัพธ์นี้จะเป็นจริงแม้ว่าเราจะใช้หลักความไม่แน่นอน แบบ Minimum Length Uncertainty Relations และขยายผลไปยังมิติ $D$ ใดๆ อย่างไรก็ตามหลุมดำแบบ MULRs จะมีคุณสมบัติของฮอโลกราฟีแค่ในช่วงที่มวลของหลุมดำมีค่ามาก หรือมีค่าเท่ากับเศษเหลือเท่านั้น ในท้ายที่สุดนี้เราจะแสดงว่ารังสีฮอว์คิงเป็นปรากฏการณ์ที่เกิดขึ้นเพื่อหักล้างความผิดปกติทางแรงโน้มถ่วง โดยการหักล้างนี้จะส่งผลให้ทฤษฎีสนามควอนตัมไม่ขัดแย้งกับทฤษฎีสัมพัทธภาพทั่วไป นอกจากนี้เราจะศึกษาถึงความเป็นไปได้ในการแก้พาราดอกซ์ข้อมูลสูญหายของแต่ละวิธีการคำนวณรังสีฮอว์คิง

ภาควิชา: . . . . . . . . . . ฟิสิกส์ . . . . . . . . . . ลายมือชื่อนิสิต . . . . . . . . . . . . . . . . . . . . . .

สาขาวิชา: . . . . . . . . . . ฟิสิกส์ . . . . . . . . . . ลายมือชื่อ อ.ที่ปรึกษาหลัก . . . . . . . . . . . . . .

ปีการศึกษา: . . . . . . . . 2558 . . . . . . .

## 567 19858 23 : MAJOR PHYSICS
KEYWORDS: HAWKING RADIATION/ HOLOGRAPHY/ HOLOGRAPHIC PRINCIPLE

TAUM WUTHICHARN : VARIOUS METHODS OF CALCULATING HAWKING RADIATION AND HOLOGRAPHY. THESIS ADVISOR : PIYABUT BURIKHAM, Ph.D., 69 pp.

There is a number of different methods to calculate Hawking radiation and Hawking temperature. Each method has its own slightly different interpretation of the emitted radiation. However, every method yields the same expression for the Hawking temperature under some approximation limits ($T = \hbar c^3/8\pi GMk_B$ in 4 dimensions). In this work, we will explore four of these methods, starting with the oldest derivation of Hawking radiation i.e., Unruh effect where the ground state or vacuum state of an observer free falling into a black hole becomes a radiation ensemble at fixed temperature for a distant observer. Hawking radiation is then being considered as a radiation composed of virtual particle escaping from vicinity of the horizon region via the quantum tunneling effect. Then, we derive Hawking radiation by utilizing various forms of uncertainty principle. A momentum uncertainty of virtual particles emerging near the horizon gives us the energy of the emitted particle which leads to the temperature of the radiation emitted from the black hole. Additionally, implications of various mass scales to the black hole evaporation process and the possibility of black hole stops radiating and becomes a remnant are being considered. The entropy of black hole remnant is found to be proportional to the surface area of the horizon in unit of the Planck area, a characteristic of holography. This statement holds even when the uncertainty relation is modified to the Minimum Length Uncertainty Relations (MLURs) and extended to arbitrary non-compact $D$-dimension. However, the entropy of black hole subjugated by MLURs possesses holography only at large mass and remnant limit. Lastly, Hawking radiation will be formulated as a cancellation term to the Einstein anomaly preserving the consistency of quantum field theory around the horizon region. We will also briefly discuss the potential solutions of information loss paradox from each method.

Department:.........Physics.........Student's Signature .....................

Field of Study:........Physics........Advisor's Signature .....................

Academic Year:..........2015.........

# Contents

**Chapter** **page**

# List of Figures

**Figure** **page**

# Chapter I

## Introduction

Hawking radiation is thermal radiation emitted from an event horizon of a black hole. We used to believe that every black hole has zero temperature. Since black hole has such an extreme gravitational pull that any thermal radiation from the black hole would simply be consumed back, effectively the black hole would have zero temperature. However, in 1974, a new mechanism that would allow the event horizon of the black hole to radiate thermal radiation was discovered by Stephen Hawking [1].

Although there are many methods of calculating Hawking radiation, most methods rely on the same physical interpretation; many pairs of virtual particles emerge near the event horizon. A single particle from each pair falls into the black hole while its partner escapes from the vicinity of the outer region of horizon and becomes a real particle. More detail may vary between each method.

Most methods agree that Hawking radiation is composed of completely random and featureless sub-atomic particles. The only differences between each black hole are its mass, angular momentum and electric charge. As a black hole gradually loses its mass over time via Hawking radiation, any information contained within the black hole will be lost forever. This process is called 'black hole evaporation process'. The loss of information in the black hole evaporation process is very problematic. According to the quantum mechanics, if we have a complete information of an isolated system at any given time, we will be able to correctly predict the state of that system at any other time. The time evolution of state is determined via a unitary operator. This property in the quantum mechanics is called 'unitarity'. It implies that the information of system must be conserved. If black holes are capable of destroying any information, then the 'law of conservation of information' would be broken. The violation of unitarity due to the quantum-gravity effect at the event horizon leads to a possibility of pure state evolving into a mixed state. However, such an evolution is impossible in the quantum mechanics. As such, we have a paradox which is called 'information loss paradox'. The paradox represents an inconsistency between general relativity and quantum mechanics. If the paradox can be resolved, it will be a good step toward the complete quantum gravity theory which is the reason why many physicists are interested in the paradox.

In quantum mechanics, there is a very important matrix called the S-matrix. This matrix describes the interaction between elementary particles. The determination of component in S-matrix relies heavily on the unitarity. Without the unitarity property, all known interactions in the quantum mechanics are in jeopardy. The unitarity breaking in the quantum mechanics was proposed in 1982 by Stephen Hawking. In order to describe interactions of elementary particle without the unitarity, a new $-matrix was proposed [2]. However, the new matrix predicts that quantum fluctuation in an empty space is capable of producing virtual microscopic black holes. The prediction implies that any empty space is capable of generating an incredible amount of heat in a fraction of second via the quantum fluctuation [3]. In order to preserve the unitarity in quantum mechanics, 't Hooft and Susskind had proposed a solution which states that the information of an infalling object would be scrambled and stored at the event horizon. Once a Hawking radiation particle is radiated from the horizon, it carries away some information stored on the horizon [4]. The idea that informations of any three-dimensional system are being stored on two-dimensional surface is called 'Holographic Principle' or 'Holography'.

There are three main types of black holes. The first one is a black hole without an angular momentum and possess a neutral electric charge. This type of black hole is called 'Schwarzschild black hole'. The second type is a black hole with a non-zero angular momentum, but with a neutral electric charge. This type is called 'Kerr black hole'. The third type is a black hole with a non-neutral electric charge, but with a zero angular momentum. This type is called 'Reissner-Nordstrom black hole'. The temperature of a Schwarzschild black hole in asymptoticly flat 4-dimensional spacetime is known to inversely proportional to the black hole mass. If a Schwarzschild black hole is colder than its ambient environment, it will absorb more energy than it radiates. The process increases the mass of the black hole. As a result, the black hole becomes colder, moving away from the thermal equilibrium. However, even if a black hole had consumed all masses in the universe, the black hole will have a very low, but non zero kelvin temperature. This fact is in accordance with the third law of black hole thermodynamics. On the other hand, if the black hole is hotter than its ambient environment, then the opposite is true. The black hole will keep getting hotter and radiating away its mass in the process which might lead to a complete evaporation. These facts also hold for all known types of black hole.

The possibility of black hole destroying information via the evaporation process is the main reason why the information loss paradox is a very glaring problem in the quantum mechanics. Supposed black holes are in very stable $final$ states. All informations that being consumed by black holes are stored within black holes forever. Even if we are unable to retrieve any information from black holes, the total information of our universe is still conserved. One possible solution to the paradox is the black hole remnant whereby every black hole will not undergo the complete evaporation process, but rather leaves behind a remnant. Even though we cannot retrieve any information from the remnant, the information would still be stored somewhere and thus preserves total information. Another possible solution is that Hawking radiation is not a featureless and completely random radiation i.e., Hawking radiation is not blackbody thermal radiation. In other words, the differences between each black hole are not merely its mass, angular momentum and electric charge. Here, the resolution of the paradox is rather straightforward [7].

In this thesis, we will generally use natural unit ($G = c = \hbar = k_B = 1$) unless states otherwise.

## 1.1 Properties of Black Hole

The following properties and components are unique to black holes. They will be referred throughout this thesis.

**Singularity:** An infinitely dense point mass locates at the center of every black hole where geodesic ends and spacetime curvature becomes infinite. Some theories suggest that the singularity contains all masses of the black hole. For a Schwarzschild black hole (no charge and non-rotating), according to these theories, all infalling masses would form a point mass with infinite density [1]. However, there are other theories suggest otherwise [21]. Since it is impossible to observe any event beyond the horizon, the existence of singularity is open for conjecture.

It is possible to avoid arriving the singularity of a Kerr or Reissner-Nordstrom black hole. However, once an infalling object crosses the event horizon, even if it escapes the singularity, it will emerge into a completely different universe. The aforementioned phenomenon implies the possibility of using black holes as wormholes to the parallel universe [1].

At the singularity, all known theories break down and no longer valid since the quantum effects and gravitational effects become equally important [5]. Although there have been many attempts to formulate a theory of quantum gravity, such a theory is still far from completion.

**Event Horizon:** A surface enveloping every black hole. A Light travels away from the black hole in a radial direction is trapped on this surface; unable to leave the surface in either away from or into the singularity direction.

If we drop an object into a black hole, that object will appear to be moving slower as it reaches the event horizon; effectively, it would take an infinite amount of time to actually touch the event horizon. Additionally due to gravitational redshift, the infalling object will appear to be redder in color and dimmer. However, if we are falling into a black hole, we will be unable to know whether or not we pass the event horizon. In other words, we will be unable to perceive the existence of event horizon and simply fall pass the horizon [1]. The aforementioned phenomenon stems from the fact that every event horizon locates at the 'coordinate singularity' which is a type of singularity emerging from a poor choice of coordinates. This type of singularity is removable by simply using better coordinates such as 'Painleve coordinates' [8].

As an example, we will find the location of event horizon for a Schwarzschild black hole whose horizon is in a perfect spherical shape. Let us start with Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2.$$

By setting $ds = 0$ for the light worldline and $d\Omega = 0$ for the light travels in radial direction, the light worldline becomes,

$$\left(1 - \frac{2M}{r}\right)^2 dt^2 = dr^2.$$

By being 'trapped' on the horizon, the light stays at the same location forever. That is, $dr \to 0$ and $dt \to \infty$, which implies that, $r = 2M$. This resultant radius is called 'Schwarzschild radius' ($R_S$). Although the definition of this surface is drastically difference from a conventional surface of any other astronomical object, it is, nevertheless, regarded as a surface of Schwarzschild black hole.

**Surface Gravity ($\kappa$):** Generally, surface gravity of an astronomical object is an acceleration of testing body at the surface of interested object. However, every black hole processes an infinitely high gravitational pull, it is only natural that every black hole has an infinite gravitational acceleration. Thus, a renormalized surface gravity is used, instead. This renormalized surface gravity is an acceleration of infalling object perceived by an observer whose location has sufficient distance from the black hole such that gravitational pull is negligible.

For a Schwarzschild black hole, we can calculate the surface gravity by multiply an acceleration of infalling object at the very moment it passes the event horizon with the gravitational redshift of interested black hole to obtain,

$$\kappa = \frac{1}{4M},$$

where $M$ is the mass of black hole [1]. The above parameter is regarded as a gravitational constant for the Schwarzschild black hole.

**Entropy:** Every object with a non-zero temperature must possesses an amount of entropy. As such, every black hole must has a non-zero entropy. However, before the discovery of Hawking radiation, we believed that every black hole have zero temperature and entropy. This property proves to be problematic, as every matter being consumed by any black hole must possess some amount of entropy. If every black hole truly processes a zero entropy, then it is possible to decrease the total entropy of the universe by simply allowing some amount of entropy to be consumed by any black hole. Even now after the discovery of Hawking radiation and temperature, the entropy of black hole is quite unusual.

In order to find the entropy of black hole, we start with the following equation,

$$dS = \frac{dQ}{T}.$$

Due to the immense gravitation of black hole, all heat energies that enter is stored as masses. It will be shown later on that the temperature of Schwarzschild black hole is equal to $1/8\pi M$. For now, the aforementioned fact is used to find that the entropy of black hole is,

$$dS = 8\pi M dM = d(4\pi M^2),$$
$$S = 4\pi M^2 = \pi R_S^2 = \frac{A}{4},$$

where $A$ is a surface area of the event horizon. The above equation tells us that the entropy of black hole is stored on the event horizon surface, not volume. According to the statistical mechanics, the entropy and the information of a system is highly related to each other. The aforementioned equation shows a characteristic of holography in the Schwarzschild black hole which roughly states that a complete information of system is being encoded on the boundary of that system.

## 1.2 Motivation and Thesis Outline

The purpose of this work is to explore the derivation of Hawking radiation from four different methods. Each method will present a unique physical interpretation of the radiation. Although each method presents a different emergence of the radiation, the resultant temperature is widely accepted to be equal to, $T \sim 1/2\pi M$. While every method relies on the same basis mechanics (Hawking radiation is composed of escaped virtual particles), each method still needs another mechanics to function properly. The fact that supplementary mechanics has no baring on the final result is rather intriguing.

Additionally, new mass scales have been discovered. This new mass scale seems to have an implication on black hole evaporation process and the existence of black hole remnant.

In the next chapter, Hawking radiation will be derived by utilizing Unruh effect phenomenon where ground state of accelerated observer becomes a thermal ensemble for an inertial observer. In chapter three, a virtual particle is found to be able to escape from the vicinity of the horizon region via quantum tunneling effect. This phenomenon leads to an emergence of hawking radiation. The temperature of Reissner-Nordstrom black hole will be calculated as well. In chapter four, various uncertainty principles will also be utilized to calculate Hawking temperature. New mass scales and their implication on black hole evaporation process will also be considered. Additionally, black hole lifetime and existence of black hole remnant will be explored. We have also found that the remnant from a MLUR type of black hole possesses the holography in arbitrary $D$-dimension. Lastly, in chapter five, Hawking radiation will be derived as a cancellation phenomenon of the gravitational anomaly. In addition, a potential solution to the information loss paradox from each method will be considered.

# Chapter II

## Unruh Effect

According to the special relativity, we know that the maximum speed of every object is the speed of light in vacuum. As a massive object undergoes a constant acceleration process, its speed becomes closer and closer to the speed of light in vacuum, but never reaches the ultimate speed. If we plotted the geodesic of that accelerated object, it will appear to be a hyperbola curve in spacetime diagram with light-like geodesic as its asymptotic line (as shown in Figure 2.1). Any event occurring in a region with time coordinate higher than the asymptotic line (Region A) cannot send any physical signal to the accelerated object. Naturally, if the object stopped accelerating, it will cross the asymptotic line and perceives those event. Because of this reason, the accelerated object perceives an apparent event horizon. This horizon is called a 'Rindler horizon'. Due to an effect where a ground state of inertial observer appears to be a thermal equilibrium state for an accelerated observer, this horizon is known to radiate thermal radiation. The aforementioned effect is called 'Unruh Effect'.

The physical interpretation of Unruh effect is as follows; the ground state of inertial observer contains many virtual particles. However, as another observer undergoes constant acceleration process, observes this ground state he or she is only able to observed some of the virtual particle due to the existence of apparent horizon. If a pair of virtual particles emerges at the vicinity of apparent horizon region, it is possible that a single virtual particle from the pair will cross the apparent horizon and 'lost' to the accelerated observer. Thus, the accelerated observer perceives the ground state of inertial observer to be a thermal equilibrium state.

The Rindler horizon is quite similar to the event horizon of black hole, albeit some physical differences (we will address later on). The Unruh temperature is proportional to the acceleration of constant accelerated observer. The proper acceleration of every object free falling into the black hole is equal to the surface gravity of black hole. By setting the acceleration of constant acceleration observer

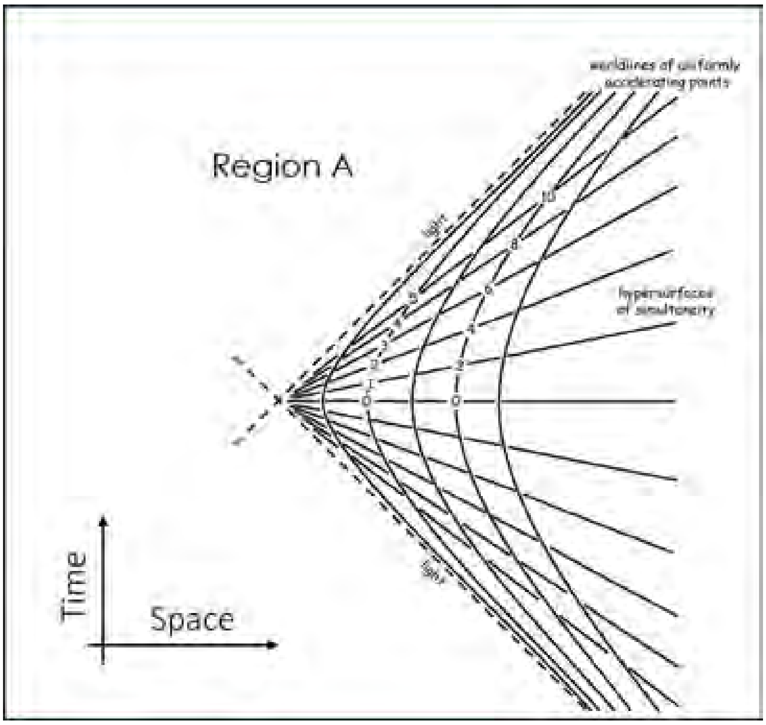


Figure 2.1: Space time diagram of constant acceleration observers (Rindler space).

equal to the surface gravity of black hole, the Unruh effect can be utilized to calculate Hawking radiation.

This method is one of the very first derivations of Hawking radiation. In this method, we utilize two observers. One observer is free falling into the black hole. For this observer, he or she will not be able to perceive the existence of black hole via gravitational pull or any physical experiment. The observer is also unable to perceive the existence of event horizon. In his or her view, all virtual particles emerge and completely destroy each other which makes his ground state well defined. Another observer is located far away sufficiently for any gravitational effect to be negligible. Once this 'distant' observer observes the ground state of 'free falling' observer, he or she will perceive a thermal radiation of perfect blackbody. This radiation is *Hawking radiation*. We will identify a temperature of blackbody which radiates the same amount of thermal radiation as the temperature of black hole [1].

For simplicity, we can assume that both observer and black hole are all aligning on the same straight line without losing generality. This assumption reduces our spacetime to 1+1 dimensional spacetime. For the free falling observer, he or she lives in a Minkowski spacetime whose metric is,

$$ds^2 = -dt^2 + dx^2.$$

Since he or she is free falling into the black hole, he or she has constant acceleration. The worldline of a constant acceleration object is,

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau), \tag{2.1a}$$

$$x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau), \tag{2.1b}$$

where $\tau$ is a proper time of free falling observer.

We will verify that this worldline has constant acceleration. The proper acceleration of any object is,

$$a^{\mu} = \frac{d^2}{d\tau^2}x^{\mu}.$$

From the above equation, we find the acceleration of free falling observer to be,

$$a^t = \alpha\sinh(\alpha\tau),$$

$$a^x = \alpha\cosh(\alpha\tau).$$

Hence, the total acceleration of free falling observer is,

$$\sqrt{a_{\mu}a^{\mu}} = \sqrt{-\alpha^2\sinh^2(\alpha\tau) + \alpha^2\cosh^2(\alpha\tau)} = \alpha.$$

The above equation clearly shows that the proper acceleration of the aforementioned worldline is, indeed, constant.

We will now construct new coordinates $(\eta,\, \xi)$,

$$t(\eta,\xi) = \frac{1}{a}e^{a\xi}\sinh(a\eta), \tag{2.3a}$$

$$x(\eta,\xi) = \frac{1}{a}e^{a\xi}\cosh(a\eta). \tag{2.3b}$$

The reason for establishment of these new coordinates $(\eta,\, \xi)$ is to take account of free falling observers with different initial condition. $\eta$ will act as though it is the proper time of each observer while $\xi$ contains the initial position of free falling observer. By reversing time, the free falling observers with a different non-zero velocity become zero-velocity observers whose initial position is difference. Hence, only $\xi$ is needed to describe every possible free falling observer.

However, these coordinates only describe observers traveling to the positive direction of $x$-axis. In order to describe observers traveling to the negative direction of $x$-axis, we add negative signs on both $x$ and $t$.

