CHAPTER IV

INTEGRATION

This chapter reviews known results in integration theory. The new results concern the integration of quaternion functions and integration with respect to quaternion measures.

4.1 <u>Definition</u> Let $[a,b]$ be an interval in $\mathbb{R}$. A finite set of points $P = \{x_o, x_1, \ldots, x_n\}$ is called a <u>subdivision</u> of $[a,b]$ if $a = x_o < x_1 < \ldots < x_n = b$. Let

$$\mathcal{P}[a,b] = \{P / \text{ P is a subdivision of } [a,b]\}$$

$$\|P\| = \max\{x_{i+1} - x_i / \ i = o, 1, \ldots, n-1\}.$$

Let μ be a σ-algebra in X. Let $f: E \longrightarrow \mathbb{R}$ be a measurable bounded function and $E \in \mathcal{M}$. Then there exist a' and b' such that

$$a' < f(x) < b'$$

$\forall x \in E$. Let μ be a positive measure on $\mathcal{M}$ and $\mu(E) < \infty$. For a subdivision $P = \{y_o, y_1, \ldots, y_n\}$ of $[a', b']$, let

$$L(P,f) = \sum_{i=o}^{n-1} y_i \mu(f^{-1}[y_i, y_{i+1}))$$

$$U(P,f) = \sum_{i=o}^{n-1} y_{i+1} \mu(f^{-1}[y_i, y_{i+1}))$$

$$\bar{I}(f) = \inf\{U(P,f) / \ P \in \mathcal{P}[a', b']\}$$

$$\underline{I}(f) = \sup\{L(P,f) / \ P \in \mathcal{P}[a', b']\}.$$

Note that $f^{-1}[y_i, y_{i+1}) \cap f^{-1}[y_j, y_{j+1}) = \emptyset$ if $i \neq j$ and $E = \bigcup_{i=o}^{n-1} f^{-1}[y_i, y_{i+1})$. Hence $\mu(E) = \sum_{i=o}^{n-1} \mu(f^{-1}[y_i, y_{i+1}))$, so

$$0 \leq U(P,f)-L(P,f) \leq \|P\|\mu(E).$$

4.2 <u>Theorem</u> Let $P=\{y_0,y_1,\ldots,y_n\}$ be a subdivision of $[a,b]$ If we add a new point, $\bar{y}$, in P, then we have

$$L(P,f) \leq L(P\cup\{\bar{y}\},f), \quad U(P\cup\{\bar{y}\},f) \leq U(P,f).$$

<u>Proof</u>[5] Suppose that

$$(*) \qquad y_k < \bar{y} < y_{k+1}$$

for some k. Then the half-open interval $[y_k,y_{k+1})$ is replaced by the two half-open intervals

$$[y_k,\bar{y}) \quad , \quad [\bar{y},y_{k+1}).$$

The set $f^{-1}[y_k,y_{k+1})$ is divided into two sets

$$f^{-1}[y_k,\bar{y}), \quad f^{-1}[\bar{y},y_{k+1}).$$

It is obvious that

$$f^{-1}[y_k,y_{k+1}) = f^{-1}[y_k,\bar{y}) \cup f^{-1}[\bar{y},y_{k+1})$$

and

$$f^{-1}[y_k,\bar{y}) \cap f^{-1}[\bar{y},y_{k+1}) = \emptyset$$

so that

$$(**) \quad \mu(f^{-1}[y_k,y_{k+1})) = \mu(f^{-1}[y_k,\bar{y}))+\mu(f^{-1}[\bar{y},y_{k+1})).$$

It is now clear that $L(P\cup\{\bar{y}\},f)$ is obtained from $L(P,f)$ by replacing the term

$$y_k\mu(f^{-1}[y_k,y_{k+1}))$$

by the two terms

$$y_k\mu(f^{-1}[y_k,\bar{y}))+\bar{y}\mu(f^{-1}[\bar{y},y_{k+1})),$$

together with (*) and (**), it follows that

$$L(P,f) \leq L(P\cup\{\bar{y}\},f).$$

The reasoning is analogous for $U(P\cup\{\bar{y}\},f) \leq U(P,f)$. #

4.3 Corollary For any subdivisions P_1 and P_2 of $[a,b]$,

$$L(P_1,f) \leq U(P_2,f).$$

Proof[5] Applying Theorem 4.2, we have

$$L(P_1,f) \leq L(P_1 \cup P_2,f) \text{ and } U(P_1 \cup P_2,f) \leq U(P_2,f).$$

But $L(P_1 \cup P_2,f) \leq U(P_1 \cup P_2,f)$. Then $L(P_1,f) \leq U(P_2,f)$ #

4.4 Definition Recall $\bar{I}(f)$ and $\underline{I}(f)$ as defined in Definition 4.1 . If $\bar{I}(f) = \underline{I}(f)$, then we shall call $\underline{I}(f)$ is the Lebegue integral of f over E, with respect to μ and we shall denote it by $\int_E f d\mu$. (We shall soon show that $\int_E f d\mu$ is independent of the choice of a and b such that $a < f(x) < b$ for all $x \in E$)

4.5 Theorem Every bounded measurable function defined on a set of finite measure has a Lebesgue integral.

Proof[5] Let f be a bounded measurable function defined on a measurable set E. Then there exist a and b such that

$$a < f(x) < b$$

for all $x \in E$. Let μ be a positive measure and $\mu(E) < \infty$.

Choose any fixed $P_0 \in \mathcal{P}[a,b]$. By Corollary 4.3,

$$L(P,f) \leq U(P_0,f),$$

for all $P \in \mathcal{P}[a,b]$, so $\{L(P,f)/ \ P \in \mathcal{P}[a,b]\}$ is bounded above. Let $\underline{I}(f) = \sup\{L(P,f)/ \ P \in \mathcal{P}[a,b]\}$. Then $\underline{I}(f) \leq U(P_0,f)$. Since P_0 is arbitrary, $\{U(P,f)/ \ P \in \mathcal{P}[a,b]\}$ is bounded below by $\underline{I}(f)$. Let $\bar{I}(f) = \inf\{U(P,f)/ \ P \in \mathcal{P}[a,b]\}$. Then $\underline{I}(f) \leq \bar{I}(f)$. For $P \in \mathcal{P}[a,b]$, we have

$$L(P,f) \leq \underline{I}(f) \leq \bar{I}(f) \leq U(P,f).$$

But, as note above, $0 \leq U(P,f) - L(P,f) \leq \|P\| \mu(E)$, and hence

$$0 \leq \bar{I}(f) - \underline{I}(f) \leq \|P\| \mu(E).$$

Since $\|P\|$ can be made arbitrary small, we have

$$\bar{I}(f) = \underline{I}(f). \ \#$$

4.6 Theorem If P be a subdivision of $[a,b] \ni \|P\| \to 0$, then $U(P,f)$ and $L(P,f)$ approach the integral $\int_E f \, d\mu$.

Proof[5] Since $L(P,f) \leq \underline{I}(f) \leq \bar{I}(f) \leq U(P,f)$ and f is a bounded positive measurable function defined on a measrable set E such that $\mu(E) < \infty$, we have $L(P,f) \leq \int_E f d\mu \leq U(P,f)$. Since $0 \leq U(P,f) - L(P,f) \leq \|P\| \mu(E)$, $0 \leq \lim_{\|P\| \to 0} (U(P,f) - L(P,f)) \leq 0$, hence $\lim_{\|P\| \to 0} (U(P,f) - L(P,f)) = 0$. Since

$$0 \leq \int_E f d\mu - L(P,f) \leq U(P,f) - L(P,f),$$

we have

$$\int_E f d\mu - \lim_{\|P\| \to 0} L(P,f) = 0,$$

that is

$$\lim_{\|P\| \to 0} L(P,f) = \int_E f d\mu .$$

Since $\lim_{\|P\| \to 0} (U(P,f) - L(P,f)) = 0$, $\lim_{\|P\| \to 0} U(P,f) = \lim_{\|P\| \to 0} L(P,f) = \int_E f d\mu$.#

4.7 Theorem Let f be a measurable function defined on a measurable set E such that $\mu(E) < \infty$ and there exist a and b such that

$$a < f(x) < b$$

for all $x \in E$. Then $\int_E f d\mu$ is independent of the bounds a and b.

<u>Proof</u>[5] Suppose that

$$a < f(x) < b^*$$

with $b^* < b$. Let $P = \{y_0, y_1, \ldots, y_n\}$ be a subdivision of $[a,b]$ where we include the point b^* in P, say $b^* = y_m$. Since $y_n = b$ and $b^* < b$, we have $m < n$, hence $f^{-1}[y_i, y_{i+1}) = \emptyset$ if $i \geq m$. This implies that

$$L(P,f) = \sum_{i=0}^{n-1} y_i \mu(f^{-1}[y_i, y_{i+1})) = \sum_{i=0}^{m-1} y_i \mu(f^{-1}[y_i, y_{i+1}))$$

$$= L(\{y_0, y_1, \ldots, y_m\}, f).$$

Hence

$$\lim_{\|P\| \to 0} L(P,f) = \lim_{\|P\| \to 0} L(\{y_0, y_1, \ldots, y_m\}, f) .$$

By Theorem 4.6, $\lim\limits_{\|P\| \to 0} L(P,f) = \int_E f d\mu$, hence $\int_E f d\mu =$ $\lim\limits_{\|P\| \to 0} L(\{y_0, y_1, \ldots, y_m\}, f)$. Thus changing the number b to b^* has no effect on the value of integral. The corresponding fact is true of the number a. #

4.8 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. If the bounded measurable function f satisfies the inequalities

$$a \leq f(x) \leq b,$$

on the measurable set E such that $\mu(E) < \infty$, then

$$a\mu(E) \leq \int_E f d\mu \leq b\mu(E).$$

<u>Proof</u>[5] Let m be a natural number. If we set

$$\bar{a} = a - \frac{1}{m} , \quad \bar{b} = b + \frac{1}{m} ,$$

then it is obvious that

$$\bar{a} < f(x) < \bar{b}.$$

Let $P = \{y_0, y_1, \ldots, y_n\}$ be a partition of $[a,b]$. Since $\bar{a} \leq y_i \leq \bar{b}$ for all $i = 0,1,2,\ldots,n$, we have

$$\bar{a}\sum_{i=0}^{n-1}\mu(f^{-1}[y_i,y_{i+1}))\leq \sum_{i=0}^{n-1} y_i\mu(f^{-1}[y_i,y_{i+1}))$$
$$\leq \bar{b}\sum_{i=0}^{n-1}\mu(f^{-1}[y_i,y_{i+1}))$$

or, equivalently,

$$\bar{a}\mu(E) \leq L(P,f) \leq \bar{b}\mu(E).$$

Since P is arbitrary, $\lim_{\|P\|\to 0} L(P,f) = \int_E f d\mu$, hence

$$\bar{a}\mu(E) \leq \int_E f d\mu \leq \bar{b}\mu(E).$$

Then $(a-\frac{1}{m})\mu(E) \leq \int_E f d\mu \leq (b+\frac{1}{m})\mu(E)$. Since m is arbitrary, we have

$$a\mu(E)\leq \int_E f d\mu \leq b\mu(E). \#$$

4.9 <u>Corollary</u> If the function f is constant on the measurable set E such that $\mu(E)<\infty$, $f(x) = c$, then

$$\int_E f d\mu = c\mu(E).$$

4.10 <u>Corollary</u> If the function f is non negative (non positive), then its integral is non negative (non positive)

4.11 <u>Corollary</u> If $\mu(E) = 0$, we have

$$\int_E f d\mu = 0$$

for all bounded measurable functions f defined on E.

4.12 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let a bounded measurable function f be defined on a measurable set E such that $\mu(E)<\infty$. If $E_1, E_2, \ldots$ are pairwise disjoint measurable sets such that $E = \bigcup_{i=1}^{\infty} E_i$, then

$$\int_E f d\mu = \sum_{i=1}^{\infty}\int_{E_i} f d\mu .$$

<u>Proof</u>[5] <u>Case I</u> Assume E' and E'' are disjoint measurable

sets such that $E = E' \cup E''$. Suppose that

$$a < f(x) < b$$

on the set E. Let $P = \{y_0, y_1, \ldots, y_n\}$ be a subdivision of $[a,b]$. We define the set

$$e_i = f^{-1}[y_i, y_{i+1}), \quad e_i' = (f^{-1}[y_i, y_{i+1})) \cap E',$$

$$e_i'' = (f^{-1}[y_i, y_{i+1})) \cap E'',$$

then we obviously have

$$e_i = e_i' \cup e_i'' \quad \text{and} \quad e_i' \cap e_i'' = \emptyset.$$

Since $\mu(e_i) = \mu(e_i') + \mu(e_i'')$, it follows that

$$\sum_{i=0}^{n-1} y_i \mu(e_i) = \sum_{i=0}^{n-1} y_i \mu(e_i') + \sum_{i=0}^{n-1} y_i \mu(e_i'')$$

and P is arbitrary, we obtain

$$\int_E f d\mu = \int_{E'} f d\mu + \int_{E''} f d\mu .$$

Hence if $E_1, E_2, \ldots, E_n$ are pairwise disjoint measurable sets such that $E = \bigcup_{i=1}^{n} E_i$, then

$$\int_E f d\mu = \sum_{i=1}^{n} \int_{E_i} f d\mu .$$

<u>Case II</u> Assume $E_1, E_2, \ldots$ are pairwise disjoint measurable sets such that $E = \bigcup_{i=1}^{\infty} E_i$. Since $\sum_{i=1}^{\infty} \mu(E_i) = \mu(E) < \infty$, we have $\sum_{i=n+1}^{\infty} \mu(E_i) \to 0$ as $n \to \infty$. Let

$$\bigcup_{i=n+1}^{\infty} E_i = R_n .$$

Hence

$$\int_E f d\mu = \sum_{i=1}^{n} \int_{E_i} f d\mu + \int_{R_n} f d\mu .$$

By Theorem 4.8,

$$a \mu(R_n) \leq \int_{R_n} f d\mu \leq b \mu(R_n).$$

Since $\lim_{n\to\infty} \mu(R_n) = 0$, we have $\lim_{n\to\infty} \int_{R_n} f d\mu = 0$. Therefore

$$\int_E f d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu \ . \ \#$$

4.13 Corollary Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If the bounded measurable functions f and g , both defined on E, f = g a.e. on E, then

$$\int_E f d\mu = \int_E g d\mu \ .$$

Proof[5] Let $A = \{x \in E / \ f(x) \neq g(x)\}$. Then $E \setminus A = \{x \in E / \ f(x) = g(x)\}$. So $\mu(A) = 0$. By Corollary 4.11,

$$\int_A f d\mu = \int_A g d\mu = 0.$$

Since f = g on $E \setminus A$, $\int_{E \setminus A} f d\mu = \int_{E \setminus A} g d\mu$. Since $E = A \cup (E \setminus A)$ and $A \cap (E \setminus A) = \emptyset$, by Theorem 4.12,

$$\int_E f d\mu = \int_A f d\mu + \int_{E \setminus A} f d\mu = \int_A g d\mu + \int_{E \setminus A} g d\mu = \int_E g d\mu \ . \ \#$$

4.14 Corollary Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f is a bounded non negative measurable function on E and $E_0 \in \mathcal{M}$ such that $E_0 \subsetneq E$, then

$$\int_{E_0} f d\mu \leq \int_E f d\mu \ .$$

Proof Follows from Theorem 4.12 . #

Remark: The converse of Corollary 4.13 is false. For example if f is defined on $[-1,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Let m be a Lebesgue measure. Then

$$\int_{-1}^{1} f\,dm = \int_{-1}^{0} f\,dm + \int_{0}^{1} f\,dm = -1+1 = 0,$$

but $f \neq 0$ a.e. on $[-1,1]$.

