CHAPTER IV

SOME PROPERTIES OF THE MAGIC SQUARES

4.1 We shall now discuss the relation between magic squares and Latin squares. First of all we shall state the following definition:

Definition.[1] A Latin square of side n is an arrangement of n letters (or n integers from 0 to n-1) into n^2 cells of a square in such a way that every row and every column contains every letter (integer) exactly once. Two Latin squares are termed orthogonal if, when one is superimposed upon the other, every ordered pair of symbols occurs exactly once in the resulting square.

Now consider the magic square obtained by our method. From 2.6 we can see that in each row and in each column both values of q and s are distinct. Since the values of q and s run through the complete set of integers 0, 1,..... ....., n-1 then every row and every column contains every integer exactly once. Therefore the values of q and s both form Latin squares. And since there is one and only one s for each q and each s is distinct from all the others in each row and in each column, we can see that when q and s are written together in each cell as ordered pairs (q,s)

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[1] H.B. Mann. Analysis and Design of Experiments. (New York: Dover Publications, Inc., 1949), p.87.

every ordered pair (q, s) occurs exactly once in the square. Thus the Latin squares of q and s are orthogonal.

Therefore from the magic squares of odd order obtained by our method we can find pairs of orthogonal Latin squares by writing the values of q and s in place of the integers m where q, s and m satisfy the relation $m = nq+s+1$ as in 2.5.

Since the numbers in the diagonals of the Latin squares and orthogonal Latin squares are subject to no special conditions, we can obtain orthogonal Latin squares from the quasi-magic squares of 2.8.

It follows that from a given pair of orthogonal Latin squares we can always obtain a quasi-magic square. This holds both for odd and even order squares. The converse is not always true for there may be magic squares such that the values of q and s do not run through the complete set of integers from 0 to n-1, as our discussion of the diagonals in the various cases of chapter II shows. The following magic squares are examples of even and odd orders which do not give pairs of orthogonal Latin squares.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

3	8	22	25	7
2	16	9	14	24
21	11	13	15	5
20	12	17	10	6
19	18	4	1	23

Figure 4.1. Magic squares which do not give orthogonal Latin squares.

However, every quasi-magic square and every magic square of any odd order obtained by our method always gives a pair of orthogonal Latin squares.

Example. If in the magic square of order 7 on page 17 in 2.11 of chapter II, we replace every integer m in each cell by the ordered pair (q,s) which satisfies the relation m = nq+s+1, we have the following pair of orthogonal Latin squares of order 7:

53	00	24	41	65	12	36
06	23	40	64	11	35	52
22	46	63	10	34	51	05
45	62	16	33	50	04	21
61	15	32	56	03	20	44
14	31	55	02	26	43	60
30	54	01	25	42	66	13

Figure 4.2. A pair of orthogonal Latin squares of order 7.

The underlined set of integers is the Latin square formed by the values of q and the other set of integers is the Latin square formed by the values of s.

4.2 If we add the same value to the integer in each cell of a magic square we again get a magic square with the common sum increased by the order of the square times the added value.

From this fact we can see that the set of n^2 consecutive integers starting from any number can be arranged in the form of a magic square the same manner as we have discussed in chapter II. The sum of the integers in each row and each column, as well as in each main diagonal is $\frac{n(n^2+2x+1)}{2}$, where n is the order of the magic square and the consecutive integers start from the number x+1. This is because starting from x+1 n^2 consecutive numbers reach the number $x+n^2$. Now since the sum of all numbers 1, 2, ..., $x+n^2$ amounts to $\frac{(x+n^2)(x+n^2+1)}{2}$ and the sum of all numbers 1, 2, ..., x amounts to $\frac{x(x+1)}{2}$, the sum of all the numbers x+1, x+2,.... ...., $x+n^2$ is $\frac{(x+n^2)(x+n^2+1)}{2} - \frac{x(x+1)}{2}$ or $\frac{n^2(n^2+2x+1)}{2}$.

Hence the common value of the sum in each row, each column and each main diagonal is $\frac{n(n^2+2x+1)}{2}$.

Now if the set of integers is not consecutive but is the set of arithmetical progression with common difference d and the first term is A; that is, this set of integers is in the form A, A+d, A+2d,.........., $A+(n^2-2)d$, $A+(n^2-1)d$. We can show that by arrangements similar to those of the magic square discussed in chapter II we may also form a magic square.

Since if we put all the integers of this set into the square in a way similar to that of the magic squares we constructed formerly every cell will have one of the values listed above. We can then subtract the common value A from every cell without losing the magic property of the sums in the columns, rows and main diagonals, for we have subtracted the same value from each sum. The set of integers is now $0, d, 2d, \ldots\ldots, (n^2-1)d$. We can see that every integer has d as common divisor. We may therefore divide every integer by d without losing the magic property of the sums. The set of integers will now be $0, 1, 2, \ldots\ldots, n^2-1$ which is the set of consecutive numbers from 0 up to n^2-1. Therefore these consecutive integers form a magic square. Hence above arithmetical progression forms a magic square with the common sum $\frac{nd(n^2-1)}{2} + nA$, where n is the order of the magic square, d is the common difference of the arithmetical progression and A is the first term of this progression.

We therefore can say that every arithmetical progression containing n^2 terms may be arranged in the form of a magic square the same manner as a set of consecutive numbers.

4.3 We shall now show that if we can construct magic squares of order n and m, then we can construct a magic square of order nm by dividing the square of $(mn)^2$ cells into m^2 small squares[2]. Number each of these m^2 small squares from 1 to m^2 in such a manner that the numerical labels construct a magic square of order m. Then in each small square which is divided into n^2 cells, construct a magic square of order n by arranging the consecutive integers from 1 to n^2 in the small square labelled 1, and the consecutive integers from n^2+1 to $2n^2$ in the small square of number 2, and so on, until the consecutive integers from $(m^2-1)n^2+1$ to m^2n^2 have been inserted into the small square labelled m^2. We then have a magic square of order mn.

The reason is that the common sums of the magic squares in each small square labelled 1, 2, 3, ..., m^2 can be determined as in 4.2 to be $\frac{n(n^2+1)}{2}$, $\frac{n(3n^2+1)}{2}$, $\frac{n(5n^2+1)}{2}$, ..., $\frac{n(2m^2n^2-n^2+1)}{2}$ respectively. From which we can see that these are the arithmetical progression with the common difference n^3. By 4.2 we therefore have magic square of order m composed of the integers in the arithmetical progression $\frac{n(n^2+1)}{2}$, $\frac{n(3n^2+1)}{2}$, ..., $\frac{n(2m^2n^2-n^2+1)}{2}$ and the common sum of this magic square is $\frac{mn(m^2n^2+1)}{2}$.

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[2]Collier's <u>Encyclopedia</u> vol. 13. (New York: C.F. Collier's & Son Corporation, 1959), p.4