CHAPTER I

INTRODUCTION

In this thesis, we denote $\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{Q}$ = the set of all rational numbers, $\mathbb{R}$ = the set of all real numbers.

In this chapter I, we review all the known results which are needed in this thesis. The only new result is Proposition 1.3 .

From now on whenever a result has been taken from a book we shall write the bibliography number of the book before writing the proof. If a proof was not taken from a reference, then no bibliography number will be written before the proof.

The Algebra of Quaternions

Let 1,i,j,k denote the elements of the standard basis for $\mathbb{R}^4$. Let $\mathbb{H}=\{a_1+ia_2+ja_3+ka_4 / \ a_1,a_2,a_3,a_4\in\mathbb{R}\}$. For any $x = a_1+ia_2+ja_3+ka_4$, $y = b_1+ib_2+jb_3+kb_4$ in $\mathbb{H}$, we define

$$x+y = (a_1+b_1)+i(a_2+b_2)+j(a_3+b_3)+k(a_4+b_4)$$

$$x.y = (a_1b_1-a_2b_2-a_3b_3-a_4b_4)+i(a_1b_2+a_2b_1+a_3b_4-a_4b_3)$$
$$+j(a_1b_3+a_3b_1+a_4b_2-a_2b_4)+k(a_1b_4+a_4b_1+a_2b_3-a_3b_2).$$

Then $i^2=j^2=k^2=-1$, $ij=k=-ji$, $jk=i=-kj$, and $ki=j=-ik$.

Let $x = a_1+ia_2+ja_3+ka_4\in\mathbb{H}$. The <u>conjugate</u> of x, $\overline{x}$, is defined to be the quaternion $a_1-(ia_2+ja_3+ka_4)$. $|x|=(x.\overline{x})^{\frac{1}{2}} =(\sum_{n=1}^{4}a_n^2)^{\frac{1}{2}}$ is called the <u>absolute value</u> of the quaternion x. Hence $\overline{s+t} = \overline{s}+\overline{t}$, $\overline{\lambda s} = \lambda\overline{s}$, $\overline{\overline{s}} = s$ and $\overline{s.t} = \overline{t}.\overline{s}$ for all $s,t\in\mathbb{H}$, $\lambda\in\mathbb{R}$. See [6].

1.1 <u>Proposition</u> $\mathbb{H}$ is a division ring.

<u>Proof</u> See [6]. #

1.2 <u>Proposition</u> $|a.b| = |a||b|$ for all $a, b \in \mathbb{H}$.

<u>Proof</u> See [6]. #

1.3 <u>Proposition</u> If $q_1, q_2, \ldots, q_n$ are quaternion numbers, then there is a subset S of $\{1,2,3,\ldots,n\}$ such that

$$\left|\sum_{i \in S} q_i\right| \geq \frac{1}{4(2^5)} \sum_{i=1}^{n} |q_i| .$$

<u>Proof</u> For each $i' \in \{1,2,\ldots,n\}$, let $q_{i'} = a_{i'} + ib_{i'} + jc_{i'} + kd_{i'}$, where $a_{i'}, b_{i'}, c_{i'}, d_{i'} \in \mathbb{R}$. Let $w = |q_1| + |q_2| + |q_3| + \ldots + |q_n|$ and let

$A_1 = \{x_1 + ix_2 + jx_3 + kx_4 / x_i \geq 0, i = 1,2,3,4\}$,

$A_2 = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_{i'} \geq 0, i' = 2,3,4\}$,

$A_3 = \{x_1 + ix_2 + jx_3 + kx_4 / x_2 \leq 0, x_{i'} \geq 0, i' = 1,3,4\}$,

$A_4 = \{x_1 + ix_2 + jx_3 + kx_4 / x_3 \leq 0, x_{i'} \geq 0, i' = 1,2,4\}$,

$A_5 = \{x_1 + ix_2 + jx_3 + kx_4 / x_4 \leq 0, x_{i'} \geq 0, i' = 1,2,3\}$,

$A_6 = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_2 \leq 0, x_3 \geq 0, x_4 \geq 0\}$,

$A_7 = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0\}$,

$A_8 = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_2 \geq 0, x_3 \geq 0, x_4 \leq 0\}$,

$A_9 = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \geq 0, x_2 \leq 0, x_3 \geq 0, x_4 \leq 0\}$,

$A_{10} = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, x_4 \leq 0\}$,

$A_{11} = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \geq 0, x_2 \geq 0, x_3 \leq 0, x_4 \leq 0\}$,

$A_{12} = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_2 \leq 0, x_3 \leq 0, x_4 \geq 0\}$,

$A_{13} = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_2 \leq 0, x_3 \geq 0, x_4 \leq 0\}$,

$A_{14} = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \leq 0, x_2 \geq 0, x_3 \leq 0, x_4 \leq 0\}$,

$A_{15} = \{x_1 + ix_2 + jx_3 + kx_4 / x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, x_4 \leq 0\}$, and

$$A_{16}=\{x_1+ix_2+jx_3+kx_4/x_1\le 0, x_2\le 0, x_3\le 0, x_4\le 0\}.$$

The quaternion numbers are distributed into 16 sets. Then there exists $t\in\{1,2,\ldots,16\}$ such that A_t has the property that the sum of $|q_i'|$ for which $q_i'\in A_t$ is at least $\frac{w}{16}$. We can distribute A_t into 4 sets, that is,

$$A_{t1}=\{x_1+ix_2+jx_3+kx_4/\ |x_1|\ge|x_2|,\ |x_1|\ge|x_3|,\ |x_1|\ge|x_4|\},$$

$$A_{t2}=\{x_1+ix_2+jx_3+kx_4/\ |x_2|\ge|x_1|,\ |x_2|\ge|x_3|,\ |x_2|\ge|x_4|\},$$

$$A_{t3}=\{x_1+ix_2+jx_3+kx_4/\ |x_3|\ge|x_1|,\ |x_3|\ge|x_2|,\ |x_3|\ge|x_4|\},$$

and

$$A_{t4}=\{x_1+ix_2+jx_3+kx_4/\ |x_4|\ge|x_1|,\ |x_4|\ge|x_2|,\ |x_4|\ge|x_3|\}.$$