4.15 Corollary Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. If $f : X \to [0,\infty]$ is bounded measurable function such that $\int_E f\,d\mu = 0$, then $f = 0$ a.e. on E.

Proof[5] We have $\{x \in E/\ f(x) > 0\} = \bigcup_{n=1}^{\infty}\{x \in E/\ f(x) > \frac{1}{n}\}$. Suppose $f \neq 0$ a.e. on E. Then there exists $n_o \in \mathbb{N}$ such that

$$\mu(\{x \in E/\ f(x) > \tfrac{1}{n_o}\}) > 0.$$

Let $\mu(\{x \in E/\ f(x) > \frac{1}{n_o}\}) = \lambda$, so $\lambda > 0$. Let

$$A = \{x \in E/\ f(x) > \tfrac{1}{n_o}\},\quad B = E \setminus A.$$

So we have

$$\int_A f\,d\mu \geq \frac{1}{n_o}\lambda \quad \text{and} \quad \int_B f\,d\mu \geq 0 \text{ (since f non negative).}$$

Hence

$$\int_E f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu \geq \frac{1}{n_o}\lambda > 0,$$

a contradiction. Therefore $f = 0$ a.e. on E. #

4.16 Theorem Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. If two measurable bounded functions f and g are defined on E, then

$$\int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu .$$

Proof[5] Let $a < f(x) < b$ and $c < g(x) < d$ for all $x \in E$. Let $P = \{y_o, y_1, \dots, y_n\}$ be a subdivision of $[a,b]$ and $Q = \{z_o, z_1, \dots, z_n\}$ be a subdivision of $[c,d]$. We define the sets

$$e_k = \{x \in E/\ y_k \leq f(x) < y_{k+1}\},\ e'_i = \{x \in E/\ z_i \leq g(x) < z_{i+1}\}$$

$$T_{i,k} = e'_i \cap e_k \quad (i = 0,1,\dots,m-1;\ k = 0,1,\dots,n-1).$$

Obviously,

$$E = \bigcup_{i,k} T_{i,k} ,$$

and the sets $T_{i,k}$ are pairwise disjoint. By Theorem 4.12,

(1) $$\int_E (f+g)\,d\mu = \sum_{i,k} \int_{T_{i,k}} (f+g)\,d\mu .$$

On the set $T_{i,k}$, we have

$$y_k + z_i \leq f(x) + g(x) < y_{k+1} + z_{i+1} .$$

By Theorem 4.8, we have

$$(y_k + z_i)\mu(T_{i,k}) \leq \int_{T_{i,k}} (f+g)\,d\mu \leq (y_{k+1} + z_{i+1})\mu(T_{i,k}).$$

Combining all these inequalities, we obtain

$$\sum_{i,k} (y_k + z_i)\mu(T_{i,k}) \leq \sum_{i,k} \int_{T_{i,k}} (f+g)\,d\mu \leq \sum_{i,k} (y_{k+1} + z_{i+1})\mu(T_{i,k}).$$

By (1), we have

(2) $$\sum_{i,k} (y_k + z_i)\mu(T_{i,k}) \leq \int_E (f+g)\,d\mu \leq \sum_{i,k} (y_{k+1} + z_{i+1})\mu(T_{i,k}).$$

We evaluate the sum

(3) $$\sum_{i,k} y_k \mu(T_{i,k}).$$

This sum can be written in the form

$$\sum_{k=0}^{n-1} y_k \sum_{i=0}^{m-1} \mu(T_{i,k}).$$

But

$$\sum_{i=0}^{m-1} \mu(T_{i,k}) = \mu\Big(\bigcup_{i=0}^{m-1} T_{i,k}\Big) = \mu\Big(\bigcup_{i=0}^{m-1} (e'_i \cap e_k)\Big) = \mu\Big(e_k \cap \Big(\bigcup_{i=0}^{m-1} e'_i\Big)\Big)$$

$$= \mu(e_k \cap E) = \mu(e_k),$$

so that the sum (3) can also be written as

$$\sum_{k=0}^{n-1} y_k \mu(e_k).$$

But

$$L(P,f) = \sum_{k=0}^{n-1} y_k \mu(e_k), \text{ so } \sum_{i,k} y_k \mu(T_{i,k}) = L(P,f).$$

The other sums in the inequality (2) are evaluated analogously, so that the inequality can be written in the form

$$L(P,f)+L(Q,g) \leq \int_E (f+g)d\mu \leq U(P,f)+U(Q,g).$$

Let $\lambda = \max\{(y_{k+1}-y_k),(z_{i+1}-z_i)/\ i = 0,1,\ldots,m-1;k=0,1,\ldots,n-1\}$. Since P and Q are arbitrary, we have

$$\lim_{\lambda\to o} L(P,f)+\lim_{\lambda\to o} L(Q,g) \leq \int_E (f+g)d\mu \leq \lim_{\lambda\to o} U(P,f)+\lim_{\lambda\to o} U(Q,g),$$

that is

$$\int_E fd\mu + \int_E gd\mu \leq \int_E (f+g)d\mu \leq \int_E fd\mu + \int_E gd\mu .$$

Thus

$$\int_E (f+g)d\mu = \int_E fd\mu + \int_E gd\mu . \#$$

4.17 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f is a bounded measurable function defined on E and c is a finite constant, then

$$\int_E cfd\mu = c\int_E fd\mu .$$

<u>Proof</u>[5] If c = 0, the theorem is trivial. Consider the case $c > 0$. Let

$$a < f(x) < b$$

for all $x \in E$. Let $P = \{y_o, y_1, \ldots, y_n\}$ be a subdivision of $[a,b]$ and $e_k = f^{-1}[y_k, y_{k+1})$. Then $E = \bigcup_{k=o}^{n-1} e_k$ and $e_i \cap e_j = \emptyset$ if $i \neq j$. By Theorem 4.12,

$$\int_E cfd\mu = \sum_{k=o}^{n-1} \int_{e_k} cfd\mu .$$

On the set e_k, the inequalities

$$cy_k \leq cf(x) < cy_{k+1}$$

hold, so that by Theorem 4.8,

$$cy_k\mu(e_k) \le \int_{e_k} cf\,d\mu \le cy_{k+1}\mu(e_k).$$

Combining all these inequalities, we obtain

$$cL(P,f) \le \int_E cf\,d\mu \le cU(P,f).$$

Since P is arbitrary, we have

$$c\lim_{\|P\|\to 0} L(P,f) \le \int_E cf\,d\mu \le c\lim_{\|P\|\to 0} U(P,f),$$

that is

$$c\int_E f\,d\mu \le {}_E\, cf\,d\mu \le c\int_E f\,d\mu .$$

Hence

$$\int_E cf\,d\mu = c\int_E f\,d\mu .$$

Finally, consider the case $c < 0$. Here

$$\begin{aligned} 0 = \int_E (cf+(-c)f)\,d\mu &= \int_E cf\,d\mu + \int_E (-c)f\,d\mu \\ &= \int_E cf\,d\mu + (-c)\int_E f\,d\mu . \end{aligned}$$

Then

$$\int_E cf\,d\mu = c\int_E f\,d\mu . \ \#$$

4.18 <u>Corollary</u> Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. If f and g are bounded measurable functions on the set E, then

$$\int_E (f-g)\,d\mu = \int_E f\,d\mu - \int_E g\,d\mu .$$

4.19 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. If f and g are bounded measurable functions on the set E and $f \le g$ on E, then

$$\int_E f\,d\mu \le \int_E g\,d\mu .$$

Proof[5] The function $g(x)-f(x)$ is non negative for all $x \in E$, so that

$$\int_E g\,d\mu - \int_E f\,d\mu = \int_E (g-f)\,d\mu \geq 0. \#$$

4.20 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f is a bounded measurable function on E, then

$$\left|\int_E f\,d\mu\right| \leq \int_E |f|\,d\mu .$$

Proof[5] Let

$$A = \{x \in E / \ f(x) \geq 0\}, \ B = \{x \in E / \ f(x) < 0\}.$$

Then

$$\int_E f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu = \int_A |f|\,d\mu - \int_B |f|\,d\mu ,$$

$$\int_E |f|\,d\mu = \int_A |f|\,d\mu + \int_B |f|\,d\mu .$$

Since $|a-b| \leq a+b$ for all $a \geq 0$, $b \geq 0$, we have

$$\left|\int_E f\,d\mu\right| = \left|\int_A |f|\,d\mu - \int_B |f|\,d\mu\right|$$

$$\leq \int_A |f|\,d\mu + \int_B |f|\,d\mu = \int_E |f|\,d\mu . \#$$

4.21 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Let a sequence $(f_n)_{n \in \mathbb{N}}$ of bounded measurable functions, converging in measure to the bounded measurable function f, be defined on the measurable set E. If there exists a constant K such that for all n and for all x in E,

$$|f_n(x)| < K,$$

then

(1) $$\lim_{n\to\infty}\int_E f_n d\mu = \int_E f d\mu .$$

Proof[5] First of all, we shall show that

(2) $$|f(x)| \leq K,$$

for almost all $x \in E$. To prove this, by Theorem 3.25, it is possible to extract a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ from the sequence $(f_n)_{n\in\mathbb{N}}$ which converges to f almost everywhere. Hence

$$\lim_{k\to\infty}|f_{n_k}(x)| = |f(x)|$$

for almost all $x \in E$. But $|f_n(x)| < K$ for all $x \in E$ for all $n \in \mathbb{N}$, so $|f_{n_k}(x)| < K$ for all $x \in E$ for all $k \in \mathbb{N}$. Hence $\lim_{k\to\infty}|f_{n_k}(x)| \leq K$ for all $x \in E$. Hence (2) holds. Now let σ be a positive number. Set

$$A_n(\sigma) = \{x \in E / |f_n(x)-f(x)| \geq \sigma\},$$

$$B_n(\sigma) = \{x \in E / |f_n(x)-f(x)| < \sigma\}.$$

Then

$$\left|\int_E f_n d\mu - \int_E f d\mu\right| \leq \int_E |f_n - f|\, d\mu = \int_{A_n(\sigma)} |f_n - f|\, d\mu + \int_{B_n(\sigma)} |f_n - f|\, d\mu .$$

By the inequality $|f_n(x)-f(x)| \leq |f_n(x)| + |f(x)|$, we have that

$$|f_n(x)-f(x)| < 2K$$

for almost all $x \in A_n(\sigma)$. Then there exists $A' \in \mathcal{M}$ and $A' \subseteq A_n(\sigma)$ such that $|f_n(x)-f(x)| < 2K$ for all $x \in A'$ and $\mu(A_n(\sigma) \smallsetminus A') = 0$. By Theorem 4.8, we have

$$\int_{A'} |f_n - f|\, d\mu \leq 2K\mu(A').$$

Since $\mu(A_n(\sigma) \smallsetminus A') = 0$, $\int_{A_n(\sigma)\smallsetminus A'} |f_n - f|\, d\mu = 0 = 2K\mu(A_n(\sigma)\smallsetminus A')$.

Hence

$$\int_{A_n(\sigma)\smallsetminus A'} |f_n - f|\, d\mu + \int_{A'} |f_n - f|\, d\mu \leq 2K\mu(A') + 2K\mu(A_n(\sigma)\smallsetminus A'),$$

so we have

(3) $$\int_{A_n(\sigma)} |f_n - f|\, d\mu \le 2K\mu(A_n(\sigma)).$$

By Theorem 4.8 agian, we have

(4) $$\int_{B_n(\sigma)} |f_n - f|\, d\mu \le \sigma\mu(B_n(\sigma)) \le \sigma\mu(E).$$

Combining (3) with (4), we find that

(5) $$\left|\int_E f_n d\mu - \int_E f d\mu\right| \le 2K\mu(A_n(\sigma)) + \sigma\mu(E).$$

Now let $\varepsilon > 0$ be given, and select a $\sigma > 0$ so small that

$$\sigma\mu(E) < \frac{\varepsilon}{2} .$$

Having fixed this σ, the definition of convergence in measure implies that

$$\mu(A_n(\sigma)) \longrightarrow 0$$

as $n \longrightarrow \infty$ and therefore there exists $N \in \mathbb{N}$ such that

$$2K\mu(A_n(\sigma)) < \frac{\varepsilon}{2}$$

for all $n \geq N$. From (5), we have $\left|\int_E f_n d\mu - \int_E f d\mu\right| < \varepsilon$ for all $n \geq N$. This proves the theorem. #

<u>Remark</u>: It is clear that Theorem 4.21 retains its validity if the inequality

$$|f_n(x)| < K$$

is satisfied only almost everywhere on E. The proof remain the same.

4.22 <u>Theorem</u> Let μ be a finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If f is a bounded measurable function on X, then

$$\int_E f d\mu = \int_X \chi_E f d\mu$$

for all $E \in \mathfrak{M}$.

__Proof__ Since $X = E \cup (X \setminus E)$ which is a disjoint union of measurable sets in $\mathcal{M}$, by Theorem 4.12, we have

$$\int_X \chi_E f d\mu = \int_E \chi_E f d\mu + \int_{X \setminus E} \chi_E f d\mu$$

$$= \int_E f d\mu + \int_{X \setminus E} 0 d\mu .$$

$$= \int_E f d\mu ,$$

since $\chi_E f = 0$ on $X \setminus E$, by Theorem 4.8, we have $\int_{X \setminus E} 0 d\mu = 0$.#

4.23 __Definition__ Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. If s is a simple measurable function on X, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\alpha_1, \ldots, \alpha_n$ are distinct values of s and $A_i = s^{-1}(\alpha_i)$ for all $i = 1, \ldots, n$, and if $E \in \mathcal{M}$ is such that $\mu(E) < \infty$, we defind

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E).$$

4.24 __Definition__ Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Suppose $f : E \to [0, \infty]$ is a bounded function (we do not assume that f is measurable).

Let $A = \{\int_E s d\mu$ / s is a simple measurable function such that $s \le f\}$. Let $s \equiv 0$ on E, so $0 \le s \le f$ and s is a simple measurable function. Then $A \neq \phi$. We define the __lower Lebesgue__ __integral__ $\underline{\int}_E f d\mu$ by

$$\underline{\int}_E f d\mu = \sup \{\int_E s d\mu \text{ / s is a simple measurable function} \ni s \le f\}.$$

Let $B = \{\int_E s\,d\mu$ / s is a simple measurable function such that $s \geq f\}$. Since f is bounded, there exists b such that $f(x) \leq b$ for all $x \in E$. Let $s = b\chi_E$, so $s \geq f$ and s is a simple measurable function. Then $B \neq \emptyset$. We define the <u>upper Lebesgue integral</u> $\overline{\int_E} f\,d\mu$ by

$$\overline{\int_E} f\,d\mu = \inf\{\int_E s\,d\mu \ / \ s \text{ is a simple measurable function} \ni s \geq f\}.$$

If $\overline{\int_E} f\,d\mu = \underline{\int_E} f\,d\mu$, we say that f is <u>integrable</u> and denote it by $\int_E f\,d\mu$.

4.25 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. Let $f: E \to [0,\infty)$ be a bounded function. Then

(1) f is integrable if and only if f is measurable a.e. on E.

(2) If f is measurable, then the definition of f integrable above = the definition of f integrable as already given using parition .

