CHAPTER III

NUMBER OF MAGIC SQUARES OF A GIVEN ODD ORDER

3.1 There is an unsolved problem about magic squares. That is to find the number of magic squares of any order. It is clear that there is no magic square of order 2. For the odd order it is known that there is only one magic square of order 3, although rotations and reflections give eight different aspects of this square. The number of higher order squares has never been determined, although the number of fifth order magic squaree is known to exceed three-quarters of a million[1].

We shall not go through this problem but only determine how many magic squares can be constructed by the method discussing in chapter II.

For order 3 we have two integers 1, 2 which are relatively prime to 3. We therefore can choose a value for a, b, α, β from these two integers. We have two choices in choosing a, b, α, but for β we have only one choice since the value of $a\beta - b\alpha$ must be different from zero modulo 3. Therefore we have altogether 8 sets of values for a, b, α and β. So we can construct 8 magic squares of 3^{th} order. For each

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[1]Collier's Encyclopedia vol. 13, (New York: C.F. Collier's & Son Corporation, 1959), p.5.

set of values for a, b, α, β, we can find the starting cell from case I of 2.15, from which we have the following values.

a	2,	2,	2,	2,	1,	1,	1,	1
b	1,	2,	1,	2,	2,	2,	1,	1
α	1,	2,	2,	1,	2,	1,	2,	1
β	1,	1,	2,	2,	2,	1,	1,	2
a_0	2,	1,	1,	2,	2,	3,	2,	3
b_0	3,	2,	2,	1,	1,	2,	3,	2

The eight magic squares constructed by these sets of values for a, b, α, β are as follows:

6	1	8
7	5	3
2	9	4

8	3	4
1	5	9
6	7	2

6	7	2
1	5	9
8	3	4

2	9	4
7	5	3
6	1	8

4	9	2
3	5	7
8	1	6

4	3	8
9	5	1
2	7	6

8	1	6
3	5	7
4	9	2

2	7	6
9	5	1
4	3	8

Figure 3.1 The 8 magic squares of order 3.

We therefore can construct all of the eight magic squares of order 3 by our method.

3.2. Now consider the number of magic squares of other odd orders n constructed by our method. In doing this we have first to determine the number of integers less than n which are relatively prime to n. Then we find the number of sets of values for a, b, α, β that we can choose from the integers relatively prime to n in such a way that a, b, α, β satisfy one of the cases of 2.15.

The number of integers less than n and relatively prime to n is given by Euler's totient function[2],

$$P = \phi(n) = n \prod_{p|n} (1-\frac{1}{p}),$$

where the notation indicates a product over all the distinct primes which divide n and an empty product is 1.

In finding the number of sets of values for a, b, α, β which satisfy case I of 2.15 we may choose a, b, α at will from P integers which are relatively prime to n. For each such choice we have to choose β in such a manner that β and $a\beta - b\alpha$ are relatively prime to n. Our problem is now to find the number of choices for β.

Write $a\beta - b\alpha \equiv k \pmod{n}$. Since a, b, and α are relatively prime to n we are to find the number of β's which are also relatively prime to n as k runs through a set of values relatively prime to n. As b and α are

[2] Niven, Ivan. Zuckerman, Herbert S. An Introduction to the Theory of Numbers. (New York: John Wiley & Sons, Inc., 1962), p.35.

relatively prime to n, $b\alpha$ is also relatively prime to n. If β is relatively prime to n so is $a\beta$. Thus we may write $a\beta \equiv x \pmod{n}$ and $b\alpha \equiv \gamma \pmod{n}$ where x and γ are relatively prime to n. Since b and α can be chosen at will from the integers less than n and relatively prime to n, we may choose $b\alpha \equiv \gamma \equiv n-1 \equiv -1 \pmod{n}$ without loss of generality. Hence our problem of finding the number of choices for β is equivalent to that of finding the number Q of integers x such that $1 \leq x \leq n$ and $(x,n) = (x+1,n) = 1$. The solution of this problem is known to be[3]:

$$Q = n \prod_{p|n} \left(1 - \frac{2}{p}\right)$$

We can obtain this result in a way similar to that for obtaining Euler's totient function by means of the principle of cross classification[4] as follows:

Consider the complete set of integers 1, 2, ...,n. If p_1 is any prime divisor of n, the number of integers less than n which are multiples of p_1 is $\frac{n}{p_1}$. In finding the number of integers which are relatively prime to n we therefore eliminate first all the $\frac{n}{p_1}$ multiples of p_1.

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[3] Ibid p. 37.

[4] Le Veque, William Judson. Topic in Number Theory vol. I. (Massechusetts: Addison-wesley Publishing Company Inc., 1958). pp.84-85.

Now when 1 (or in general δ) is added to each integer in the set 1, 2, ...,n we get again a complete set of incongruent integers modulo n from which again there are $\frac{n}{p_1}$ multiples of p_1 to be eliminated. For the prime p_1 we have therefore to eliminate $\frac{2n}{p_1}$ integers from the complete set of integers 1, 2, ...,n. The argument is the same as that for obtaining Euler's totient function but we have to repeat each step twice. Our result is therefore as follows:

$$Q = n - \sum \frac{2n}{p_1} + \sum \frac{4n}{p_1 p_2} - \sum \frac{8n}{p_1 p_2 p_3} + \ldots\ldots\ldots$$

which may be written in the form

$$Q = n \prod_{p|n} (1 - \frac{2}{p}).$$

Notice that for one restriction on β , that is β is relatively prime to n, we have $n \prod_{p|n} (1 - \frac{1}{p})$ choices for choosing β. For two restrictions on β, that is β and $a\beta - b\alpha$ are relatively prime to n, we have $n \prod_{p|n} (1 - \frac{2}{p})$ choices for choosing β. Continuing the same process as above we find $n \prod_{p|n} (1 - \frac{3}{p})$ choices for β with three restrictions and $n \prod_{p|n} (1 - \frac{4}{p})$ choices for four restrictions.