By comparing (2.1) and (2.3), we obtain the following relations,

$$\eta(\tau) = \frac{\alpha}{a}\tau, \tag{2.4a}$$

$$\xi(\tau) = \frac{1}{a}\log\left(\frac{a}{\alpha}\right). \tag{2.4b}$$

From the above relations, a new metric in these new coordinates is found to be as follows,

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2, \\
&= -e^{2a\xi}\sinh^2(a\eta)d\xi^2 - e^{2a\xi}\cosh^2(a\eta)d\eta^2 - 2e^{2a\xi}\sinh(a\eta)\cosh(a\eta)d\xi d\eta \\
&\quad +e^{2a\xi}\cosh^2(a\eta)d\xi^2 + e^{2a\xi}\sinh^2(a\eta)d\eta^2 + 2e^{2a\xi}\sinh(a\eta)\cosh(a\eta)d\xi d\eta, \\
&= e^{2a\xi}(-d\eta^2 + d\xi^2).
\end{aligned}
$$

The redshift factor in terms of these new coordinates is,

$$V = \sqrt{\mid g_{00} \mid} = e^{a\xi}.$$

We can calculate the surface gravity in these new coordinates from the redshift factor by the following equation [1],

$$\kappa = \sqrt{\nabla_\mu V \nabla^\mu V} = \sqrt{\partial_\mu V \partial^\mu V} = a. \tag{2.5}$$

Take note that this property is independent of $(\eta,\, \xi)$ which implies that the black hole has a constant surface gravity regardless of which coordinate we use. These new coordinates that we just finish constructed are belonged to the spacetime called *Rindler Space* (See Figure 2.1[6]). These coordinates are used by the free falling observer while the distant observer use $(t,\, x)$-coordinates.

Annihilation/creation operators for both observer will now be constructed. Considering virtual particles emerge near the event horizon, any particle that capable of escaping from the black hole at the vicinity of event horizon region must be traveling at near light speed. Thus, we may assume massless condition. Massless scalar field is,

$$\Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0,$$

with the following plane wave solutions,

$$g_k = \frac{1}{\sqrt{4\pi\omega}} e^{\pm i\omega\eta + ik\xi}. \tag{2.6}$$

The scalar field can be written as,

$$\phi = \int dk(\hat{b}_k^{(1)} g_k^{(1)} + \hat{b}_k^{(1)\dagger} g_k^{(1)*} + \hat{b}_k^{(2)} g_k^{(2)} + \hat{b}_k^{(2)\dagger} g_k^{(2)*}),$$

where $\hat{b}_k$ and $\hat{b}_k^\dagger$ are annihilation and creation operators respectively. These operators must have the following properties,

$$(g_{k_1}^{(i)}, g_{k_2}^{(j)}) = \delta_{ij}\delta(k_1 - k_2), \tag{2.7a}$$

$$\hat{b}_k^{(i)} \mid 0_R\rangle = 0, \tag{2.7b}$$

where $\mid 0_R\rangle$ is a ground state of free falling observer.

For $g_k^{(1)}$ which travels to the positive direction of $x$-axis, we have relationships between $(t,\, x)$-coordinates and $(\eta,\, \xi)$-coordinates as follows,

$$e^{-a(\eta-\xi)} = a(x-t), \tag{2.8a}$$

$$e^{a(\eta+\xi))} = a(x+t). \tag{2.8b}$$

Likewise, for $g_k^{(2)}$ which travels to the negative direction of $x$-axis, we have the relationships between $(t,\, x)$-coordinates and $(\eta,\, \xi)$-coordinates as follows,

$$e^{-a(\eta-\xi)} = -a(x-t), \tag{2.9a}$$

$$e^{a(\eta+\xi))} = -a(x+t). \tag{2.9b}$$

With above relations, we can now find annihilation/creation operators of distant observer in terms of operator from the free falling observer. Firstly, the plane wave solutions of free falling observer is needed to be described in $(t,\, x)$-coordinates,

$$\sqrt{4\pi\omega}g_k^{(1)} = e^{-i\omega\eta+ik\xi} = e^{-i\omega(\eta-\xi)} = a^{i\omega/a}(-t+x)^{i\omega/a}. \tag{2.10}$$

However, as $g_k^{(2)}$ is being described in $(t,\, x)$-coordinates, it becomes,

$$\sqrt{4\pi\omega}g_k^{(2)} = a^{-i\omega/a}(-t-x)^{-i\omega/a}.$$

The above form does not compatible with our previous form of $g_k^{(1)}$. In order to express $g_k^{(2)}$ in a form that compatible with that of $g_k^{(1)}$, considering $g_{-k}^{(2)*}$,

$$\sqrt{4\pi\omega}g_{-k}^{(2)*} = e^{-i\omega\eta+ik\xi} = a^{i\omega/a}(t-x)^{i\omega/a}.$$

The above form seems to be compatible with our form of $g_k^{(1)}$. However, further modification is needed,

$$\begin{aligned}
\sqrt{4\pi\omega}g_{-k}^{(2)*} &= a^{i\omega/a}(-1)^{i\omega/a}(-t+x)^{i\omega/a}, \\
&= a^{i\omega/a}e^{i\pi/a}(-t+x)^{i\omega/a}, \\
\sqrt{4\pi\omega}e^{-i\pi/a}g_{-k}^{(2)*} &= a^{i\omega/a}(-t+x)^{i\omega/a}. \qquad (2.11)
\end{aligned}$$

These results are now perfectly compatible. (2.10) and (2.11) are combined to obtain,

$$\sqrt{\pi\omega}(g_k^{(1)} + e^{-i\pi/a}g_{-k}^{(2)*}) = a^{i\omega/a}(-t+x)^{i\omega/a}.$$

With the above equation, we can now find plane wave solutions of the distant observer (whose coordinates are $(t,\, x)$) in terms of plane wave solutions of the free falling observer ($g_k^{(1)}$ and $g_k^{(2)}$). To simplify the relation, a factor $e^{\pi\omega/2a}$ is multiplied into the afore mentioned equation. Moreover, in order to preserve orthogonal property $(h_{k_1}^{(i)}, h_{k_2}^{(j)}) = \delta_{ij}\delta(k_1 - k_2)$, a factor $\frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}$ is also needed. Thus we get,

$$h_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}(e^{\pi\omega/2a}g_k^{(1)} + e^{-\pi\omega/2a}g_{-k}^{(2)*}), \tag{2.12a}$$

$$h_k^{(2)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}(e^{\pi\omega/2a}g_k^{(2)} + e^{-\pi\omega/2a}g_{-k}^{(1)*}). \tag{2.12b}$$

Additionally the massless scalar field for distant observer is known to be,

$$\phi = \int dk(\hat{c}_k^{(1)}h_k^{(1)} + \hat{c}_k^{(1)\dagger}h_k^{(1)*} + \hat{c}_k^{(2)}h_k^{(2)} + \hat{c}_k^{(2)\dagger}h_k^{(2)*}).$$

By utilizing Bogolubov transformation, the relationship between operators from both observer is found to be,

$$\hat{c}_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}(e^{\pi\omega/2a}\hat{b}_k^{(1)} + e^{-\pi\omega/2a}\hat{b}_{-k}^{(2)\dagger}), \tag{2.13a}$$

$$\hat{c}_k^{(2)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}(e^{\pi\omega/2a}\hat{b}_k^{(2)} + e^{-\pi\omega/2a}\hat{b}_{-k}^{(1)\dagger}). \tag{2.13b}$$

With these relations, it is possible to calculate the number of particles that the distant observer perceives on the ground state of free falling observer,

$$\begin{aligned}
\langle 0_R \mid \hat{n}^{(1)}(k) \mid 0_R\rangle &= \langle 0_R \mid \hat{c}_k^{(1)\dagger}\hat{c}_k^{(1)} \mid 0_R\rangle, \\
&= \frac{e^{-\pi\omega/a}}{2\sinh\left(\frac{\pi\omega}{a}\right)}\langle 0_R \mid \hat{b}_k^{(1)}\hat{b}_k^{(1)\dagger} \mid 0_R\rangle, \\
&= \frac{e^{-\pi\omega/a}}{2\sinh\left(\frac{\pi\omega}{a}\right)}\delta(0), \\
&= \frac{1}{e^{2\pi\omega/a} - 1}\delta(0).
\end{aligned}$$

By comparing the above result with the Planck spectrum, the above result is found to be thermal radiation of a perfect blackbody with temperature of $\frac{a}{2\pi}$. From (2.5), it is clear that,

$$T = \frac{\kappa}{2\pi}. \tag{2.14}$$

Note that, there is a $\delta(0)$ in our result because we did not normalize our plane wave solutions. this factor can be considered as a normalizing factor which does not affect our final result. Additionally, the Unruh temperature is observed by a constant acceleration observer which means the Unruh temperature should stem from the number of particle a constant acceleration observer (free falling observer) observes on the ground state of the inertial observer (distant observer). However in general relativity, it is the free falling observer who is actually the inertial observer. The acceleration of free falling observer stems from the curvature of spacetime not force. The distant observer, on the other hand, has a constant acceleration to the opposite direction of the black hole. The distant observer needs this constant acceleration in order to maintain his or her position. Interestingly, we can calculate the temperature an inertial observer perceives on the ground state of accelerated observer. The resultant temperature is the same which means the inertial observer must also perceive a thermal radiation ensemble of the same temperature from the ground state of accelerated observer.

This method is relatively simpler than other methods. However, the method only shows us that the ground state of constant acceleration observer would appear to be warm thermal radiation for the distant observer. In other words, any observer subjugated to a constant acceleration from any astronomical object would give us a similar result. Arguably, any astronomical object is capable of emitting some form of Hawking radiation. However, this radiation would only be observable by an inertial observer. The lack of distinction between which observer is able to observe the radiation needs further investigation. Each observer will be able to observe the radiation from the ground state of other observer. They, of course, observe their own ground state to be void of any real particle. As such, we need to establish which ground state is the background. In other words, as the observer tries to measure the temperature of his or her surrounding which ground state is being observed. For Unruh effect where both observer are in Minkowski space, the ground state of inertial observer is the background, which means that only accelerated observer is able to perceive the Unruh temperature. This result is only natural since the accelerated observer is one who perceives the apparent event horizon. For the case of an observer free falling into the black hole, however, the ground state of *apparent* acceleration observer (free falling observer) is the background. As stated earlier, the ground state of free falling observer is voided of event horizon and well defined. Additionally according to the general relativity, the free falling observer is an inertial observer. If we assume that the ground state of free falling observer is the background, then only distant observer is able to perceive Hawking temperature. Naturally, this assumption gives us a result in agreement with what we expect to happen; Hawking radiation is observable by our equipments located far away from the black hole.

The weakness of this method is that it does not reflect any quantum gravity nature of Hawking radiation. We derived this result in flat space background. This flat space assumption works because for the free falling observer, the spacetime is flat. As for the distant observer, the gravitational effect of black hole is negligible. As a result, we have completely ignored the gravitational nature of black hole. While the quantum nature of Hawking radiation is shown in this method, it does not seem to be working in combination with gravity. As we will be shown in other methods, Hawking radiation needs both quantum and gravitational effect in order to work properly. Nevertheless, this method shows us the possibility of Hawking radiation emitted from the event horizon.

This method have given us a very important message; if an observer perceives an event horizon i.e., a region where any physical signal within cannot be perceived by the observer, then that horizon will emit thermal radiation, but only observable by particular observers who share the same horizon. Even though Rindler horizon is emerged from the acceleration and black hole horizon is emerged from the curvature in spacetime, the origin of horizon itself have little to no effect on the calculation. As such, any form of event horizon is capable of radiating thermal radiation.

# Chapter III

## Quantum Tunneling

According to the previous chapter, Hawking radiation seems to be perfect blackbody thermal radiation which means the radiation only depends on mass, angular momentum and electric charge of black holes. This fact is rather problematic. As a black hole undergoes a complete evaporation process, this process can be regarded as the black hole becomes thermal radiation. All information contained within the black hole except information about its mass, angular momentum and electric charge, will be lost forever. In an attempt to resolve this issue, another theory proposes that Hawking radiation is stemmed from quantum tunneling effect. According to this theory, Hawking radiation has statistical nature. In other words, only average of Hawking radiation spectrum satisfies blackbody radiation. As such, it is now possible for Hawking radiation particle to carry some information of the black hole and potentially solve information loss paradox.

In this method, we will be working at region near the event horizon. As a pair of virtual particles emerges near the horizon, there is a chance that quantum tunneling effect will separate the pair, leaving one particle outside the horizon and the other particle inside (If the pair of virtual particles emerges outside the horizon, one of the virtual particles will tunnel across the horizon into the black hole. If the pair emerges inside the horizon, one of the particles will tunnel across the horizon away from the black hole). The infalling particle is usually antiparticle. By absorbing these particles, black hole loses some of its mass equal to the mass of escaping particles, effectively conserves total mass. Thus, the rate by which a particle tunnels away from a black hole is equal to decay rate of the black hole ($\Gamma$). As such, it is the possibility of lowering mass of the black hole that drives the evaporation process which supports an idea in quantum gravity that every black hole is in a highly excited states [7].

Since we are going to work in an area around the horizon, it is necessary to eliminate the coordinate singularity at the horizon. The singularity is removed by introducing a new time coordinate,

$$t = t_s + 2\sqrt{2Mr} + 2M \log \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}},$$

where $t_s$ is the Schwarzschild time [8].

With these new coordinates, our line element becomes,

$$\begin{aligned} dt &= dt_s + \sqrt{\frac{2M}{r}} dr + \frac{M}{\sqrt{r}} \left( \frac{1}{\sqrt{r} - \sqrt{2M}} - \frac{1}{\sqrt{r} + \sqrt{2M}} \right) dr, \\ &= dt_s + \sqrt{\frac{2M}{r}} dr + \frac{2M\sqrt{2M}}{\sqrt{r}(r - 2M)} dr, \\ &= dt_s + \sqrt{\frac{2M}{r}} \left(1 - \frac{2M}{r}\right)^{-1} dr. \\ \therefore dt_s &= dt - \sqrt{\frac{2M}{r}} \left(1 - \frac{2M}{r}\right)^{-1} dr. \end{aligned}$$

For simplicity, all particles are assumed to travel in radial direction. This assumption is possible because every particle whose position is at the near horizon region must travel in almost perfect radial direction in order to avoid being pulled into the black hole. In addition, as particles are tunneling across the horizon, it will not emerge far from its original position. As such, even if the particles do not cross the horizon in the radial direction, the deviation is negligible. With this assumption, the $r^2 d\Omega^2$ term is omitted. The new metric under these new coordinates is,

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) \left( dt - \sqrt{\frac{2M}{r}} \left(1 - \frac{2M}{r}\right)^{-1} dr \right)^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2, \\ &= \left(1 - \frac{2M}{r}\right) dt^2 - 2\sqrt{\frac{2M}{r}} dt dr + \frac{2M}{r} \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2. \end{aligned}$$

Hence, our new line element is,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - 2\sqrt{\frac{2M}{r}} dt dr - dr^2 - r^2 d\Omega^2. \tag{3.1}$$

These new coordinates are called *Painleve coordination*.

For any particle to escape a black hole from the vicinity of horizon region, it must be travel at the near light speed in radial direction. Thus, we assume $ds = 0$,

$$\begin{aligned} 0 = ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - 2\sqrt{\frac{2M}{r}} dt dr - dr^2, \\ &= \left( \left(1 - \sqrt{\frac{2M}{r}}\right) dt - dr \right) \left( \left(1 + \sqrt{\frac{2M}{r}}\right) dt + dr \right). \\ \therefore \frac{dr}{dt} &= \pm 1 - \sqrt{\frac{2M}{r}}. \end{aligned}$$

The positive sign in the above equation represents geodesic of the outgoing particle and the negative sign represents geodesic of the ingoing particle.

The above equation will be modified to account for self-gravitational effect of escaping particles. Let the outgoing particle (normal particle) possesses energy of '$\omega$' whilst the ingoing particle (anti-particle) possesses energy of '$-\omega$'. By regrading virtual particles as two mass shells of energy, outgoing particle perceives the black hole with a mass of $M-\omega$ while ingoing particle perceives the universe under an influent of mass $M+\omega$ [9].

Firstly, considering the case of outgoing particle,

$$\frac{dr}{dt} = 1 - \sqrt{\frac{2(M-\omega)}{r}}. \tag{3.2}$$

By using WKB approximation, the imaginary part of action (Im$S$) of an outgoing particle as it crosses the horizon is,

$$\mathrm{Im}S = \mathrm{Im}\int_{r_{in}}^{r_{out}} p_r dr = \mathrm{Im}\int_{r_{in}}^{r_{out}}\int_0^{p_r} dp'_r dr.$$

According to the Hamilton's equation,

$$\begin{aligned} H &= p_r\dot{r} - L. \\ \therefore \dot{r} &= \frac{dH}{dp_r}. \end{aligned}$$

From the above equation, we can change variable from a momentum to an energy. Since all energies of black hole are in the form of mass, integration limits are initial mass and final mass. Additionally, $H = M - \omega'$, which leads us to, $dH = -d\omega'$. By using $dp_r = \frac{dH}{\dot{r}}$, the imaginary part of action can be modified into the following form,

$$\begin{aligned} \mathrm{Im}\int_{r_{in}}^{r_{out}}\int_0^{p_r} dp'_r dr &= \mathrm{Im}\int_{r_{in}}^{r_{out}}\int_M^{M-\omega}\frac{dH}{\dot{r}}dr, \\ &= \mathrm{Im}\int_0^{+\omega}\int_{r_{in}}^{r_{out}}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1} dr(-d\omega'). \end{aligned}$$

At the last step, we were using (3.2). For simplicity, the term, $\int\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1} dr$, will be calculated first.

Before a black hole absorbs anti-particle of energy $(-\omega)$, mass of the black hole was $M$. After the absorption, mass of the black hole becomes $(M-\omega)$. Hence, $r_{in}$ (the horizon radius before the absorption) is $2M$ while $r_{out}$ (the horizon radius after the absorption) is $2(M-\omega)$. This fact leads us to a relation, $r_{in} > r_{out}$.

Additionally, there must be some value of $r$ between $r_{in}$ and $r_{out}$ which satisfies, $1-\sqrt{\frac{2(M-\omega')}{r}}=0$. Let $r_o$ denotes such value.

From Cauchy's integral formula,

$$2\pi i f(a)=\oint \frac{f(z)}{z-a}dz.$$

However, our integration is on a straight line or half of a close contour,

$$\pi i f(a)=\int_{a-\varepsilon}^{a+\varepsilon} \frac{f(z)}{z-a}dz.$$

Applied the above formula, our $\int_{r_{out}}^{r_{in}}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}dr$ term becomes,

$$\begin{aligned}
&= \text{Im}\int_{r_{out}}^{r_o-\varepsilon}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}dr+\text{Im}\int_{r_o-\varepsilon}^{r_o+\varepsilon}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}dr \\
&\quad +\text{Im}\int_{r_{o+\varepsilon}}^{r_{in}}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}dr, \\
&= \text{Im}\int_{r_o-\varepsilon}^{r_o+\varepsilon}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}dr.
\end{aligned}$$

The first and third terms contain no singularity. Even though we did not fully calculate these terms, we know that the results must be real numbers. A new variable, $r'=\frac{r}{2(M-\omega')}$, is needed to further solve our integration. Applied this variable into our integration term, the second term, $\text{Im}\int_{r_o-\varepsilon}^{r_o+\varepsilon}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}dr$, becomes,

$$\begin{aligned}
I &= 2(M-\omega')\text{Im}\int_{1-\varepsilon}^{1+\varepsilon}\left(1-\frac{1}{\sqrt{r'}}\right)^{-1}dr', \\
&= 2(M-\omega')\text{Im}\int_{1-\varepsilon}^{1+\varepsilon}\frac{\sqrt{r'}dr'}{\sqrt{r'}-1}, \\
&= 2(M-\omega')\text{Im}\int_{1-\varepsilon}^{1+\varepsilon}\frac{\sqrt{r'}(\sqrt{r'}+1)dr'}{r'-1}, \\
&= 4\pi(M-\omega').
\end{aligned} \tag{3.3}$$

Therefore, the imaginary part of our action is,

$$\begin{aligned}
\text{Im}S &= \text{Im}\int_0^{+\omega}\int_{r_{out}}^{r_{in}}\left(1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1}drd\omega', \\
&= 4\pi\int_0^{+\omega}(M-\omega')d\omega', \\
&= 4\pi\omega\left(M-\frac{\omega}{2}\right).
\end{aligned} \tag{3.4}$$

However, the above result is only from the case where a particle is tunneling away from the black hole. Another result from a case where an anti-particle is tunneling into the black hole must also be calculated.