<u>Proof of (1)</u>[7] First, assume that f is integrable. Then

$$\int_E f\,d\mu = \sup\{\int_E s\,d\mu \ / \ s \text{ is a simple measurable function such that } s \leq f\}$$
$$= \inf\{\int_E \varphi\,d\mu \ / \ \varphi \text{ is a simple measurable function such that } \varphi \geq f\}.$$

Let $n \in \mathbb{N}$ be given. Then there exists a simple measurable function s_n such that $s_n \leq f$ and $\int_E s_n\,d\mu > \int_E f\,d\mu - \frac{1}{2n}$ and there exists a simple measurable function φ_n such that $\varphi_n \geq f$ and

$\int_E \varphi_n d\mu < \int_E f d\mu + \frac{1}{2n}$. Therefore $\int_E \varphi_n d\mu - \int_E s_n d\mu < \frac{1}{n}$ and $s_n(x) \leq f(x) \leq \varphi_n(x)$ for all $x \in E$ and $n \in \mathbb{N}$. Let

$$\varphi^* = \inf_{n \in \mathbb{N}} \varphi_n \quad \text{and} \quad s^* = \sup_{n \in \mathbb{N}} s_n .$$

By Theorem 3.10, we have φ^* and s^* are measurable, and

$$s^*(x) \leq f(x) \leq \varphi^*(x)$$

for all $x \in E$. Let $\Delta = \{x \in E / \ s^*(x) < \varphi^*(x)\}$. Then

$$\Delta = \bigcup_{m=1}^{\infty} \Delta_m$$

where $\Delta_m = \{x \in E / \ s^*(x) < \varphi^*(x) - \frac{1}{m}\}$. For each $m \in \mathbb{N}$, Δ_m is contained in the set $\{x \in E / \ s_n(x) < \varphi_n(x) - \frac{1}{m}\}$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be arbitrary. Then

$$\begin{aligned} \frac{1}{m}\mu(\{x \in E / \ s_n(x) < \varphi_n(x) - \tfrac{1}{m}\}) &\leq \int_{\{x \in E / \ s_n(x) < \varphi_n(x) - \frac{1}{m}\}} (\varphi_n - s_n) \, d\mu \\ &\leq \int_E (\varphi_n - s_n) d\mu \\ &< \frac{1}{n} \end{aligned}$$

for all $n \in \mathbb{N}$ which implies that

$$\mu(\{x \in E / \ s_n(x) < \varphi_n(x) - \tfrac{1}{m}\}) < \frac{m}{n}$$

for all $n \in \mathbb{N}$. Then

$$\mu(\Delta_m) < \frac{m}{n}$$

for all $n \in \mathbb{N}$. Since n is arbitrary, $\mu(\Delta_m) = 0$, and so $\mu(\Delta) = 0$. Thus $s^* = \varphi^*$ a.e. on E and $s^* = f$ a.e. on E. Since s^* is measurable, we have f is measurable a.e. on E.

Conversely, assume that f is measurable a.e. on E. Then there exists $A \in \mathfrak{M}$ such that $A \subseteq E$ and $\mu(A) = 0$ and f is measurable on $E \setminus A$. Since f is bounded, there exists $M > 0$ such that $f(x) < M$ for all $x \in E$. Let $n \in \mathbb{N}$ be given. Then the sets

$$E_k = \left\{x \in E \setminus A / \frac{(k-1)M}{n} \leq f(x) < \frac{kM}{n}\right\}, \quad 1 \leq k \leq n,$$

are measurable, disjoint, and have union $E \setminus A$. Thus

$$\sum_{k=1}^{n} \mu(E_k) = \mu(E \setminus A).$$

The simple functions defined by

$$\psi_n(x) = \frac{M}{n} \sum_{k=1}^{n} k\, \chi_{E_k}(x) + M\, \chi_A(x)$$

and

$$\varphi_n(x) = \frac{M}{n} \sum_{k=1}^{n} (k-1)\, \chi_{E_k}(x)$$

satisfy

$$\varphi_n(x) \leq f(x) \leq \psi_n(x)$$

for all $x \in E$. Let

$$\alpha = \sup\left\{\int_E \varphi\, d\mu \,/\, \varphi \text{ is a simple measurable function} \ni \varphi \leq f\right\},$$

$$\beta = \inf\left\{\int_E \psi\, d\mu \,/\, \psi \text{ is a simple measurable function} \ni \psi \geq f\right\}.$$

It is clear that $0 \leq \beta - \alpha$. Thus

$$\beta \leq \int_E \psi_n\, d\mu = \frac{M}{n} \sum_{k=1}^{n} k\, \mu(E_k)$$

and

$$\alpha \geq \int_E \varphi_n\, d\mu = \frac{M}{n} \sum_{k=1}^{n} (k-1)\, \mu(E_k),$$

which implies that

$$0 \leq \beta - \alpha \leq \frac{M}{n} \sum_{k=1}^{n} \mu(E_k) = \frac{M}{n} \mu(E \setminus A).$$

Since n is arbitrary, we have

$$\beta - \alpha = 0.$$

Proof of (2) Assume f is a bounded measurable function. Then there exists a and b such that

$$a < f(x) < b$$

for all $x \in E$. Let

$\alpha = \sup\{\int_E s\,d\mu \,/\, s$ is a simple measurable function $\ni s \leq f\}$.

$\beta = \sup\{L(P,f) \,/\, P \in \mathcal{P}[a,b]\}$.

Claim I that $\beta \leq \alpha$. To prove this, let $\varepsilon > 0$ be given. Then there exists a subdivision $P = \{y_o, y_1, \ldots, y_n\}$ of $[a,b]$ such that

$$\beta - \varepsilon < L(P,f) = \sum_{i=o}^{n-1} y_i \mu(f^{-1}[y_i, y_{i+1})).$$

Since $E = \bigcup_{i=o}^{n-1} f^{-1}[y_i, y_{i+1})$ which is a disjoint union, we have

$$\sum_{i=o}^{n-1} y_i \chi_{f^{-1}[y_i, y_{i+1})}$$

is a simple measurable function and $\sum_{i=o}^{n-1} y_i \chi_{f^{-1}[y_i, y_{i+1})} \leq f$.

Hence $\beta - \varepsilon < \sum_{i=o}^{n-1} y_i \mu(f^{-1}[y_i, y_{i+1})) \leq \alpha$. Since $\varepsilon > 0$ is arbitrary, we have claim I.

Next, claim II that $\alpha \leq \beta$. To prove this, let $\varepsilon > 0$ be given. Then there exists a simple measurable function $s_o = \sum_{i=o}^{n-1} \alpha_i \chi_{E_i}$ on E such that $s_o \leq f$ and $\alpha - \varepsilon < \int_E s_o d\mu = \sum_{i=o}^{n-1} \alpha_i \mu(E_i)$. Assume $\alpha_i < \alpha_{i+1}$ for all $i = 0, 1, \ldots, n-1$. Then $P_o = \{\alpha_o - 1, \alpha_o, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, b\}$ is a subdivision of $[\alpha_o - 1, b]$ and $\alpha_o - 1 < f(x) < b$ for all $x \in E$. Since $\alpha_o \leq f(x)$ for all $x \in E$, $f^{-1}[\alpha_o - 1, \alpha_o) = \phi$. Then

$$L(P_o,f) = (\alpha_o - 1)\mu(f^{-1}[\alpha_o - 1, \alpha_o)) + \sum_{i=o}^{n-1} \alpha_i \mu(f^{-1}[\alpha_i, \alpha_{i+1}))$$

(let $\alpha_n = b$)

$$= \sum_{i=o}^{n-1} \alpha_i \mu(f^{-1}[\alpha_i, \alpha_{i+1}))$$

$$= \int_E s' d\mu$$

where

$$s' = \sum_{i=0}^{n-1} \alpha_i \chi_{f^{-1}[\alpha_i, \alpha_{i+1})}$$

is a simple measurable function. Subclaim that $s_o \le s'$ on E. To prove this, let $x \in E$. Then $x \in E_i$ for some $i \in \{0,1,\ldots,n-1\}$. Then $s_o(x) = \alpha_i \le f(x)$, so there exists $j \in \mathbb{N}$ such that $\alpha_i \le \alpha_j \le f(x) < \alpha_{j+1}$. Hence $x \in f^{-1}[\alpha_j, \alpha_{j+1})$ and $\alpha_i \le \alpha_j$. Thus we have subclaim. Hence $\int_E s_o d\mu \le \int_E s' d\mu$, i.e., $\sum_{i=0}^{n-1} \alpha_i \mu(E_i) \le \sum_{i=0}^{n-1} \alpha_i \mu(f^{-1}[\alpha_i, \alpha_{i+1})) = L(P_o, f)$. Since

$\beta = \sup\{L(P,f) / P \in \mathcal{P}[a,b]\} = \sup\{L(P,f) / P \in \mathcal{P}[\alpha_o - 1, b]\}$,

we have $\alpha - \varepsilon \le L(P_o, f) \le \beta$. Since $\varepsilon > 0$ is arbitrary, $\alpha \le \beta$. By claim I and II, we have $\alpha = \beta$. This shows that (2) is proved. #

4.26 <u>Definition</u> Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. Let $f : X \longrightarrow [0, \infty]$ be a measurable unbounded function on E and let N be a natural number. If the function $[f(x)]_N$ is defined in the following manner:

$$[f(x)]_N = \begin{cases} f(x) & \text{if } f(x) \le N, \\ N & \text{if } f(x) > N. \end{cases}$$

Then $[f(x)]_N$ is a bounded measurable function. Since

$$[f(x)]_1 \le [f(x)]_2 \le [f(x)]_3 \le \cdots,$$

we have

$$\int_E [f]_1 d\mu \le \int_E [f]_2 d\mu \le \int_E [f]_3 d\mu \le \ \cdots .$$

Then we define

$$\int_E f d\mu = \lim_{N \to \infty} \int_E [f]_N d\mu \quad \text{(may be infinite)}.$$

Thus, every non negative measurable function has a Lebesgue integral. Those functions having a finite Lebesgue integral will be called summable.

It is not difficult to see that for a bounded measurable non negative function f , the new definition of the integral coincides with that given earlier, because for sufficiently large N, we have

$$[f]_N \equiv f .$$

It follows that every bounded measurable non negative function is summable.

4.27 Theorem Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. If a function f is summable on E, then it is finite almost everywhere on E.

Proof[5] Set $A = \{x \in E / \ f(x) = \infty\}$. Let $N \in \mathbb{N}$ be given. Then on the set A, $[f]_N = N$, so that

$$\int_E [f]_N d\mu \geq \int_A [f]_N d\mu = N\mu(A),$$

and if $\mu(A) > 0$, then $\int_E [f]_N d\mu \longrightarrow \infty$ as $N \longrightarrow \infty$, which contradicts the hypothesis that f is summable. #

4.28 Theorem Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. Then every non negative function f is summable on E and

$$\int_E f d\mu = 0.$$

Proof Obvious. #

4.29 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If the non negative measurable functions f and g are equivalent on E, then

$$\int_E f d\mu = \int_E g d\mu .$$

Proof[5] There exists $A \in \mathcal{M}$ such that f = g on A and $\mu(E \setminus A) = 0$. Then $[f]_N = [g]_N$ on A for all $N \in \mathbb{N}$, and so for $N \in \mathbb{N}$

$$\begin{aligned}\int_E [f]_N d\mu &= \int_A [f]_N d\mu + \int_{E \setminus A} [f]_N d\mu \\ &= \int_A [f]_N d\mu \\ &= \int_A [g]_N d\mu + \int_{E \setminus A} [g]_N d\mu \\ &= \int_E [g]_N d\mu .\end{aligned}$$

Since N is arbitrary, $\int_E f d\mu = \int_E g d\mu$. #

4.30 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f is a non negative measurable function on E and $E_o \in \mathcal{M}$ is such that $E_o \subseteq E$, then

$$\int_{E_o} f d\mu \leq \int_E f d\mu .$$

In particular, it follows that if the function f is summable on E then it is summable on every measurable subset of E.

Proof It follows from Corollary 4.14 . #

4.31 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Let f and g are non negative measurable functions on E. If $f \leq g$ on E, then

$$\int_E f d\mu \leq \int_E g d\mu \ .$$

Proof[5] It follows from Theorem 4.19 and fact that

$$[f]_N \leq [g]_N$$

for all $N \in \mathbb{N}$. #

4.32 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f is a non negative measurable function f and $\int_E f d\mu = 0$, then $f = 0$ a.e. on E.

Proof[5] Since

$$0 \leq \int_E [f]_N d\mu \leq \int_E f d\mu$$

for all $N \in \mathbb{N}$, we have $\int_E [f]_N d\mu = 0$ for all $N \in \mathbb{N}$. Let

$$A = \bigcup_{N=1}^{\infty} \{x \in E / \ [f(x)]_N \neq 0\}.$$

By Corollary 4.15, $[f]_N = 0$ a.e. on E for all $N \in \mathbb{N}$, hence $\mu(A) = 0$. Let $x_0 \in E \setminus A$, so $[f(x_0)]_N = 0$ for all $N \in \mathbb{N}$. Since $\lim_{N \to \infty} [f(x)]_N = f(x)$ for all $x \in E$, $\lim_{N \to \infty} [f(x_0)]_N = f(x_0) = 0$. #

4.33 Theorem Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f and g are non negative measurable functions on E, then

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu \ .$$

Proof[5] Since, for each n ,

$$[f]_N + [g]_N \leq f+g$$

on E, it is clear that

$$\int_E [f]_N d\mu + \int_E [g]_N d\mu \leq \int_E (f+g)d\mu \ .$$

Taking the limit as $N \to \infty$, we obtain

(1) $$\int_E f d\mu + \int_E g d\mu \leq \int_E (f+g)d\mu \ .$$

In order to verify the inverse inequality, we shall show that

(2) $$[(f+g)(x)]_N \leq [f(x)]_N + [g(x)]_N$$

for all $N \in \mathbb{N}$. Let $x_o \in E$ and let $N \in \mathbb{N}$. If

$$f(x_o) \leq N, \quad g(x_o) \leq N,$$

then

$$[(f+g)(x_o)]_N \leq f(x_o)+g(x_o) = [f(x_o)]_N + [g(x_o)]_N \ .$$

If at least one of the numbers $f(x_o)$ and $g(x_o)$ is greater than N, then

$$[(f+g)(x_o)]_N = N \leq [f(x_o)]_N + [g(x_o)]_N \ .$$

This establishes (2). Hence

$$\int_E [f+g]_N d\mu \leq \int_E [f]_N d\mu + \int_E [g]_N d\mu \ .$$

Taking the limit as $N \to \infty$, we get that

(3) $$\int_E (f+g)d\mu \leq \int_E f d\mu + \int_E g d\mu \ .$$

Combining (1) and (3), we obtain the theorem. #

4.34 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If f is a non negative measurable function on E and if k is a finite non negative number, then

$$\int_E kf d\mu = k\int_E f d\mu \ .$$

<u>Proof</u>[5]The throrem is trivial for $k = 0$. Assume $k > 0$. If $k = \frac{1}{m}$, where m is a natural number, then again,

by Theorem 4.33,

$$\int_E f\,d\mu = m\int_E \frac{1}{m} f\,d\mu \quad .$$

so

$$\int_E \frac{1}{m} f\,d\mu = \frac{1}{m}\int_E f\,d\mu \quad .$$

From this, the validity of the theorem for every non negative rational value of k follows. Finally, let k be a positive irrational number. Take positive rational numbers r and s such that $r < k < s$. By Theorem 4.31,

$$\int_E r f\,d\mu \leq \int_E k f\,d\mu \leq \int_E s f\,d\mu \quad .$$

Hence

$$r\int_E f\,d\mu \leq \int_E k f\,d\mu \leq s\int_E f\,d\mu \quad .$$

Taking the limit as $r \to k$, we obtain

$$k\int_E f\,d\mu \leq \int_E k f\,d\mu \quad ,$$

and taking the limit $s \to k$, we obtain $\int_E k f\,d\mu \leq k\int_E f\,d\mu$.