Then at least one of these sets, w.l.o.g., say A_{t1} has the property that the sum of $|q_i'|$ for which $q_i'\in A_{t1}$ is at least $\frac{w}{4(16)}$. For $q_i'=a_i'+ib_i'+jc_i'+kd_i'\in A_{t1}$, we have $|a_i'|\ge\frac{|q_i'|}{2}$ since $|q_i'|=\sqrt{a_i'^2+b_i'^2+c_i'^2+d_i'^2}\le\sqrt{4a_i'^2}=2\sqrt{a_i'^2}=2|a_i'|$. Let $S=\{i\in\{1,2,\ldots,n\}/ q_i'\in A_{t1}\}$. It follows that

$$\Big|\sum_{i\in S}q_i'\Big|\ge\Big|\sum_{i\in S}a_i'\Big|=\sum_{i\in S}|a_i'|\ge\tfrac{1}{2}\sum_{i\in S}|q_i'|\ge\frac{\frac{1}{2}w}{4(16)}=\frac{w}{4(2^5)}=\frac{1}{4(2^5)}\sum_{i=1}^{n}|q_i'| \quad .\#$$

<u>Remark</u>: In general, if $y_1,y_2,\ldots,y_n\in\mathbb{R}^p$, $p\in\mathbb{N}$, then there is a subset S of $\{1,2,\ldots,n\}\ni\Big\|\sum_{i\in S}y_i\Big\|\ge\frac{1}{(p\sqrt{p})2^p}\sum_{i=1}^{n}\|y_i\|$ where $\|x\|=\|(x_1,x_2,\ldots,x_p)\|=(\sum_{i=1}^{p}x_i^2)^{\frac{1}{2}}\ \forall x\in\mathbb{R}^p$.

<u>Linear Algebra over $\mathbb{H}$</u>

1.4 <u>Definition</u> A <u>left</u> <u>vector</u> <u>space</u> V over $\mathbb{H}$ is a set of elements in which the operation of addition and scalar multiplication on the left defined. If x and y are in V and $\alpha,\beta\in\mathbb{H}$. Then

1. V is an abelian group under addition,

2. $\alpha(x+y) = \alpha x+\alpha y$

3. $\alpha(\beta x) = (\alpha\beta)x$

4. $1.x = x$

5. $(\alpha+\beta)x=\alpha x+\beta x$

A <u>right vector space</u> V over $\mathbb{H}$ is defined dually. A <u>vector space</u> V over $\mathbb{H}$ is a left vector and right vector space over $\mathbb{H}$ such that $\alpha(x\beta)=(\alpha x)\beta$ for all $x\in V$ and for all $\alpha,\beta\in\mathbb{H}$.

<u>Example</u>: Let $\mathbb{H}^n=\{(x_1,x_2,\ldots,x_n)/x_i\in\mathbb{H}, i=1,2,\ldots,n\}$ and define $(x_1,\ldots,x_n)+(y_1,\ldots,y_n)=(x_1+y_1,\ldots,x_n+y_n)$, $\alpha(x_1,\ldots,x_n)=(\alpha x_1,\ldots,\alpha x_n)$ and $(x_1,\ldots,x_n)\alpha=(x_1\alpha,\ldots,x_n\alpha)$, $\alpha\in\mathbb{H}$. Then $\mathbb{H}^n$ is a vector space over $\mathbb{H}$.

1.5 <u>Definition</u> Let V be a left vector space over $\mathbb{H}$ and $A\subseteq V$. Then A is said to be <u>left linear subspace</u> of V if and only if $\alpha x+\beta y\in A$ for all $x,y\in A$ and for all $\alpha,\beta\in\mathbb{H}$.

A <u>right linear subspace</u> is defined dually. If V is a vector space over $\mathbb{H}$. Then A is said to be <u>linear subspace</u> of V if and only if A is a left and right linear subspace.

<u>Example</u>: $\mathbb{H}^n$ is a vector space over $\mathbb{H}$. Then $A=\{(x,0,\ldots,0)\in\mathbb{H}^n/x\in\mathbb{H}\}$ is linear subspace of $\mathbb{H}^n$.

1.6 <u>Definition</u> Let V,W be left vectorspaces over $\mathbb{H}$ and $f:V\longrightarrow W$ a map. Then f is said to be a <u>left linear map</u> if and only if $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$ for all $x,y\in V$ and for all $\alpha,\beta\in\mathbb{H}$.

If V,W are right vector spaces over $\mathbb{H}$, Then right linear is defined dually.

If V,W are vector spaces over $\mathbb{H}$, $f:V\to W$ is a map, then f is said to be a linear map if and only if f is both left and right linear.

Example: Define $f:\mathbb{H}^2\to\mathbb{H}^2$ by

$$f(x,y)=(x+y,x+y).$$

Then f is a linear map.

Arithematic in $[-\infty,\infty]$

We establish the folllowing laws of operation for $a\in\mathbb{R}$, ∞ and $-\infty$:

$$\infty\pm a=\infty,\quad \infty+\infty=\infty,\quad \infty-(-\infty)=\infty\ ,$$

$$-\infty\pm a=-\infty,\quad -\infty+(-\infty)=-\infty\ ,\quad |\infty|=|-\infty|=\infty,$$

$$\infty.a=a.\infty=\infty,\quad -\infty.a=a.(-\infty)=-\infty,\ \text{if } a>0,$$

$$\infty.a=a.\infty=-\infty,\quad -\infty.a=a.(-\infty)=\infty\ ,\ \text{if } a<0,$$

$$\infty.\infty=(-\infty).(-\infty)=\infty,\quad \infty.(-\infty)=(-\infty).\infty=-\infty,$$

$$0.(\pm\infty)=(\pm\infty).0=0\ ,\quad \frac{a}{\pm\infty}=0\ .$$

Here a designates an arbitrary finite real number. The expressions

$$\infty-\infty\ ,\ -\infty-(-\infty)\ ,\ \infty+(-\infty)\ ,\ -\infty+\infty,\quad \frac{\pm\infty}{\pm\infty},\quad \frac{a}{0}$$

are not defined at all.

Set Theory

Let X be any nonempty set.

1.7 Definition A ring (or Boolean ring) of sets is a nonempty set $\mathcal{R}$ of subsets of X such that if

$E\in\mathcal{R}$ and $F\in\mathcal{R}$, then

$$E \cup F \in \mathcal{R} \quad \text{and} \quad E \setminus F \in \mathcal{R} .$$

<u>Remarks</u>: 1.7.1 The empty set belongs to every ring $\mathcal{R}$, for if $E \in \mathcal{R}$, then $\emptyset = E \setminus E \in \mathcal{R}$.