In this case, we consider a particle which travels back in time. According to (3.1), the time reversal will only result in, $\sqrt{\frac{2M}{r}} \to -\sqrt{\frac{2M}{r}}$, which changes (3.2) into, $\dot{r} = \pm 1 + \sqrt{\frac{2M}{r}}$. Additionally, for an anti-particle, the universe will be under the influence of mass $M+\omega$. While the Hamiltonian's equation is still, $H = M-\omega$, this equation is modified into, $H = M + \omega'$, where $\omega'$ ranges from 0 to $-\omega$. Hence, the imaginary part of action for the anti-particle is,

$$
\begin{aligned}
\mathrm{Im}S &= \mathrm{Im}\int_0^{-\omega}\int_{r_{out}}^{r_{in}} \left(-1+\sqrt{\frac{2(M+\omega')}{r}}\right)^{-1} dr d\omega', \\
&= \mathrm{Im}\int_0^{+\omega}\int_{r_{out}}^{r_{in}} \left(1-\sqrt{\frac{2(M-\omega'')}{r}}\right)^{-1} dr d\omega'', \\
&= \mathrm{Im}\int_0^{+\omega}\int_{r_{in}}^{r_{out}} \left(1-\sqrt{\frac{2(M-\omega'')}{r}}\right)^{-1} dr(-d\omega'').
\end{aligned}
$$

The above term is exactly the same as what we had in our previous case. Therefore, the imaginary part of action for the anti-particle is also,

$$
\mathrm{Im}S = 4\pi\omega\left(M - \frac{\omega}{2}\right). \tag{3.5}
$$

The amplitudes from both channels (particle and anti-particle) are needed to be square before they can be added together. In a more detailed calculation, this statement means that the term Re$S$ is needed to be calculate. However, such detailed only affects the pre-factor. According to the WKB approximation, the exponential part of the semi-classical emission rate in either case is,

$$
\Gamma \sim e^{-2\mathrm{Im}S} = e^{-8\pi\omega\left(M-\frac{\omega}{2}\right)}. \tag{3.6}
$$

From Planck's spectrum, $\Gamma \sim \frac{1}{e^{\omega/T}-1} \sim e^{-\omega/T}$, we can read off temperature from (3.6),

$$
T = \frac{1}{8\pi}\left(M - \frac{\omega}{2}\right)^{-1}. \tag{3.7}
$$

Previously in 'Unruh Effect' chapter, we obtained, $T = 1/8\pi M$. Our 'Quantum Tunneling' method seems to be more accurate. A correction term, $\omega^2$, emerges from self-gravitational effect of Hawking radiation. This term also preserves the conservation of total mass and energy. Generally, an energy of radiation will be very small in comparison to that of black hole and can be neglected. However, as a black hole becomes hotter than its ambient temperature and starts the evaporation process, this correction term will start to be of significant. If the above resultant temperature formula still holds until the last moment of black hole life, $\omega$ will be in the same order with that of $M$. The temperature of black hole as it radiates its last radiation will no longer equal to $1/8\pi\epsilon$, but instead equal to $1/4\pi\epsilon$, where $\epsilon$ is an energy of the black hole at its final moment which should be in order of sub-atomic particle energy.

In this method, Hawking radiation emerges from quantum tunneling effect. Interestingly, while this method still relies on virtual particles to explain physical interpretation of Hawking radiation, a region where virtual particles can escape from the black hole in this method have to be wider than the previous method. However, if we redo the calculation and neglect the self- gravitational effect, the resultant temperature will be the same as that of previous method. This result suggests that there might be something wrong with our interpretation of Hawking radiation in the previous chapter. Perhaps, even though, we did all the calculations without any regard for the quantum tunneling effect, the underlying mechanics of Unruh temperature need quantum tunneling in order to function properly.

According to the quantum mechanics, if any particle is tunneling, it does so across a barrier or classically forbidden region. However, if we consider black hole gravitational potential, there is no such barrier exists [10]. The source of barrier is the Hawking radiation particle itself. As a black hole adsorbs the negative energy particle, it loses some of its energy and shrinks. This act of horizon shrinking effect is the source of barrier [11]. As the horizon is shrinking, it causes the potential energy of black hole to shift its value in $r$-coordinate. This shifting effect appears to be a barrier for the particle which is also why we set our integration limits from $r_{in} = 2M$ to $r_{out} = 2(M - \omega)$.

Moreover in this method, the mechanics which drives the radiation process is the possibility of lowering the black hole mass. At first glance, it seems we have only accounted for quantum nature of the black hole and neglect its gravitational nature. However, if we look closely at our calculation steps, the pole which gives us the imaginary part of action is corresponding to the location of horizon. Interestingly, this statement means that only quantum tunneling effect at vicinity of the horizon gives us the emission rate which in turn gives us the Hawking radiation. This fact shows us that Hawking radiation is emerged from both quantum and gravity effect combine.

From statistical mechanics, the Planck spectral flux of a gray body is,

$$\rho(\omega) = \frac{d\omega}{2\pi} \frac{|T(\omega)|^2}{e^{\omega/T} - 1},$$

where $|T(\omega)|^2$ is the gray body frequency dependent transmission coefficient. This equation tells us that the pre-factor from real part of action effects the transmission coefficient. In a more detailed calculation, the other properties of Schwarzschild black hole will affect $|T(\omega)|^2$ which suggests how we might be able to extract information from black holes. Additionally, the $\omega^2$ term in our resultant temperature (3.7) strongly suggests that Hawking temperature is no longer perfect blackbody temperature.

Moreover the resultant emission rate from (3.6) can be written as,

$$\Gamma \sim e^{8\pi M\omega} \sim e^{\Delta S},$$

where $S$ is a entropy of the black hole. The above equation is modified under the assumption that the black hole temperature is constant over the emission process of a single Hawking radiation particle. The black hole loses energy ($\Delta Q$) of $\omega$ in the process. As such, the entropy change in this process is, $\Delta Q/T = 8\pi M\omega$. The above equation can be interpreted as follows; the decay rate is also the possibility of black hole evolving into another state. There are $e^S$ states in total. The possibility of finding a shell containing all the mass of black hole is proportional to $e^{-S}$. This interpretation shows us that the evaporation process can be regarded as state evolution in statistical mechanics.

The fact that Hawking radiation has statistical nature means that we might be able to extract informations from black holes via Hawking radiation. This statement is the most interesting idea that we learn from this method.

## 3.1 A Generalization to Charged Black Hole

In this section, we will repeat the previous calculation for a Reissner-Nordstrom black hole. However, if the radiation has electric charge, then we need to account for electromagnetic forces which would further complicate our calculation. As such, we assume that the radiation has neutral electric charge. Another crucial different is that for a Reissner-Nordstrom black hole, we have two event horizons as can be seen from equation, $1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$, which has two solutions. Only the outer horizon contributes to Hawking radiation. According to the Penrose diagram any radiation from inner horizon emerges in a new universe.

In order to calculate Hawking radiation of Reissner-Nordstrom black hole, we need to switch Schwarzschild background with Reissner-Nordstrom background,

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt_{R-N}^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2. \quad (3.8)$$

The coordinate singularity at the horizon is eliminated by using the following time coordinate,

$$dt_{R-N} = dt - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2.$$

Hence, the charge *Painleve* line element is,

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} dtdr - dr^2 - r^2 d\Omega^2. \quad (3.9)$$

This equation has almost the same form as (3.1). Thus, the equation of motion for massless particle is,

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}.$$

By substituting $M$ with $M - \omega$ to account for self-gravitational effect, we find the imaginary part of the action for a positive energy outgoing particle to be,

$$\mathrm{Im}S = \mathrm{Im} \int_0^{+\omega} \int_{r_{out}}^{r_{in}} \left(1 - \sqrt{\frac{2(M-\omega')}{r} - \frac{Q^2}{r^2}}\right)^{-1} dr(-d\omega').$$

We will also do the integration with respect to $r$ first. Here, however, a new variable $r'$ is set to $\sqrt{2(M-\omega')r - Q^2}$. Additionally, there are two poles in this calculation, but one of the pole is associated with the inner horizon radius. Our integration limits $(r_{out}, r_{in})$ cover only the outer horizon. Thus, we have only one pole to be concern with,

$$\begin{aligned} \mathrm{Im} \int_{r_{out}}^{r_{in}} \left(1 - \sqrt{\frac{2(M-\omega')}{r} - \frac{Q^2}{r^2}}\right)^{-1} dr &= \frac{1}{M-\omega'} \int_\beta^\alpha \left(1 - 2\frac{r'(M-\omega')}{r'^2 + Q^2}\right)^{-1} r' dr', \\ &= \frac{1}{M-\omega'} \int_\beta^\alpha \frac{(r'^2 + Q^2) r' dr'}{r'^2 - 2(M-\omega')r' + Q^2}. \end{aligned}$$

From the denominator, the value of $r'$ at our poles is found to be, $r'_\pm = (M-\omega') \pm \sqrt{(M-\omega')^2 - Q^2}$. Only the residue of pole associate with outer horizon will be calculated,

$$\begin{aligned} \mathrm{Res}(r'_+) &= \frac{\pi}{(M-\omega')} \frac{2(M-\omega')^2 + 2(M-\omega')\sqrt{(M-\omega')^2 - Q^2}}{2\sqrt{(M-\omega')^2 - Q^2}} \\ &\quad \times ((M-\omega') + \sqrt{(M-\omega')^2 - Q^2}), \\ &= \pi \frac{((M-\omega') + \sqrt{(M-\omega')^2 - Q^2})^2}{\sqrt{(M-\omega')^2 - Q^2}}, \\ &= \pi \frac{(2(M-\omega')^2 - Q^2) + 2(M-\omega')\sqrt{(M-\omega')^2 - Q^2}}{\sqrt{(M-\omega')^2 - Q^2}}. \end{aligned}$$

From this result, it is clear that as, $Q \to 0$, our residue is the same as that of previous case. We can now calculate the imaginary part of action,

$$\begin{aligned}
\mathrm{Im}S &= 2\pi \int_0^{+\omega} \frac{(M-\omega')^2 - Q^2 + (M-\omega')\sqrt{(M-\omega')^2 - Q^2}}{\sqrt{(M-\omega')^2 - Q^2}} d\omega, \\
&= 2\pi \int_0^{+\omega} (\sqrt{(M-\omega')^2 - Q^2} + M - \omega') d\omega, \\
&= 2\pi\omega \left(M - \frac{\omega}{2}\right) + \pi \left[M\sqrt{M^2 - Q^2} - (M-\omega)\sqrt{(M-\omega)^2 - Q^2}\right] \\
&\quad + \pi Q^2 \ln \left(\frac{(M-\omega) + \sqrt{(M-\omega)^2 - Q^2}}{M + \sqrt{M^2 - Q^2}}\right). \qquad (3.10)
\end{aligned}$$

Since the logarithm term only contributes to magnitude of decay rate, it will be ignored. Naturally, we will need to also calculate decay rate from anti-particle channel, however as seen in the previous case, this channel will also yield the same resultant decay rate. Before the temperature of black hole can be read off, the second term of (3.10) needs further treatment. This term will now be expanded with respect to $\omega$, by utilizing the following relation,

$$(M-x)\sqrt{(M-x)^2 - Q^2} = M\sqrt{M^2 - Q^2} - \frac{2M^2 - Q^2}{\sqrt{M^2 - Q^2}} x + \frac{M}{2} \frac{2M^2 - 3Q^2}{(M^2 - Q^2)^{3/2}} x^2 - ...,$$

the (3.10) becomes,

$$\mathrm{Im}S = 2\pi\omega \left(M - \frac{\omega}{2} + \frac{M^2 - Q^2/2}{\sqrt{M^2 - Q^2}} - \frac{M}{2} \frac{2M^2 - 3Q^2}{(M^2 - Q^2)^{3/2}} \omega + ...\right). \qquad (3.11)$$

As shown in the previous case, the temperature of black hole is,

$$\begin{aligned}
T &= \frac{\omega}{2\mathrm{Im}S}, \\
&= \frac{1}{4\pi} \left[M + \frac{M^2 - Q^2/2}{\sqrt{M^2 - Q^2}} - \frac{1}{2}\left(1 + M\frac{2M^2 - 3Q^2}{(M^2 - Q^2)^{3/2}}\right)\omega + ...\right]^{-1}. \qquad (3.12)
\end{aligned}$$

The zeroth order of $\omega$ can be modified into,

$$T = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} \qquad (3.13)$$

The above solution is agreed with well known temperature of a Reissner-Nordstrom black hole. Interestingly, if we redo the calculation with integration limits ranging across both horizon, the resultant temperature would be equal to that of Schwarzschild black hole. This implies that as a value of $Q$ becomes smaller with respect to $M$ i.e., $Q/M \to 0$, and the two horizons are starting to converge, the Reissner-Nordstrom black hole will behavior like Schwarzschild black hole. This statement is also supported by substituting, $Q = 0$, into (3.13) which yields the same resultant temperature as Schwarzschild black hole temperature.

According to (3.10), if a black hole keep radiates neutral radiation until, $M \sim \omega$, the second term under square root sign will be negative. However, the emission rate must be real number which implies that the Reissner-Nordstrom black hole must radiate some charge radiation in order to preserve a relation, $M \geq Q + \omega$ or $M > Q$. This statement is a manifestation of third law of black hole thermodynamics; surface gravity ($\kappa = \sqrt{M^2 - Q^2}/(M + \sqrt{M^2 - Q^2})^2$) must never equal to zero.

Moreover, from (3.13), our result is no longer perfect blackbody radiation. Especially as a black hole shrinks to its final state, the values of $M$ and $\omega$ are starting to be of the same order. The terms with large order of $\omega$ will also effect the Hawking radiation in the final state heavily. The radiation will diverge from blackbody radiation even more than our previous case. This statement is under the assumption that the black hole still processes some electric charge at this point in its life time.

# Chapter IV

## Uncertainty Principle

From the particle-wave duality nature of quantum objects, it is impossible to measure the precise position and momentum of any quantum object at the same time. There will always be some uncertainty in either position or momentum. However, the afore mentioned fact has neglected the act of measurement or some other property of the interested object. By considering these neglected point, the uncertainty of either position or momentum is increased and thus, modifies the Heisenberg's uncertainty principle. These modifications are rather important to our work in this chapter. With different modifications, we obtain different resultant temperature. Some resultant temperature also predicts an existence of black hole remnant. An object left behind by a black hole as uncertainty mechanics prevent the black hole from evaporating any further. The existence of remnant is very interesting as it has a potential to resolve the information loss paradox.

In this chapter, the uncertainty principle will be used to calculate Hawking radiation. By considering a position uncertainty of virtual particle emerges near the horizon, we can calculate Hawking temperature. Since we do not know exactly where the pair of virtual particle emerges, there is a position uncertainty ($\Delta x$) equal to Schwarzschild diameter ($4GM/c^2$) or diameter of the horizon. From the position uncertainty, the corresponding minimum momentum uncertainty is found. If we assume that the escaping virtual particle possesses a momentum in the order of its own momentum uncertainty, we can calculate temperature of radiation composes of these escaping virtual particles. The resultant temperature is then identified as the temperature of black hole. Naturally, the aforementioned assumption is rather ambiguous and the resultant temperature needs further calibration [12].

In this chapter, we will no longer use natural unit. The natural unit will be used again in the next chapter. Our calculation is started with the Heisenberg's uncertainty principle,

$$\begin{aligned} \Delta x \Delta p &\geq \hbar, \\ \Delta p &= \frac{\hbar}{\Delta x} = \frac{\hbar c^2}{4GM}. \end{aligned} \tag{4.1}$$

In the last step, we set $\Delta x$ equal to Schwarzschild diameter. The energy of radiation made of these escaping virtual particles is then,

$$E = pc = \frac{\hbar c^3}{4GM}.$$

From the following equation, $E = k_B T$, where $k_B$ is a Boltzmann constant, the temperature of these escaping virtual particles is found to be,

$$T = \frac{\hbar c^3}{4GMk_B}. \tag{4.2}$$

In analogous to previous methods, we assume massless condition for these escaping virtual particles because they can escape from black hole gravity at the vicinity of horizon region.

The above result agrees with the Bekenstein-Hawking temperature up to a factor of $1/2\pi$. Interestingly, this method also shows us that Hawking radiation stems from combination of gravity and quantum effect. The position uncertainty is set to Schwarzschild diameter which shows an indirect involvement from the curvature of spacetime due to the gravitation of black hole. The involvement of quantum mechanics is obviously from the Heisenberg's uncertainty principle. This method relies heavily on other method to properly calculate Hawking temperature. Not only the Hawking radiation has already been known to compose of escaping virtual particle, but the Hawking temperature is also known before hand in order to calibrate the resultant temperature from this method. Nevertheless, the advantage of this method is its ability to quickly calculate Hawking temperature up to a numerical factor.

Different uncertainty principles may be used in order to obtain some interesting results. In this thesis only MLURs (Minimum Length Uncertainty Relations) and GUP (Generalized Uncertainty Principle) type of modification are used,

$$\Delta x \geq \frac{\hbar}{2\Delta p} + \frac{2R}{Mc}\Delta p, \tag{4.3a}$$

$$\Delta x \geq \frac{\hbar}{\Delta p} + \frac{G}{c^3}\Delta p. \tag{4.3b}$$

The above uncertainty principles are MLURs and GUP uncertainty principle respectively. $R$ is an apparatus size or distance between detector and observed particle. The MLURs modification bases on an idea that there exists an interval of time between moments a signal was sent and returned. The observed particle gains some position uncertainty equal to a distance it may or may not travel within the interval which equal to the 'velocity uncertainty' multiplied by the time interval. More detail can be found in an Appendix A.3.

By using above uncertainty relations as starting points instead of (4.1), we need to multiply both uncertainty principles by $\Delta p$ and treat them as quadratic equations. These following results are obtained,

$$\begin{aligned}\Delta p_{MLURs} &= \frac{Mc\Delta x}{4R}\left(1 \pm \sqrt{1 - \frac{4R\hbar}{Mc(\Delta x)^2}}\right),\\ \Delta p_{GUP} &= \frac{c^3\Delta x}{2G}\left(1 \pm \sqrt{1 - \frac{4G\hbar}{c^3(\Delta x)^2}}\right).\end{aligned}$$

By setting $\Delta x$ equal to Schwarzschild diameter, the corresponding temperature from each equation is found to be,

$$\begin{aligned}T_{MLURs} &= \frac{GM^2}{Rk_B}\left(1 \pm \sqrt{1 - \frac{R\hbar c^3}{4G^2M^3}}\right),\\ T_{GUP} &= \frac{Mc^2}{2k_B}\left(1 \pm \sqrt{1 - \frac{\hbar c}{GM^2}}\right).\end{aligned}$$

If the negative signs were chosen, at large mass limit ($M \to \infty$), these resultant temperatures would agree with the Bekenstein-Hawking temperature up to a factor of $1/\pi$ and $1/2\pi$ respectively. Thus, the temperature of black hole from each modification is found to be,

$$T_{MLURs} = \frac{GM^2}{\pi Rk_B}\left(1 - \sqrt{1 - \frac{R\hbar c^3}{4G^2M^3}}\right), \tag{4.4a}$$

$$T_{GUP} = \frac{Mc^2}{4\pi k_B}\left(1 - \sqrt{1 - \frac{\hbar c}{GM^2}}\right). \tag{4.4b}$$

With these resultant temperatures, a black hole (in perfect vacuum) would no longer keep evaporating to oblivion. The evaporation process is stopped when the value of $M$ becomes sufficiently small to render a term under the square root sign negative and meaningless. The final mass of black hole remnant from each result is,

$$M_{Remnant,(MLURs)} = \left(\frac{R\hbar c^3}{4G^2}\right)^{1/3}, \tag{4.5a}$$

$$M_{Remnant(GUP)} = \sqrt{\frac{\hbar c}{G}} = M_P, \tag{4.5b}$$

where $M_P$ is the Planck's mass.

From (4.4a), it is important to define physical interpretation of $R$ in this resultant temperature. $R$ is the distance between an observer and the black hole. As the black hole radiates Hawking radiation, there must be a back reaction on the black hole to conserve total momentum. Since the Hawking radiation particle possesses a momentum uncertainty, the black hole must gain some momentum uncertainty as well. This fact means that as the black hole radiates Hawking radiation, it gains a momentum uncertainty and thus, a velocity uncertainty. This velocity uncertainty is then causes the black hole to gain a position uncertainty. The position uncertainty causes the exact location of black hole to be unknown. The size of horizon is effected as a result. As such, the distance between an observer and the black hole affects the black hole temperature.