Hence $\quad \int_E k f\,d\mu = k\int_E f\,d\mu$. #

4.35 Theorem Let

$$\lim_{n\to\infty} f_n(x_o) = f(x_o)$$

at the point x_o . Then for every integer N

$$\lim_{n\to\infty} \left[f_n(x_o)\right]_N = \left[f(x_o)\right]_N \, .$$

Proof[5] Let $N \in \mathbb{N}$ be given. If $f(x_o) > N$, for every sufficiently large n, we will have $f_n(x_o) > N$ and for these n

$$\left[f_n(x_o)\right]_N = N = \left[f(x_o)\right]_N \, .$$

In exactly the same way if $f(x_o) < N$, for sufficiently large n, we will have $f_n(x_o) < N$, and hence

$$[f_n(x_o)]_N = f_n(x_o) \longrightarrow f(x_o) = [f(x_o)]_N .$$

It remains to consider the case when $f(x_o) = N$. In this case, let $\varepsilon > 0$ be given. Then there exists n_o such that

$$f_n(x_o) > N-\varepsilon$$

for all $n \geq n_o$, and hence

$$N-\varepsilon < [f_n(x_o)]_N \leq N$$

for all $n \geq n_o$, that is

$$\left| [f_n(x_o)]_N - [f(x_o)]_N \right| < \varepsilon$$

for all $n \geq n_o$. Thus, the theorem holds in all case. #

4.36 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. If $(f_n)_{n\in \mathbb{N}}$ is a sequence of non negative measurable functions converging to the function f almost everywhere on E, then

$$\int_E f d\mu \leq \sup\left\{ \int_E f_n d\mu \,/\, n \in \mathbb{N} \right\}$$

<u>Proof</u>[5] By Theorem 4.35, we have for each $N \in \mathbb{N}$

$$[f_n(x)]_N \longrightarrow [f(x)]_N$$

as $n \to \infty$, almost everywhere on E. By Theorem 3.17 and the definition of convergence in measure, the sequence $([f_n]_N)_{n\in\mathbb{N}}$ converges in measure to $[f]_N$ for all $N \in \mathbb{N}$. Inasmuch as each of the functions $[f_n]_N$ is bounded by the number N, we can apply Theorem 4.21 on passage to the limit under the integral sign, so that

$$\int_E [f]_N d\mu = \lim_{n\to\infty} \int_E [f_n]_N d\mu .$$

But for all n,

$$\int_E [f_n]_N d\mu \leq \int_E f_n d\mu \leq \sup\{\int_E f_n d\mu \ / \ n \in \mathbb{N}\},$$

so that upon taking the limit $n \to \infty$, we have

$$\int_E [f]_N d\mu \leq \sup\{\int_E f_n d\mu \ / \ n \in \mathbb{N}\}.$$

Taking the limit as $N \to \infty$, we obtain the theorem #

4.37 <u>Corollary</u> If, under the hypotheses of Theorem 4.36, the limit

(1) $$\lim_{n\to\infty} \int_E f_n d\mu ,$$

exists, then

(2) $$\int_E f d\mu \leq \lim_{n\to\infty} \int_E f_n d\mu .$$

<u>Proof</u>[5] The inequality under consideration is trivial if the limit (1) equals ∞. Suppose then that

$$\lim_{n\to\infty} \int_E f_n d\mu = l' < \infty .$$

Then, for an arbitrary $\varepsilon > 0$, there exists a natural number n_o such that for $n \geqslant n_o$,

$$\int_E f_n d\mu < l' + \varepsilon .$$

Applying Theorem 4.36 to the sequence of functions f_{n_o}, $f_{n_o+1}, \ldots$, we obtain

$$\int_E f d\mu \leq l' + \varepsilon ,$$

and this implies (2), since ε is arbitrary. #

4.38 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Let an increasing sequence of non negative measurable functions

$$f_1 \leq f_2 \leq f_3 \leq \ldots$$

be defined on E. If

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in E$, then

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu .$$

<u>Proof</u>[5] First of all, the limit

$$\lim_{n\to\infty} \int_E f_n d\mu$$

exists and, by the preceding corollary,

$$\int_E f d\mu \leq \lim_{n\to\infty} \int_E f_n d\mu .$$

On the other hand, we have $f_n \leq f$ for every n for every $x \in E$, whence

$$\int_E f_n d\mu \leq \int_E f d\mu ,$$

and this implies that

$$\lim_{n\to\infty} \int_E f_n d\mu \leq \int_E f d\mu .$$

This completes the proof. #

4.39 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Let $(u_n)_{n\in \mathbb{N}}$ be a sequence of non negative measurable functions be defined on E. If

$$\sum_{k=1}^{\infty} u_k(x) = f(x)$$

for all $x \in E$, then

$$\sum_{k=1}^{\infty} \int_E u_k d\mu = \int_E f d\mu .$$

<u>Proof</u>[5] For each $n \in \mathbb{N}$, let

$$f_n(x) = \sum_{k=1}^{n} u_k(x)$$

for all $x \in E$. Hence $(f_n)_{n \in N}$ is an increasing sequence of non negative measurable functions and $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in E$. By Theorem 4.38, we have

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu = \lim_{n\to\infty} \int_E \sum_{k=1}^{n} u_k d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E u_k d\mu$$
$$= \sum_{k=1}^{\infty} \int_E u_k d\mu \quad . \#$$

4.40 <u>Corollary</u> Under the hypotheses of Theorem 4.39, suppose that

$$\sum_{k=1}^{\infty} \int_E u_k d\mu < \infty .$$

Then

(1) $\qquad \lim_{k\to\infty} u_k(x) = 0$

almost everywhere on E.

<u>Proof</u>[5] Since $\sum_{k=1}^{\infty} \int_E u_k d\mu = \int_E f d\mu$, f is summable, hence, by Theorem 4.27, f is finite almost everywhere on E, so $\sum_{k=1}^{\infty} u_k(x)$ converges almost everywhere on E. This implies that $\lim_{k\to\infty} u_k(x) = 0$ almost everywhere on E. #

4.41 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$ and E be the union of a family of pairwise disjoint measurable sets E_k:

$$E = \bigcup_{k=1}^{\infty} E_k .$$

For every non negative measurable function f defined on E, we have

$$\int_E f d\mu = \sum_{k=1}^{\infty} \int_{E_k} f d\mu .$$

Proof[5] Introduce the functions u_k ($k = 1,2,\ldots$) , letting

$$u_k(x) = \begin{cases} f(x) & \text{if } x \in E_k, \\ 0 & \text{if } x \in E \setminus E_k. \end{cases}$$

It is easy to see that

$$f(x) = \sum_{k=1}^{\infty} u_k(x),$$

for all $x \in E$ and hence, by Theorem 4.39,

$$(*) \qquad \int_E f\,d\mu = \sum_{k=1}^{\infty} \int_E u_k\,d\mu .$$

We now evaluate the integral $\int_E u_k\,d\mu$. To do this, we note that

$$[u_k(x)]_N = \begin{cases} [f(x)]_N & \text{if } x \in E_k, \\ 0 & \text{if } x \in E \setminus E_k. \end{cases}$$

This implies that

$$\int_E [u_k]_N\,d\mu = \int_{E_k} [f]_N\,d\mu .$$

Taking the limit as $N \to \infty$, we find that

$$\int_E u_k\,d\mu = \int_{E_k} f\,d\mu ,$$

which, together with (*), proves the theorem. #

4.42 Theorem Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Suppose $E \in \mathfrak{M}$ such that $\mu(E) = \infty$. If $(E_i)_{i \in \mathbb{N}}$ and $(F_j)_{j \in \mathbb{N}}$ are both disjoint sequences of members of $\mathfrak{M}$ such that

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{\infty} F_j \ , \ \mu(E_i) < \infty \ , \ \mu(F_j) < \infty$$

for all $i, j \in \mathbb{N}$ and if f is a non negative measurable function defined on E, then

$$\sum_{i=1}^{\infty} \int_{E_i} f\,d\mu = \sum_{j=1}^{\infty} \int_{F_j} f\,d\mu .$$

<u>Proof</u> We have $\sum_{i=1}^{\infty}\int_{E_i} f\,d\mu = \sum_{i=1}^{\infty}\int_{\bigcup_{j=1}^{\infty}(E_i\cap F_j)} f\,d\mu =$ $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{E_i\cap F_j} f\,d\mu$ and $\sum_{j=1}^{\infty}\int_{F_j} f\,d\mu = \sum_{j=1}^{\infty}\int_{\bigcup_{i=1}^{\infty}(E_i\cap F_j)} f\,d\mu =$ $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\int_{E_i\cap F_j} f\,d\mu$. Since $\int_{E_i\cap F_j} f\,d\mu \geq 0$ for all $i,j \in \mathbb{N}$, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{E_i\cap F_j} f\,d\mu = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\int_{E_i\cap F_j} f\,d\mu$, hence we have the theorem, #

4.43 <u>Definition</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Then for each $E \in \mathfrak{M}$, we can choose a disjoint countable collection $(E_n)_{n\in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{n=1}^{\infty} E_n \;,\quad \mu(E_n) < \infty \;,\quad n = 1,2,\ldots \;.$$

Let f be a non negative measurable function defined on X. We define $\int_E f\,d\mu$ by

$$\int_E f\,d\mu = \sum_{n=1}^{\infty}\int_{E_n} f\,d\mu \;.$$

If $\int_E f\,d\mu < \infty$, then f is called summable.

4.44 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If f is a summable function on a measurable set E, then f is finite almost everywhere.

<u>Proof</u> By Definition 4.43, there exists a disjoint countable collection $(E_n)_{n\in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{n=1}^{\infty} E_n, \quad \mu(E_n) < \infty \;,\quad n \in \mathbb{N} \;.$$

Then $\int_E f\,d\mu = \sum_{n=1}^{\infty}\int_{E_n} f\,d\mu < \infty$, so $\int_{E_n} f\,d\mu < \infty$ for all $n \in \mathbb{N}$.

By Theorem 4.27, f is finite a.e. on E_n for all $n \in \mathbb{N}$, hence f is finite a.e. on $\bigcup_{n=1}^{\infty} E_n = E$. #

4.45 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If the non negative measurable functions f and g are equivalent on $E \in \mathfrak{M}$, then

$$\int_E f d\mu = \int_E g d\mu .$$

<u>Proof</u> There exists a disjoint countable collection $(E_n)_{n \in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{n=1}^{\infty} E_n \ , \ \mu(E_n) < \infty \ , \ n \in \mathbb{N} .$$

By Theorem 4.29, $\int_{E_n} f d\mu = \int_{E_n} g d\mu$ for all $n \in \mathbb{N}$. This implies that the theorem is proved. #

4.46 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If f is a non negative measurable function on a measurable set E and if $E_o \in \mathfrak{M}$ such that $E_o \subsetneq E$, then

$$\int_{E_o} f d\mu \leq \int_E f d\mu .$$

<u>Proof</u> There exists a disjoint countable collection $(E_n)_{n \in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{n=1}^{\infty} E_n \ , \ \mu(E_n) < \infty \ , \ n \in \mathbb{N} .$$

Hence $E_o = \bigcup_{n=1}^{\infty} (E_o \cap E_n)$ which is a disjoint union. By Theorem 4.30, $\int_{E_o \cap E_n} f d\mu \leq \int_{E_n} f d\mu$ for all $n \in \mathbb{N}$. Hence

$$\sum_{n=1}^{\infty} \int_{E_o \cap E_n} f d\mu \leq \sum_{n=1}^{\infty} \int_{E_n} f d\mu ,$$

which implies that

$$\int_{E_0} f\,d\mu \leq \int_E f\,d\mu \ . \ \#$$

4.47 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Let f and g be non negative measurable function on $E \in \mathfrak{M}$. If $f \leq g$ on E, then

$$\int_E f\,d\mu \leq \int_E g\,d\mu \ .$$

<u>Proof</u> Follows from Theorem 4.31 and Definition 4.43.#

4.48 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If f is a non negative measurable function on $E \in \mathfrak{M}$ and $\int_E f\,d\mu = 0$, then $f = 0$ a.e. on E.

<u>Proof</u> There exists a disjoint countable collection $(E_n)_{n \in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{n=1}^{\infty} E_n \ , \ \mu(E_n) < \infty \ , \ n \in \mathbb{N} \ .$$

Since $0 = \int_E f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$, $\int_{E_n} f\,d\mu = 0$ for all $n \in \mathbb{N}$.

By Theorem 4.32, $f = 0$ a.e. on E_n for all $n \in \mathbb{N}$. Hence $f = 0$ a.e. on $\bigcup_{n=1}^{\infty} E_n = E$. #

4.49 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If $f = 0$ on $E \in \mathfrak{M}$, then $\int_E f\,d\mu = 0$.

<u>Proof</u> Follows from Corollary 4.9 and Definition 4.43.#

4.50 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If f is a non negative measurable function defined on $E \in \mathfrak{M}$ and if k is a finite non negative number, then

$$\int_E kf\,d\mu = k\int_E f\,d\mu \quad .$$

Proof Follows from Theorem 4.34 and Definition 4.43.#

4.51 Lebesgue's Monotone Convergence Theorem Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Let

$(f_n)_{n\in \mathbb{N}}$ be an increasing sequence of non negative measurable functions

$$f_1 \leq f_2 \leq f_3 \leq \ldots$$

defined on $E \in \mathfrak{M}$. If

$$\lim_{n\to\infty} f_n(x) = f(x),$$

then

$$\lim_{n\to\infty}\int_E f_n\,d\mu = \int_E f\,d\mu \quad .$$

Proof If $\mu(E)<\infty$, then this theorem is true by Theorem 4.38 . Assume $\mu(E) = \infty$, so there exists a disjoint countable collection $(E_i)_{i\in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{i=1}^{\infty} E_i \quad , \mu(E_i)<\infty \quad , i\in \mathbb{N} \; .$$

We shall show that

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\int_{E_i} f_n\,d\mu = \sum_{i=1}^{\infty}\lim_{n\to\infty}\int_{E_i} f_n\,d\mu \quad .$$

This proves the theorem. To prove this, first suppose that $\sum_{i=1}^{\infty}\lim_{n\to\infty}\int_{E_i} f_n\,d\mu = \infty$. Let $M > 0$, so there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^{m}\lim_{n\to\infty}\int_{E_i} f_n\,d\mu > M$$

for all $m \geq N_1$. If there exists $m_0 \in \{N_1, N_1+1, N_1+2, \ldots\}$

such that $\sum_{i=1}^{m_0} \lim_{n\to\infty} \int_{E_i} f_n d\mu = \infty$, then $\lim_{n\to\infty} \sum_{i=1}^{\infty} \int_{E_i} f_n d\mu \geq$

$\lim_{n\to\infty} \sum_{i=1}^{m_0} \int_{E_i} f_n d\mu = \sum_{i=1}^{m_0} \lim_{n\to\infty} \int_{E_i} f_n d\mu = \infty$, i.e.,

$\lim_{n\to\infty} \sum_{i=1}^{\infty} \int_{E_i} f_n d\mu = \infty$. Now assume that $\sum_{i=1}^{m} \lim_{n\to\infty} \int_{E_i} f_n d\mu < \infty$

for all $m \geq N_1$, so there exists $\alpha > 0$ such that

$$M+\alpha = \sum_{i=1}^{N_1} \lim_{n\to\infty} \int_{E_i} f_n d\mu = \lim_{n\to\infty} \sum_{i=1}^{N_1} \int_{E_i} f_n d\mu .$$

Then there exists $N_2 \in \mathbb{N}$ such that

$$\left| \sum_{i=1}^{N_1} \int_{E_i} f_n d\mu - (M+\alpha) \right| < \frac{\alpha}{2}$$

for all $n \geq N_2$. Hence

(1) $$M < -\frac{\alpha}{2} + M + \alpha < \sum_{i=1}^{N_1} \int_{E_i} f_n d\mu$$

for all $n \geq N_2$. But

(2) $$\sum_{i=1}^{N_1} \int_{E_i} f_n d\mu \leq \sum_{i=1}^{\infty} \int_{E_i} f_n d\mu$$

for all $n \geq N_2$. From (1) and (2) implies that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \int_{E_i} f_n d\mu = \infty .$$

Finally, suppose that $\sum_{i=1}^{\infty} \lim_{n\to\infty} \int_{E_i} f_n d\mu < \infty$. By Theorem 4.38, we have $\sum_{i=1}^{\infty} \int_{E_i} f d\mu < \infty$, i.e., $\int_E f d\mu < \infty$.