1.7.2 The difference $E \setminus F$ is called <u>proper</u> if $F \subseteq E$. Since

$$E \setminus F = (E \cup F) \setminus F,$$

it follows that a non empty set of subsets of X closed under the formation of unions and proper differences is a ring.

1.7.3 The <u>symmetric</u> <u>difference</u> of two sets E and F is denoted by $E \Delta F$. Since

$$E \Delta F = (E \setminus F) \cup (F \setminus E)$$

and

$$E \cap F = (E \cup F) \setminus (E \Delta F),$$

it follows that a ring is closed under the formation of symmetric differences and intersections. Hence if $\mathcal{R}$ is a ring and

$$E_i \in \mathcal{R} , \; i=1,\dots,n,$$

then

$$\bigcup_{i=1}^{n} E_i \in \mathcal{R} \quad \text{and} \quad \bigcap_{i=1}^{n} E_i \in \mathcal{R} .$$

1.7.4 If a non empty set of subsets of X is closed under the formation of intersections, proper differences and disjoint unions, then it is a ring , since

$$E \cup F = [E \setminus (E \cap F)] \cup [F \setminus (E \cap F)] \cup (E \cap F).$$

1.7.5 If a non empty set of subsets of X is closed under the formation of intersections and symmetric differences, then it is a ring, since

$$E \cup F = (E \Delta F) \Delta (E \cap F)$$

and

$$E \setminus F = E \Delta (E \cap F).$$

1.7.6 If a non empty set of subsets of X is closed under the formation of unions and symmetric differences, then it is a ring, since

$$E \setminus F = F \Delta (E \cup F).$$

1.8 <u>Theorem</u> If $(E_i)_{i \in \mathbb{N}}$ is a seguence in a ring $\mathcal{R}$, then there exists a disjoint seguence $(F_i)_{i \in \mathbb{N}}$ in $\mathcal{R}$ such that

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i \text{ and } F_i \subseteq E_i, \ i \in \mathbb{N}.$$

<u>Proof</u> Let $F_1 = E_1$, $F_i = E_i \setminus (F_1 \cup F_2 \cup \ldots \cup F_{i-1})$ for all $i \geq 2$. Then $F_i \in \mathcal{R}$ and $F_i \subseteq E_i$ for all $i \in \mathbb{N}$ and $F_i \cap F_j = \emptyset$ if $i \neq j$. Claim that $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$. To prove this, clearly $\bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} E_i$. Next, let $x \in E_i$ for some $i \in \mathbb{N}$. Assume that j is smallest such that $x \in E_j$. Then $x \notin F_k$ for all $k<j$ (If $x \in F_k$ for some $k<j$, then $x \in E_k$ (since $F_k \subseteq E_k$), a contradiction). Hence $x \in E_j \setminus (F_1 \cup \ldots \cup F_{j-1}) = F_j$. So we have the claim. #

<u>Example of a ring</u> Consider $\mathbb{R}^n$.

A <u>rectangle</u> in $\mathbb{R}^n$ is a set of the form $I_1 \times \ldots \times I_n$ where I_j is an open or closed or half open interval in $\mathbb{R}$ (the intervals may be infinite). A <u>paving</u> of $\mathbb{R}^n$ is a finite collection of pairwise disjoint rectangles whose union is $\mathbb{R}^n$. Let D be a bounded subset of $\mathbb{R}^n$ and let $P=\{\Delta_i\}_{i=1,\ldots,n}$ be a paving of $\mathbb{R}^n$. Let

$$I_P(D)=\{\Delta_i\in P/\ \Delta_i\subseteq D\}$$

(it's possible that $I_P(D)=\emptyset$). Let

$$O_P(D)=\{\Delta_i\in P/\ \Delta_i\cap D\neq\emptyset\}.$$

The <u>volume</u> of a rectangle is the product the length of the component intervals (the volume may be infinite). Let

$$\mu(\Delta)$$

denote the volume of the rectangle Δ. Now for each bounded subset D of $\mathbb{R}^n$ we define

$$\mu_-(D)=\sup_P\{\textstyle\sum_i\mu(\Delta_i)/\ \Delta_i\in I_P(D)\},$$

and define

$$\mu_+(D)=\inf_P\{\textstyle\sum_i\mu(\Delta_i)/\ \Delta_i\in O_P(D)\}.$$

If $\mu_-(D)=\mu_+(D)$, then D is said to have Jordan content and $\mu_-(D)=\mu_+(D)$ is called the Jordan content of D. The set of all sets having Jordan content is a ring of sets. See [10].

1.9 <u>Definition</u> An <u>algebra</u> (or <u>Boolean</u> <u>algebra</u>) of sets is a non empty set $\mathcal{R}$ of subsets of X such that

(a) if $E\in\mathcal{R}$ and $F\in\mathcal{R}$, then $E\cup F\in\mathcal{R}$, and

(b) if $E\in\mathcal{R}$, then $E^c\in\mathcal{R}$.

Since

$$E\setminus F=E\cap F^c=(E^c\cup F)^c,$$

it follows that every algebra is a ring. Hence an algebra may be characterized as a ring containing X.

1.10 <u>Theorem</u> If $\mathcal{E}$ is any set of subsets of X, then there exists a unique ring $\mathcal{R}_0$ such that $\mathcal{R}_0\supseteq\mathcal{E}$ and such that if $\mathcal{R}$ is any other ring containing $\mathcal{C}$, then $\mathcal{R}_o \subseteq \mathcal{R}$.

<u>Proof</u> Standard. #

1.11 <u>Definition</u> The ring $\mathcal{R}_o$ in Theorem 1.10, the smallest ring containing $\mathcal{C}$, is called the <u>ring generated by</u> $\mathcal{C}$; it will be denoted by $\mathcal{R}(\mathcal{C})$.

1.12 <u>Theorem</u> If $\mathcal{C}$ is any set of subsets of X, then every set in $\mathcal{R}(\mathcal{C})$ may be covered by a finite union of sets in $\mathcal{C}$.