It is interesting to notice that just before the evaporation process stops, the black hole emits very high thermal energy from the horizon. However, once the evaporation process is stopped, the black hole remnant should no longer possess such a high temperature any longer. It is important to remind ourself that Hawking radiation is not a conventional thermal radiation emits from a matter possessing heat energy. Once the evaporation process stops, the black hole remnant must possess other temperature via other mechanics.

This method is possibly the easiest one. However, this method alone is not enough to find the exact temperature of black hole and there are some ambiguity in the assumption. Our resultant temperature is needed to be compare with resultant temperature in other method to find calibrating factor. The most important point we receive from this method is the prediction of black hole remnant. This prediction could potentially solve the information loss paradox. If black holes leave something behind instead of completely evaporate to sub-atomic particles and ground states, then it is possible that any information within black holes is stored in the remnant. There are some theories state that black hole remnant is inert and only interact via gravitation. These properties are quite similar to dark matter which suggests that primordial black hole remnant can be a warm dark matter candidate. Interestingly, the existence of black hole remnant stems from the modification term of uncertainty principle. With different modification term, we obtain different remnant mass. Another interesting point is that the remnant mass is corresponding to the minimum position uncertainty of its respective uncertainty principle. This fact shows us that the horizon must have the minimum size. The minimum diameter of horizon is the minimum position uncertainty of black hole. As such, black hole remnant mass can also be calculated by setting horizon diameter equal to the minimum position uncertainty. The mass of black hole with that size of horizon is the remnant mass.

## 4.1 Mass Scale and Black Hole Remnant

We are all familiar with Planck's mass which constructed from three fundamental constants ($G$, $\hbar$, $c$). However, it is possible to construct three other forms by adding another fundamental constant, that is, cosmological constant ($\Lambda$). These masses are,

$$\frac{\hbar}{c}\sqrt{\frac{\Lambda}{3}} \quad , \quad \frac{c^2}{G}\sqrt{\frac{3}{\Lambda}} \quad , \quad \left(\frac{\hbar^2\sqrt{\Lambda}}{G}\right)^{1/3} .$$

Above masses are $M_W$, $M'_W$, and $M_T$ respectively each of them has different physical meaning. $M_W$ is the smallest possible quantity of mass an object can have. $M'_W$ is the maximum mass any universe can possess without collapsing into a black hole. The discovery of $M_T$ is rather new and its physical interpretation is still being studied [13]. Together these masses form a mass scale,

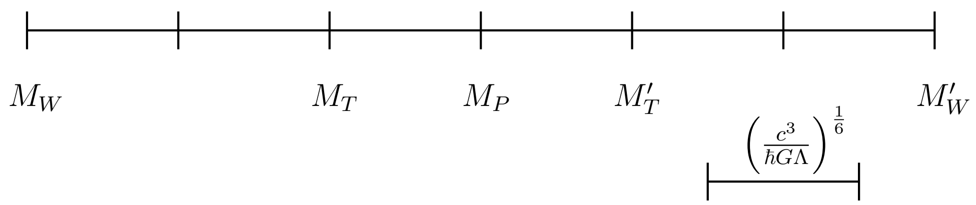


Figure 4.1: Hierarchy of masses on the logarithmic scale.

Interestingly, these masses are related by a dimensionless factor $c^3/\hbar G\Lambda$. This factor can be interpreted as the maximum quantum bits of the observable universe. The radius of observable universe is in order of $1/\Lambda$. By dividing surface of the universe with Planck's length ($\sqrt{\hbar G/c^3}$) squares, we obtain the surface area of our observable universe in the unit of Planck's area which equal to the dimensionless factor [14].

The black hole temperature we find previously can be expressed in terms of these mass scales as,

$$T_{BH} = \frac{\hbar c^3}{8\pi G M k_B} = \frac{1}{8\pi}\frac{M_P c^2}{k_B}\left(\frac{M_P}{M}\right), \tag{4.6a}$$

$$T_{MLURs} = \frac{1}{\pi}\frac{Mc^2}{k_B}\left(\frac{R_W}{R}\right)\left(\frac{M}{M'_W}\right)\left(1-\sqrt{1-\frac{A}{M^3}}\right), \tag{4.6b}$$

$$T_{GUP} = \frac{1}{4\pi}\frac{Mc^2}{k_B}\left(1-\sqrt{1-\left(\frac{M_P}{M}\right)^2}\right), \tag{4.6c}$$

where $R_W$ is a Compton wave length of $M_W$ and $A = \frac{1}{4}\left(\frac{R}{R_W}\right)M_T'^3$. Interestingly, temperatures of black holes (just before the evaporation process stops) are,

$$T_{MLURs,(Remnant)} = \frac{M_T c^2}{2\pi k_B}\left(\frac{R_W}{2R}\right)^{1/3} = \frac{T_T}{2^{4/3}\pi}\left(\frac{R_W}{R}\right)^{1/3}, \tag{4.7a}$$

$$T_{GUP,(Remnant)} = \frac{M_P c^2}{4\pi k_B} = \frac{T_P}{4\pi}, \tag{4.7b}$$

where $T_P$ is a temperature associate with $M_P$ and $T_T$ is a temperature associate with $M_T$. We can also express masses of remnant as,

$$M_{Rem(MLURs)} = \frac{1}{2^{2/3}}\left(\frac{R}{R_W}\right)^{1/3} M'_T = \frac{1}{2^{2/3}}\left(\frac{R}{R'_W}\right)^{1/3} M_T, \tag{4.8a}$$

$$M_{Rem(GUP)} = M_P. \tag{4.8b}$$

Although, we have quite different formula for Hawking temperature, at Cosmic-Microwave-Background limit ($T = 2.7$ K), all resultant temperatures agree that the black hole should possess mass of $4.5 \times 10^{22}$ kg. This fact is not surprising as we set calibrating factor such that every result agrees at large mass limit.

From (4.8a), if we set $R$ to be as large as physically possible, that is, $R_W$ which is in the order of our observable universe radius, the remnant mass is then in the order of $M'_T$. In other words, we are observing a black hole locates at the edge of our observable universe. If we set $R$ to be as small as $R'_W$ which is Compton wave length of our observable universe, the remnant mass is in the order of $M_T$. This setting means that we are observing an extreme small black hole. This length is even smaller than Planck's length which might be problematic as physics at this scale have not been completely established. Nevertheless, we use this limit as the minimum limit. These limits mean that the remnant mass, for the MLURs type of black hole, is ranging from $M_T$ to $M'_T$. Interestingly, if we set $R$ to be Planck's length, the remnant is in the order of $M_P$. This result is in agreement with the GUP type of black hole. Additionally, this limit is the lowest limit that our current physics (2016) still working properly.

The entropy of black hole subject to the MLURs can be calculated by using $dS = dQ/T = c^2 dM/T$,

$$S = c^2 \int T^{-1} dM \quad = \quad k_B \left[ \frac{3\pi}{4} \left( \frac{M'_W}{M} \right) \left( \frac{R}{R_W} \right) {}_2F_1 \left( \frac{1}{3}, \frac{1}{2}, \frac{4}{3}; \frac{A}{M^3} \right) + 2\pi \left( \frac{M}{M_P} \right)^2 \left( 1 + \sqrt{1 - \frac{A}{M^3}} \right) \right]. \tag{4.9}$$

From the above equation, the entropy of remnant is determined to be,

$$S_0 = \frac{\pi k_B}{4} \left( 2 + \frac{3\sqrt{\pi}\Gamma(4/3)}{\Gamma(5/6)} \right) \left( \frac{R_H}{R_P} \right)^2, \tag{4.10}$$

where $R_H$ is a Schwarzschild radius of the remnant,

$$R_H = \frac{2GM}{c^2} = \frac{2G}{c^2} \left( \frac{R\hbar c^3}{4G^2} \right)^{1/3} \sim \left( \frac{\hbar G R}{c^3} \right)^{1/3}.$$

The term $(R_H/R_P)^2$ clearly shows us that even though we modified the uncertainty relation with MLURs, the entropy still obeys Holography i.e., the entropy is proportional to the horizon area ($\sim R_H^2$) in the unit of Planck's area ($R_P^2$).

The lifetime of a black hole (in perfect vacuum) with different initial mass is now being determined. Starting with Stefan-Boltzmann law of radiation,

$$A_H \epsilon \sigma T^4 = -\frac{dQ}{dt} = -c^2 \frac{dM}{dt}.$$

By integrating both side in respect to $t$, we obtain,

$$\begin{aligned} t_{ev} &= \int_{M_f}^{M} \frac{c^6 dM'}{16\pi\epsilon\sigma (GM')^2 T^4}, \\ &= \frac{320\pi}{\epsilon} \frac{R_P}{c} \left( \frac{M}{M_P} \right)^3 \left[ -\left( \frac{A}{M^3} \right)^2 + \frac{A}{M^3} \left( 8\sqrt{1 - \frac{A}{M^3}} - 7 \right) \right. \\ &\quad \left. + 8 \left( \sqrt{1 - \frac{A}{M^3}} + 1 \right) + 16 \frac{A}{M^3} \ln \left( 1 + \sqrt{1 - \frac{A}{M^3}} \right) \right]. \end{aligned} \tag{4.11}$$

By setting $R = R_W$, the time it takes for any black hole with mass greater than the remnant mass ($A^{1/3} \simeq 2.517 \times 10^{12}$ kg) to finish evaporating would be longer than the present age of the universe (13.8 billion years or $4.35 \times 10^{17}$ seconds) as shown in Figure 4.2.

While we were setting $R = R_W$, we have made $R$ as large as possible. A case where $R$ is set to be as small as possible is now being considered. Since it is impossible for $R$ to be smaller than the horizon, the smallest possible value of $R$ should be in the same order as Schwarzschild radius,

$$R = \frac{2GM}{c^2}.$$

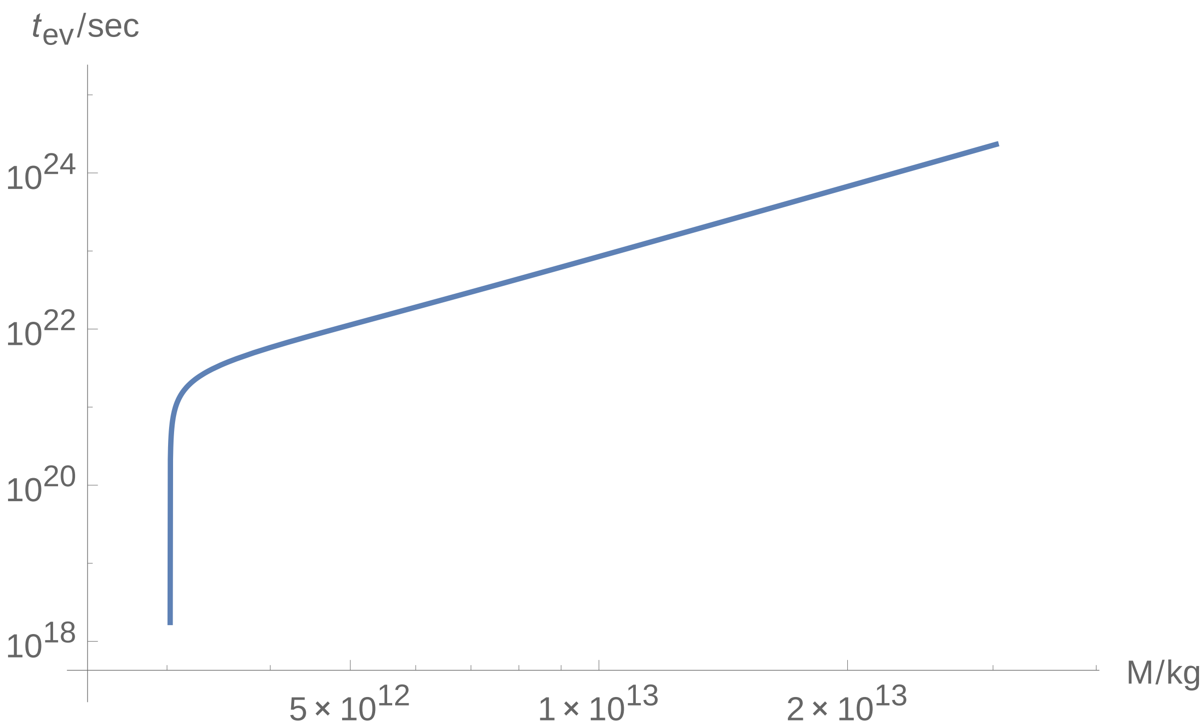


Figure 4.2: Evaporation lifetime of black hole for $R = R_W$.

By substituting this value into (4.8), the remnant mass becomes,

$$M_{Rem(MLURs)} = \frac{1}{2^{1/3}}\left(\frac{\hbar c M}{G}\right)^{1/3} = \frac{1}{2^{1/3}}\left(M_P^2 M\right)^{1/3}. \tag{4.12}$$

This is a weighed geometric mean between the original mass ($M$) and the Planck's mass.


Figure 4.3: Relation between Planck's mass, remnant mass, and original mass on the logarithmic scale.

From the above equation, the remnant mass is known to be varying between $M_P$ and $M'_T$ with the maximum value achieves when original mass equals to $M'_W$. The temperature and entropy of the remnant are,

$$T_0 = \frac{M_P c^2}{2^{5/3}\pi k_B}\left(\frac{M_P}{M}\right)^{1/3}, \tag{4.13}$$

$$S_0 = \frac{\pi k_B}{2^{2/3}}\left(\frac{M}{M_P}\right)^{2/3}\left(2 + \frac{3\sqrt{\pi}\,\Gamma(4/3)}{\Gamma(5/6)}\right). \tag{4.14}$$

From the above equation, we may not immediately see holography. However, by changing $M$ (the original mass) into the remnant mass $\sim (M_P^2 M)^{1/3}$ (the above entropy is the entropy of remnant), then changing $M_{Rem}$ into its correspond Schwarzschild radius, and $M_P$ into Planck's length. The term $(M/M_P)^{2/3}$ becomes,

$$\begin{aligned}
\left(\frac{M}{M_P}\right)^{2/3} &= \left(\frac{2M_{Rem}^3}{M_P^3}\right)^{2/3} = 2^{2/3}\left(\frac{M_{Rem}}{M_P}\right)^2, \\
&= \frac{1}{2^{1/3}}\left(\frac{c^2 R_H}{G}\frac{R_P c}{\hbar}\right)^2, \\
&= \frac{1}{2^{1/3}}\left(\frac{R_H}{R_P}\right)^2. \\
\therefore S_0 &= \frac{\pi k_B}{2}\left(\frac{R_H}{R_P}\right)^2\left(2+\frac{3\sqrt{\pi}\Gamma(4/3)}{\Gamma(5/6)}\right). \qquad (4.15)
\end{aligned}$$

Again, holographic principle holds for our black hole remnant.

The above limit only applies to black hole whose mass is higher than $M_T' \sim 10^{12} kg$. This limit seems to be fine for general black hole. However, for a miniature black hole such as the primordial black hole whose mass can be smaller than $M_T'$, we need to consider an even smaller limit $R \to R_W'$. The existence of smaller limit means that the previous limit is the smallest possible limit for a *naturally* created black hole in the present universe. Under this new limit, the remnant mass is,

$$A^{1/3} = \frac{M_T}{2^{2/3}} \qquad (4.16)$$

The remnant entropy can be calculated from (4.10), by setting $R = R_W'$,

$$S_0 = \frac{\pi k_B}{4}\left(2+\frac{3\sqrt{\pi}\Gamma(4/3)}{\Gamma(5/6)}\right)\left(\frac{\hbar G\Lambda}{c^3}\right)^{2/9} \qquad (4.17)$$

It may be hard to see holography from the above equation. However, by setting $R = R_W'$, $R_H$ becomes Schwarzschild radius of our remnant. In other words, by setting a new limit for $R$, we do not violate the holography. The lifetime of this black hole can be calculated by using (4.11). The life time of this type of black hole is very short as shown in Figure 4.4.

### 4.1.1 Black Hole in Schwarzschild-(Anti) de Sitter Background

In this subsection, we will find the temperature of black hole in Schwarzschild-(Anti) de Sitter space to determine whether or not the result can be arranged in

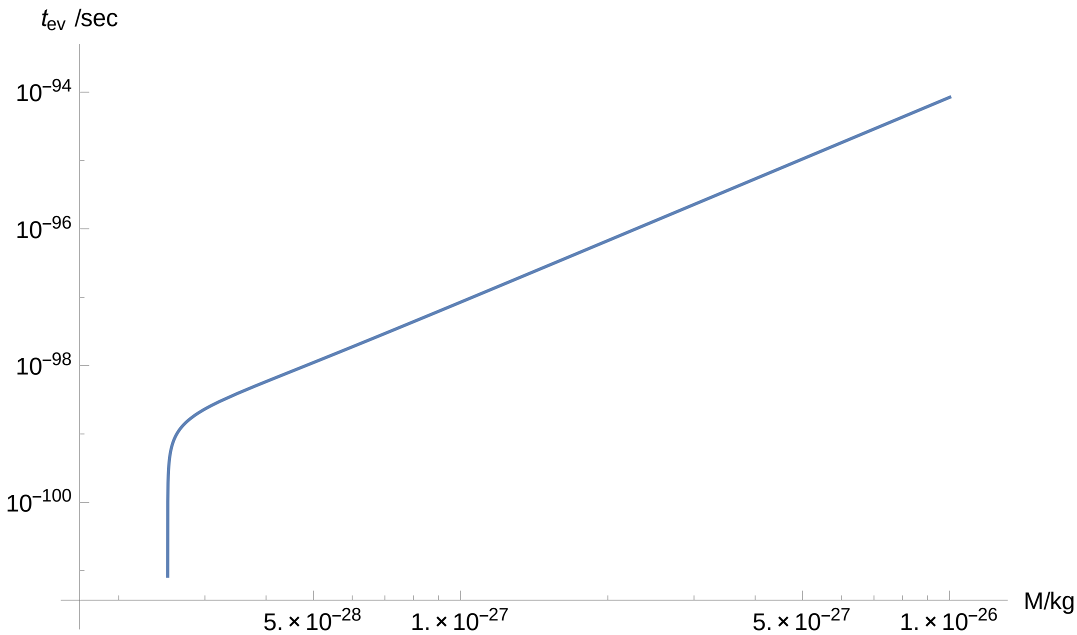


Figure 4.4: Evaporation lifetime of black hole for $R = R'_W$.

terms of any mass scale. Starting with the Schwarzschild-(Anti) de Sitter metric,

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} \pm \frac{\Lambda}{3} r^2\right) dt^2 - \left(1 - \frac{2GM}{c^2 r} \pm \frac{\Lambda}{3} r^2\right)^{-1} dr^2 - r^2 d\Omega^2,$$

where the plus (minus) sign is for the Schwarzschild-(Anti) de Sitter space. The horizon radius can be found by setting $\left(1 - \frac{2GM}{c^2 r} \pm \frac{\Lambda}{3} r^2\right)$ equal to zero,

$$r_{dS} = \frac{2}{\sqrt{\Lambda}} \cos\left[\frac{\pi}{3} + \frac{1}{3}\cos^{-1}\left(3\frac{GM\sqrt{\Lambda}}{c^2}\right)\right], \tag{4.18a}$$

$$r_{AdS} = \frac{2}{\sqrt{\Lambda}} \sinh\left[\frac{1}{3}\sinh^{-1}\left(3\frac{GM\sqrt{\Lambda}}{c^2}\right)\right], \tag{4.18b}$$

where $r_{dS}$ and $r_{AdS}$ are horizon radius in the Schwarzschild-de Sitter and Schwarzschild-Anti-de Sitter space respectively [15][16]. The black hole temperature can be calculated from,

$$T = \frac{\hbar c}{k_B}\frac{g'_{00}(r_H)}{4\pi} = \frac{\hbar c}{4\pi k_B r_H}(1 \pm \Lambda r_H^2). \tag{4.19}$$

By simply substituting horizon radius from (4.18) and expand the temperature in terms of $M$ to obtain,

$$T_{dS} = \frac{M_W c^2}{8\pi k_B}\left(\frac{M'_W}{M} - 16\frac{M}{M'_W} - 80\left(\frac{M}{M'_W}\right)^3 - ...\right), \tag{4.20a}$$

$$T_{AdS} = \frac{M_W c^2}{8\pi k_B}\left(\frac{M'_W}{M} + 16\frac{M}{M'_W} - 80\left(\frac{M}{M'_W}\right)^3 + ...\right). \tag{4.20b}$$

These resultant temperatures do not predict an existence of black hole remnant. However, by looking closer at $r_{dS}$, we see that the value of $M$ can not exceed $M'_W/3$ as an arc-cos of any value higher than one is physically meaningless. Additionally in $AdS$ case, we have the term, $1-\Lambda r_H^2$. If this term is set to zero, the value of $M$ is found to be $\frac{2}{3}M'_W$. These results give us the maximum mass any black hole can possess.