By hypotheses we have that $0 \leq f_n(x) \leq f_{n+1}(x) \leq f(x)$ for all $n \in \mathbb{N}$, so $0 \leq \int_{E_i} f_n d\mu \leq \int_{E_i} f_{n+1} d\mu \leq \int_{E_i} f d\mu < \infty$ for all

$i,n \in \mathbb{N}$. Now, let

$$a_{in} = \int_{E_i} f_n d\mu$$

for all $i,n \in \mathbb{N}$. Then $0 \leq a_{in} \leq a_{i(n+1)}$ for all $i,n \in \mathbb{N}$. Let

$$b_{in} = a_{in} - a_{i(n-1)}$$

for all $i,n \in \mathbb{N}$ (let $a_{io} = 0$). Hence $0 \leq b_{in}$ for all $i,n \in \mathbb{N}$ and $\sum_{n=1}^{m} b_{in} = a_{im}$. By Theorem 1.33,

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} b_{in} = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} b_{in} .$$

Hence

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} b_{in} = \lim_{m \to \infty} \sum_{n=1}^{m} \sum_{i=1}^{\infty} b_{in} = \lim_{m \to \infty} \sum_{i=1}^{\infty} \sum_{n=1}^{m} b_{in} = \lim_{m \to \infty} \sum_{i=1}^{\infty} a_{im}$$

$= \lim_{n \to \infty} \sum_{i=1}^{\infty} \int_{E_i} f_n d\mu$, and $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} b_{in} = \sum_{i=1}^{\infty} \lim_{m \to \infty} \sum_{n=1}^{m} b_{in} =$ $\sum_{i=1}^{\infty} \lim_{m \to \infty} a_{im} = \sum_{i=1}^{\infty} \lim_{n \to \infty} \int_{E_i} f_n d\mu$. Thus

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} \int_{E_i} f_n d\mu = \sum_{i=1}^{\infty} \lim_{n \to \infty} \int_{E_i} f_n d\mu \ . \ \#$$

4.52 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of non negative measurable functions defined on $E \in \mathfrak{M}$. If

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for all $x \in E$, then

$$\sum_{n=1}^{\infty} \int_E f_n d\mu = \int_E f d\mu .$$

<u>Proof</u> There exists a disjoint countable collection $(E_i)_{i \in \mathbb{N}}$ of members of $\mathfrak{M}$ such that

$$E = \bigcup_{i=1}^{\infty} E_i \ , \ \mu(E_i) < \infty \ , \ i \in \mathbb{N} .$$

By Theorem 4.39, we have

$$\sum_{n=1}^{\infty}\int_{E_i} f_n d\mu = \int_{E_i} f d\mu .$$

for all $i \in \mathbb{N}$. Hence

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\int_{E_i} f_n d\mu = \sum_{i=1}^{\infty}\int_{E_i} f d\mu = \int_E f d\mu .$$

By Theorem 1.33, we get that

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\int_{E_i} f_n d\mu = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\int_{E_i} f_n d\mu = \sum_{n=1}^{\infty}\int_E f_n d\mu .$$

Hence

$$\sum_{n=1}^{\infty}\int_E f_n d\mu = \int_E f d\mu . \#$$

4.53 <u>Corollary</u> Under the hypotheses of Theorem 4.52, suppose that

$$\sum_{n=1}^{\infty}\int_E f_n d\mu < \infty .$$

Then

$$\lim_{n\to\infty} f_n(x) = 0$$

almost everywhere on E.

<u>Proof</u> It is similar to the proof of Corollary 4.4o.#

4.54 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Let a measurable set E be the union of a family of pairwise disjoint measurable sets E_n:

$$E = \bigcup_{n=1}^{\infty} E_n$$

For every non negative measurable function f defined on E, we have

$$\int_E f d\mu = \sum_{n=1}^{\infty}\int_{E_n} f d\mu .$$

<u>Proof</u> For each $n \in \mathbb{N}$ there exists a family of pairwise disjoint measurable set E_{ni} such that

$$E_n = \bigcup_{i=1}^{\infty} E_{ni} \ , \ \mu(E_{ni}) < \infty \ , \ i \in \mathbb{N} \ .$$

Hence

$$E = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} E_{ni} \text{ and } \mu(E_{ni}) < \infty \quad \text{for all } n, i \in \mathbb{N} \ .$$

For each $n \in \mathbb{N}$, by Definition 4.43, we have

$$\int_{E_n} f d\mu = \sum_{i=1}^{\infty} \int_{E_{ni}} f d\mu \ .$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_{E_n} f d\mu &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \int_{E_{ni}} f d\mu \\
&= \int_{\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} E_{ni}} f d\mu \\
&= \int_E f d\mu \ . \ \#
\end{aligned}$$

4.55 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If f is a non negative measurable function defined on X and $E \in \mathfrak{M}$, then

$$\int_E f d\mu = \int_X \chi_E f d\mu \ .$$

<u>Proof</u> Since $X = E \cup (X \setminus E)$, by Theorem 4.54, we have

$$\begin{aligned}
\int_X \chi_E f d\mu &= \int_E \chi_E f d\mu + \int_{X \setminus E} \chi_E f d\mu \\
&= \int_E f d\mu + \int_{X \setminus E} 0 d\mu \\
&= \int_E f d\mu \ . \ \#
\end{aligned}$$

4.56 <u>Fatou's Lemma</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If $f_n : X \to [0, \infty]$ is measurable for all $n \in \mathbb{N}$, then

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu \ .$$

Proof[9] For each $k \in \mathbb{N}$, let

$$g_k(x) = \inf\{f_k(x), f_{k+1}(x), \ldots\} = \inf_{i \geq k} f_i(x) \text{ for all } x \in X.$$

Then $g_k \leq f_k$ for all $k \in \mathbb{N}$, so for each k

(*) $$\int_X g_k d\mu \leq \int_X f_k d\mu .$$

Also, we have

$$g_1 \leq g_2 \leq g_3 \leq \ldots,$$

and each g_k is measurable and by definition

$$\lim_{n\to\infty} g_n(x) = \liminf_{n\to\infty} f_n(x)$$

for all $x \in X$. Then, by Lebesgue's Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_X g_n d\mu = \int_X (\liminf_{n\to\infty} f_n(x)) d\mu .$$

From (*), we get that

$$\lim_{n\to\infty} \int_X g_n d\mu = \liminf_{n\to\infty} \int_X g_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu .$$

Hence

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu . \ \#$$

4.57 Theorem Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Suppose $f: X \to [0, \infty]$ is measurable, $\int_X f d\mu < \infty$ and

(1) $$\varphi(E) = \int_E f d\mu$$

for all $E \in \mathfrak{M}$. Then φ is a finite positive measure on $\mathfrak{M}$ and if $g: X \to [0, \infty]$ is measurable, then

(2) $$\int_X g d\varphi = \int_X gf d\mu .$$

Proof[9] Let $E_1, E_2, \ldots$ be disjoint members of $\mathfrak{M}$. Let $E = \bigcup_{n=1}^{\infty} E_n$. Then

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f ,$$

so

$$\varphi(E) = \int_E f d\mu = \int_X \chi_E f d\mu = \int_X \sum_{j=1}^{\infty} \chi_{E_j} f d\mu =$$

$$\sum_{j=1}^{\infty} \int_X \chi_{E_j} f d\mu \quad \text{(by Theorem 4.52)} = \sum_{j=1}^{\infty} \int_{E_j} f d\mu = \sum_{j=1}^{\infty} \varphi(E_j).$$

Also, $\varphi(\phi) = 0$. Hence φ is a positive measure on $\mathfrak{M}$. Since $\varphi(X) = \int_X f d\mu < \infty$, φ is a finite positive measure on $\mathfrak{M}$.

Next, (1) shows that (2) holds whenever $g = \chi_E$ for some $E \in \mathfrak{M}$, i.e., $\int_X g d\varphi = \int_X \chi_E d\varphi = \int_E d\varphi = \varphi(E) = \int_E f d\mu$ $= \int_X \chi_E f d\mu = \int_X g f d\mu$.

Assume g is a simple measurable functions. Then

$$g = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

where $\alpha_1, \ldots, \alpha_n$ are distinct values of g and $E_i = g^{-1}(\alpha_i)$ for all $i = 1,2,\ldots,n$. Then

$$\int_X g d\varphi = \int_X \sum_{i=1}^{n} \alpha_i \chi_{E_i} d\varphi = \sum_{i=1}^{n} \alpha_i \int_X \chi_{E_i} d\varphi =$$

$$\sum_{i=1}^{n} \alpha_i \int_X \chi_{E_i} f d\mu = \int_X \left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right) f d\mu = \int_X g f d\mu .$$

Assume g is a measurable function. By Theorem 3.15, there exists a non decreasing sequence $(s_n)_{n \in \mathbb{N}}$ of simple measurable functions such that $\lim_{n \to \infty} s_n(x) = g(x)$ for all $x \in X$. By Lebesgue's Monotone Convergence Theorem, $\lim_{n\to\infty} \int_X s_n d\varphi = \int_X g d\varphi$ Since $(s_n f)_{n \in \mathbb{N}}$ is a non decreasing sequence and $\lim_{n\to\infty} (s_n f)(x)$ $= (gf)(x)\ \forall x \in X$, by Lebesgue's Monotone Convergence Theorem,

$$\lim_{n \to \infty} \int_X s_n f d\mu = \int_X g f d\mu .$$

Hence

$$\int_X g\,d\varphi = \lim_{n\to\infty}\int_X s_n\,d\varphi = \lim_{n\to\infty}\int_X s_n f\,d\mu = \int_X gf\,d\mu \ . \ \#$$

4.58 <u>Definition</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ in X. Let $f:X\longrightarrow[-\infty,\infty]$ be a measurable function. For any measurable set E, we define

$$\int_E f\,d\mu = \int_E f^+d\mu - \int_E f^-d\mu$$

provided that at least one of the integrals on the right is finite. Let $g:X\longrightarrow\mathbb{H}$ be a measurable function. Then $g = g_1+ig_2+jg_3+kg_4$ for some real measurable functions $g_{i'}$, $i'\leq 4$. For any measurable set E, we define

$$(*)\ \int_E g\,d\mu = \int_E g_1 d\mu + i\int_E g_2 d\mu + j\int_E g_3 d\mu + k\int_E g_4 d\mu$$

if $\left|\int_E g_{i'}d\mu\right|<\infty$ for all $i'\leq 4$. Thus (*) defines the integral on the left as a quaternion number.

4.59 <u>Definition</u> Let μ be a quaternion measure on a σ-algebra $\mathcal{M}$ in X. Then $\mu = \mu_1+i\mu_2+j\mu_3+k\mu_4$ for some real measures $\mu_{i'}$, $i'\leq 4$.

Let λ be a real measure on a σ-algebra $\mathcal{M}$ in X. Let $f:X\longrightarrow[0,\infty]$ be measurable since $|\lambda(X)|<\infty$, $\lambda^+(X)<\infty$ and $\lambda^-(X)<\infty$. We define integration with respect to a real measure λ by defining

$$\int_E f\,d\lambda = \int_E f\,d\lambda^+ - \int_E f\,d\lambda^- \qquad (E\in\mathcal{M})$$

provided that at least one of the integrals on the right is finite.

Then we define integration with respect to a quaternion measure μ as before by defining

$$(*)\quad \int_E f d\mu = \int_E f d\mu_1 + i\int_E f d\mu_2 + j\int_E f d\mu_3 + k\int_E f d\mu_4 \quad (E \in \mathfrak{M}).$$

if $\left|\int_E f d\mu_i\right| < \infty$ for all $i \leq 4$. Thus (*) defines the integral on the left as a quaternion number.

If $f: X \longrightarrow [-\infty, \infty]$ is measurable, then we define integration with respect to a quaternion measure μ by defining

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \qquad (E \in \mathfrak{M})$$

if $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ exist, so $\int_E f d\mu$ is a quaternion number.

If $f: X \longrightarrow \mathbb{H}$ is measurable, then there exist real measurable functions f_i, $i \leq 4$ such that

$$f = f_1 + if_2 + jf_3 + kf_4.$$

We define the <u>left</u> <u>integral</u> of f with respect to a quaternion measure μ by defining

$$\int_E f d\mu = \int_E f_1 d\mu + i\int_E f_2 d\mu + j\int_E f_3 d\mu + k\int_E f_4 d\mu \qquad (E \in \mathfrak{M})$$

if $\int_E f_i d\mu$ exists for all $i \leq 4$, so $\int_E f d\mu$ is a quaternion number.

Also, we define the <u>right</u> <u>integral</u> of f with respect to a quaternion measure μ, denoted by $\left[\int_E (d\mu) f\right]$, by defining

$$\left[\int_E (d\mu) f\right] = \int_E f d\mu_1 + i\int_E f d\mu_2 + j\int_E f d\mu_3 + k\int_E f d\mu_4 \quad (E \in \mathfrak{M}).$$

if $\int_E f d\mu_i$ exists for all $i \leq 4$, so $\left[\int_E (d\mu) f\right]$ is a quaternion number.

Remarks: (1) Let μ be a real measure on a σ-algebra $\mathcal{M}$ in X. Let f be a quaternion measurable function defined on $E \in \mathcal{M}$. If $\int_E f d\mu$ and $[\int_E (d\mu) f]$ exist, then

$$\int_E f d\mu = [\int_E (d\mu) f] .$$

(2) Let μ be a quaternion measure on a σ-algebra $\mathcal{M}$ in X and f a real measurable function defined on $E \in \mathcal{M}$. If $\int_E f d\mu$ and $[\int_E (d\mu) f]$ exist, then

$$\int_E f d\mu = [\int_E (d\mu) f] .$$

(3) Let μ be a quaternion measure on a σ-algebra $\mathcal{M}$ in X and f a quaternion measurable function defined on $E \in \mathcal{M}$. Let $\int_E f d\mu$ and $[\int_E (d\mu) f]$ exist. Then they may not be equal. For example:

Let μ' be a real measure on a σ-algebra $\mathcal{M}$ in X. Define $\mu = i\mu'$, so μ is a quaternion measure on $\mathcal{M}$. Let f' be a bounded real measurable function defined on $E \in \mathcal{M}$. Define $f = jf'$, so f is a bounded quaternion measurable function on E. We have

$$\int_E f d\mu = j\int_E f' d\mu = j(i\int_E f' d\mu') = -k\int_E f' d\mu'$$

and

$$[\int_E (d\mu) f] = i\int_E f d\mu' = i(j\int_E f' d\mu') = k\int_E f' d\mu',$$

so

$$\int_E f d\mu \neq [\int_E (d\mu) f] .$$

In this chapter, from now, an arbitrary measure means a σ-finite positive or a quaternion measure.