<u>Proof</u> Clear. #

1.13 <u>Theorem</u> If $\mathcal{C}$ is a countable set of subsets of X, then $\mathcal{R}(\mathcal{C})$ is countable.

<u>Proof</u> For any set $\mathcal{C}$ of subsets of X, we write $\mathcal{C}^*$ for the set of all finite unions of differences of sets of $\mathcal{C}$. It is clear that if $\mathcal{C}$ is countable, then so is $\mathcal{C}^*$, and if

$$\emptyset \in \mathcal{C},$$

then

$$\mathcal{C} \subseteq \mathcal{C}^*.$$

To prove the theorem we assume, as we may without any loss of generality, that

$$\emptyset \in \mathcal{C},$$

and we write

$$\mathcal{C}_o = \mathcal{C}, \quad \mathcal{C}_n = \mathcal{C}^*_{n-1}, \ n=1,2,3,\dots .$$

Then $\mathcal{C} \subseteq \mathcal{C}^*_o = \mathcal{C}_1$ and $\mathcal{C}_n \subseteq \mathcal{R}(\mathcal{C})$ and $\mathcal{C}_n$ is countable for all $n \in \mathbb{N}$, so

$$\mathcal{C} \subseteq \bigcup_{n=1}^{\infty} \mathcal{C}_n \subseteq \mathcal{R}(\mathcal{C})$$

and the set

$$\bigcup_{n=1}^{\infty} \mathscr{C}_n$$

is countable. We shall complete the proof by showing that $\bigcup_{n=1}^{\infty} \mathscr{C}_n$ is a ring. Since

$$\mathscr{C} = \mathscr{C}_o \subseteq \mathscr{C}_1 \subseteq \mathscr{C}_2 \subseteq \ldots,$$

it follows that if A and B are any two sets in $\bigcup_{n=1}^{\infty} \mathscr{C}_n$, then there exists an $n \in \mathbb{N}$ such that both A and B belong to $\mathscr{C}_n$. We have

$$A \setminus B \in \mathscr{C}_{n+1},$$

and, since

$$\emptyset \in \mathscr{C}_o \subseteq \mathscr{C}_n,$$

it follows also that

$$A \cup B = (A \setminus \emptyset) \cup (B \setminus \emptyset) \in \mathscr{C}_{n+1}.$$

We have proved therefore that both $A \setminus B$ and $A \cup B$ belong to $\bigcup_{n=1}^{\infty} \mathscr{C}_n$, i.e., $\bigcup_{n=1}^{\infty} \mathscr{C}_n$ is a ring. #

1.14 <u>Definition</u> A <u>σ-ring</u> is a non empty set $\mathscr{S}$ of subsets of X such that

(a) if $E \in \mathscr{S}$ and $F \in \mathscr{S}$, then $E \setminus F \in \mathscr{S}$, and

(b) if $E_i \in \mathscr{S}$, $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{S}$.

Equivalently a σ-ring is a ring closed under the formation of countable unions. If $\mathscr{S}$ is a σ-ring and if

$$E_i \in \mathscr{S},\ i \in \mathbb{N},\ \text{and } E = \bigcup_{i=1}^{\infty} E_i,$$

then

$$\bigcap_{i=1}^{\infty} E_i = E \setminus \bigcup_{i=1}^{\infty} (E \setminus E_i)$$

shows that

$$\bigcap_{i=1}^{\infty} E_i \in \mathscr{S}.$$

Since the truth and proof of Theorem 1.10 remain unaltered if we replace "ring" by "σ-ring" throughout, we may define the σ-ring $\mathcal{S}(\mathcal{E})$ <u>generated</u> <u>by</u> any set $\mathcal{E}$ of subsets of X as the smallest σ-ring containing $\mathcal{E}$.

1.15 <u>Theorem</u> If $\mathcal{E}$ is any set of subsets of X and E_o is any set in $\mathcal{S} = \mathcal{S}(\mathcal{E})$, then there exists a countable subset $\mathcal{D}$ of $\mathcal{E}$ such that $E_o \in \mathcal{S}(\mathcal{D})$.

<u>Proof</u>[2] Let $\mathcal{F} = \{\mathcal{D} / \mathcal{D}$ is a countable subset of $\mathcal{E}\}$. Claim that $\bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$ is a σ-ring containing $\mathcal{E}$ and contained in $\mathcal{S}$. Since $\mathcal{D} \subseteq \mathcal{E}$, $\mathcal{S}(\mathcal{D}) \subseteq \mathcal{S}(\mathcal{E})$ for all $\mathcal{D} \in \mathcal{E}$, so $\bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D}) \subseteq \mathcal{S}$. For $E \in \mathcal{E}$, $\{E\} \in \mathcal{F}$, so $\{E\} \in \mathcal{S}(\{E\}) \subseteq \bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$, hence $\mathcal{E} \subseteq \bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$. Next, we shall show that $\bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$ is a σ-ring. To prove this, let $E, F \in \bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$, then $E \in \mathcal{S}(\mathcal{D}_1)$ and $F \in \mathcal{S}(\mathcal{D}_2)$ for some $\mathcal{D}_1, \mathcal{D}_2$ belong to $\mathcal{F}$. So $\mathcal{D}_1 \cup \mathcal{D}_2 \in \mathcal{F}$ and $E, F \in \mathcal{S}(\mathcal{D}_1 \cup \mathcal{D}_2)$, hence $E \smallsetminus F \in \mathcal{S}(\mathcal{D}_1 \cup \mathcal{D}_2)$, that is, $E \smallsetminus F \in \bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$. Let $E_i \in \bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$ for all $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $E_i \in \mathcal{S}(\mathcal{D}_i)$ for some $\mathcal{D}_i \in \mathcal{F}$. Since $\bigcup_{i=1}^{\infty} \mathcal{D}_i \in \mathcal{F}$ and $E_n \in \mathcal{S}(\bigcup_{i=1}^{\infty} \mathcal{D}_i)$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}(\bigcup_{i=1}^{\infty} \mathcal{D}_i)$, that is $\bigcup_{n=1}^{\infty} E_n \in \bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$. Therefore $\bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D})$ is a σ-ring containing $\mathcal{E}$ and contained in $\mathcal{S} = \mathcal{S}(\mathcal{E})$. Hence $\bigcup_{\mathcal{D}\in\mathcal{F}} \mathcal{S}(\mathcal{D}) = \mathcal{S}$. #

For any set $\mathcal{E}$ of subsets of X and every fixed subset A of X, we shall denote by

$$\mathcal{E} \cap A$$

the set of all sets of the form $E \cap A$ with $E \in \mathcal{E}$.