We therefore have $Q = n \prod_{p|n} (1 - \frac{2}{p})$ choices in choosing β such that β and $a\beta - b\alpha$ are relatively prime to n.

Now we have P ways of choosing each of the numbers a, b and α and Q ways of choosing β. By the principle of permutations and combinations[5] we therefore get P^3Q sets of values for a, b, α, β such that $(a,n) = (b,n) = (\alpha,n) = (\beta,n) = (a\beta - b\alpha, n) = 1$.

That is there are P^3Q sets of values for a, b, α, β that satisfy case I of 2.15.

3.3 In finding the number of sets of values for a, b, α, β which satisfy case II of 2.15 we may choose b, α at will from P integers less than n and relatively prime to it. Then β must be chosen so that β, $\alpha+\beta$ and $\alpha-\beta$ are relatively prime to n. Here again we can determine the number of choices for β by using the principle of cross classification as in 3.2 and we have here $R = n \prod_{p/n} (1 - \frac{3}{p})$ choices for β. Finally a must be chosen so that a and $a\beta - b\alpha$ are relatively prime to n. This number of choices for a is Q as in 3.2. Therefore we have P choices for b and α, Q choices for a and R choices for β. Hence we get P^2QR sets of values for a, b, α, β that satisfy case II of 2.15.

Similarly for case III we also have P^2QR sets of values for a, b, α and β.

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[5]Hall, H.S and Knight S.R. Higher Algebra (London: Macmillan and Co., Limited, 1950) p.115

3.4 For case IV of 2.15 we have to find the number of sets of values for a, b, α, β such that a, b, α, β, $a\beta - b\alpha$, $a \pm b$ and $\alpha \pm \beta$ are relatively prime to n.

We may choose a, α at will from P integers less than n and relatively prime to it. So we have P choices for choosing a and α. The b must be chosen so that b, $a+b$ and $a-b$ are relatively prime to n. This is the same as in 3.3; we have R choices for b. Finally β must be chosen so that β, $\alpha+\beta$, $\alpha-\beta$ and $a\beta - b\alpha$ shall be relatively prime to n. Here again we can approach our problem for finding the number of choices for β in a way similar to that in 3.2. By further applying the principle of cross classification we get $n \prod_{p|n} (1 - \frac{4}{p})$ choices for this β. Let this number be denoted by S. For case IV, we therefore have P choices for a and α, R choices for b and S choices for β. Hence the number of sets of values for a, b, α, β in case IV is P^2RS.

3.5 We shall now find the number of magic squares of any odd order n which can be constructed by our method.

Since we have P^2RS sets of values for a, b, α, β in case IV, and since for each set of values for a, b, α, β in this case we can construct n^2 magic squares, we can therefore construct n^2P^2RS magic squares in case IV.

Consider the total number of sets of values for a, b, α, β in case II and case III. We can see that the sets of values for a, b, α, β in these two cases include the sets of values for a, b, α, β in case IV. Therefore the total number of sets of values for a, b, α, β in cases II and III is $2P^2QR - P^2RS$ or $P^2R(2Q-S)$.

Now since the sets of values for a, b, α, β in cases II and III also occur in case I, the number of sets of values for a, b, α, β that satisfy case I but not cases II and III is $P^3Q - P^2R(2Q-S)$. Hence the number of magic squares obtainable by case I is $P^3Q - P^2R(2Q-S)$.

Again, the sets of values for a, b, α, β in case IV also occur in cases II and III. Therefore the number of sets of values for a, b, α, β that satisfy cases II and III but not case IV is $P^2R(2Q-S) - P^2RS$ or $2P^2R(Q-S)$. The number of magic squares obtainable by cases II and III but not case IV is therefore $2nP^2R(Q-S)$.

3.6 The total number of magic squares is of course the sum of the numbers of magic squares obtained by cases I, II, III and IV. Let N denote the total number of magic squares obtainable by our method.

$$\text{Then } N = P^3Q - P^2R(2Q-S) + 2nP^2R(Q-S) + n^2P^2RS$$
$$= P^2\left[PQ + (n-1)R\left\{2Q+(n-1)S\right\}\right]$$

The formula, therefore, for the number of magic squares of any odd order n obtainable by our method is

$$N = P^2\left[PQ + (n-1)R\left\{2Q + (n-1)S\right\}\right]$$

where
$$P = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$$
$$Q = n\prod_{p|n}\left(1 - \frac{2}{p}\right)$$
$$R = n\prod_{p|n}\left(1 - \frac{3}{p}\right)$$
$$S = n\prod_{p|n}\left(1 - \frac{4}{p}\right)$$

and the notation indicates a product over all the distinct primes which divide n.

Notice that for an odd n containing the factor 3, the number R will be zero. Therefore our formula in this special case will be reduced to

$$N = P^3Q.$$

This points out the fact which we noted before in 2.35, that the conditions in cases II, III, IV can never be fulfilled when n is 3 or when n contains the factor 3. For such an odd magic squares can be obtained only from case I.