## 4.2 A Generalization to $D$-Dimension and Holography

In this section, we will repeat the calculation of entropy and temperature to determined whether or not holographic principle holds in arbitrary dimension. Starting with the MLURs equation,

$$\Delta p = \frac{Mc\Delta x}{4R}\left(1-\sqrt{1-\frac{4R\hbar}{Mc(\Delta x)^2}}\right).$$

The Schwarzschild radius in $D$-dimension is $2\chi M^{1/(d-3)}$, where $\chi$ is the gravitational constant ($(\kappa c^2)^{1/(d-3)}$) whose dimension is dependent with $\kappa = 8\pi G_d/c^4$. By setting $\Delta x$ equal to Schwarzschild radius in $D$-dimension, the Hawking temperature is found to be,

$$T = \frac{Mc^2}{4\pi k_B}\left(\frac{R_S}{R}\right)\left(1-\sqrt{1-\frac{A_d}{M^{(d-1)/(d-3)}}}\right), \tag{4.21}$$

where $A_d = \frac{\hbar R}{c\chi^2}$ and $R_S$ is the Schwarzschild radius. The black hole will stops evaporate when,

$$M = A_d^{(d-3)/(d-1)}. \tag{4.22}$$

The corresponding Schwarzschild radius of the remnant mass is,

$$R_{S-min} = 2\left(\frac{\hbar R\chi^{(d-3)}}{c}\right)^{1/(d-1)} = 2(\zeta^2\chi^{(d-3)})^{1/(d-1)}, \tag{4.23}$$

where $\zeta = \sqrt{\hbar R/c}$. The above value is also the minimum position uncertainty. The entropy of black hole can then be calculated as follows,

$$\begin{aligned} S &= c^2\int T^{-1}dM, \\ &= 4\pi k_B\left(\frac{d-3}{d-2}\right)\left(\frac{R}{R_S}\right)\left[\frac{(d-1)}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{d-1},\frac{d}{d-1};\frac{A_d}{M^{(d-1)/(d-3)}}\right)\right. \\ &\quad \left. +\left(\frac{M^{(d-1)/(d-3)}}{A_d}\right)\left(1+\sqrt{1-\frac{A_d}{M^{(d-1)/(d-3)}}}\right)\right]. \end{aligned} \tag{4.24}$$

From the above equation, the remnant entropy can also be calculated,

$$S_0 = 2\pi k_B \left(\frac{R}{R_{S-min}}\right)\left(\frac{d-3}{d-1}\right)\left(1 + \frac{\sqrt{\pi}}{2}\frac{\Gamma(1/(d-1))}{\Gamma((d+1)/2(d-1))}\right). \tag{4.25}$$

The above solution does not appear to obey holographic principle. However, if we modify the term $R/R_{s-min}$, the holography is shown as follow,

$$\begin{aligned}
\because \left(\frac{R_{S-min}}{2}\right)^{(d-1)} &= \frac{\hbar R}{c}\chi^{(d-3)}. \\
\left(\frac{R_{S-min}}{2}\right)^{(d-1)} &= \frac{\hbar R}{c}\kappa c^2, \\
&= \hbar R \kappa c, \\
\therefore \frac{R}{R_{S-min}} &= \frac{1}{2^{(d-1)}}\frac{R_{S-min}^{(d-2)}}{\hbar\kappa c} = \frac{1}{2^{(d-1)}}\frac{R_{S-min}^{(d-2)}}{R_{P,d}^{(d-2)}},
\end{aligned} \tag{4.26}$$

where $R_{P,d}$ is the $D$-dimension Planck's length. With the above modification, we see that the remnant entropy is, indeed, proportional to the horizon area ($R_{S-min}^{(d-2)}$) in the unit of $D$-dimensional Planck's area ($R_{P,d}^{(d-2)}$). In other words, we have holographic principle in arbitrary $D$-dimensional space for the MLURs type of black hole.

# Chapter V

## Gravitational Anomaly

According to the quantum field theory, anomalies represent the violations of classical law in the quantum theory. These violations are evidents of the inconsistency in quantum physics. These anomalies stem from local symmetry breaking in the quantum theory. In order to preserve the consistency in the quantum theory, these anomalies must be eliminated. One possible method of eliminating these anomalies is done via the quantum hall effect where an additional magnetic field is introduced into an electric field. The gauge anomaly term found in the electric field is then being canceled by the magnetic field in the total action [17]. This fact suggests that the gauge symmetry breaking in the electric field results in an induction of the magnetic field.

At the small region in the vicinity of horizon, there exists an Einstein (gravitational) anomaly which is an anomaly stemmed from a general coordinate transformation in a fermion graviton loop. This anomaly represents the violation of energy conservation. Analogous to the previous example, Hawking radiation can be derived as a cancellation term to the Einstein anomaly in order to preserve the energy conservation law.

In this method, the Einstein anomaly at the vicinity of horizon region is studied. Hawking radiation is derived as a cancellation term to the Einstein anomaly [18]. By considering the gravitational anomaly, an extra term in the energy-momentum tensor is found. The extra term appears to be a pure radiation flux. A temperature of blackbody with a corresponding thermal radiation flux equivalent to the extra term is defined as a temperature of the black hole.

We will start with the Einstein anomaly,

$$\nabla_\mu T^\mu_\nu = \frac{1}{96\pi\sqrt{-g}}\epsilon^{\alpha\beta}\partial_\alpha\partial_\rho\Gamma^\rho_{\beta\nu}. \tag{5.1}$$

The Einstein anomaly holds only in the small region at the vicinity of horizon region. Otherwise, $\nabla_\mu T^\mu_\nu = 0$.

From this anomaly, new parameter $A_\nu$ and $N^\mu_\nu$ are defined as,

$$\nabla_\mu T^\mu_\nu \equiv A_\nu \equiv \frac{1}{\sqrt{-g}}\partial_\mu N^\mu_\nu. \tag{5.2}$$

These variables will help us later on when we calculate the effect of Einstein anomaly on the energy-momentum tensor. In order to calculate the anomalous term, we start with the metric of spacetime. Our metric is assumed to be in the following form,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{(d-2)},$$

where $d\Omega^2_{(d-2)}$ is a line element on the ($d$-2)-sphere and $f(r)$ depends on the structure of our spacetime. The above form is a representation of many interesting spacetime such as Schwarzschild, de Sitter and Reissner-Nordstrom. For simplicity, spherical symmetry is assumed which truncates our dimension into 1+1 dimensional spacetime. This truncation is an important detail when we compare the thermal radiation flux from our result with that of a blackbody.

With the above metric, we are able to identify $N^\mu_\nu$ in term of $f(r)$. Firstly, a formula of $N^\mu_\nu$ is needed to be extract from (5.2),

$$N^\mu_\nu = \frac{1}{96\pi}\epsilon^{\alpha\beta}\partial_\alpha \Gamma^\mu_{\beta\nu}. \tag{5.3}$$

From the above formula, all Christoffel symbols are calculated in term of $f(r)$ before $N^\mu_\nu$ can be evaluated,

$$\begin{aligned}
&\Gamma^r_{rr} = \frac{1}{2}g^{rr}\partial_r g_{rr} = -\frac{1}{2}\frac{f'(r)}{f(r)}, \qquad \Gamma^r_{rt} = 0, \qquad \Gamma^t_{tr} = \frac{1}{2}g^{tt}\partial_r g_{tt} = \frac{1}{2}\frac{f'(r)}{f(r)},\\
&\Gamma^r_{tt} = -\frac{1}{2}g^{rr}\partial_r g_{tt} = \frac{1}{2}f'(r)f(r), \qquad \Gamma^t_{tt} = 0, \qquad \Gamma^t_{rr} = 0.
\end{aligned}$$

We are now ready to find the value of each $N^\mu_\nu$. Firstly, we start with the $N^t_t$,

$$\begin{aligned}
N^t_t &= \frac{1}{96\pi}\epsilon^{\alpha\beta}\partial_\alpha \Gamma^t_{\beta t} = \frac{1}{96\pi}\epsilon^{rr}\partial_r(g^{tt}\partial_r g_{tt}),\\
&= 0.
\end{aligned}$$

The value of $N^t_t$ is equal to zero because $g_{\mu\nu}$ is independent of $t$, so $\partial_t g_{\mu\nu}$ is zero.

Likewise, we can calculate other $N^\mu_\nu$ as follows,

$$\begin{aligned} N^r_r &= \frac{1}{96\pi}\epsilon^{\alpha\beta}\partial_\alpha\Gamma^r_{\beta r} = \frac{1}{96\pi}\epsilon^{r\beta}\partial_r\Gamma^r_{\beta r} = \frac{1}{96\pi}\epsilon^{rt}\partial_r\Gamma^r_{tr}, \\ &= 0, \end{aligned}$$

$$\begin{aligned} N^r_t &= \frac{1}{96\pi}\epsilon^{rt}\partial_r\Gamma^r_{tt} = -\frac{1}{192\pi}\epsilon^{rt}\partial_r(g^{rr}\partial_r g_{tt}), \\ &= \frac{1}{192\pi}\partial_r(f(r)\partial_r f(r)) = \frac{1}{192\pi}(f'^2(r) + f(r)f''(r)), \end{aligned}$$

$$\begin{aligned} N^t_r &= \frac{1}{96\pi}\epsilon^{rt}\partial_r\Gamma^t_{tr} = \frac{1}{192\pi}\epsilon^{rt}\partial_r(g^{tt}\partial_r g_{tt}), \\ &= \frac{1}{192\pi}\partial_r\left(\frac{1}{f(r)}\partial_r f(r)\right) = \frac{1}{192\pi}\frac{1}{f^2(r)}(f(r)f''(r) - f'^2(r)). \end{aligned}$$

The energy-momentum tensor can now be evaluated in terms of $f(r)$ and integral constant. From (5.2) and constancy in time ($\partial_t T^\mu_\nu = 0$), the form of $T^\mu_\nu$ is found to be,

$$\begin{aligned} \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\rho}T^\rho_\nu - \Gamma^\rho_{\mu\nu}T^\mu_\rho &= A_\nu, \\ \partial_\mu T^\mu_\nu - \Gamma^\rho_{\mu\nu}T^\mu_\rho &= A_\nu. \end{aligned}$$

The following equation, $\Gamma^\mu_{\mu\rho} = \frac{1}{\sqrt{-g}}\partial_\rho\sqrt{-g} = 0$, is utilized in the last step of the above calculation. We will calculate $T^r_r$ by setting $\nu = r$,

$$\begin{aligned} A_r &= \partial_r T^r_r + \partial_t T^t_r - \Gamma^\rho_{\mu r}T^\mu_\rho, \\ &= \partial_r T^r_r - \Gamma^r_{rr}T^r_r - \Gamma^t_{tr}T^t_t, \\ &= \partial_r T^r_r + \frac{1}{2}\frac{f'(r)}{f(r)}T^r_r - \frac{1}{2}\frac{f'(r)}{f(r)}(T^\alpha_\alpha - T^r_r), \\ \frac{1}{f(r)}\partial_r\left(f(r)T^r_r\right) &= A_r + \frac{1}{2}\frac{f'(r)}{f(r)}T^\alpha_\alpha. \end{aligned}$$

We simply need to multiply both side of the above equation by $f(r)$ and integrate out $\partial_r$. We then divide $f(r)$ out of both side and get,

$$\begin{aligned} T^r_r &= K'/f(r) + B(r)/f(r) + I(r)/f(r), \\ &= (K+Q)/f(r) + B(r)/f(r) + I(r)/f(r), \end{aligned} \tag{5.4}$$

where $B(r) = \int_{rH}^r f(x)A_r(x)dx$ and $I(r) = \frac{1}{2}\int_{rH}^r T^\alpha_\alpha(x)f'(x)dx$. From the following relation, $T^\alpha_\alpha = T^r_r + T^t_t$, $T^t_t$ is found to be,

$$T^t_t = T^\alpha_\alpha - \left((K+Q)/f(r) + B(r)/f(r) + I(r)/f(r)\right). \tag{5.5}$$

In order to find $T^r_t$, we need to set $\nu = t$,

$$\begin{aligned} A_t &= \partial_r T^r_t + \partial_t T^t_t - \Gamma^\rho_{\mu t} T^\mu_\rho, \\ &= \partial_r T^r_t - \Gamma^r_{tt} T^t_r - \Gamma^t_{tr} T^r_t, \\ &= \partial_r T^r_t - \frac{1}{2} f'(r) f(r) T^t_r - \frac{1}{2}\frac{f'(r)}{f(r)} T^r_t, \\ &= \partial_r T^r_t + \frac{1}{2}\frac{f'(r)}{f(r)} T^r_t - \frac{1}{2}\frac{f'(r)}{f(r)} T^r_t = \partial_r T^r_t. \\ \therefore T^r_t &= -K + C(r) = g^{rr} g_{tt} T^t_r = -f^2(r) T^t_r, \end{aligned} \tag{5.6}$$

where $C(r) = \int_{r_H}^r A_t(x)dx$.

Generally $\nabla_\mu T^\mu_\nu = 0$, however, at the horizon region, these conditions are no longer hold. As such, $T^\mu_\nu$ is separated into three parts as follow,

$$T^\mu_\nu = T^\mu_{\chi\nu} H + T^\mu_{o\nu}\Theta_+ + T^\mu_{i\nu}\Theta_-,$$

where $\Theta_+ = \Theta[r - (r_H + \epsilon)]$ and $\Theta_- = \Theta[(r_H - \epsilon) - r]$ are scalar step functions and $H = 1 - \Theta_+ - \Theta_-$ is a top hat function which has value of one in the region $(r_H - \epsilon, r_H + \epsilon)$ and zero elsewhere.

We have two unknown constants, $K$ and $Q$, for each $T^\mu_\nu$. In order to determine the value of these constants, we start with effective action as a function of metric and matter field,

$$W[g_{\mu\nu}] \equiv -i \ln \left( \int D[matter] e^{iS[matter, g_{\mu\nu}]} \right), \tag{5.7}$$

where $S[matter, g_{\mu\nu}]$ is a classical action function which transforms under general coordinate transformation as, $\delta_\lambda S = -\int d^d x \sqrt{-g} \lambda^\nu \nabla_\mu T^\mu_\nu$. The general covariance of effective action require that, $\delta_\lambda W = 0$. Thus, we obtain the following equation,

$$\delta_\lambda W = \frac{\int D[matter] \left( e^{iS[matter, g_{\mu\nu}]} \delta_\lambda S \right)}{\int D[matter] e^{iS[matter, g_{\mu\nu}]}} = \delta_\lambda S = 0,$$

since $\delta_\lambda S$ is a function of $g_{\mu\nu}$ and does not depend on matter field. Now, $\delta_\lambda W$ will be expanded into the following form,

$$\begin{aligned} -\delta_\lambda W &= \int d^2 x \sqrt{-g} \lambda^\nu \nabla_\mu [T^\mu_{\chi\nu} H + T^\mu_{o\nu}\Theta_+ + T^\mu_{i\nu}\Theta_-], \\ &= \int d^2 x \lambda^\nu \left[ (\nabla_\mu T^\mu_{\chi\nu}) H + (T^\mu_{\chi\nu}\partial_\mu H + T^\mu_{o\nu}\partial_\mu\Theta_+ + T^\mu_{i\nu}\partial_\mu\Theta_-) \right], \\ &= \int d^2 x \lambda^\nu \left[ \partial_\mu (N^\mu_\nu H) - N^\mu_\nu \partial_\mu H + (T^\mu_{\chi\nu}\partial_\mu H + T^\mu_{o\nu}\partial_\mu\Theta_+ + T^\mu_{i\nu}\partial_\mu\Theta_-) \right], \\ &= \int d^2 x \lambda^t \left[ \partial_r (N^r_t H) + (T^r_{ot} - T^r_{\chi t} + N^r_t)\partial_r\Theta_+ + (T^r_{it} - T^r_{\chi t} + N^r_t)\partial_r\Theta_- \right] \\ &\quad + \lambda^r \left[ (T^r_{or} - T^r_{\chi r})\partial_r\Theta_+ + (T^r_{ir} - T^r_{\chi r})\partial_r\Theta_- \right], \end{aligned} \tag{5.8}$$

In the second line, we used a property, $\sqrt{-g}=1$. We need to expand $\partial_r\Theta_\pm$ terms in order to solve (5.8). As previously stated, the forms of $\Theta_\pm$ are,

$$\Theta_\pm = \Theta(\pm r \mp r_H - \epsilon).$$

From the relation, $\partial_x\Theta(x) = \delta(x)$, where $\delta(x)$ is a Dirac delta function, we may expand $\partial_r\Theta_\pm$ using Taylor's series as follows,

$$\begin{aligned}
\partial_r\Theta_\pm &= \pm\delta(r - r_H \mp \epsilon), \\
&= \pm\left[\delta(r-r_H) + (\mp\epsilon)\partial_r\delta(r-r_H) + \frac{1}{2}\epsilon^2\partial_r^2\delta(r-r_H) + ...\right], \\
&= \left[\pm 1 - \epsilon\partial_r \pm \frac{1}{2}\epsilon^2\partial_r^2 - ...\right]\delta(r-r_H).
\end{aligned}$$

A few remark before we start solving (5.8), $f(r)$ flips sign as it crosses the horizon which means, $f_o(r) = -f_i(r)$. As for the sign of $f_\chi(r)$, it will be determined by the attached $\Theta_\pm$. We will assume that, $I(r)/f(r)|_{r_H} = \frac{1}{2}T^\alpha_\alpha|_{r_H}$, and the term $\partial_r(N^r_t H)$ vanishes under the very small $\epsilon$ limit. Thus,

$$\begin{aligned}
-\delta_\lambda W &= \int d^2x\lambda^t\left[(T^r_{ot} - T^r_{\chi t} + N^r_t)\partial_r\Theta_+ + (T^r_{it} - T^r_{\chi t} + N^r_t)\partial_r\Theta_-\right] \\
&\quad +\lambda^r\left[(T^r_{or} - T^r_{\chi r})\partial_r\Theta_+ - (T^r_{ir} - T^r_{\chi r})\partial_r\Theta_-\right].
\end{aligned}$$

We flip the sign of $(T^r_{ir} - T^r_{\chi r})$ term to cancel out the flipping sign effect of $f_i(r)$ and $f_\chi(r)$. Once we substitute the expansion of $\partial_r\Theta_\pm$ into the above equation, it is clear that there are only four different terms regardless of how high the order of expansion we use i.e., $(T^r_{ot} - T^r_{\chi t} + N^r_t) \pm (T^r_{it} - T^r_{\chi t} + N^r_t)$ and $(T^r_{or} - T^r_{\chi r}) \pm (T^r_{ir} - T^r_{\chi r})$. These terms are simplified by using the following conditions, $C(r_H) = 0$ and $B(r_H) = 0$,

$$\begin{aligned}
(T^r_{ot} - T^r_{\chi t} + N^r_t) + (T^r_{it} - T^r_{\chi t} + N^r_t) &= K_o + K_i - 2K_\chi - 2N^r_t, \\
(T^r_{ot} - T^r_{\chi t} + N^r_t) - (T^r_{it} - T^r_{\chi t} + N^r_t) &= K_o - K_i, \\
(T^r_{or} - T^r_{\chi r}) - (T^r_{ir} - T^r_{\chi r}) &= \left(\frac{K_o + Q_o - K_i - Q_i}{f(r_H)}\right), \\
(T^r_{or} - T^r_{\chi r}) + (T^r_{ir} - T^r_{\chi r}) &= \left(\frac{K_o + Q_o + K_i + Q_i - 2K_\chi - 2Q_\chi}{f(r_H)}\right).
\end{aligned}$$

By setting $\delta_\lambda W = 0$, all these terms must vanish. Thus we obtain the following conditions,

$$K_o = K_i = K_\chi + \Phi,$$

$$Q_o = Q_i = Q_\chi - \Phi,$$

where,

$$\Phi = N^r_t|_{r_H} = \frac{f'^2(r)}{192\pi} = \frac{\kappa^2}{48\pi}. \tag{5.9}$$

The last step is done by utilizing the following relation, $\kappa = \frac{1}{2}f'(r)$.

