CHAPTER III

NUMERICAL PROCEDURE

Boundary Discretization

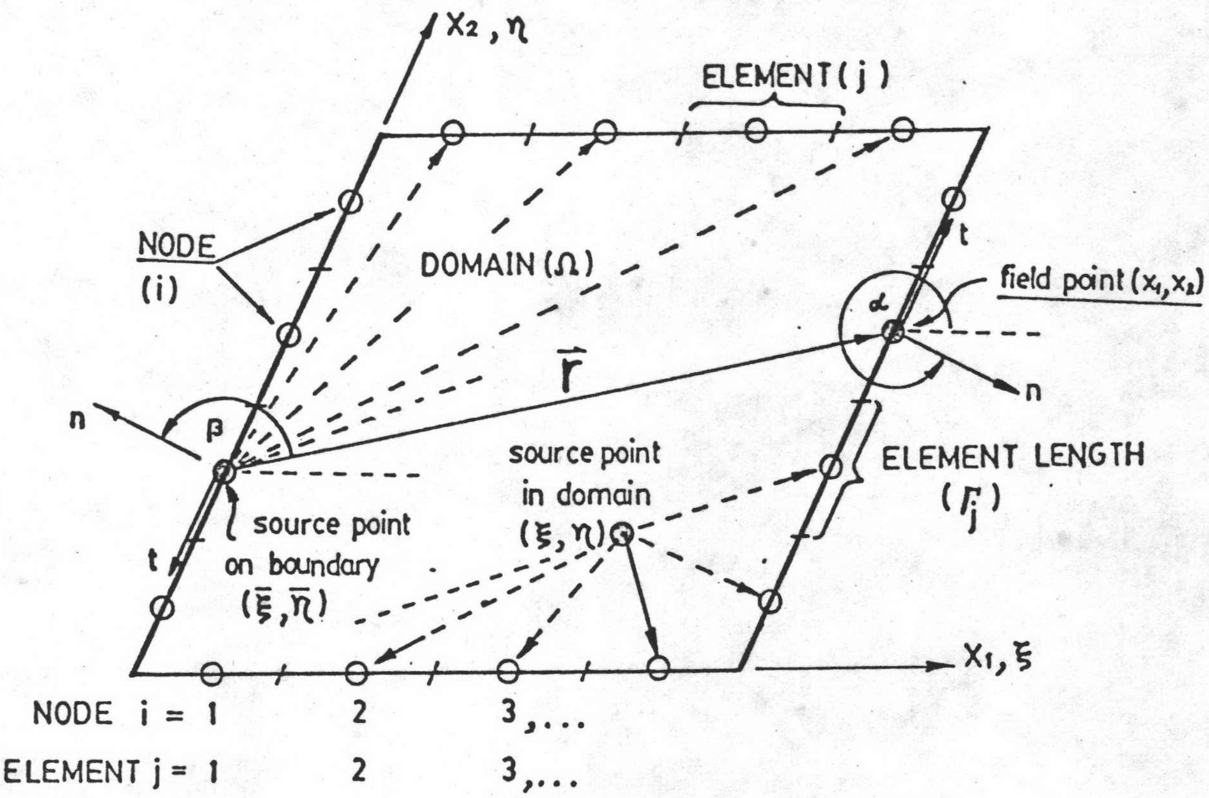

FIG.6 Boundary Nodes , Elements and Sign Convention

The integral equations (28) and (30) can be discretized into a series of elements . The boundary of plate is devided into a series of " segments " or " boundary element " as shown in FIG.6 . The points whose unknown values must be determined are called " nodes (i) " and are taken to be in the middle of each element (j) . The unknown function ($W[X_1,X_2]$, $N[X_1,X_2]$, $M[X_1,X_2]$, $V[X_1,X_2]$) are discretely assumed to be constant values over the interval of element (Γ_j) . Thus the integrals equation (27) , (28) and (30) can be approximate by summations. Hence , the values at nodes ($\bar{\xi}_j,\bar{\eta}_j$) in equation (28) and (30) can be calculated . In the case of skew plate , there are four corners ,therefore $j = 1,2,...,n+4$, where n is the total number of boundary elements .