4.60 <u>Definition</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X . Define

$$L^1(\mu) = \{f : X \to \mathbb{H} \,/\, f \text{ is measurable and } \int_X |f|\, d|\mu| < \infty\}.$$

The members of $L^1(\mu)$ are called <u>Lebesgue</u> <u>integrable</u> <u>functions</u> with respect to μ .

4.61 <u>Theorem</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. Then $f \in L^1(\mu)$ iff $\int_E f d\mu$ ($[\int_E (d\mu) f]$) exists for all $E \in \mathfrak{M}$.

<u>Proof</u> Let $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions $f_{i'}$, $i' \leq 4$. and let $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures $\mu_{i'}$, $i' \leq 4$. Let $E \in \mathfrak{M}$. To show that $\int_E f_1 d\mu \in \mathbb{H}$. It suffices to show that $\int_E f_1 d\mu_1$, $\int_E f_1 d\mu_2$, $\int_E f_1 d\mu_3$ and $\int_E f_1 d\mu_4$ belong to $\mathbb{R}$. Claim that $\int_E f_1 d\mu_1 \in \mathbb{R}$. It suffices to prove that $\int_E f_1^+ d\mu_1^+$, $\int_E f_1^- d\mu_1^+$, $\int_E f_1^+ d\mu_1^-$ and $\int_E f_1^- d\mu_1^-$ belong to $\mathbb{R}$. Since $f_1^+ = |f_1^+| \leq |f_1| = \sqrt{(f_1)^2} \leq \sqrt{f_1^2 + f_2^2 + f_3^2 + f_4^2} = |f|$, we have $f_1^+ \leq |f|$. Similarly $f_1^- \leq |f|$. Since $\mu_1{}^+ = \frac{1}{2}(|\mu_1| + \mu_1)$ and $|\mu_1|(E') \geq |\mu_1(E')|$ for all $E' \in \mathfrak{M}$, $|\mu_1^+(E')| = |\frac{1}{2}(|\mu_1|(E') + \mu_1(E'))| \leq \frac{1}{2}(|\mu_1|(E') + |\mu_1(E')|) \leq \frac{1}{2}(|\mu_1|(E') + |\mu_1|(E')) = |\mu_1|(E')$ for all $E' \in \mathfrak{M}$. Hence $\mu_1^+ \leq |\mu_1|$. Similarly, $\mu_1^- \leq |\mu_1|$. For each $E' \in \mathfrak{M}$

$$|\mu|(E') = \sup\{\sum_{i=1}^{\infty} |\mu(E_i')| \,/\, (E_i')_{i \in \mathbb{N}} \text{ is a partition of } E'\}$$

$$= \sup\{\sum_{i=1}^{\infty} |\mu_1(E_i') + i\mu_2(E_i') + j\mu_3(E_i') + k\mu_4(E_i')| \,/\, (E_i')_{i \in \mathbb{N}} \text{ is a partition of } E'\}$$

$$\geq \sup\left\{\sum_{l=1}^{\infty}|\mu_1(E'_{l'})| \,/\, (E'_{l'})_{l'\in\mathbb{N}} \text{ is a partition of } E'\right\}$$

$$= |\mu_1|(E') .$$

Hence $\mu_1^+ \leq |\mu_1| \leq |\mu|$ and $\mu_1^- \leq |\mu_1| \leq |\mu|$. Then $\int_E f_1^+ d\mu_1^+ \leq \int_E |f|\, d\mu_1^+ \leq \int_E |f|\, d|\mu| < \infty$. Similarly, we have $\int_E f_1^- d\mu_1^+$, $\int_E f_1^+ d\mu_1^-$ and $\int_E f_1^- d\mu_1^-$ are finite. So we have the claim, i.e., $\int_E f_1 d\mu_1 \in \mathbb{R}$. Similarly, $\int_E f_1 d\mu_2$, $\int_E f_1 d\mu_3$ and $\int_E f_1 d\mu_4$ belong to $\mathbb{R}$. Hence

$$\int_E f_1 d\mu = \int_E f_1 d\mu_1 + i\int_E f_1 d\mu_2 + j\int_E f_1 d\mu_3 + k\int_E f_1 d\mu_4 \in \mathbb{H} .$$

Similarly, we have $\int_E f_2 d\mu$, $\int_E f_3 d\mu$ and $\int_E f_4 d\mu$ belong to $\mathbb{H}$. Hence $\int_E f d\mu = \int_E f_1 d\mu + i\int_E f_2 d\mu + j\int_E f_3 d\mu + k\int_E f_4 d\mu \in \mathbb{H}$.

Conversely, let $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions $f_{l'}, l' \leq 4$; $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures $\mu_{l'}, l' \leq 4$, and assume that $\int_E f d\mu$ exists for all $E \in \mathfrak{M}$. For $E \in \mathfrak{M}$,

$$|\mu|(E) = \sup\left\{\sum_{n=1}^{\infty}|\mu(E_n)| \,/\, (E_n)_{n\in\mathbb{N}} \text{ is a partition of } E\right\}$$

$$\leq \sup\left\{\sum_{n=1}^{\infty}(|\mu_1(E_n)| + |\mu_2(E_n)| + |\mu_3(E_n)| + |\mu_4(E_n)| \,/\, (E_n)_{n\in\mathbb{N}} \text{ is a partition of } E\right\}$$

$$\leq \sum_{l'=1}^{4}\left[\sup\left\{\sum_{n=1}^{\infty}|\mu_{l'}(E_n)| \,/\, (E_n)_{n\in\mathbb{N}} \text{ is a partition of } E\right\}\right]$$

$$= |\mu_1|(E) + |\mu_2|(E) + |\mu_3|(E) + |\mu_4|(E) .$$

Hence $\int_X |f|\, d|\mu| \leq \int_X |f|\, d(|\mu_1| + |\mu_2| + |\mu_3| + |\mu_4|)$

$$= \int_X |f|\, d|\mu_1| + \int_X |f|\, d|\mu_2| + \int_X |f|\, d|\mu_3| + \int_X |f|\, d|\mu_4|.$$

We must show that $\int_X |f|\, d|\mu_{i'}| < \infty$ for all $i' = 1,2,3,4$. This proves the theorem. To prove this, let $i' \in \{1,2,3,4\}$. Since $|\mu_{i'}| = \mu_{i'}^+ + \mu_{i'}^-$, $\int_X |f|\, d|\mu_{i'}| = \int_X |f|\, d\mu_{i'}^+ + \int_X |f|\, d\mu_{i'}^-$. Since $|f| = \sqrt{f_1^2+f_2^2+f_3^2+f_4^2} \le |f_1| + |f_2| + |f_3| + |f_4|$,

$$\int_X |f|\, d\mu_{i'}^+ \le \int_X |f_1|\, d\mu_{i'}^+ + \int_X |f_2|\, d\mu_{i'}^+ + \int_X |f_3|\, d\mu_{i'}^+ + \int_X |f_4|\, d\mu_{i'}^+ .$$

Since $|f_1| = f_1^+ + f_2^-$, $\int_X |f_1|\, d\mu_{i'}^+ = \int_X f_1^+ d\mu_{i'}^+ + \int_X f_1^- d\mu_{i'}^+$. But $\int_X f_1^+ d\mu_{i'}^+$ and $\int_X f_1^- d\mu_{i'}^+$ are finite becasue $\int_X f d\mu$ exists, so $\int_X |f_1|\, d\mu_{i'}^+ < \infty$. Similarly, $\int_X |f_2|\, d\mu_{i'}^+$, $\int_X |f_3|\, d\mu_{i'}^+$ and $\int_X |f_4|\, d\mu_{i'}^+$ are finite. Hence $\int_X |f|\, d\mu_{i'}^+ < \infty$. Similarly, $\int_X |f|\, d\mu_{i'}^- < \infty$. Thus $\int_X |f|\, d|\mu_{i'}| < \infty$. #

4.62 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $f,g: X \longrightarrow [-\infty, \infty]$ be measurable and $E \in \mathfrak{M}$ such that $\int_E f d\mu$ and $\int_E g d\mu$ exist. Then

(a) If $f \le g$, then $\int_E f d\mu \le \int_E g d\mu$.

(b) For all $c \in \mathbb{R}$, $\int_E cf d\mu = c\int_E f d\mu$.

(c) If $\mu(E) = 0$, then $\int_E f d\mu = 0$.

(d) $\int_E f d\mu = \int_X \chi_E f d\mu$.

<u>Proof of (a)</u>[9] Since $f \le g$, $0 \le f^+ \le g^+$ and $0 \le g^- \le f^-$.

By Theorem 4.47, $\int_E f^+ d\mu \leq \int_E g^+ d\mu$ and $\int_E g^- d\mu \leq \int_E f^- d\mu$, so $-\int_E f^- d\mu \leq -\int_E g^- d\mu$. It follows that $\int_E f d\mu \leq \int_E g d\mu$.

<u>Proof of (b)</u>[9] <u>Case $c \geq 0$</u> Since $(cf)^+ = cf^+$ and $(cf)^- = cf^-$, by Theorem 4.50, we have $c\int_E f d\mu =$ $c(\int_E f^+ d\mu - \int_E f^- d\mu) = \int_E cf^+ d\mu - \int_E cf^- d\mu = \int_E (cf)^+ d\mu - \int_E (cf)^- d\mu$ $= \int_E cf d\mu$.

<u>Case $c < 0$</u> Since $-c > 0$, $(cf^+) = ((-c)(-f))^+ = (-c)(-f)^+ =$ $(-c)f^-$ and similarly, $(cf)^- = (-c)f^+$. Then, by Theorem 4.50, $c\int_E f d\mu = c(\int_E f^+ d\mu - \int_E f^- d\mu) = (-c)(\int_E f^- d\mu - \int_E f^+ d\mu) =$ $\int_E (-c)f^- d\mu - \int_E (-c)f^+ d\mu = \int_E (cf)^+ d\mu - \int_E (cf)^- d\mu = \int_E cf d\mu$.

<u>Proof of (c)</u>[9] By Theorem 4.28, we have $\int_E f^+ d\mu =$ $0 = \int_E f^- d\mu$, hence $\int_E f d\mu = 0$.

<u>Proof of (d)</u>[9] Since $(\chi_E f)^+ = \chi_E f^+$ and $(\chi_E f)^- = \chi_E f^-$, by Theorem 4.55, $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = \int_X \chi_E f^+ d\mu -$ $\int_X \chi_E f^- d\mu = \int_X (\chi_E f)^+ d\mu - \int_X (\chi_E f)^- d\mu = \int_X \chi_E f d\mu$. #

4.63 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If $f \in L^1(\mu)$ and $E \in \mathfrak{M}$, then the following hold:

(a) If $f = 0$ on E, then $\int_E f d\mu = 0$.

(b) If $\mu(E) = 0$, then $\int_E f d\mu = 0$.

(c) $\int_E f d\mu = \int_X \chi_E f d\mu$.

Proof Since $f \in L^1(\mu)$, $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions f_l, $l \le 4$. To prove (a), let $f = 0$ on E. Then $f_l = 0$ on E for all $l \le 4$, so by Theorem 4.49 it follows that $\int_E f\,d\mu = 0$. To prove (b), let $\mu(E) = 0$. By Theorem 4.62 (c), $\int_E f_l\,d\mu = 0$ for all $l \le 4$, hence $\int_E f\,d\mu = 0$. To prove (c), we have $\chi_E f = \chi_E f_1 + i(\chi_E f_2) + j(\chi_E f_3) + k(\chi_E f_4)$ and for each $l \le 4$, $\chi_E f_l : X \longrightarrow \mathbb{R}$. By Theorem 4.62(d) we have $\int_E f\,d\mu = \int_E f_1\,d\mu + i\int_E f_2\,d\mu + j\int_E f_3\,d\mu + k\int_E f_4\,d\mu =$ $\int_X \chi_E f_1\,d\mu + i\int_X \chi_E f_2\,d\mu + j\int_X \chi_E f_3\,d\mu + k\int_X \chi_E f_4\,d\mu = \int_X \chi_E f\,d\mu$. #

4.64 Theorem Let μ be a quaternion measure on a σ-algebra $\mathfrak{M}$ in X. If $f \in L^1(\mu)$ and $E \in \mathfrak{M}$, then the following hold:

(a) If $f = 0$ on E, then $\int_E f\,d\mu = 0$ $\left([\int_E (d\mu) f] = 0\right)$.

(b) $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$ $\left([\int_E (d\mu) f] = [\int_X (d\mu)(\chi_E f)]\right)$.

Proof Let $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions f_l, $l \le 4$ and $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures μ_l, $l \le 4$.

To prove (a), let $f = 0$ on E. Claim that $\int_E f_l\,d\mu = 0$ for all $l \le 4$. Fix $t \le 4$. By Theorem 4.63 (a), $\int_E f_t\,d\mu_l^+ = 0 = \int_E f_t\,d\mu_l^-$ for all $l \le 4$. Hence $\int_E f_t\,d\mu_l = \int_E f_t\,d\mu_l^+ - \int_E f_t\,d\mu_l^-$ $= 0$ for all $l \le 4$. Hence $\int_E f_t\,d\mu = \int_E f_t\,d\mu_1 + i\int_E f_t\,d\mu_2 +$ $j\int_E f_t\,d\mu_3 + k\int_E f_t\,d\mu_4 = 0$. So we have the claim. Hence $\int_E f\,d\mu = \int_E f_1\,d\mu + i\int_E f_2\,d\mu + j\int_E f_3\,d\mu + k\int_E f_4\,d\mu = 0$.