The total energy-momentum tensor becomes, under very small $\epsilon$ limit,

$$\begin{aligned} T^{\mu}_{\nu} &= T^{\mu}_{o\nu}\Theta_{+} + T^{\mu}_{i\nu}\Theta_{-} + T^{\mu}_{\chi\nu}H, \\ &= T^{\mu}_{c\nu} + T^{\mu}_{\Phi\nu}, \end{aligned}$$

where $T^{\mu}_{c\nu}$ is a conserved energy-momentum tensor which arises from a combination of $T^{\mu}_{o\nu}$, $T^{\mu}_{i\nu}$ and parts of $T^{\mu}_{\chi\nu}$. This tensor is a tensor that which we would have had without any gravitational anomaly effect. By substituting $K_{\chi}$ with $K_O - \Phi$ and $Q_{\chi}$ with $Q_O + \Phi$, every component of $T^{\mu}_{\chi\nu}$ that is not stemmed from the gravitational anomaly is taken into $T^{\mu}_{c\nu}$. All other components are put into $T^{\mu}_{\Phi\nu}$. The component of $T^{\mu}_{\Phi\nu}$ tensor is found to be a pure flux of $\Phi$. Another way to interpret the above equation is as follows; since we can regard (5.2) as differential equations, $T^{\mu}_{c\nu}$ can be regraded as complementary solutions while $T^{\mu}_{\Phi\nu}$ can be regarded as particular solutions.

Since a blackbody with temperature $T$ emits a massless thermal radiation flux in the form of, $\Phi = \frac{\pi}{12}T^2$, in the direction of $r^{+}$. The flux of $T^{\mu}_{\Phi\nu}$ is equivalent to the blackbody radiation flux with the temperature of, $T = \kappa/2\pi$. This result is, indeed, agreed with the Bekenstein-Hawking temperature.

In this method, Hawking radiation is an anomalous cancellation phenomenon emerges in order to preserve the consistency in the quantum field theory and the general relativity [19]. The Einstein anomaly is emerged from a general coordinate transformation in one-loop fermion-graviton diagram. This fact implies that Hawking radiation has both quantum and gravity nature. Since, $\nabla_{\mu}T^{\mu\nu} = 0$, is a representation of the energy-momentum conservation in the general relativity, the Einstein anomaly represents the violation of the conservation of energy and momentum due to the quantum field effect. In analogous to the quantum hall effect, Hawking radiation cancels the Einstein anomaly at the vicinity of horizon region [18].

Hawking radiation is shown to strongly relate with both quantum and gravity effect. In this method, however, Hawking radiation does not seem to rely heavily with the existence of virtual particle as other methods do. However, in order to try and explain how the radiation emerges to cancel the Einstein anomaly, the existence of virtual particle is needed. In the end, this method also needs virtual particle in order to explain the existence of Hawking radiation. The difference is that the radiation compose of virtual particle emerges in the area with the Einstein anomaly as a consequence of local symmetry breaking. In contrast with the usual explanation of Hawking radiation where a single virtual particle from each pair is being consumed by the black hole whilst its partner escapes and becomes a real particle.

# Chapter VI

## Conclusion

Throughout this thesis, we have explored and discussed four different methods of calculating Hawking radiation. Each method yields a unique and interesting interpretation of Hawking radiation. Weaknesses and strengths of each method were also discussed. Some methods have offered a potential solution to information loss paradox. In this chapter, we summarize and compare our results from all previous methods.

Let us start with the Unruh effect. Relatively speaking, this method is easy to derive. In this method, we found that a ground state of an inertial observer appears to be a thermal radiation ensemble for a constant acceleration observer. As an observer is subjected to a constant acceleration, he or she will perceive an apparent event horizon. The existence of horizon induces thermal radiation via the Unruh effect. The aforementioned phenomenon suggests that an event horizon of any origin is capable of emitting thermal radiation. On one hand, this property of the Unruh effect is useful. We utilize the property to calculate thermal radiation radiated by the black hole horizon i.e., Hawking radiation. This property also implies that the event horizon at the edge of our observable universe is capable of emitting thermal radiation. On the other hand, this property suggests that any astronomical object is capable of emitting thermal radiation, even if it does not possess any event horizon of its own. Additionally, this method neglected quantum and gravity nature of the black hole. Hawking radiation is shown to be perfect blackbody radiation which gives rise the information loss paradox [1].

Secondly, quantum tunneling effect is used to calculate Hawking radiation. This method utilizes quantum effect at the vicinity of horizon region (whose existence relies on the gravitational effect) which indicates the quantum and gravity nature of Hawking radiation. At first glance, there seems to be no classical barrier or classically forbidden region in the black hole gravitational potential. However, we learned that as a black hole radiates a Hawking radiation particle, it shrinks. The shrinking effect shifts gravitational potential which creates an apparent barrier for the particle. The transmission rate across the apparent barrier is then being recognized as a black hole decay rate. This fact implies that as a black hole decays, it radiates thermal radiation which is Hawking radiation. Interestingly, the decay rate can be modified into the following form, $e^{\Delta S}$, where $S$ is an entropy of the black hole. According to the Statistical mechanics, the aforementioned form shows us that the black hole decomposition can be regarded as a black hole evolving into a new state. As such, Hawking radiation has Statistic nature which means that properties other than mass, angular momentum and electric charge of black holes effect Hawking radiation spectrum. This fact could potentially solve the information loss paradox. A self-gravitational effect of the Hawking radiation particle is found as a correction term to standard Bekenstein-Hawking temperature. While negligible, the correction term implies that the Hawking radiation is somewhat deviate from blackbody radiation. Moreover, in 'Quantum Tunneling' chapter, our calculation only gives us an approximation solution. We had neglected the pre-factor of decay rate. In a more detailed calculation, the pre-factor is found to be depending on frequency of Hawking radiation as well which further supports that the radiation fits poorly with blackbody radiation. A charged black hole is also being studied [7].

Afterward, uncertainty principles are adapted to calculate Hawking temperature. The method gives us Hawking temperature rather easily. However, the resultant temperatures need calibrating factors which means that we need to already obtain Hawking temperature from other methods. Nevertheless, the fact that this method yields Hawking temperature in a quick and easy manner is quite useful. By modifying Heisenberg uncertainty principle, the existence of black hole remnant is shown [12]. These modifications are done by taking account of the actual measuring process. With these modifications, the minimum position uncertainty is no longer zero as the momentum uncertainty diverges to infinity. Due to the minimum position uncertainty, the black hole diameter is known to have a minimum size in length at which point the evaporation process must stop. The existence of remnant has a potential to solve the information loss paradox. Since a black hole does not undergo a complete evaporation process, the information stored within the black hole is preserved in the remnant.

Additionally, physical interpretations of new mass scales ($M_T$, $M_T'$) which are constructed from dimensional analysis are known to be the maximum ($M_T'$) and the minimum ($M_T$) mass limits of black hole remnant subjugated by the 'Minimum Length Uncertainty Relation' (MLUR). We had also studied the black hole entropy where we find that, under the MLUR modification, the entropy of black hole obeys holography only at either the large mass limit or the remnant mass limit. The holography in the MLUR type of black hole seems to be lost at other limits. The holographic property at the remnant mass limit was also shown to be hold at arbitrary non-compact $D$-dimension. Life time of the MLUR type of black hole is also being studied which is found to have the minimum life time varying between $10^{-99}$ and $10^{21}$ [13].

Lastly, Hawking radiation is identified as a cancellation term to the Einstein anomaly. Anomalies in quantum physics are representations of the inconsistency in quantum theory. The Einstein anomaly is derived from the general coordinate transformation in a fermion graviton loop. This anomaly violates the conservation of total energy and momentum. In analogous to the quantum Hall effect where the gauge anomaly in electric field is known to induce magnetic field, the Einstein anomaly is shown to induce thermal radiation, that is, Hawking radiation [19]. The interpretation of Hawking radiation in this method is rather interesting. The Hawking radiation is derived as a cancellation term which preserves the consistency in quantum gravity. Although the quantum gravity nature of Hawking radiation is very clear in this method, the calculation steps are rather complicated. [18].

## 5.1 Future study

In the chapter three, we have neglect the pre-factor term and utilized WKB approximation. It will be quite interesting to derive Hawking radiation in a full detail without using WKB approximation. In the chapter four, new mass scales were introduced. These mass scale were only discovered recently (2016). It will be interesting to discover other possible physical meaning and implications. Due to an incompletion of quantum-gravity theory, we cannot find any potential solution to information loss paradox in the chapter five. With better a understanding of gravitational anomaly and quantum-gravity, it will be worth our time to re derive Hawking radiation with better details via the method as shown in the chapter five.

# References


[1] S. M. Carroll, "Spacetime and geometry: An introduction to general relativity," San Francisco, USA: Addison-Wesley (2004) 513 p ISBN 0-8053-8732-3

[2] S. W. Hawking, "The Unpredictability of Quantum Gravity," Commun. Math. Phys. **87**, 395 (1982). doi:10.1007/BF01206031

[3] T. Banks, L. Susskind and M. E. Peskin, "Difficulties for the Evolution of Pure States Into Mixed States," Nucl. Phys. B **244**, 125 (1984). doi:10.1016/0550-3213(84)90184-6

[4] P. Mao, X. Wu and H. Zhang, "Soft hairs on isolated horizon implanted by electromagnetic fields," arXiv:1606.03226 [hep-th].

[5] R. M. Wald, "General Relativity," Chicago, Usa: Univ. Pr. (1984) 491p doi:10.7208/chicago/9780226870373.001.0001 ISBN 0-226-87033-2

[6] http://physics.stackexchange.com/questions/158195/thought-experiment-poking-a-stick-across-a-black-holes-event-horizon

[7] M. K. Parikh and F. Wilczek, "Hawking radiation as tunneling," Phys. Rev. Lett. **85**, 5042 (2000) doi:10.1103/PhysRevLett.85.5042 [hep-th/9907001].

[8] P. Painleve "La mecanique classique et la theorie de la relativite," C. R. Acad. Sci. (Paris) 173, 677–680(1921)

[9] P. Kraus and F. Wilczek, "Selfinteraction correction to black hole radiance," Nucl. Phys. B **433**, 403 (1995) doi:10.1016/0550-3213(94)00411-7 [gr-qc/9408003].

[10] Y. K. Ha, "The Gravitational energy of a black hole," Submitted to: Gen.Rel.Grav.35:2045-2050 [gr-qc/0508041].

[11] M. K. Parikh, "A Secret tunnel through the horizon," Int. J. Mod. Phys. D **13**, 2351 (2004) [Gen. Rel. Grav. **36**, 2419 (2004)] doi:10.1142/S0218271804006498 [hep-th/0405160].

[12] R. J. Adler, P. Chen and D. I. Santiago, "The Generalized uncertainty principle and black hole remnants," Gen. Rel. Grav. **33**, 2101 (2001) doi:10.1023/A:1015281430411 [gr-qc/0106080].

[13] P. Burikham, R. Dhanawittayapol and T. Wuthicharn, "A new mass scale, implications on black hole evaporation and holography," Int. J. Mod. Phys. **31**, no. 16, 1650089 (2016) doi:10.1142/S0217751X16500895 [arXiv:1605.05866 [gr-qc]].

[14] P. Burikham, K. Cheamsawat, T. Harko and M. J. Lake, "The minimum mass of a spherically symmetric object in $D$-dimensions, and its implications for the mass hierarchy problem," Eur. Phys. J. C **75**, no. 9, 442 (2015) doi:10.1140/epjc/s10052-015-3673-5 [arXiv:1508.03832 [gr-qc]].

[15] M. A. Rahman and M. I. Hossain, "Hawking Radiation of Schwarzschild-de Sitter Black Hole by Hamilton-Jacobi method," Phys. Lett. B **712**, 1 (2012) doi:10.1016/j.physletb.2012.04.049 [arXiv:1205.1216 [gr-qc]].

[16] N. Cruz, M. Olivares and J. R. Villanueva, "The Geodesic structure of the Schwarzschild anti-de Sitter black hole," Class. Quant. Grav. **22**, 1167 (2005) doi:10.1088/0264-9381/22/6/016 [gr-qc/0408016].

[17] A. Bilal, "Anomaly cancellations on lower-dimensional hypersurfaces by inflow from the bulk," In *Shifman, M. (ed.) et al.: From fields to strings, vol. 3* 2051-2091 [hep-th/0409028].

[18] S. P. Robinson and F. Wilczek, "A Relationship between Hawking radiation and gravitational anomalies," Phys. Rev. Lett. **95**, 011303 (2005) doi:10.1103/PhysRevLett.95.011303 [gr-qc/0502074].

[19] L. Alvarez-Gaume and E. Witten, "Gravitational Anomalies," Nucl. Phys. B **234**, 269 (1984). doi:10.1016/0550-3213(84)90066-X

[20] J. M. Bardeen, Non-singular general-relativistic gravitational collapse, in Proc. Int. Conf. GR5, Tbilisi, p. 174, 1968.

[21] P. Burikham, "An introduction to general relativity," Bangkok, Thailand:Danex (2014) 206p ISBN 978-616-305-529-3

[22] S. L. Adler, "Axial vector vertex in spinor electrodynamics," Phys. Rev. **177**, 2426 (1969). doi:10.1103/PhysRev.177.2426

[23] G. 't Hooft, "Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle," Phys. Rev. D **14**, 3432 (1976) Erratum: [Phys. Rev. D **18**, 2199 (1978)]. doi:10.1103/PhysRevD.18.2199.3, 10.1103/PhysRevD.14.3432

[24] R. A. Bertlmann and E. Kohlprath, "Two-dimensional gravitational anomalies, Schwinger terms and dispersion relations," Annals Phys. **288**, 137 (2001) doi:10.1006/aphy.2000.6110 [hep-th/0011067].

# APPENDICES

# Appendix A

## Detailed Calculation

### A.1 Bogolubov Transformation

From special relativity, every observer with different velocity will have his or her own different spacetime coordinate. As such, it is not unusual to have two different observers, each observer with his or her own different annihilation and creation ($\hat{a}_i$,$\hat{b}_i$) operators as well as different free wave modes ($f_i$,$g_i$). In order to, compare a state between different observer, it is useful to establish a transformation of operators and free wave modes between each observer. Bogolubov transformation is such transformation [1].

While these operators and wave modes belong to different observers, it still have the same orthonormal properties for these different free wave modes, that is,

$$\begin{aligned} (f_i, f_j) &= \delta_{ij}, & (g_i, g_j) &= \delta_{ij}, \\ (f_i^*, f_j^*) &= -\delta_{ij}, & (g_i^*, g_j^*) &= -\delta_{ij}. \end{aligned}$$

Additionally, these operators have the following commutation relations,

$$\begin{aligned} [\hat{a}_i, \hat{a}_j] &= 0, & [\hat{a}_i^\dagger, \hat{a}_j^\dagger] &= 0, & [\hat{a}_i, \hat{a}_j^\dagger] &= \delta_{ij}. \\ [\hat{b}_i, \hat{b}_j] &= 0, & [\hat{b}_i^\dagger, \hat{b}_j^\dagger] &= 0, & [\hat{b}_i, \hat{b}_j^\dagger] &= \delta_{ij}. \end{aligned}$$

Our matter field can also be expanded as,

$$\phi = \sum_i (\hat{a}_i f_i + \hat{a}_i^\dagger f_i^*), \tag{A.1}$$

$$\phi = \sum_i (\hat{b}_i g_i + \hat{b}_i^\dagger g_i^*). \tag{A.2}$$

Let us define a transformation from one wave modes to another wave modes as,

$$g_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*), \tag{A.3}$$

where $\alpha_{ij}$ and $\beta_{ij}$ are Bogolubov coefficients. We shall implement the following normalization condition on these coefficients,

$$\sum_k (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = \delta_{ij}, \tag{A.4a}$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0. \tag{A.4b}$$

Before we move forward, it will be easier to work in the matrix notation. Let $G = \begin{bmatrix} g_i \\ g^*_i \end{bmatrix}$, $F = \begin{bmatrix} f_i \\ f^*_i \end{bmatrix}$, $A = [\alpha_{ij}]$, and $B = [\beta_{ij}]$. In addition, (A.3) is needed to be complex conjugated which yields $g^*_i = \sum_j (\beta^*_{ij} f_j + \alpha^*_{ij} f^*_j)$.

By transforming (A.3) and (A.4) into matrix notation, we have,

$$G = \begin{bmatrix} A & B \\ B^* & A^* \end{bmatrix} F, \tag{A.5a}$$

$$AA^\dagger - BB^\dagger = [1], \tag{A.5b}$$

$$AB^T - BA^T = [0]. \tag{A.5c}$$

With these properties, it is possible to construct a matrix which transforms $F$ into $G$. The matrix is equaled to $\begin{bmatrix} A & B \\ B^* & A^* \end{bmatrix}^{-1}$. An inverse matrix can easily be constructed by utilizing properties (A.5b) and (A.5c) which yield, $\begin{bmatrix} A^\dagger & -B^T \\ -B^\dagger & A^T \end{bmatrix}$. From the constructed matrix, the following expansions are known,

$$f_i = \sum_j (\alpha^*_{ji} g_j - \beta_{ji} g^*_j), \tag{A.6a}$$

$$f^*_i = \sum_j (-\beta^*_{ji} g_j + \alpha_{ji} g^*_j). \tag{A.6b}$$

By substituting (A.6) into (A.1), the following result is obtained,

$$\begin{aligned} \phi &= \sum_i (\hat{a}_i f_i + \hat{a}^\dagger_i f^*_i), \\ &= \sum_i (\hat{a}_i \sum_j (\alpha^*_{ji} g_j - \beta_{ji} g^*_j) + \hat{a}^\dagger_i \sum_j (-\beta^*_{ji} g_j + \alpha_{ji} g^*_j)), \\ &= \sum_j (\sum_i (\hat{a}_i \alpha^*_{ji} - \hat{a}^\dagger_i \beta^*_{ji}) g_j - \sum_i (\hat{a}_i \beta_{ji} - \hat{a}^\dagger_i \alpha_{ji}) g^*_j), \end{aligned}$$

The above result must agree with (A.2). Hence, the transformation between operators are,

$$\hat{b}_j = \sum_i (\alpha^*_{ji} \hat{a}_i - \beta^*_{ji} \hat{a}^\dagger_i). \tag{A.7}$$

Similarly, by substituting (A.3) into (A.2), the following result is obtained,

$$\hat{a}_j = \sum_i (\alpha^*_{ij}\hat{b}_i + \beta^*_{ij}\hat{b}^\dagger_i). \tag{A.8}$$

From (A.7) and (A.8), we have a transformation of operator between different observers. Additionally, we can transform free wave modes between each observer with (A.3) and (A.6). This concludes Bogolubov transformation.

## A.2 Radiation in 1+1 Dimensions

Generally, from Stefan-Boltzmann's law, blackbody radiation flux is,

$$\Phi = \frac{\pi^2}{60}T^4. \tag{A.9}$$

The above formula describes a blackbody radiation flux in 3+1 dimension. However, most of our work is in 1+1 dimension. Naturally, the radiation flux in different dimension would be in difference form due to directions by which the radiation flux propagates. As such, it is useful to re-derive blackbody radiation flux in 1+1 dimension.

Starting from total energy of massless boson,

$$U = \int_0^\infty \bar{U}(\omega)dn(\omega),$$

where $\bar{U}(\omega)$ is an average thermal energy of boson gas and $dn(\omega)$ is multiplicity of energy states between frequency interval $\omega$ to $\omega + d\omega$. In order to find $dn(\omega)$, we need to work in mode of frequency $(k)$; $dn(k)$ is simply $dk$ in 1+1 dimensions. From $\omega = \frac{k\pi}{L}$, $dn(\omega) = (L/\pi)d\omega$ ($c \equiv 1$; natural unit). $L$ will also equal to volume of the box containing our interested boson gas,

$$\begin{aligned} U &= \int_0^\infty \frac{\omega}{e^{\beta\omega}-1}\frac{L}{\pi}d\omega, \\ \frac{U}{L} &= \frac{1}{\beta^2\pi}\int_0^\infty \frac{x}{e^x-1}dx. \end{aligned}$$

Here, a new variable is defined, $x = \beta\omega$. The integration term can be solved by,

$$\begin{aligned} \int_0^\infty \frac{x}{e^x-1}dx &= \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}dx = \sum_{k=1}^\infty \int_0^\infty xe^{-kx}dx, \\ &= \left(\sum_{k=1}^\infty \frac{1}{k^2}\right)\left(\int_0^\infty ye^{-y}dy\right) = \xi(2)\Gamma(2), \\ &= \frac{\pi^2}{6}, \end{aligned}$$

where $y = kx$. Substitute the result back, energy density is found to be,

$$\begin{aligned} u &= \frac{1}{\beta^2\pi}\frac{\pi^2}{6}, \\ &= \frac{\pi}{6}T^2. \end{aligned}$$

However, the result is a total energy emitted by a blackbody which radiates to both $r^+$ and $r^-$ directions. Hence, the blackbody radiation emitted to $r^+$ direction is only $\pi T^2/12$.

It is worth noticing that $\pi T^2/12$ is an energy density of the blackbody radiation (emission energy density, $u$) not the blackbody radiation flux (emission power density, $\Phi$). In order to find the blackbody radiation flux, we will need to do small calculation steps. The emission power density will equal to the rate that the energy density passes through some fixed area ($uAc$) divided by that area ($A$). Thus, $\Phi = uc = \pi T^2/12$, under natural unit.