The source points on the boundary node (i) can also be replaced numerically by a set of 2n+4 equations (n+4 equations for equation (28) and the others for equation (30)) with 4n+8 unknowns as follow :

deflection	W_i	; at nodes $i = 1,2,...,n+4$
normal slope	N_i	; at nodes $i = 1,2,...,n$
bending moment	M_i	; at nodes $i = 1,2,...,n$
kirchoff's shear	V_i	; at nodes $i = 1,2,...,n$
corner force	R_i	; at nodes $i = 1,2,...,4$

But 2n+4 of these unknowns can found from boundary conditions. Thus , the number of 2n+4 algebraic equations are sufficient .

Rewriting of equation (27) , (28) and (30) yields

$$W(\xi,\eta) = -\sum_{j=1}^{n} W_j[X_1,X_2]\int_{\Gamma j} \overset{*}{V}[X_1,X_2;\xi,\eta]d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} N_j[X_1,X_2]\int_{\Gamma_j}\overset{*}{M}[X_1,X_2;\xi,\eta]d\Gamma_j[X_1,X_2]$$

$$-\sum_{j=1}^{n} M_j[X_1,X_2]\int_{\Gamma_j}\overset{*}{N}[X_1,X_2;\xi,\eta]d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} V_j[X_1,X_2]\int_{\Gamma_j}\overset{*}{W}[X_1,X_2;\xi,\eta]d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=n+1}^{n+4} R_j[X_1,X_2]\overset{*}{W}_j[X_1,X_2;\xi,\eta]$$

$$-\sum_{j=n+1}^{n+4} W_j[X_1,X_2]\overset{*}{R}_j[X_1,X_2;\xi,\eta]$$

$$+\int_{\Omega} q\overset{*}{W}[X_1,X_2;\xi,\eta]d\Omega[X_1,X_2] \tag{31}$$

$$\frac{\gamma}{2\pi} W(\bar{\xi}_1,\bar{\eta}_1) = -\sum_{j=1}^{n} W_j[X_1,X_2]\int_{\Gamma_j}\overset{*}{V}[X_1,X_2;\bar{\xi}_1,\bar{\eta}_1]d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} N_j[X_1,X_2]\int_{\Gamma_j}\overset{*}{M}[X_1,X_2;\bar{\xi}_1,\bar{\eta}_1]d\Gamma_j[X_1,X_2]$$

$$-\sum_{j=1}^{n} M_j[X_1\ X_2]\int_{\Gamma_j}\overset{*}{N}[X_1,X_2;\xi_i,\eta_i]d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} V_j[X_1,X_2]\int_{\Gamma_j}\overset{*}{W}[X_1,X_2;\xi_i,\eta_i]d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=n+1}^{n+4} R_j[X_1,X_2]\overset{*}{W}_j[X_1,X_2;\xi_i,\eta_i] \quad - \quad \sum_{j=n+1}^{n+4} W_j[X_1,X_2]\overset{*}{R}_j[X_1,X_2;\xi_i,\eta_i]$$

$$+\int_{\Omega} q\overset{*}{W}[X_1,X_2;\xi_i,\eta_i]d\Omega[X_1,X_2] \qquad ; \quad i = 1,2,\ldots,n+4 \qquad (32)$$

$$\frac{\gamma}{2\pi}N(\xi_i,\eta_i) = -\sum_{j=1}^{n} W_j[X_1,X_2]\int_{\Gamma_j}\frac{\partial\overset{*}{V}[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} N_j[X_1,X_2]\int_{\Gamma_j}\frac{\partial\overset{*}{M}[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} M_j[X_1,X_2]\int_{\Gamma_j}\frac{\partial\overset{*}{N}[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=1}^{n} V_j[X_1,X_2]\int_{\Gamma_j} \frac{\partial \overset{*}{W}[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}\, d\Gamma_j[X_1,X_2]$$