1.16 <u>Theorem</u> If $\mathcal{C}$ is any set of subsets of X and A is any subset of X, then

$$\mathcal{S}(\mathcal{C})\cap A = \mathcal{S}(\mathcal{C}\cap A).$$

<u>Proof</u>[2] Let $\mathcal{E} = \{B\cup(D\setminus A)/B\in\mathcal{S}(\mathcal{C}\cap A),\ D\in\mathcal{S}(\mathcal{C})\}$. Claim that $\mathcal{E}$ is a σ-ring. To prove this, let $B_i\in\mathcal{S}(\mathcal{C}\cap A)$ and $D_i\in\mathcal{S}(\mathcal{C})$ for all $i\in\mathbb{N}$. Then $\bigcup_{i=1}^{\infty}[B_i\cup(D_i\setminus A)] = (\bigcup_{i=1}^{\infty}B_i)\cup[\bigcup_{i=1}^{\infty}(D_i\setminus A)] = (\bigcup_{i=1}^{\infty}B_i)\cup[(\bigcup_{i=1}^{\infty}D_i)\setminus A]\in\mathcal{E}$, since $\bigcup_{i=1}^{\infty}B_i\in\mathcal{S}(\mathcal{C}\cap A)$ and $(\bigcup_{i=1}^{\infty}D_i)\in\mathcal{S}(\mathcal{C})$. Note that for all $B\in\mathcal{S}(\mathcal{C}\cap A)$, $B\subseteq A$. $[B_1\cup(D_1\setminus A)]\setminus[B_2\cup(D_2\setminus A)] = [B_1\setminus(B_2\cup(D_2\setminus A))]$ $\cup[(D_1\setminus A)\setminus(B_2\cup(D_2\setminus A))]$. $B_1\setminus(B_2\cup(D_2\setminus A)) =$ $(B_1\setminus B_2)\cap(B_1\setminus(D_2\setminus A)) = (B_1\setminus B_2)\cap B_1$ (since $B_1\subseteq A$) $=$ $B_1\setminus B_2\in\mathcal{S}(\mathcal{C}\cap A)$, so $B_1\setminus(B_2\cup(D_2\setminus A))\in\mathcal{E}$.

$(D_1\setminus A)\setminus(B_2\cup(D_2\setminus A)) = (D_1\setminus A)\cap B_2^{c}\cap(D_2\setminus A)^{c} =$ $[(D_1\setminus A)\setminus B_2]\cap(D_2\setminus A)^{c} = (D_1\setminus A)\cap(D_2\setminus A)^{c}$ (since $B_2\subseteq A$) $=$ $(D_1\cap A^{c})\cap(D_2^{c}\cup A) = D_1\cap D_2^{c}\cap A^{c} = (D_1\setminus D_2)\setminus A\in\mathcal{E}$ (since $D_1\setminus D_2\in\mathcal{S}(\mathcal{C})$). Therefore $[B_1\cup(D_1\setminus A)]\setminus[B_2\cup(D_2\setminus A)]\in\mathcal{E}$, hence $\mathcal{E}$ is a σ-ring. If $E\in\mathcal{C}$, then the relation

$$E = (E\cap A)\cup(E\setminus A),$$

together with

$$E\cap A\in\mathcal{C}\cap A\subseteq\mathcal{S}(\mathcal{C}\cap A),$$

shows that $E\in\mathcal{E}$, and therefore that

$$\mathcal{C}\subseteq\mathcal{E}.$$

It follows that

$$\mathcal{S}(\mathcal{C})\subseteq\mathcal{E},$$

and therefore that

$$\mathcal{S}(\mathcal{C}) \cap A \subseteq \mathcal{C} \cap A.$$

If $B \in \mathcal{S}(\mathcal{C} \cap A)$ and $D \in \mathcal{S}(\mathcal{C})$, $[B \cup (D \setminus A)] \cap A =$ $(B \cap A) \cup ((D \setminus A) \cap A) = B \cap A = B$. Now we have

$$\mathcal{C} \cap A = \mathcal{S}(\mathcal{C} \cap A),$$

so

$$\mathcal{S}(\mathcal{C}) \cap A \subseteq \mathcal{S}(\mathcal{C} \cap A).$$

From the fact that $\mathcal{S}(\mathcal{C}) \cap A$ is a σ-ring and

$$\mathcal{C} \cap A \subseteq \mathcal{S}(\mathcal{C}) \cap A,$$

we get that

$$\mathcal{S}(\mathcal{C} \cap A) \subseteq \mathcal{S}(\mathcal{C}) \cap A.$$

Therefore

$$\mathcal{S}(\mathcal{C}) \cap A = \mathcal{S}(\mathcal{C} \cap A). \ \#$$

1.17 <u>Definition</u> A <u>σ-algebra</u> is a non empty set $\mathfrak{M}$ of subsets of X such that

(a) if $E \in \mathfrak{M}$, then $E^c \in \mathfrak{M}$, and

(b) if $E_i \in \mathfrak{M}$, $i \in \mathbb{N}$, Then $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{M}$.

Equivalently a σ-algebra is an algebra closed under the formation of countable unions.

<u>Remark</u>: If $\mathfrak{M}$ is a σ-algebra in X, then

(a) $\emptyset, X \in \mathfrak{M}$,

(b) if $A_1, \ldots, A_n \in \mathfrak{M}$, then $\bigcup_{i=1}^{n} A_i \in \mathfrak{M}$,

(c) if $A_1, A_2, \ldots \in \mathfrak{M}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathfrak{M}$ (since $\bigcap_{i=1}^{\infty} A_i = (\bigcup_{i=1}^{\infty} A_i^c)^c$), so if $A_1, A_2, \ldots, A_n \in \mathfrak{M}$, then $\bigcap_{i=1}^{n} A_i \in \mathfrak{M}$, and

(d) if $A, B \in \mathfrak{M}$, then $A \setminus B \in \mathfrak{M}$.