## A.3 Minimum Length Uncertainty Relations

Previously in 'Uncertainty Principle' chapter, we used a new modification to uncertainty principle called MLURs without proper introduction. In this section, we will derive MLURs in more detail [13]. We will also restore the natural constants in this section.

Starting with Heisenberg's uncertainty principle,

$$\Delta x \geq \frac{\hbar}{2\Delta p}.$$

The above relation tells us that there is a natural limit on how precise we can measure position and momentum of a quantum object. However, the above equation neglects the process of actual measurement. For example, if uncertainty from our probe is being considered, then an extra term equals to the size of our probe ($\Delta x \geq l$) must be added. If the object in question interact with its own energy/momentum, then another uncertainty proportional to $\Delta p$ will be added ($\beta \Delta p$). Thus, the uncertainty principle is modified into the following form,

$$\Delta x \geq \frac{\hbar}{2\Delta p} + l + \beta \Delta p. \tag{A.10}$$

Formerly, $\Delta x$ can have a value as low as zero, but with these additional terms it is clear that as $\Delta p \to \infty$, $\Delta x$ also diverges to infinity. Hence, there is a non-zero minimum value of position uncertainty $\Delta x_{min}$ which can be calculated by minimizing $\Delta x$ with respect to $\Delta p$ and obtained,

$$\Delta x_{min} = l + \sqrt{2\hbar\beta}.$$

The corresponding $\Delta p$ which gives $\Delta x_{min}$ is,

$$\Delta p_c = \sqrt{\frac{\hbar}{2\beta}}.$$

Since we can make the probe size to be arbitrary small, therefore $\Delta x_{min}$ is roughly equaled to $\sqrt{2\hbar\beta}$.

Another form of additional term is $\Delta x \geq 2R\Delta p/Mc$. The term arises from the following thought experiment; suppose we have a particle traveling parallel to a mirror. Let R be the distance between the particle and the mirror. The time it would require for photon to be emitted from the particle, reflected by the mirror and absorbed back by the same particle is roughly $t = 2R/c$. During that period, the particle would have gain additional position uncertainty of $\Delta x = (\Delta p/M)t = 2R\Delta p/Mc$. Thus, we have a new uncertainty principle,

$$\Delta x \geq \frac{\hbar}{2\Delta p} + \frac{2R\Delta p}{Mc}. \tag{A.11}$$

The above uncertainty principle is called 'Minimum length uncertainty relations' (MLURs). By comparing the above relation with (A.10), we see that it is the case where $l = 0$ and $\beta = 2R/Mc$. From the relation, if $\Delta x$ is minimized with respect to $\Delta p$, the following result is obtained,

$$\Delta x_{min} = 2\sqrt{\frac{R\hbar}{Mc}}. \tag{A.12}$$

Certainly, the minimum value of $l$ is equaled to Schwarzschild radius of the particle that we measure. Thus,

$$\Delta x_{min} = \frac{GM}{c^2} + 2\sqrt{\frac{R\hbar}{Mc}}. \tag{A.13}$$

From the above equation, $\Delta x_{min}$ is minimized with respect to $M$ in order to find the lowest possible value of $\Delta x_{min}$. The lowest value is,

$$\Delta x_{min} = \frac{3}{2}\left(\frac{RG\hbar}{c^3}\right)^{1/3} = \frac{3}{2}(RR_P^2)^{1/3}, \tag{A.14}$$

where the corresponding mass which gives $\Delta x_{min}$ is,

$$M_c = \frac{1}{2}\left(\frac{R\hbar c^3}{G^2}\right)^{1/3} = \frac{1}{2}(R\sqrt{\Lambda})^{1/3}M_T'. \tag{A.15}$$

Interestingly, the result is exactly the remnant mass we obtained from MLURs type of Hawking radiation.

# Appendix B

## Structure of Black Hole

## B.1 Singularity and Event Horizon

Generally, we believe that a black hole is composed of a point material mass (singularity) and an encompassed-surface (event horizon) where light traveled away from the black hole trapped on. However, there are some theories suggest that when a black hole was born, material masses could not move across the horizon. Additionally, at the length smaller than Planck's length, gravitation can behave as a repulsive force as well. Thus, there is no singularity. All the material masses are distributed around event horizon. This type of black hole is called 'Bardeen black hole' or 'Regular black hole' [20].

We will look at the creation of a black hole and consider how the above statement might have been true [21]. Starting from a dust sphere under a process of collapsing into a black hole, line element within said dust sphere is,

$$ds^2 = dt^2 - R^2(t)\left(d\chi^2 + \chi^2 d\Omega_2\right), \tag{B.1}$$

where $R(t)$ is a radius of the dust sphere. $\chi$ is a scaling coordinate within the dust sphere with value ranging from zero to one.

Assuming Schwarzschild space background, line element outside the dust sphere is,

$$ds^2 = \left(1 - \frac{a}{r}\right)dt^2 - \left(\frac{dr^2}{1 - \frac{a}{r}} + r^2 d\Omega_2\right). \tag{B.2}$$

Considering the case where $r$ and $\chi$ are constants, the following equations are hold,

$$Inside: \qquad ds^2 = dt^2 - R^2(t)\chi^2 d\Omega_2, \tag{B.3a}$$

$$Outside: \qquad ds^2 = \left(1 - \frac{a}{r}\right)dt^2 - r^2 d\Omega_2. \tag{B.3b}$$

From Lagrangian outside the dust sphere, the following result is obtained,

$$\begin{aligned} L &= \left(1-\frac{a}{r}\right)\dot{t}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2, \\ 0=\frac{\partial L}{\partial t} &= \partial_\tau\left[2\left(1-\frac{a}{r}\right)\dot{t}\right]. \\ \therefore \dot{t} &= \frac{1}{\left(1-\frac{a}{r}\right)}. \end{aligned}$$

By substituting the above result into original Lagrangian, and assuming spherical symmetry, we have,

$$\begin{aligned} L=1 &= \left(1-\frac{a}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{a}{r}}, \\ &= \frac{1}{1-\frac{a}{r}} - \frac{\dot{r}^2}{1-\frac{a}{r}}. \\ \therefore \dot{r}^2 &= \frac{a}{r}. \end{aligned} \tag{B.4}$$

Since $\dot{R}$ has the same form as $\dot{r}$ ($\dot{R}^2 = A/R$), parameter from inside and outside can be connected by simply setting $a = A$ and $r = R$.

To help with further calculation, let us set $R = f(x)$ as follow,

$$\begin{aligned} R &= Ax^2, \\ t &= \frac{2}{3}Ax^3. \end{aligned}$$

It is very straightforward to verify that, $\dot{R}^2 = A/R$, still hold.

From equation of state inside the dust sphere ($w = 0$),

$$\rho = kR^{-3},$$

where $k$ is some constant. Since mass of the dust sphere must equal to $\rho V$, value of $k$ can be calculated by,

$$\begin{aligned} M &= \frac{k}{R^3}\frac{4}{3}\pi R^3 = \frac{4\pi k}{3}. \\ \therefore k &= \frac{3M}{4\pi} = \frac{3A}{8\pi}. \end{aligned}$$

Properties, $2M = a = A$, are used in the last line. By substituting the above solution back into the equation of state, the density of dust sphere is found to be,

$$\rho = \frac{3A}{8\pi}\frac{1}{r^3}. \tag{B.5}$$

We shall now find a moment when the creation of the event horizon occurs. The time when our dust sphere supposes to completely collapse into singularity will be identified as zero. The worldline of light that would be 'freeze' on the horizon is being retraced back in time. This worldline will show us exactly where the event horizon is. The condition at the horizon is, $r = a$. This condition, along with previous continuous conditions, lead us to new conditions of, $R = r = a = A = 2M$. The only value of $x$ which satisfies both $R = A$ and $t < 0$ is, $x = -1$. The event horizon can now be retraced from the moment it envelops our dust sphere to its creation,

$$\begin{aligned} 0 &= dt^2 - R^2 d\chi^2, \\ \int \frac{1}{R} dt &= \int d\chi, \\ \int_{x_0}^{-1} \frac{1}{Ax^2} 2Ax^2 dx &= \int_0^1 d\chi = 1, \\ \therefore x_0 &= -\frac{3}{2}, \end{aligned} \tag{B.6}$$

where $x_0$ is a parameter corresponding to the creation of the event horizon. Thus, the time when the horizon emerges is,

$$t = -\frac{9}{4} A. \tag{B.7}$$

This shows us that the event horizon is created $9A/4$ before our dust sphere actually finishes collapsing into singularity. The time interval between horizon creation $(-9A/4)$ and all material masses cross horizon $(-2A/3)$ is $\frac{19}{12}A$. This period is rather interesting as we have horizon even before the black hole is actually born. Also, at the very moment the surface of our dust sphere touches the horizon, the horizon has a size equals to that of Schwarzschild radius. Some theory suggests that phenomenon prevents the collapsing process from completion.

### B.1.1 A Generalization to Schwarzschild dS and AdS Space

In previous section, we had assumed Schwarzschild space. In order to obtain more generalized solutions, we will now repeat our calculation under Schwarzschild AdS/dS space. In the new back ground, our metric becomes,

$$ds^2 = \left(1 - \frac{a}{r} - \frac{\Lambda}{3} r^2\right) dt^2 - \left(\frac{dr^2}{1 - \frac{a}{r} - \frac{\Lambda}{3} r^2} + r^2 d\Omega_2\right). \tag{B.8}$$

By simply repeating our previous calculation steps, a new $\dot{r}$ is found to be,

$$\dot{r}^2 = \frac{a}{r} + \frac{\Lambda}{3} r^2 \tag{B.9}$$

However, our new $\dot{R}$ will equals to $A/R + \Lambda R^2/3$. Thus, the continuous condition still yields the same restriction, $A = a$ and $R = r$. The other differences are our conditions at the horizon and our form of $R$. For now, we will set,

$$\begin{aligned} R &= f(x), \\ t &= g(x), \end{aligned}$$

In order to find what form $f(x)$ and $g(x)$ should take, we will try to determine the moment a horizon emerges,

$$\begin{aligned} \int \frac{1}{R} dt &= \int d\chi, \\ \int \frac{1}{f(x)} g'(x) dx &= 1. \end{aligned}$$

From $\dot{R}$, a relation between $f(x)$ and $g(x)$ is found to be,

$$\begin{aligned} \dot{R}^2 = \left(\frac{dR/dx}{dt/dx}\right)^2 &= \left(\frac{f'(x)}{g'(x)}\right)^2 = \frac{A}{f(x)} + \frac{\Lambda}{3} f^2(x), \\ g'(x) &= \frac{f'(x) f^{1/2}(x)}{\left(A + \frac{\Lambda}{3} f^3(x)\right)^{1/2}}. \end{aligned}$$

Substitute the relation back into our previous equation, we get,

$$1 = \int \frac{f'(x)}{f^{1/2}\left(A + \frac{\Lambda}{3} f^3(x)\right)^{1/2}} dx.$$

We can now guess the form of $f(x)$,

$$\begin{aligned} &\text{Schwarzschild de Sitter :} && f^3(x) = \frac{3A}{\Lambda} \tan^2 x, \\ &\text{Schwarzschild Anti-de Sitter :} && f^3(x) = \frac{3A}{\Lambda} \sin^2 x. \end{aligned}$$

Consequently, our $g(x)$ becomes, under initial condition $g(0) = 0$,

$$\begin{aligned} &\text{Schwarzschild de Sitter :} && g(x) = 2\frac{A^{1/3}}{(3\Lambda)^{1/2}} \ln(\sec x + \tan x), \\ &\text{Schwarzschild Anti-de Sitter :} && g(x) = 2\frac{A^{1/3}}{(3\Lambda)^{1/2}} x. \end{aligned}$$

Hence, our integration becomes,

$$\text{Schwarzschild de Sitter :} \qquad 1 = \frac{2}{3^{5/6}} \frac{1}{\Lambda^{1/6}} \int_{x_0}^{x_e} \frac{\sec x}{\tan^{2/3} x} dx, \tag{B.10a}$$

$$\text{Schwarzschild Anti-de Sitter :} \qquad 1 = \frac{2}{3^{5/6}} \frac{1}{\Lambda^{1/6}} \int_{x_0}^{x_e} \frac{dx}{\sin^{2/3} x}, \tag{B.10b}$$

where $x_e$ is a parameter corresponding to the moment horizon envelops the dust sphere. The value of $x_e$ is needed to be evaluated. At that particular moment, the following conditions are hold, $R = r = r_H$, where $r_H$ is a horizon radius. We know that radii of the horizon are,

$$r_{Schwarzschild-dS} = \frac{2}{\sqrt{\Lambda}} \cos\left[\frac{\pi}{3} + \frac{1}{3}\cos^{-1}(3M\sqrt{\Lambda})\right], \tag{B.11a}$$

$$r_{Schwarzschild-AdS} = \frac{2}{\sqrt{\Lambda}} \sinh\left[\frac{1}{3}\sinh^{-1}(3M\sqrt{\Lambda})\right]. \tag{B.11b}$$

By solving for values of $x_e$ for each case, $x_0$ can then be evaluated. These steps must be done by computational evaluation. For a dust sphere with solar mass, we find,

$$\text{Schwarzschild de Sitter :} \qquad x_e = -3.92403 \times 10^{-27}, \tag{B.12a}$$

$$\text{Schwarzschild Anti-de Sitter :} \qquad x_e = -4.64483 \times 10^{-37}, \tag{B.12b}$$

while $x_0$ have values of,

$$\text{Schwarzschild de Sitter :} \qquad x_0 = -3.92399 \times 10^{-27}, \tag{B.13a}$$

$$\text{Schwarzschild Anti-de Sitter :} \qquad x_0 = -1.05367 \times 10^{-8}. \tag{B.13b}$$

From these results, the interval of time between the creation of horizon and the moment our horizon completely envelops the dust sphere can be evaluated,

$$\text{Schwarzschild de Sitter :} \qquad t = 6.5001, \tag{B.14a}$$

$$\text{Schwarzschild Anti-de Sitter :} \qquad t = 1.74541 \times 10^{19}. \tag{B.14b}$$

These results are in second unit.

# Appendix C

## Einstein Anomaly

In field theory, an anomaly occurs as an evident that a symmetry of action or corresponding conservation law which was valid in classical theory is violated in quantized version of the theory. These anomalies are indication of inconsistency of the theory. However, there is a different between global and local symmetry breaking. The global symmetry breaking is good for the consistency of the theory as it provides, for example, the physical explanation for the $\pi^0$-decay [22]or the $U(1)$ problem in QCD [23]. While the local symmetry breaking leads to the inconsistency of the quantum theory. The anomalous terms can cause the theory to lose re-normalizeability or $S$-matrix to lose its unitarity property.

An Einstein anomaly is corresponding to the violation of classical conservation law of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} \neq 0$. This anomaly occurs from general coordinate transformations in fermion graviton loop. Other important anomalies are Lorentz anomaly which is an asymmetry of the energy-momentum tensor, $T^{\mu\nu} \neq T^{\nu\mu}$, and Weyl anomaly which is a non-vanishing of the trace of energy-momentum tensor, $T^\alpha_\alpha \neq 0$.

In 'Gravitational Anomaly' chapter, we simply introduce the Einstein anomaly (5.1) without any proof. We will now verify that the aforementioned equation holds [24]. Starting with two-dimension Lagrangian describing a Weyl fermion in a gravitational back ground,

$$\begin{aligned} L &= \frac{ie}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{D}_\mu \frac{1 \pm \gamma_5}{2}\psi, \\ &= \frac{ie}{2}E^{a\mu}\bar{\psi}\gamma_a \overleftrightarrow{D}_\mu \frac{1 \pm \gamma_5}{2}\psi, \end{aligned} \tag{C.1}$$

where $e^a_\mu$ is the zweibein and $E^\mu_a$ is inverse zweibein ($E^\mu_a e^a_\nu = \delta^\mu_\nu$), $e$ is the determinant of the zweibein $e = |\det e^a_\mu|$, and $D_\mu = \partial_\mu + \omega_\mu$ is the covariant derivative with spin connections $\omega_\mu$.

The Einstein anomaly and the Weyl anomaly can be determined by one loop fermion graviton. It is sufficient to use linearized gravitational field,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{mu\nu} + O(\kappa^2), \qquad g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + O(\kappa^2),$$
$$e^a_\mu = \eta^a_\mu + \frac{1}{2}\kappa h^a_\mu + O(\kappa^2), \qquad E^\mu_a = \eta^\mu_a - \frac{1}{2}\kappa h^\mu_a + O(\kappa^2).$$

In two-dimension, $\omega_\mu = 0$. $\kappa$ is absorbed into $h_{\mu\nu}$ for convenience. A linearized interaction Lagrangian is found to be,

$$L_I^{lin} = -\frac{i}{4}\left(h^{a\mu}\bar{\psi}\gamma_a \frac{1 \pm \gamma_5}{2}\overleftrightarrow{\partial^\psi_\mu}\psi + h^\mu_\mu \bar{\psi}\gamma^a \frac{1 \pm \gamma_5}{2}\overleftrightarrow{\partial^\psi_a}\psi\right), \tag{C.2}$$

where $\overleftrightarrow{\partial^\psi_\mu}$ only acts on $\psi$. The second term comes from, $e = 1 - h^\mu_\mu + O(h^2)$.

We have the following properties,

$$\begin{aligned} L_I^{lin} &= -\frac{1}{2}h_{\mu\nu}T^{\mu\nu}, \\ T^{\mu\nu} &= \frac{1}{2}(T^\mu_a E^{a\nu} + T^\nu_a E^{a\mu}). \end{aligned}$$

From these properties, we can read off $T^{\mu\nu}$ as,

$$T^{\mu\nu} = \frac{i}{4}\left(\bar{\psi}E^{a\nu}\gamma_a \frac{1 \pm \gamma_5}{2}\overleftrightarrow{\partial^\mu}\psi + \bar{\psi}E^{a\mu}\gamma_a \frac{1 \pm \gamma_5}{2}\overleftrightarrow{\partial^\nu}\psi\right). \tag{C.3}$$

From the above expression, vertices in the loop diagram is found to be,

$$-\frac{i}{4}(\gamma_\mu(k_1 - k_2)_\nu + \gamma_\nu(k_1 - k_2)_\mu)\frac{1 \pm \gamma_5}{2}. \tag{C.4}$$

The whole amplitude can be given by two-point function,

$$T_{\mu\nu\rho\sigma}(p) = i\int d^2x e^{ipx}\langle 0|T[T_{\mu\nu}(x)T_{\rho\sigma}(0)]|0\rangle.$$

This amplitude can be used to calculate many anomalies; however, only one anomaly interested us; that is the Einstein anomaly ($\nabla^\mu T_{\mu\nu}$). It is quite clear that with some integration by part, differential operator ($\nabla^\mu$) can be moved from $T_{\mu\nu}$ term to $e^{ipx}$ term. In other word, $\nabla^\mu T_{\mu\nu}$ is equivalent to $p^\mu T_{\mu\nu\rho\sigma}$. If $p^\mu T_{\mu\nu\rho\sigma} = 0$, then the Einstein anomaly is vanished. But if $p^\mu T_{\mu\nu\rho\sigma}$ have non-zero value, then that value is our Einstein anomaly.