$$+\sum_{j=n+1}^{n+4} R_j[X_1,X_2]\frac{\partial \overset{*}{W}_j[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}$$

$$-\sum_{j=n+1}^{n+4} W_j[X_1,X_2]\frac{\partial \overset{*}{R}_j[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}$$

$$+\int_{\Omega} q\, \frac{\partial \overset{*}{W}[X_1,X_2;\xi_i,\eta_i]}{\partial n[\xi,\eta]}\, d\Omega[X_1,X_2] \quad ; \; i = 1,2,\ldots,n \qquad (33)$$

All of the virtual plate terms ($\overset{*}{V},\overset{*}{M},\ldots$,etc.) are determined by substitute equation (26) into equations (17) , (18) , (19) and (20) and the normal derivative of equations (17) , (18) , (19) and (20) (as shown in appendix(C)) as a functions of $r^2\ln(r)$, $r\ln(r)$, $\ln(r)$, $1/r$ and $1/r^2$. The integral terms of equations (31) , (32) and (33) ($\int_{\Gamma_j}\overset{*}{W}[X_1,X_2;\xi_i,\eta_i]d\Gamma_j[X_1,X_2]$, $\int_{\Gamma_j}\overset{*}{N}[X_1,X_2;\xi_i,\eta_i]d\Gamma_j[X_1,X_2]$,...,etc.) can be calculated numerically by using Simple Gauss Quadrature rules [11] for all elements , except the elements where the source point and the field point coincide which are called " singular element " . Considering equation (31) , (32) and (33) for singular element that one wish to integrate $r^2\ln(r)$, $r\ln(r)$, $\ln(r)$, $1/r$ and $1/r^2$ along lines which includes $r = 0$. No problems arise in the integration of $r^2\ln(r)$, $r\ln(r)$, along the line which includes $r = 0$.

However ,these terms ; $\ln(r)$, $1/r$ and $1/r^2$ also occur in the expressions for $\overset{*}{M}$, $\overset{*}{V}$, $\partial\overset{*}{N}/\partial n$, $\partial\overset{*}{M}/\partial n$ and $\partial\overset{*}{V}/\partial n$. In order to cope with these integral problems , the treatment of singularities by equilibrium equation and analysis method are introduced in CHAPTER IV.

Matrix Formulation

Since the equation (32) will be expressed in n+4 equations (total boundary elements " n " and 4 corners) and also another n equations from expression (33) ,therefore it will be more convenient to write these equations in matrix form respectively :

$$[\overset{*}{V}][W]-[\overset{*}{M}][N]+[\overset{*}{N}][M]-[\overset{*}{W}][V]+[\overset{*}{R}][W_c]-[\overset{*}{W}_c][R] = [\overset{*}{Q}] \qquad (34)$$

$$[\overset{**}{V}][W]-[\overset{**}{M}][N]+[\overset{**}{N}][M]-[\overset{**}{W}][V]+[\overset{**}{R}][W_c]-[\overset{**}{W}_c][R] = [\overset{**}{Q}] \qquad (35)$$

where

$[W],[N],[M],[V]$ = nx1 unknown matrices of deflection , normal slope, bending moment and Kirchoff's shear force at boundary of real plate respectively.

$[W_c],[R]$ = 4x1 unknown matrices of deflection at corner and corner force of real plate.

$[\overset{*}{W}],[\overset{*}{N}],[\overset{*}{M}],[\overset{*}{V}]$ = (n+4)xn known matrices of deflection,normal slope, bending moment and Kirchoff's shear force at boundary of virtual plate respectively.

$[\overset{*}{W}_c],[\overset{*}{R}]$ = (n+4)x4 known matrices of deflection at corner and corner force of real plate.

$[\overset{**}{W}],[\overset{**}{N}],[\overset{**}{M}],[\overset{**}{V}]$ = nxn known matrices of derivative terms of deflection , normal slope , bending moment and Kirchoff's shear force at boundary of virtual plate respectively.