1.18 Theorem If $\mathscr{C}$ is any set of subsets of X, then there exists a smallest σ-algebra $\mathfrak{M}_0$ in X such that $\mathscr{C} \subseteq \mathfrak{M}_0$.

This $\mathfrak{M}_0$ is called the σ-algebra generated by $\mathscr{C}$.

Proof Stardard.#

1.19 Definition Let Y be a topological space(see definition 1.24). By Theorem 1.18, $\exists$ a smallest σ-algebra $\mathscr{B}$ in Y $\ni$ every open set in Y belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the Borel sets in Y.

Remarks:(a) Every open set is a Borel set.

(b) Every closed set is a Borel set.

(c) A countable union of closed sets is a Borel set.

(d) A countable intersection of open sets is a Borel set.

1.20 Definition A non empty set $\mathcal{M}$ of subsets of X is said to be monotone class if

(a) $E_i \in \mathcal{M}$ for all $i \in \mathbb{N}$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$,

(b) $E_i \in \mathcal{M}$ for all $i \in \mathbb{N}$ and $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots \rightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$.

Since the set of all subsets of X is a monotone class, and the intersection of any collection of monotone class is a monotone class, we may define the monotone class $\mathcal{M}(\mathscr{C})$ generated by any set $\mathscr{C}$ of subsets of X as the smallest monotone class containing $\mathscr{C}$.

1.21 <u>Theorem</u> A σ-ring and σ-algebra are monotone classes; a monotone ring is a σ-ring.

<u>Proof</u>[2] The first assertion is obvious. To prove the second assertion we must show that a monotone ring is closed under the formation of countable unions. If $\mathcal{M}$ is a monotone ring and if

$$E_i \in \mathcal{M}\ ,\ i=1,2,\ldots,$$

then, since $\mathcal{M}$ is a ring,

$$\bigcup_{i=1}^{n} E_i \in \mathcal{M},\ n=1,2,\ldots .$$

Since $(\bigcup_{i=1}^{n} E_i)_{n\in N}$ is an increasing sequence of sets whose union is $\bigcup_{i=1}^{\infty} E_i$, the fact that $\mathcal{M}$ is a monotone class implies that

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{M} . \ \#$$

<u>Remark</u>: Also, a monotone algebra is a σ-algebra.

1.22 <u>Theorem</u> If $\mathcal{R}$ is a ring, then $\mathcal{M}(\mathcal{R}) = \mathcal{S}(\mathcal{R})$. Hence if a monotone class contains a ring $\mathcal{R}$, then it contains $\mathcal{S}(\mathcal{R})$.

<u>Proof</u>[2] Since a σ-ring is a monotone class and since $\mathcal{S}(\mathcal{R}) \supseteq \mathcal{R}$, it follows that

$$\mathcal{S}(\mathcal{R}) \supseteq \mathcal{M}(\mathcal{R}).$$

The proof will be complete by showing that $\mathcal{M}(\mathcal{R})$ is a σ-ring; it will then follow, since $\mathcal{M}(\mathcal{R}) \supseteq \mathcal{R}$, that $\mathcal{M}(\mathcal{R}) \supseteq \mathcal{S}(\mathcal{R})$.

For any set $F \subseteq X$, let

$$\mathcal{K}(F) = \{E \subseteq X /\ E \setminus F, F \setminus E, E \cup F \in \mathcal{M}(\mathcal{R})\}.$$

We observe that, because of the symmetric roles of E and F in the definition of $\mathcal{K}(F)$, the relations

$$E \in \mathcal{K}(F) \text{ and } F \in \mathcal{K}(E)$$

are equivalent. If $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{K}(F)$, then $\bigcup_{n=1}^{\infty} E_n \setminus F = \bigcup_{n=1}^{\infty}(E_n \setminus F) \in \mathcal{M}(\mathcal{R})$, since $(E_n \setminus F)_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{M}(\mathcal{R})$; $F \setminus (\bigcup_{n=1}^{\infty} E_n) = \bigcap_{n=1}^{\infty}(F \setminus E_n) \in \mathcal{M}(\mathcal{R})$, since $(F \setminus E_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets in $\mathcal{M}(\mathcal{R})$; $(\bigcup_{n=1}^{\infty} E_n) \cup F = \bigcup_{n=1}^{\infty}(E_n \cup F) \in \mathcal{M}(\mathcal{R})$, since $(E_n \cup F)_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{M}(\mathcal{R})$, hence $\bigcup_{n=1}^{\infty} E_n \in \mathcal{K}(F)$. Also, if $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets in $\mathcal{K}(F)$ then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{K}(F)$. Therefore if $\mathcal{K}(F)$ is not empty then it is a monotone class.

If $E \in \mathcal{R}$ and $F \in \mathcal{R}$, then, by the definition of a ring, $E \in \mathcal{K}(F)$. Since this is true for every E in $\mathcal{R}$, it follows that $\mathcal{R} \subseteq \mathcal{K}(F)$ for all $F \in \mathcal{R}$ and therefore

$$\mathcal{M}(\mathcal{R}) \subseteq \mathcal{K}(F)$$

for all $F \in \mathcal{R}$. Hence if $E \in \mathcal{M}(\mathcal{R})$ and $F \in \mathcal{R}$, then $E \in \mathcal{K}(F)$, and therefore $F \in \mathcal{K}(E)$. Since this is true for every $F \in \mathcal{R}$, it follows that $\mathcal{R} \subseteq \mathcal{K}(E)$ for all $E \subseteq \mathcal{M}(\mathcal{R})$ and so

$$\mathcal{M}(\mathcal{R}) \subseteq \mathcal{K}(E)$$

for all $E \in \mathcal{M}(\mathcal{R})$. Claim that $\mathcal{M}(\mathcal{R})$ is a ring . To prove this, let E, $F \in \mathcal{M}(\mathcal{R})$. Then $E \in \mathcal{K}(F)$, hence $E \setminus F$ and $E \cup F$ belong to $\mathcal{M}(\mathcal{R})$. So we have claim. By Theorem 1.21, we have that $\mathcal{M}(\mathcal{R})$ is a σ-ring. #

1.23 Definition A non empty set $\mathcal{C}$ of subsets of X is hereditary if whenever $E \in \mathcal{C}$ and $F \subseteq E$ then $F \in \mathcal{C}$.