To prove (b), we have $\chi_E f = \chi_E f_1 + i(\chi_E f_2) + j(\chi_E f_3) + k(\chi_E f_4)$. Claim $\int_X (\chi_E f_{l'}) d\mu = \int_E f_{l'} d\mu$ for all $l' \leq 4$. Fix $t \leq 4$. By Theorem 4.63 (c), $\int_X \chi_E f_t d\mu_{l'}^+ = \int_E f_t d\mu_{l'}^+$ and $\int_X \chi_E f_t d\mu_{l'}^- = \int_E f_t d\mu_{l'}^-$ for all $l' \leq 4$. Hence $\int_X \chi_E f_t d\mu_{l'} = \int_X \chi_E f_t d\mu_{l'}^+ - \int_X \chi_E f_t d\mu_{l'}^- = \int_E f_t d\mu_{l'}^+ - \int_E f_t d\mu_{l'}^- = \int_E f_t d\mu_{l'}$ for all $l' \leq 4$. Hence

$$\begin{aligned}\int_X \chi_E f_t d\mu &= \int_X \chi_E f_t d\mu_1 + i\int_X \chi_E f_t d\mu_2 + j\int_X \chi_E f_t d\mu_3 + \\ &\quad k\int_X \chi_E f_t d\mu_4 \\ &= \int_E f_t d\mu_1 + i\int_E f_t d\mu_2 + j\int_E f_t d\mu_3 + k\int_E f_t d\mu_4 \\ &= \int_E f_t d\mu .\end{aligned}$$

So we have the claim. Hence

$$\begin{aligned}\int_X \chi_E f d\mu &= \int_X \chi_E f_1 d\mu + i\int_X \chi_E f_2 d\mu + j\int_X \chi_E f_3 d\mu + k\int_X \chi_E f_4 d\mu \\ &= \int_E f_1 d\mu + i\int_E f_2 d\mu + j\int_E f_3 d\mu + k\int_E f_4 d\mu \\ &= \int_E f d\mu . \ \#\end{aligned}$$

4.65 <u>Lebesgue's Monotone Convergence Theorem for a Quaternion Measure.</u> Let μ be a quaternion measure on a σ-algebra $\mathcal{M}$ in X. Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of non negative measurable functions

$$f_1 \leq f_2 \leq f_3 \leq \cdots$$

defined on X. If

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and $f \in L^1(\mu)$, then

$$\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu .$$

<u>Proof</u> By assumption we have $0 \le f_n \le f$ for all $n\in\mathbb{N}$. Hence $|f_n| \le |f|$ for all $n\in\mathbb{N}$. It follows that $f_n \in L^1(\mu)$ for all $n\in\mathbb{N}$. $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures $\mu_{i'}$, $i' \le 4$. So $\mu_{i'}^+$ and $\mu_{i'}^-$ are bounded positive measures, $i' \le 4$. By Theorem 4.51,

$$\lim_{n\to\infty}\int_X f_n d\mu_{i'}^+ = \int_X f d\mu_{i'}^+ , \quad \lim_{n\to\infty}\int_X f_n d\mu_{i'}^- = \int_X f d\mu_{i'}^-$$

for all $i' \le 4$. Hence

$$\lim_{n\to\infty}\int_X f_n d(\mu_{i'}^+ - \mu_{i'}^-) = \int_X f d(\mu_{i'}^+ - \mu_{i'}^-)$$

i.e.,

$$\lim_{n\to\infty}\int_X f_n d\mu_{i'} = \int_X f d\mu_{i'}$$

for all $i' \le 4$. By Theorem 1.31,

$$\lim_{n\to\infty}\left(\int_X f_n d\mu_1 + i\int_X f_n d\mu_2 + j\int_X f_n d\mu_3 + k\int_X f_n d\mu_4\right)$$
$$= \int_X f d\mu_1 + i\int_X f d\mu_2 + j\int_X f d\mu_3 + k\int_X f d\mu_4 .$$

Hence

$$\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu . \ \#$$

4.66 <u>Theorem</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. If $f,g \in L^1(\mu)$ and $f,g: X \to [0,\infty]$, then $f+g \in L^1(\mu)$ and $\int_X (f+g)d\mu = \int_X f d\mu + \int_X g d\mu$.

<u>Proof</u> Since $|f+g| \le |f| + |g|$, $\int_X |f+g| d|\mu| \le \int_X (|f| + |g|) d|\mu| = \int_X |f| d|\mu| + \int_X |g| d|\mu| < \infty$, hence $f+g \in L^1(\mu)$. Let φ be a real measure such that $f,g \in L^1(\varphi)$. Then $f+g \in L^1(\varphi)$. Hence $\int_X (f+g)d\varphi = \int_X (f+g)d\varphi^+ - \int_X (f+g)d\varphi^- =$

$\int_X f\,d\varphi^+ + \int_X g\,d\varphi^+ - \int_X f\,d\varphi^- - \int_X g\,d\varphi^- = (\int_X f\,d\varphi^+ - \int_X f\,d\varphi^-) +$ $(\int_X g\,d\varphi^+ - \int_X g\,d\varphi^-) = \int_X f\,d\varphi + \int_X g\,d\varphi$. Since $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures μ_l for all $l \leq 4$,

$$\begin{aligned}\int_X (f+g)\,d\mu &= \int_X (f+g)\,d\mu_1 + i\int_X (f+g)\,d\mu_2 + j\int_X (f+g)\,d\mu_3 + k\int_X (f+g)\,d\mu_4 \\ &= \int_X f\,d\mu_1 + \int_X g\,d\mu_1 + i(\int_X f\,d\mu_2 + \int_X g\,d\mu_2) + \\ &\quad j(\int_X f\,d\mu_3 + \int_X g\,d\mu_3) + k(\int_X f\,d\mu_4 + \int_X g\,d\mu_4) \\ &= \int_X f\,d\mu + \int_X g\,d\mu \text{ . \#}\end{aligned}$$

4.67 Theorem Let μ be a quaternion measure on a σ-algebra $\mathfrak{M}$ in X. Let $f_n : X \to [0,\infty]$ be measurable functions for all $n \in \mathbb{N}$. If

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for all $x \in X$ and $f \in L^1(\mu)$, then

$$\sum_{n=1}^{\infty} \int_X f_n\,d\mu = \int_X f\,d\mu \text{ .}$$

Proof By hypotheses we have $f_n \leq f$ for all $n \in \mathbb{N}$, so $f_n \in L^1(\mu)$ for all $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let

$$g_k(x) = \sum_{n=1}^{k} f_n(x)$$

for all $x \in X$. Hence $(g_k)_{k \in \mathbb{N}}$ is an increasing sequence of non negative measurable functions and $\lim_{k\to\infty} g_k(x) = f(x)$ for all $x \in X$. By Lebesgue's Monotone Convergence Theorem, we have

$$\begin{aligned}\int_X f\,d\mu &= \lim_{k\to\infty} \int_X g_k\,d\mu = \lim_{k\to\infty} \int_X \sum_{n=1}^{k} f_n\,d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int_X f_n\,d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n\,d\mu \text{ . \#}\end{aligned}$$

4.68 <u>Theorem</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. If $f,g \in L^1(\mu)$ and $f,g: X \longrightarrow \mathbb{H}$, then $f+g \in L^1(\mu)$ and

$$\int_X (f+g)d\mu = \int_X f d\mu + \int_X g d\mu .$$

$$\left(\left[\int_X (d\mu)(f+g)\right] = \left[\int_X (d\mu)f\right] + \left[\int_X (d\mu)g\right] \right).$$

<u>Proof</u> Since $|f+g| \leq |f| + |g|$, $\int_X |f+g| d|\mu| \leq$ $\int_X (|f|+|g|) d|\mu| = \int_X |f| d|\mu| + \int_X |g| d|\mu| < \infty$, hence $f+g \in L^1(\mu)$. Let $f = f_1+if_2+jf_3+kf_4$ and $g = g_1+ig_2+jg_3+kg_4$ for some real measurable functions $f_{i'}, g_{i'}$ for all $i' \leq 4$. Let $h_1 = f_1+g_1$, so h_1 is a real measurable function and $h_1^+ - h_1^- = f_1^+ - f_1^- + g_1^+ - g_1^-$, hence $h_1^+ + f_1^- + g_1^- = h_1^- + f_1^+ + g_1^+$. By Theorem 4.66, we get that

$$\int_X h_1^+ d\mu + \int_X f_1^- d\mu + \int_X g_1^- d\mu = \int_X h_1^- d\mu + \int_X f_1^+ d\mu + \int_X g_1^+ d\mu .$$

It follows that $\int_X h_1 d\mu = \int_X f_1 d\mu + \int_X g_1 d\mu$,i.e., $\int_X (f_1+g_1)d\mu = \int_X f_1 d\mu + \int_X g_1 d\mu$. Similarly, we have $\int_X (f_{i'}+g_{i'})d\mu = \int_X f_{i'} d\mu + \int_X g_{i'} d\mu$ for all $i' = 2,3,4$. Hence

$$\begin{aligned}
\int_X (f+g)d\mu &= \int_X (f_1+g_1)d\mu + i\int_X (f_2+g_2)d\mu + j\int_X (f_3+g_3)d\mu + \\
&\quad k\int_X (f_4+g_4)d\mu \\
&= \left(\int_X f_1 d\mu + \int_X g_1 d\mu\right) + i\left(\int_X f_2 d\mu + \int_X g_2 d\mu\right) + \\
&\quad j\left(\int_X f_3 d\mu + \int_X g_3 d\mu\right) + k\left(\int_X f_4 d\mu + \int_X g_4 d\mu\right) \\
&= \int_X f d\mu + \int_X g d\mu . \ \#
\end{aligned}$$

Remarks: Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X.

(1) If $f_i \in L^1(\mu)$ for all $i = 1,2,\ldots,n$, then

$$\int_X (f_1+f_2+\ldots+f_n)d\mu = \int_X f_1 d\mu + \ldots + \int_X f_n d\mu .$$

$$\left(\left[\int_X (d\mu)(f_1+f_2+\ldots+f_n)\right] = \left[\int_X (d\mu)f_1\right] + \ldots + \left[\int_X (d\mu)f_n\right]\right).$$

(2) Let $A,B \in \mathfrak{M}$ be such that $A \cap B = \emptyset$ and $f \in L^1(\mu)$. Then

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu .$$

$$\left(\left[\int_{A\cup B} (d\mu)f\right] = \left[\int_A (d\mu)f\right] + \left[\int_B (d\mu)f\right]\right).$$

Proof of (2) Since $A\cap B = \emptyset$, $\chi_{A\cup B}f = \chi_A f + \chi_B f$. Then, by Theorem 4.64 (b) and Theorem 4.68,

$$\int_{A\cup B} f d\mu = \int_X \chi_{A\cup B} f d\mu = \int_X (\chi_A f + \chi_B f)d\mu$$
$$= \int_X \chi_A f d\mu + \int_X \chi_B f d\mu$$
$$= \int_A f d\mu + \int_B f d\mu . \ \#$$

4.69 Theorem Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Suppose $f \in L^1(\mu)$ and

(1) $$\varphi(E) = \int_E f d\mu \qquad (E \in \mathfrak{M}).$$

Then φ is a quaternion measure on $\mathfrak{M}$ and if $g \in L^1(\mu)$, then

(2) $$\int_X g d\varphi = \int_X g f d\mu \quad \left(\left[\int_X (d\varphi)g\right] = \int_X f g d\mu\right).$$

Proof Assume $f: X \longrightarrow [-\infty,\infty]$, so f^+ and f^- belong to $L^1(\mu)$ and $\varphi(E) = \int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$. By Theorem 4.57, $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite positive measures. Then φ is a real measure on $\mathfrak{M}$. Let $g \in L^1(\mu)$ be such that

$g:X\to[-\infty,\infty]$. Since $gf=(g^+-g^-)(f^+-f^-)=g^+f^+-g^-f^+-g^+f^-+g^-f^-$, we get that

$$\int_X gf\,d\mu = \int_X g^+f^+d\mu - \int_X g^-f^+d\mu - \int_X g^+f^-d\mu + \int_X g^-f^-d\mu$$
$$= \int_X g^+d\varphi_1 - \int_X g^-d\varphi_1 - \int_X g^+d\varphi_2 + \int_X g^-d\varphi_2$$

(by Theorem 4.57, let $\varphi_1(E)=\int_E f^+d\mu$, $\varphi_2(E)=\int_E f^-d\mu$, $E\in\mathfrak{M}$)

$$= \int_X g\,d\varphi_1 - \int_X g\,d\varphi_2$$
$$= \int_X g\,d\varphi \quad (\text{since } \varphi = \varphi_1 - \varphi_2).$$

Next, let $f:X\to\mathbb{H}$. Then $f=f_1+if_2+jf_3+kf_4$ for some real measurable functions f_i, $i\le 4$. Then $\varphi(E)=\int_E f\,d\mu = \int_E f_1d\mu + i\int_E f_2d\mu + j\int_E f_3d\mu + k\int_E f_4d\mu$ for all $E\in\mathfrak{M}$. For each $E\in\mathfrak{M}$, let $\int_E f_1d\mu=\varphi_1(E)$, $\int_E f_2d\mu=\varphi_2(E)$, $\int_E f_3d\mu=\varphi_3(E)$ and $\int_E f_4d\mu=\varphi_4(E)$. So φ_1, φ_2, φ_3 and φ_4 are real measures on $\mathfrak{M}$. Hence $\varphi(E)=\varphi_1(E)+i\varphi_2(E)+j\varphi_3(E)+k\varphi_4(E)$ for all $E\in\mathfrak{M}$, i.e., φ is a quaternion measure on $\mathfrak{M}$.

Let $g\in L^1(\mu)$. Then $g=g_1+ig_2+jg_3+kg_4$ for some real measurable functions g_i, $i\le 4$. Thus

$$gf=(g_1f_1-g_2f_2-g_3f_3-g_4f_4)+i(g_1f_2+g_2f_1+g_3f_4-g_4f_3)+$$
$$j(g_1f_3+g_3f_1+g_4f_2-g_2f_4)+k(g_1f_4+g_4f_1+g_2f_3-g_3f_2).$$

Hence

$$\int_X gf\,d\mu = \int_X(g_1f_1-g_2f_2-g_3f_3-g_4f_4)d\mu + i\int_X(g_1f_2+g_2f_1+g_3f_4-g_4f_3)d\mu$$
$$+j\int_X(g_1f_3+g_3f_1+g_4f_2-g_2f_4)d\mu + k\int_X(g_1f_4+g_4f_1+g_2f_3-g_3f_2)d\mu$$
$$=\left(\int_X g_1f_1d\mu - \int_X g_2f_2d\mu - \int_X g_3f_3d\mu - \int_X g_4f_4d\mu\right)$$
$$+i\left(\int_X g_1f_2d\mu + \int_X g_2f_1d\mu + \int_X g_3f_4d\mu - \int_X g_4f_3d\mu\right)$$

$$+j\left(\int_X g_1 f_3\,d\mu + \int_X g_3 f_1\,d\mu + \int_X g_4 f_2\,d\mu - \int_X g_2 f_4\,d\mu\right)$$
$$+k\left(\int_X g_1 f_4\,d\mu + \int_X g_4 f_1\,d\mu + \int_X g_2 f_3\,d\mu - \int_X g_3 f_2\,d\mu\right)$$
$$= \left(\int_X g_1\,d\varphi_1 - \int_X g_2\,d\varphi_2 - \int_X g_3\,d\varphi_3 - \int_X g_4\,d\varphi_4\right)$$
$$+i\left(\int_X g_1\,d\varphi_2 + \int_X g_2\,d\varphi_1 + \int_X g_3\,d\varphi_4 - \int_X g_4\,d\varphi_3\right)$$
$$+j\left(\int_X g_1\,d\varphi_3 + \int_X g_3\,d\varphi_1 + \int_X g_4\,d\varphi_2 - \int_X g_2\,d\varphi_4\right)$$
$$+k\left(\int_X g_1\,d\varphi_4 + \int_X g_4\,d\varphi_1 + \int_X g_2\,d\varphi_3 - \int_X g_3\,d\varphi_2\right).$$

$$\int_X g\,d\varphi = \int_X g_1\,d\varphi + i\int_X g_2\,d\varphi + j\int_X g_3\,d\varphi + k\int_X g_4\,d\varphi$$
$$= \int_X g_1\,d\varphi_1 + i\int_X g_1\,d\varphi_2 + j\int_X g_1\,d\varphi_3 + k\int_X g_1\,d\varphi_4$$
$$+i\left(\int_X g_2\,d\varphi_1 + i\int_X g_2\,d\varphi_2 + j\int_X g_2\,d\varphi_3 + k\int_X g_2\,d\varphi_4\right)$$
$$+j\left(\int_X g_3\,d\varphi_1 + i\int_X g_3\,d\varphi_2 + j\int_X g_3\,d\varphi_3 + k\int_X g_3\,d\varphi_4\right)$$
$$+k\left(\int_X g_4\,d\varphi_1 + i\int_X g_4\,d\varphi_2 + j\int_X g_4\,d\varphi_3 + k\int_X g_4\,d\varphi_4\right)$$
$$= \left(\int_X g_1\,d\varphi_1 - \int_X g_2\,d\varphi_2 - \int_X g_3\,d\varphi_3 - \int_X g_4\,d\varphi_4\right)$$
$$+i\left(\int_X g_1\,d\varphi_2 + \int_X g_2\,d\varphi_1 + \int_X g_3\,d\varphi_4 - \int_X g_4\,d\varphi_3\right)$$
$$+j\left(\int_X g_1\,d\varphi_3 - \int_X g_2\,d\varphi_4 + \int_X g_3\,d\varphi_1 + \int_X g_4\,d\varphi_2\right)$$
$$+k\left(\int_X g_1\,d\varphi_4 + \int_X g_2\,d\varphi_3 - \int_X g_3\,d\varphi_2 + \int_X g_4\,d\varphi_1\right)$$

Hence

$$\int_X g\,d\varphi = \int_X gf\,d\mu \quad . \#$$

4.70 <u>Theorem</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. If $f, g \in L^1(\mu)$, $\alpha, \beta \in \mathbb{H}$, then $\alpha f + \beta g \in L^1(\mu)$ and

(1) $$\int_X (\alpha f + \beta g)\,d\mu = \alpha\int_X f\,d\mu + \beta\int_X g\,d\mu .$$

$$\left(\left[\int_X (d\mu)(f\alpha + g\beta)\right] = \left[\int_X (d\mu)f\right]\alpha + \left[\int_X (d\mu)g\right]\beta\right).$$

<u>Proof</u> We have $\alpha f+\beta g$ is measurable. Since $|\alpha f+\beta g| \leq |\alpha||f|+|\beta||g|$, $\int_X |\alpha f+\beta g|\,d|\mu| \leq \int_X |\alpha||f|\,d|\mu| + \int_X |\beta||g|\,d|\mu| < \infty$ hence $\alpha f+\beta g \in L^1(\mu)$.