$[\overset{**}{W}_c],[\overset{**}{R}]$ = 4x4 known matrices of derivative terms of deflection at corner and corner force of real plate.

$[\overset{*}{Q}]$ = $(n+4)\times 1$ known matrices of the domain integral term of equation (32)

$[\overset{**}{Q}]$ = $n\times 1$ known matrices of the domain integral term of equation (33)

Notes : the element of matrices

$$\left.\begin{aligned} \overset{*}{V}_{ij} &= \overset{*}{V}_{ij} \\ \overset{**}{M}_{ij} &= \overset{**}{M}_{ij} \end{aligned}\right\} \text{ when } i \neq j$$

and

$$\left.\begin{aligned} \overset{*}{V}_{ij} &= \overset{*}{V}_{ij} + \gamma/2\pi \\ \overset{**}{M}_{ij} &= \overset{**}{M}_{ij} - \gamma/2\pi \end{aligned}\right\} \text{ when } i = j$$

The expressions (34) and (35) respresent the sets of boundary integral equations relating the boundary factors on boundary to the values of deflection ,normal slope , bending moment and Kirchoff's shear at any point to one another , according to the boundary condition of the plate as follows :

1) Simply-Supported Edges : [M] , [W] = 0

equation (34) and (35) can be replaced by

$$[\overset{*}{M}][N]+[\overset{*}{W}][V]+[\overset{*}{W}_c][R] = -[\overset{*}{Q}] \qquad (36)$$

$$[\overset{**}{M}][N]+[\overset{**}{W}][V]+[\overset{**}{W}_c][R] = -[\overset{**}{Q}] \qquad (37)$$

2) Clamped Edges : [W] , [N] = 0

equation (34) and (35) can be replaced by

$$[\overset{*}{N}][M]-[\overset{*}{W}][V]-[\overset{*}{W}_c][R] = [\overset{*}{Q}] \qquad (38)$$

$$[\overset{**}{N}][M]-[\overset{**}{W}][V]-[\overset{**}{W}_c][R] = [\overset{**}{Q}] \qquad (39)$$

3) Free Edges : [M] , [V] = 0

equation (34) and (35) can be replaced by

$$[\overset{*}{V}][W]-[\overset{*}{M}][N]+[\overset{*}{R}][W_c] = [\overset{*}{Q}] \qquad (40)$$

$$[\overset{**}{V}][W]-[\overset{**}{M}][N]+[\overset{**}{R}][W_c] = [\overset{**}{Q}] \qquad (41)$$

A similar technique can be used for mixed boundary conditions as shown in the example III and IV . Therefore , for any boundary condition only 2n+4 algebraic equations are required to cope with the 2n+4 unknowns . Hence, one can re-replace these set of equations in the form of

$$[A][B] = [C] \qquad (42)$$

where

[A] = coefficient matrix [(2n+4)x(2n+4)] of linear equations

[B] = unknown matrix [(2n+4)x1]

[C] = known matrix [(2n+4)x1] of domain integral

For example , in the case of all edges are clamped , one can write equation (42) as :

$$\begin{bmatrix} [\overset{*}{N}] & -[\overset{*}{W}] & -[\overset{*}{W}_c] \\ & & \\ [\overset{**}{N}] & -[\overset{**}{W}] & -[\overset{**}{W}_c] \end{bmatrix} \begin{bmatrix} [M] \\ [V] \\ [R] \end{bmatrix} = \begin{bmatrix} [\overset{*}{Q}] \\ \\ [\overset{**}{Q}] \end{bmatrix}$$

The unknowns in equation (42) can be calculated using " Gauss Elimination Method " .

Domain Solution

By the " Gauss Elimination Method " , all of the boundary function are known . Therefore , one can easily determine the values of deflections at any point inside the domain by equation (31).

Finally , the desired stress resultants inside the domain can be obtained by appropriate differentiating of the deflection $W[\xi,\eta]$ in equation (31).