Since the set of all subsets of X is a hereditary set, and the intersection of any collection of hereditary sets is a hereditary set, and a hereditary set is a σ-ring if and only if it is closed under the formation of countable unions, we may define the hereditary σ-ring $\mathcal{H}(\mathcal{C})$ generated by any set $\mathcal{C}$ of subsets of X as the smallest hereditary σ-ring containing $\mathcal{C}$. The hereditary σ-ring generated by $\mathcal{C}$ is, in fact, the set of all sets which can be covered by countably many sets in $\mathcal{C}$, i.e.,

$$\mathcal{H}(\mathcal{C}) = \{H \subseteq X / \; H \subseteq \bigcup_{i=1}^{\infty} E_i, E_i \in \mathcal{C}\}.$$

Topological Prerequisites

1.24 Definition A topology $\mathcal{T}$ on a set X is a collection of subsets of X such that

(a) $\emptyset$ and X belong to $\mathcal{T}$,

(b) any union of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.

(c) any finite intersection of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.

A topological space is a set X together with a topology $\mathcal{T}$ on X. A topological space is denoted by $(X, \mathcal{T})$. The elements of $\mathcal{T}$ are called the open sets of X.

The extended real line $[-\infty, \infty]$ is a topological space by defining (a,b), $[-\infty, a)$, $(a, \infty]$ $(a, b \in \mathbb{R})$ and any union of segments of these types' to be open sets.

1.25 <u>Definition</u> Let (X,d) be a metric space, The <u>open</u> <u>ball</u> with center p and radius $r>0$ is the set

$$B(p;r) = \{x \in X / \ d(x,p) < r\} .$$

1.26 <u>Definition</u> Let (X,d) be a metric space. The topology $\mathcal{T}_d$ on X induced by d is the topology on X generated by the set of open balls, i.e., the smallest topology containing the open balls.

1.27 <u>Definition</u> Let $x,y,z \in \mathbb{R}^4$. Then $x = (x_1,x_2,x_3,x_4)$, $y = (y_1,y_2,y_3,y_4)$, $z = (z_1,z_2,z_3,z_4)$, $x_i,y_i,z_i \in \mathbb{R}$ for $i=1,2,3,4$. We define

$$d_u(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2+(x_4-y_4)^2}$$

$$d_1(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, |x_3-y_3|, |x_4-y_4|\} .$$

1.28 <u>Theorem</u> d_u and d_1 as defined in 1.27 are metrics on $\mathbb{R}^4$ and $(\mathbb{R}^4, \mathcal{T}_{d_u}) = (\mathbb{R}^4, \mathcal{T}_{d_1})$.

<u>Proof</u> Standard. #

<u>Remark 1.28</u> For each $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$, $r>0$, we have

$$B_{d_1}((x_1,x_2,x_3,x_4);r) = (x_1-r,x_1+r)\times(x_2-r,x_2+r)\times(x_3-r,x_3+r)\times(x_4-r,x_4+r).$$

1.29 <u>Theorem</u> $\{B_{d_1}((x_1,x_2,x_3,x_4);\frac{1}{n}) / \ x_1,x_2,x_3,x_4 \in \mathbb{Q}, n \in \mathbb{N}\} \cup \emptyset$ is a countable base for the usual topology on $\mathbb{R}^4$.

<u>Proof</u> Standard. #

1.30 <u>Theorem</u> Every open set V in $\mathbb{R}^4$ is a countable union of closed balls contained in V.

Proof Standard. #

1.31 Theorem A sequence $(a_n, b_n, c_n, d_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^4$ converges to (a,b,c,d) if and only if the corresponding sequences of real numbers $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ converges to a,b,c,d, respectively.

Proof Standard. #

1.32 Theorem If f_n is continuous on a topological space X to $\mathbb{R}^q$ for each $n\in\mathbb{N}$ and if $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on X then f is continuous on X.

Proof See[1]. #

1.33 Theorem If $a_{ij} \geq 0$ for i and j = 1,2,3,..., then

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} .$$

Proof See[3]. #

1.34 Definition Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $[-\infty,\infty]$. For each $k\in\mathbb{N}$, let

$$b_k = \sup\{a_k, a_{k+1}, a_{k+2}, \ldots\}.$$

Then $b_1 \geq b_2 \geq b_3 \geq \ldots$. Let $\beta = \inf\{b_1, b_2, \ldots\}$. We call β the limit superior of $(a_n)_{n\in\mathbb{N}}$ and write

$$\beta = \limsup_{n\to\infty} a_n .$$

Then $\beta = \lim_{n\to\infty} b_n$.

Remark: If $\limsup_{n\to\infty} a_n = \beta$, then there exists a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{i\to\infty} a_{n_i} = \beta$. See[9].

The <u>limit inferior</u> of $(a_n)_{n\in\mathbb{N}}$ is defined analogously by interchanging sup. and inf., i.e.,

$$c_k = \inf\{a_k, a_{k+1}, \ldots\} \quad (k = 1,2,3,\ldots).$$

Then $c_1 \le c_2 \le c_3 \le \ldots$. Let

$$\alpha = \sup\{c_1, c_2, \ldots\} .$$

Then $\alpha = \lim_{n\to\infty} c_n$. α is call the <u>limit inferior</u> of $(a_n)_{n\in\mathbb{N}}$ and write

$$\alpha = \liminf_{n\to\infty} a_n .$$

<u>Remark</u>: Similarly, there exists a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{i\to\infty} a_{n_i} = \alpha$.

<u>Note</u> 1.34.1 $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$,

$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n)$.

1.34.2 If $(a_n)_{n\in\mathbb{N}}$ converges, then

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n .$$

1.34.3 If $(a_n)_{n\in\mathbb{N}}$ is a sequence of monnegative real numbers such that $\lim_{n\to\infty} a_n \neq 0$, then $\limsup_{n\to\infty} a_n > 0$.

1.35 <u>Definition</u> Let X be a topological space.

(a) A set $K \subseteq X$ is <u>compact</u> if every open cover of K has a finite subcover.

In particular, if X is itself compact, then X is called a <u>compact space</u>.

(b) X is <u>locally compact</u> if every point of X has a neighborhood whose closure is compact.

Obviously, every compact space is locally compact.

(c) A set $E \subseteq X$ is called <u>σ-compact</u> if E is a countable union of compact sets.