The amplitude will now be separated into a vector part and an axial part,

$$T_{\mu\nu\rho\sigma} = T^V_{\mu\nu\rho\sigma} + T^A_{\mu\nu\rho\sigma}. \tag{C.5}$$

It is possible to expand the vector part using form-factors,

$$\begin{aligned} T^V_{\mu\nu\rho\sigma}(p) &= p_\mu p_\nu p_\rho p_\sigma T_1(p^2) + (p_\mu p_\nu g_{\rho\sigma} + p_\rho p_\sigma g_{\mu\nu})T_2(p^2) \\ &\quad + (p_\mu p_\rho g_{\nu\sigma} + p_\mu p_\sigma g_{\nu\rho} + p_\nu p_\rho g_{\mu\sigma} + p_\nu p_\sigma g_{\mu\rho})T_3(p^2) \\ &\quad + g_{\mu\nu}g_{\rho\sigma}T_4(p^2) + (g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})T_5(p^2). \end{aligned} \tag{C.6}$$

Generally, the terms $p_\mu p_\nu g_{\rho\sigma}$ and $p_\rho p_\sigma g_{\mu\nu}$ should be separate and have its own form-factor, but due to symmetries ($\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$, and $\mu\nu \leftrightarrow \rho\sigma$) the two terms must have equal value of form-factor. $p^2$ is represented square of momentum amplitude. Likewise, the form factors of the axial part can be expanded as,

$$\begin{aligned} T^A_{\mu\nu\rho\sigma}(p) &= (\epsilon_{\mu\tau}p^\tau p_\nu p_\rho p_\sigma + \epsilon_{\nu\tau}p^\tau p_\mu p_\rho p_\sigma + \epsilon_{\rho\tau}p^\tau p_\mu p_\nu p_\sigma + \epsilon_{\sigma\tau}p^\tau p_\mu p_\nu p_\rho)T_6(p^2) \\ &\quad +(\epsilon_{\mu\tau}p^\tau p_\nu g_{\rho\sigma} + \epsilon_{\nu\tau}p^\tau p_\mu g_{\rho\sigma} + \epsilon_{\rho\tau}p^\tau p_\sigma g_{\mu\nu} + \epsilon_{\sigma\tau}p^\tau p_\rho g_{\mu\nu})T_7(p^2) \\ &\quad +[\epsilon_{\mu\tau}p^\tau(p_\rho g_{\nu\sigma} + p_\sigma g_{\nu\rho}) + \epsilon_{\nu\tau}p^\tau(p_\rho g_{\mu\sigma} + p_\sigma g_{\mu\rho}) \\ &\quad +\epsilon_{\rho\tau}p^\tau(p_\mu g_{\nu\sigma} + p_\nu g_{\mu\sigma}) + \epsilon_{\sigma\tau}p^\tau(p_\mu g_{\nu\rho} + p_\nu g_{\mu\rho})]T_8(p^2). \end{aligned} \tag{C.7}$$

Let us contract $p^\mu$ to identify what is needed to be calculated in order to evaluate the Einstein anomaly,

$$\begin{aligned} p^\mu T^V_{\mu\nu\rho\sigma} &= p_\nu p_\rho p_\sigma(p^2T_1 + T_2 + 2T_3) + p_\nu g_{\rho\sigma}(p^2T_2 + T_4) \\ &\quad +(p_\rho g_{\nu\sigma} + p_\sigma g_{\nu\rho})(p^2T_3 + T_5), \end{aligned} \tag{C.8}$$

$$\begin{aligned} p^\mu T^A_{\mu\nu\rho\sigma} &= \epsilon_{\nu\tau}p^\tau[p_\rho p_\sigma(p^2T_6 + 2T_8) + g_{\rho\sigma}p^2T_7] \\ &\quad +\epsilon_{\rho\tau}p^\tau[p_\nu p_\sigma(p^2T_6 + T_7 + T_8) + g_{\nu\sigma}p^2T_8] \\ &\quad +\epsilon_{\sigma\tau}p^\tau[p_\nu p_\rho(p^2T_6 + T_7 + T_8) + g_{\nu\rho}p^2T_8]. \end{aligned} \tag{C.9}$$

For a pure vector part, we will see later on that there is no anomaly. From (C.8), if left hand side is set to zero, then the following equations are obtained,

$$p^2T_1 + T_2 + T_3 = 0, \tag{C.10a}$$

$$p^2T_2 + T_4 = 0, \tag{C.10b}$$

$$p^2T_3 + T_5 = 0. \tag{C.10c}$$

These conditions are called ‘Ward Identity’.

The amplitude of massive fermion according to Feynman rules is,

$$\begin{aligned} T_{\mu\nu\rho\sigma}(p) &= -\frac{i}{16}Tr\int\frac{d^2k}{(2\pi)^2}\left[[\gamma_\mu(p+2k)_\nu + \gamma_\nu(p+2k)_\mu]\frac{1\pm\gamma_5}{2}\frac{\not{p}+\not{k}+m}{(p+k)^2 - m^2 + i\epsilon}\right. \\ &\quad \left.[\gamma_\rho(p+2k)_\sigma + \gamma_\sigma(p+2k)_\rho]\frac{1\pm\gamma_5}{2}\frac{\not{k}+m}{k^2 - m^2 + i\epsilon}\right]. \end{aligned} \tag{C.11}$$

Naturally, the amplitude has symmetry with the interchange of indexes, which would make a lot of confusion. Therefore, we need to expand $T^V_{\mu\nu\rho\sigma}$ into four different terms each with its own non-interchangeable indexes,

$$T^V_{\mu\nu\rho\sigma} = T^{ni}_{\mu\nu\rho\sigma} + T^{ni}_{\nu\mu\rho\sigma} + T^{ni}_{\mu\nu\sigma\rho} + T^{ni}_{\nu\mu\sigma\rho}.$$

Let us take a closer look at integrand of (C.11). There are two $(1 \pm \gamma_5)/2$ terms,

$$\begin{aligned}
I &= A\frac{1 \pm \gamma_5}{2}B\frac{1 \pm \gamma_5}{2}C, \\
4I &= ABC \pm A\gamma_5 BC \pm AB\gamma_5 C + A\gamma_5 B\gamma_5 C, \\
&= 2ABC \pm (A\gamma_5 BC + AB\gamma_5 C). \\
\therefore I &= \frac{1}{2}ABC \pm \frac{1}{4}(A\gamma_5 BC + AB\gamma_5 C).
\end{aligned}$$

This is possible because there are two $\gamma$-matrices in $B$. The first term will contribute to $T^V_{\mu\nu\rho\sigma}$ while the second term contributes to $T^A_{\mu\nu\rho\sigma}$. As such, $T^V_{\mu\nu\rho\sigma}$ will be twice as large as $T^A_{\mu\nu\rho\sigma}$. In two-dimension, the following property holds,

$$\gamma_\mu\gamma_5 = -\epsilon_{\mu\nu}\gamma^\nu.$$

It is clear how the term $A^{ni}_{\mu\nu}\gamma_5 B^{ni}_{\rho\sigma}C$ become $\mp\frac{1}{2}\epsilon^\tau_\mu T^{ni}_{\tau\nu\rho\sigma}$. Hence,

$$\begin{aligned}
T^A_{\mu\nu\rho\sigma} &= \mp\left[\frac{1}{2}(\epsilon^\tau_\mu T^{ni}_{\tau\nu\rho\sigma} + \epsilon^\tau_\rho T^{ni}_{\mu\nu\tau\sigma}) + \frac{1}{2}(\epsilon^\tau_\nu T^{ni}_{\tau\mu\rho\sigma} + \epsilon^\tau_\rho T^{ni}_{\nu\mu\tau\sigma})\right. \\
&\quad \left. + \frac{1}{2}(\epsilon^\tau_\mu T^{ni}_{\tau\nu\sigma\rho} + \epsilon^\tau_\sigma T^{ni}_{\mu\nu\tau\rho}) + \frac{1}{2}(\epsilon^\tau_\nu T^{ni}_{\tau\mu\sigma\rho} + \epsilon^\tau_\sigma T^{ni}_{\nu\mu\tau\rho})\right]. \qquad \text{(C.12)}
\end{aligned}$$

The form-factors will now be evaluated by utilizing Cutkosky's rules,

$$\begin{aligned}
\mathrm{Im}T^{ni}_{\mu\nu\rho\sigma} &= \frac{1}{32}\int d^2k(p+2K)_\nu(p+2k)_\sigma \\
&\quad \times\left[(p+k)_\mu k_\rho + (p+k)_\rho k_\mu - g_{\mu\rho}(p+k)^\lambda k_\lambda\right] \\
&\quad \times\delta(k^2-m^2)\delta((p+k)^2-m^2)\theta(-k_0)\theta(k_0+p_0). \qquad \text{(C.13)}
\end{aligned}$$

Since we are going to work in near-light speed limit, the term with $m^2$ will be ignored. Considering the following Dirac delta function property,

$$\int dx f(x)\delta(x) = f(x_0), \qquad \text{(C.14)}$$

where $f(x_0) = 0$. Hence $\mathrm{Im}T^{ni}_{\mu\nu\rho\sigma}$ becomes,

$$\begin{aligned}
\mathrm{Im}T^{ni}_{\mu\nu\rho\sigma} &= \frac{1}{32}(p+2K)_\nu(p+2k)_\sigma[(p+k)_\mu k_\rho \\
&\quad +(p+k)_\rho k_\mu - g_{\mu\rho}(p+k)^\lambda k_\lambda]|_{k'}J_0, \qquad \text{(C.15)}
\end{aligned}$$

where $k'$ is a solution to equations $k^2 - m^2 = 0$ and $(p+k)^2 - m^2 = 0$ and $J_0$ equals to,

$$\begin{aligned}
J_0 &= \int d^2k\delta(k^2-m^2)\delta((p+k)^2-m^2)\theta(-k_0)\theta(k_0+p_0), \\
&= \frac{1}{p^2}\left(1 - \frac{4m^2}{p^2}\right)^{-1/2}. \qquad \text{(C.16)}
\end{aligned}$$

Let us solve for $k'$ in term of $p$. Starting from, $k^2-m^2=0$. With the step function $\theta(-k_0)$, $k_0$ is found to be,

$$k_0 = -\sqrt{k_1^2+m^2}.$$

From $(p+k)^2-m^2=0$,

$$\begin{aligned} p^2+2p\cdot k+k^2-m^2 &= 0, \\ p^2+2p\cdot k = p^2+2(p_0k_0-p_1k_1) &= 0, \\ p^2-2(p_0\sqrt{k_1^2+m^2}+p_1k_1) &= 0, \\ \frac{p^2}{2}-p_1k_1 &= p_0\sqrt{k_1^2+m^2}. \end{aligned}$$

By squaring both sides of equation and treat it as a quadratic equation, the value of $k_1$ is found to be,

$$k_1 = -\frac{p_1 \pm p_0\sqrt{1-\frac{4m^2}{p^2}}}{2}. \tag{C.17}$$

Substituting the above value into $k_0=-\sqrt{k_1^2+m^2}$,

$$\begin{aligned} k_0 &= -\sqrt{\left(\frac{p_1 \pm p_0\sqrt{1-\frac{4m^2}{p^2}}}{2}\right)^2+m^2}, \\ &= -\frac{1}{2}\sqrt{\left(p_1 \pm p_0\sqrt{1-\frac{4m^2}{p^2}}\right)^2+4m^2}, \\ &= -\frac{1}{2}\sqrt{p_1^2+p_0^2\left(1-\frac{4m^2}{p^2}\right) \pm 2p_0p_1\sqrt{1-\frac{4m^2}{p^2}}+4m^2}, \\ &= -\frac{1}{2}\sqrt{p_1^2\left(1-\frac{4m^2}{p^2}\right)+p_0^2 \pm 2p_0p_1\sqrt{1-\frac{4m^2}{p^2}}}, \\ &= -\frac{1}{2}\sqrt{\left(p_0 \pm p_1\sqrt{1-\frac{4m^2}{p^2}}\right)^2}. \end{aligned}$$

Thus, $k_0$ can be written in term of $p_0$ and $p_1$ as,

$$k_0 = -\frac{p_0 \pm p_1\sqrt{1-\frac{4m^2}{p^2}}}{2}. \tag{C.18}$$

By substituting both $k_0$ and $k_1$ into $J_0$, we can verify that (C.16) holds.

We will now set $(\mu,\nu,\rho,\sigma)$ into different sets of values to solve for $T_1(p^2)$-

$T_5(p^2)$. For example,

$$
\begin{aligned}
&(0,0,0,0), \\
&\quad LHS \;:\; \frac{J_0}{32}(p+2k)_0(p+2k)_0[(p+k)_0k_0+(p+k)_0k_0-g_{00}(p+k)^\lambda k_\lambda], \\
&\qquad\qquad = -p_1^2(p_0^2+p_1^2)\frac{m^2}{p^2}\left(1-\frac{4m^2}{p^2}\right), \\
&\quad RHS \;:\; p_0^4T_1^{ni}(p^2)+2p_0^2T_2^{ni}(p^2)+4p_0^2T_3^{ni}(p^2)+T_4^{ni}(p^2)+2T_5^{ni}(p^2).
\end{aligned}
$$

However, after substituting every possible sets of ($\mu$,$\nu$,$\rho$,$\sigma$), only the $T_1^{ni}(p^2)$ can be solved,

$$\mathrm{Im}T_1(p^2) = -\frac{1}{4}J_0\frac{m^2}{p^2}\left(1-4\frac{m^2}{p^2}\right),$$

which is not surprising since it is possible to boost our reference frame and obtain different anomalous results. Nevertheless, we would obtain enough relationship between $T_2(p^2)$-$T_5(p^2)$ to verify that (C.10) would hold. Subsequently, by utilizing (C.12), the following properties are obtained,

$$T_6 = \mp\frac{1}{4}T_1, \quad T_7 = \mp\frac{1}{4}T_2, \quad T_8 = \mp\frac{1}{4}T_3.$$

Since we can verify that (C.10) holds regardless of our inability to find exact value of $T_2(p^2)$-$T_5(p^2)$, we know that $p^\mu T^V_{\mu\nu\rho\sigma} = 0$. Hence, only the anomaly in axial part is needed,

$$
\begin{aligned}
p^\mu T^A_{\mu\nu\rho\sigma}(p) &= -T_7\epsilon_{\nu\tau}p^\tau(p_\rho p_\sigma - g_{\rho\sigma}p^2) \\
&\quad -T_8(\epsilon_{\rho\tau}p^\tau(p_\nu p_\sigma - g_{\nu\sigma}p^2)+\epsilon_{\sigma\tau}p^\tau(p_\nu p_\rho - g_{\nu\rho}p^2)), \\
&= \pm\frac{1}{4}T_2\epsilon_{\nu\tau}p^\tau(p_\rho p_\sigma - g_{\rho\sigma}p^2) \\
&\quad \pm T_3(\epsilon_{\rho\tau}p^\tau(p_\nu p_\sigma - g_{\nu\sigma}p^2)+\epsilon_{\sigma\tau}p^\tau(p_\nu p_\rho - g_{\nu\rho}p^2)).
\end{aligned}
$$

Here, the relationship $\epsilon_{\nu\tau}p^\tau p_\rho p_\sigma + \epsilon_{\rho\tau}p^\tau p_\sigma p_\nu + \epsilon_{\sigma\tau}p^\tau p_\nu p_\rho = 0$ is used. Next, in flat space limit ($h \to 0$), the following properties hold,

$$\epsilon_{\nu\tau}p^\tau(p_\rho p_\sigma - g_{\rho\sigma}p^2) = \epsilon_{\rho\tau}p^\tau(p_\sigma p_\nu - g_{\sigma\nu}p^2) = \epsilon_{\sigma\tau}p^\tau(p_\nu p_\rho - g_{\nu\rho}p^2).$$

Thus, our axial anomaly is,

$$
\begin{aligned}
p^\mu T^A_{\mu\nu\rho\sigma}(p) &= \pm\frac{1}{4}(T_2+2T_3)\epsilon_{\nu\tau}p^\tau(p_\rho p_\sigma - g_{\rho\sigma}p^2), \\
&= \mp\frac{p^2}{4}T_1\epsilon_{\nu\tau}p^\tau(p_\rho p_\sigma - g_{\rho\sigma}p^2).
\end{aligned}
$$

Again, we use (C.10). In the end, we do not need to find any specific value of $T_2(p^2)$-$T_5(p^2)$ and the Einstein anomaly is still obtainable. This means our Einstein anomaly is independent of reference frame.

We only need to find the value of $T_1(p^2)$ to complete our evaluation of the Einstein anomaly. From dispersion relation for the form-factors,

$$T(p^2) = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t-p^2}\mathrm{Im}T(t).$$

$T_1(p^2)$ can be evaluated as,

$$T_1(p^2) = -\frac{1}{4\pi}\int_{4m^2}^{\infty}\frac{dt}{t-p^2}\frac{m^2}{t^2}\left(1-\frac{4m^2}{p^2}\right)^{1/2}.$$

To solve the above equation, we set $t = 4m^2\sec^2\theta$,

$$\begin{aligned} T_1(p^2) &= -\left(\frac{m^2}{4\pi}\right)\int_0^{\frac{\pi}{2}}\frac{2(2m)^2\sec^2\theta\tan\theta d\theta}{(2m)^2\sec^2\theta - p^2}\frac{2m\tan\theta}{(2m)^5\sec^5\theta}, \\ &= -\left(\frac{m^2}{4\pi}\right)\frac{2}{(2m)^4}\int_0^{\frac{\pi}{2}}\frac{(\sec^2\theta-1)d\theta}{\sec^3\theta(\sec^2\theta-(p/2m)^2)}. \end{aligned}$$

By utilizing the following identities,

$$\begin{aligned} \frac{x^2-1}{x^3(x^2-A^2)} &= \left(\frac{1-A^2}{A^4}\right)\frac{1}{x} + \frac{1}{A^2}\frac{1}{x^3} + \left(\frac{A^2-1}{A^4}\right)\frac{x}{x^2-A^2}, \\ \int_0^{\frac{\pi}{2}}\cos\theta d\theta &= 1, \\ \int_0^{\frac{\pi}{2}}\cos^3\theta d\theta &= \frac{2}{3}. \end{aligned}$$

$T_1(p^2)$ is found to be,

$$\begin{aligned} T_1(p^2) &= -\left(\frac{m^2}{4\pi}\right)\frac{2}{(2m)^4}\left[\left(\frac{4m^2}{p^2}-1\right)\frac{4m^2}{p^2} + \frac{2}{3}\left(\frac{4m^2}{p^2}\right)\right. \\ &\quad \left. - \frac{4m^2}{p^2}\left(\frac{4m^2}{p^2}-1\right)\int_0^{\frac{\pi}{2}}\frac{\sec\theta d\theta}{\sec^2\theta-(p/2m)^2}\right]. \end{aligned}$$

Next, the last integration is needed to be evaluated,

$$\begin{aligned} \int_0^{\frac{\pi}{2}}\frac{\sec\theta d\theta}{\sec^2\theta-(p/2m)^2} &= \int_0^{\frac{\pi}{2}}\frac{\cos\theta}{1-\left(\frac{p^2}{4m^2}\right)\cos^2\theta}d\theta, \\ &= \int_0^{\frac{\pi}{2}}\frac{d\sin\theta}{\left(1-\frac{p^2}{4m^2}\right)+\left(\frac{p^2}{4m^2}\right)\sin^2\theta}, \\ &= \frac{4m^2}{p^2}\sqrt{\frac{p^2}{4m^2-p^2}}\arctan\sqrt{\frac{p^2}{4m^2-p^2}}. \end{aligned}$$

By substituting the integration result back into our original equation, we obtain,

$$\begin{aligned} T_1(p^2) &= \frac{1}{p^2}\left[\frac{1}{24\pi} - \frac{1}{2\pi}\frac{m^2}{p^2}\right. \\ &\quad \left. + \frac{1}{2\pi}\frac{m^2}{p^2}\sqrt{\frac{4m^2-p^2}{p^2}}\arctan\sqrt{\frac{p^2}{4m^2-p^2}}\right]. \end{aligned} \tag{C.19}$$

Under massless limit, $T_1(p^2)$ simply becomes,

$$p^2 T_1 = \frac{1}{24\pi}.$$

Thus, our anomaly is,

$$p^\mu T_{\mu\nu\rho\sigma}(p) = \mp \frac{1}{96\pi} \epsilon_{\nu\tau} p^\tau (p_\rho p_\sigma - g_{\rho\sigma} p^2). \tag{C.20}$$

We can, then, deduce the Einstein anomaly via Fourier Transformation,

$$\partial^\mu \langle T_{\mu\nu} \rangle = \mp \frac{1}{192\pi} \epsilon_{\mu\nu} \partial^\mu (\partial_\alpha \partial_\beta h^{\alpha\beta} - \partial_\alpha \partial^\alpha h^\beta_\beta). \tag{C.21}$$

This result agrees with linearization of,

$$\nabla^\mu \langle T_{\mu\nu} \rangle = \mp \frac{1}{96\pi} \frac{1}{e} \epsilon^{\alpha\beta} \partial_\alpha \partial_\rho \Gamma^\rho_{\beta\nu}. \tag{C.22}$$

This equation is exactly (5.1) at the beginning of 'Gravitational Anomaly' chapter (Note: $e = \sqrt{-g}$).

# VITAE

Mr. Taum Wuthicharn was born in July $20^{th}$, 1990 and received his Bachelor's degree in Aerospace Engineering from King Mongkut's University of Technology North Bangkok in 2011. While he was studying at King Mongkut's University of Technology North Bangkok, he won a second prize in Mathematic for Engineering competition at Ramkhamhaeng University in November $24^{th}$, 2010. His research interests are in theoretical physics and mathematical physics.

## Presentations

1. A new mass scale, implication on black hole evaporation and holography: The $11^{th}$ annual conference of the Thai physics society; Siam Physics Congress 2016, Baansuan-Khunta Golf&Spa hotel, Ubon Ratchathani, Thailand, June $8^{th}$-$10^{th}$, 2016