(d) X is a Hausdorff space if $p\in X$, $q\in X$ and $p\neq q$, then p has a neighborhood U and q has a neighborhood V such ghat $U\cap V = \emptyset$.

1.36 Theorem Let X be a locally compact Hausdorff space, U open and K compact such that $K\subseteq U\subseteq X$. Then there exists an open set V of X such that $\bar{V}$ is compact and

$$K\subseteq V\subseteq \bar{V}\subseteq U.$$

Proof See [9]. #

1.37 Definition The support of a quaternion function f on a topological space X is the closure of the set

$$\{x\in X/\ f(x)\neq 0\}.$$

Let

$C_c(X) = \{f:X\to \mathbb{H}$ / f is continuous and support f is compact$\}$. Observe that if $f,g\in C_c(X)$ and $\alpha\in \mathbb{H}\sim\{0\}$, then

$$\text{support}(f+g)\subseteq(\text{support } f)\cup(\text{support } g),$$

$$\text{support } (\alpha f) = \text{support } f,$$

and $f+g$, αf are continuous. Hence $C_c(X)$ is a left(right) vector space over $\mathbb{H}$.

1.38 Definition A left(right) linear functional Λ on $C_c(X)$ over $\mathbb{H}$ is positive if $f\geq 0$ implies that $\Lambda f\geq 0$.

1.39 Notation The notation

(1) $K\prec f$

will mean that K is a compact subset of X, $f\in C_c(X)$, $0\leq f\leq 1$ and $f(x) = 1$ for all $x\in K$. The notation

(2) $f \prec V$

will mean that V is an open subset of X, $f \in C_c(X)$, $0 \leq f \leq 1$ and support $f \subseteq V$. The notation

$$K \prec f \prec V$$

will mean that (1) and (2) hold.

1.40 <u>Urysohn's Lemma</u> Suppose X is a locally compact Hausdorff space, V is open in X, $K \subseteq V$ and K is compact. Then there exists an $f \in C_c(X)$, such that

$$K \prec f \prec V .$$

<u>Proof</u> See [9]. #

1.41 <u>Theorem</u> Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and

$$K \subseteq V_1 \cup \ldots \cup V_n .$$

Then there exist functions $h_i \prec V_i$ $(i = 1, 2, \ldots, n)$ such that

$$h_1(x) + \ldots + h_n(x) = 1$$

for all $x \in X$.

<u>Proof</u> See [9]. #

1.42 <u>Definition</u> Let X be a locally compact Hausdorff space. A function $f: X \longrightarrow \mathbb{H}$ is said to <u>vanish at infinity</u> if for every $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in K^c$. Let

$$C_o(X) = \{ f: X \longrightarrow \mathbb{H} \,/\, f \text{ is continuous and vanishes at infinity} \}.$$

Then $C_c(X) \subseteq C_o(X)$ and $C_o(X)$ is a left(right) vector space over $\mathbb{H}$. If X is compact, then $C_c(X) = C_o(X) = \{ f: X \longrightarrow \mathbb{H} \,/\, f \text{ is continuous} \}$.

1.43 <u>Theorem</u> If X is a locally compact Hausdorff space, then $C_c(X)$ is dense in $C_o(X)$, relative to the metric defined by the supremum norm

$$\|f\| = \sup_{x \in X} |f(x)| .$$

<u>Proof</u>[9] Let $d: C_o(X) \times C_o(X) \longrightarrow \mathbb{R}$ be defined by

$$d(f,g) = \|f-g\| = \sup_{x \in X} |f(x)-g(x)| .$$

If $f \in C_o(X)$, then there exists a compact set K such that $|f(x)| < 1$ for all $x \in K^c$, so $|f(x)| \leq \max\{1, \max_{x' \in X} |f(x')|\}$ for all $x \in X$, hence $\sup_{x \in X} |f(x)| < \infty$. For $f, g \in C_o(X)$, $0 \leq d(f,g) < \infty$, $d(f,f) = 0$, $d(f,g) = d(g,f)$ and $d(f,g) = 0 \rightarrow f = g$.

If $f, g, h \in C_o(X)$, then

$$\sup_{x \in X} |f(x)-h(x)| \leq \sup_{x \in X} (|f(x)-g(x)| + |g(x)-h(x)|)$$

$$\leq \sup_{x \in X} |f(x)-g(x)| + \sup_{x \in X} |g(x)-h(x)| .$$

Hence d is a metric on $C_o(X)$. Next, we have to show that $C_c(X)$ is dense in $C_o(X)$. To prove this, let $f \in C_o(X)$ and let $\varepsilon > 0$ be given. Then there is a compact set K such that $|f(x)| < \frac{\varepsilon}{2}$ for all $x \in K^c$. By Urysohn's Lemma, there is $g \in C_c(X)$ such that $0 \leq g \leq 1$ and $g = 1$ on K. Let $h = fg$. Then $h \in C_c(X)$. Thus for all $x \in X$ $|f(x)-h(x)| = |f(x)|\,|1-g(x)| \leq |f(x)|$. If $x \in K$, then $|1-g(x)| = 0$, so $|f(x)-h(x)| = 0$. If $x \in K^c$, then $|f(x)-h(x)| \leq |f(x)| < \frac{\varepsilon}{2}$. Thus $\|f-h\| = \sup_{x \in X} |f(x)-h(x)| \leq \frac{\varepsilon}{2} < \varepsilon$. #

1.44 <u>Definition</u> A real function φ defined on a segment (a,b), where $-\infty \leq a < b \leq \infty$, is called <u>convex</u> if the inequality

(1) $\qquad \varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$

holds whenever $a<x<b$, $a<y<b$, and $0 \leq \lambda \leq 1$. Also, (1) is

equivalent to the condition that

(2) $$\frac{\varphi(t)-\varphi(s)}{t-s} \leq \frac{\varphi(u)-\varphi(t)}{u-t}$$

whenever $a<s<t<u<b$.

The mean value theorem for differentiation, combined with (2), shows immediately that a real differentiable function φ is convex in (a,b) if and only if $a<s<t<b$ implies $\varphi'(s)\leq\varphi'(t)$, i.e., if and only if the derivative φ' is a monotonically increasing function. See [9].

For example, the exponential function is convex on $(-\infty,\infty)$.

1.45 Theorem If φ is convex on (a,b), then φ is continuous on (a,b).

Proof See [9]. #