To prove (1), it is sufficient to prove that

$$\int_X (f+g)d\mu = \int_X fd\mu + \int_X gd\mu \text{ and } \int_X \alpha fd\mu = \alpha\int_X fd\mu .$$

By Theorem 4.68, we have $\int_X (f+g)d\mu = \int_X fd\mu + \int_X gd\mu$. Let $f = f_1+if_2+jf_3+kf_4$ for some real measurable functions $f_{i'}$, $i' \leq 4$. If $\alpha \in \mathbb{R}$, then $\int_X \alpha fd\mu = \int_X \alpha f_1 d\mu + i\int_X \alpha f_2 d\mu + j\int_X \alpha f_3 d\mu + k\int_X \alpha f_4 d\mu$. Claim that $\int_X \alpha f_1 d\mu = \alpha\int_X f_1 d\mu$. To prove this, let $\mu_{i'}$, $i' \leq 4$ be real measures such that $\mu = \mu_1+i\mu_2+j\mu_3+k\mu_4$. Then

$$\int_X \alpha f_1 d\mu_1 = \int_X \alpha f_1 d\mu_1^+ - \int_X \alpha f_1 d\mu_1^- = \alpha\int_X f_1 d\mu_1^+ - \alpha\int_X f_1 d\mu_1^-$$

(by Theorem 4.62 (b)) $= \alpha\int_X f_1 d\mu_1$. Similarly, $\int_X \alpha f_1 d\mu_2 = \alpha\int_X f_1 d\mu_2$, $\int_X \alpha f_1 d\mu_3 = \alpha\int_X f_1 d\mu_3$ and $\int_X \alpha f_1 d\mu_4 = \alpha\int_X f_1 d\mu_4$. Hence $\int_X \alpha f_1 d\mu = \int_X \alpha f_1 d\mu_1 + i\int_X \alpha f_1 d\mu_2 + j\int_X \alpha f_1 d\mu_3 + k\int_X \alpha f_1 d\mu_4$ $= \alpha\int_X f_1 d\mu_1 + \alpha i\int_X f_1 d\mu_2 + \alpha j\int_X f_1 d\mu_3 + \alpha k\int_X f_1 d\mu_4 =$ $\alpha(\int_X f_1 d\mu_1 + i\int_X f_1 d\mu_2 + j\int_X f_1 d\mu_3 + k\int_X f_1 d\mu_4) = \alpha\int_X f_1 d\mu$. So we have the claim. Similarly, $\int_X \alpha f_{i'} d\mu = \alpha\int_X f_{i'} d\mu$ for all $i' = 2,3,4$. Hence

$$\begin{aligned}
\int_X \alpha fd\mu &= \int_X \alpha f_1 d\mu + i\int_X \alpha f_2 d\mu + j\int_X \alpha f_3 d\mu + k\int_X \alpha f_4 d\mu \\
&= \alpha\int_X f_1 d\mu + \alpha i\int_X f_2 d\mu + \alpha j\int_X f_3 d\mu + \alpha k\int_X f_4 d\mu \\
&= \alpha\int_X fd\mu .
\end{aligned}$$

Let $\alpha = \alpha_1 + i\alpha_2 + j\alpha_3 + k\alpha_4$ for some $\alpha_i \in \mathbb{R}$ for all $i \leq 4$. Then

$$\begin{aligned}\int_X if\,d\mu &= \int_X (if_1 - f_2 + kf_3 - jf_4)\,d\mu \\ &= i\int_X f_1 d\mu - \int_X f_2 d\mu + k\int_X f_3 d\mu - j\int_X f_4 d\mu \\ &= i\left(\int_X f_1 d\mu + i\int_X f_2 d\mu + j\int_X f_3 d\mu + k\int_X f_4 d\mu\right) \\ &= i\int_X f\,d\mu .\end{aligned}$$

Similarly, we have $\int_X jf\,d\mu = j\int_X f\,d\mu$ and $\int_X kf\,d\mu = k\int_X f\,d\mu$

Hence

$$\begin{aligned}\int_X \alpha f\,d\mu &= \int_X (\alpha_1 + i\alpha_2 + j\alpha_3 + k\alpha_4) f\,d\mu \\ &= \int_X \alpha_1 f\,d\mu + \int_X i\alpha_2 f\,d\mu + \int_X j\alpha_3 f\,d\mu + \int_X k\alpha_4 f\,d\mu \\ &= \alpha_1 \int_X f\,d\mu + i\int_X \alpha_2 f\,d\mu + j\int_X \alpha_3 f\,d\mu + k\int_X \alpha_4 f\,d\mu \\ &= \alpha_1 \int_X f\,d\mu + i\alpha_2 \int_X f\,d\mu + j\alpha_3 \int_X f\,d\mu + k\alpha_4 \int_X f\,d\mu \\ &= \alpha \int_X f\,d\mu . \ \#\end{aligned}$$

4.71 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. If $f \in L^1(\mu)$, then

$$\left|\int_X f\,d\mu\right| \leq \int_X |f|\,d\mu .$$

<u>Proof</u> Let $h = \int_X f\,d\mu$, then $h \in \mathbb{H}$. Let

$$\alpha = \begin{cases} 1 & \text{if } h = 0, \\ \dfrac{|h|}{h} & \text{if } h \neq 0. \end{cases}$$

Then

(*) $\qquad \alpha h = |h|$ and $|\alpha| = 1$.

Let $\alpha f = g_1 + ig_2 + jg_3 + kg_4$ for some real measurable functions g_i, $i \leq 4$. Then $\int_X g_1 d\mu + i\int_X g_2 d\mu + j\int_X g_3 d\mu + k\int_X g_4 d\mu =$

$\int_X \alpha f d\mu = \alpha \int_X f d\mu = \left|\int_X f d\mu\right|$ (From (*)). Hence

$$0 \le \left|\int_X f d\mu\right| = \int_X g_1 d\mu \le \int_X |\alpha f| d\mu \text{ (since } g_1 \le |\alpha f| \text{ and}$$

Theorem 4.62 (a)) $= \int_X |f| d\mu$. Therefore

$$\left|\int_X f d\mu\right| \le \int_X |f| d \ . \ \#$$

4.72 <u>Lebesgue's Dominated Convergence Theorem</u>

Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ in X. Let $f_n : X \longrightarrow \mathbb{H}$ be measurable for all $n \in N$ such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$. If there exists a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \le g(x)$$

for all $n \in N$ for all $x \in X$, then

(i) $f \in L^1(\mu)$,

(ii) $\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$, and

(iii) $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

<u>Proof</u>[9] Since $|f_n| \le g$ for all $n \in N$, $|f| \le g$. Since f is measurable, $f \in L^1(\mu)$. This proves (i). To prove (ii), for each $n \in N$ $|f_n - f| \le |f_n| + |f| \le 2g$. Since $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$, $\lim_{n\to\infty} |f_n - f|(x) = 0$ for all $x \in X$. Thus $\lim_{n\to\infty}(2g - |f_n - f|)(x) = 2g(x)$ for all $x \in X$. Hence

$$\int_X 2g d\mu = \int_X \liminf_{n\to\infty}(2g - |f_n - f|) d\mu \le \liminf_{n\to\infty} \int_X (2g - |f_n - f|) d\mu$$

$$= \liminf_{n\to\infty}\left(\int_X 2g d\mu - \int_X |f_n - f| d\mu\right)$$

$$= \int_X 2g d\mu - \limsup_{n\to\infty} \int_X |f_n - f| d\mu \ .$$

Since $\int_X 2g\,d\mu < \infty$, $\limsup_{n\to\infty}\int_X |f_n - f|\,d\mu \leq 0$. By Note 1.34.3, $\lim_{n\to\infty}\int_X |f_n - f|\,d\mu = 0$. This proves (ii).

To prove (iii), by Theorem 4.71, $0 \leq \left|\int_X (f_n - f)\,d\mu\right| \leq \int_X |f_n - f|\,d\mu$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\int_X |f_n - f|\,d\mu = 0$, $\lim_{n\to\infty}\left|\int_X (f_n - f)\,d\mu\right| = 0$, i.e., $\lim_{n\to\infty}\left|\int_X f_n\,d\mu - \int_X f\,d\mu\right| = 0$, so $\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu$. This proves (iii). #

4.73 <u>Lebesgue's Dominated Convergence Theorem</u>

Let μ be a quaternion measure on a σ-algebra $\mathfrak{M}$ in X. Let $f_n : X \to \mathbb{H}$ be measurable for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x)$$

for all $n \in \mathbb{N}$ for all $x \in X$, then

(i) $f \in L^1(\mu)$,

(ii) $\lim_{n\to\infty}\int_X |f_n - f|\,d\mu = 0$, and

(iii) $\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu$.

$\left(\lim_{n\to\infty}\left[\int_X (d\mu) f_n\right] = \left[\int_X (d\mu) f\right]\right)$.

<u>Proof</u> By Theorem 4.72 shows that (i) holds. Let $f = a+ib+jc+kd$ for some real measurable functions a, b, c and d. For each $n \in \mathbb{N}$, let $f_n = a_n + ib_n + jc_n + kd_n$ for some real measurable functions a_n, b_n, c_n and d_n. Then for all $x \in X$ $a(x)+ib(x)+jc(x)+kd(x) = f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(a_n(x)+ib_n(x)+jc_n(x)+kd_n(x))$. Hence $\lim_{n\to\infty} a_n(x) = a(x)$,

$\lim_{n\to\infty} b_n(x) = b(x)$, $\lim_{n\to\infty} c_n(x) = c(x)$ and $\lim_{n\to\infty} d_n(x) = d(x)$ for all $x \in X$. For each $n \in \mathbb{N}$, $|a_n| \le |f_n| \le g$, $|b_n| \le |f_n| \le g$, $|c_n| \le |f_n| \le g$ and $|d_n| \le |f_n| \le g$. Let $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures $\mu_{l'}$, $l' \le 4$. By Theorem 4.72, $\lim_{n\to\infty}\int_X |a_n - a|\, d\mu_1^+ = 0$, $\lim_{n\to\infty}\int_X |b_n - b|\, d\mu_1^+ = 0$, $\lim_{n\to\infty}\int_X |c_n - c|\, d\mu_1^+ = 0$ and $\lim_{n\to\infty}\int_X |d_n - d|\, d\mu_1^+ = 0$. For each $n \in \mathbb{N}$,

$$\begin{aligned} |f_n - f| &= |a_n + ib_n + jc_n + kd_n - (a + ib + jc + kd)| \\ &= |(a_n - a) + i(b_n - b) + j(c_n - c) + k(d_n - d)| \\ &\le |a_n - a| + |b_n - b| + |c_n - c| + |d_n - d| . \end{aligned}$$

Hence $\lim_{n\to\infty}\int_X |f_n - f|\, d\mu_1^+ = 0$. Similarly, $\lim_{n\to\infty}\int_X |f_n - f|\, d\mu_1^- = 0$. Then $\lim_{n\to\infty}\int_X |f_n - f|\, d\mu_1 = \lim_{n\to\infty}(\int_X |f_n - f|\, d\mu_1^+ - \int_X |f_n - f|\, d\mu_1^-) = 0$. Similarly, we have $\lim_{n\to\infty}\int_X |f_n - f|\, d\mu_{l'} = 0$, $l' = 2,3,4$. Since for each $n \in \mathbb{N}$, $\int_X |f_n - f|\, d\mu = \int_X |f_n - f|\, d\mu_1 + i\int_X |f_n - f|\, d\mu_2 + j\int_X |f_n - f|\, d\mu_3 + k\int_X |f_n - f|\, d\mu_4$, we get that $\lim_{n\to\infty}\int_X |f_n - f|\, d\mu = 0$. This proves (ii)

Next, we shall prove (iii). By Theorem 4.72, we have $\lim_{n\to\infty}\int_X a_n\, d\mu_1^+ = \int_X a\, d\mu_1^+$ and $\lim_{n\to\infty}\int_X a_n\, d\mu_1^- = \int_X a\, d\mu_1^-$. Then $\lim_{n\to\infty}\int_X a_n\, d\mu_1 = \lim_{n\to\infty}(\int_X a_n\, d\mu_1^+ - \int_X a_n\, d\mu_1^-) = \lim_{n\to\infty}\int_X a_n\, d\mu_1^+ - \lim_{n\to\infty}\int_X a_n\, d\mu_1^- = \int_X a\, d\mu_1^+ - \int_X a\, d\mu_1^- = \int_X a\, d\mu_1$. Similarly,

$\lim_{n\to\infty}\int_X a_n\, d\mu_{l'} = \int_X a\, d\mu_{l'}$ for all $l' = 2,3,4$. Hence

$$\begin{aligned} \lim_{n\to\infty}\int_X a_n\, d\mu &= \lim_{n\to\infty}(\int_X a_n\, d\mu_1 + i\int_X a_n\, d\mu_2 + j\int_X a_n\, d\mu_3 + k\int_X a_n\, d\mu_4) \\ &= \int_X a\, d\mu_1 + i\int_X a\, d\mu_2 + j\int_X a\, d\mu_3 + k\int_X a\, d\mu_4 \end{aligned}$$

$$= \int_X a\,d\mu \ .$$

Similarly, we have $\lim_{n\to\infty}\int_X b_n d\mu = \int_X b\,d\mu$, $\lim_{n\to\infty}\int_X c_n d\mu = \int_X c\,d\mu$ and $\lim_{n\to\infty}\int_X d_n d\mu = \int_X d\ d\mu$. Hence

$$\lim_{n\to\infty}\int_X f_n d\mu = \lim_{n\to\infty}\left(\int_X a_n d\mu + i\int_X b_n d\mu + j\int_X c_n d\mu + k\int_X d_n d\mu\right)$$

$$= \int_X a\,d\mu + i\int_X b\,d\mu + j\int_X c\,d\mu + k\int_X d\ d\mu$$

$$= \int_X f\,d\mu \ .$$

This proves (iii). #

4.74 <u>Thoemrem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X.

(a) If $f \in L^1(\mu)$ and $\int_E f\,d\mu = 0$ for all $E \in \mathfrak{M}$, then $f = 0$ a.e. on X.

(b) If $f \in L^1(\mu)$ and $\left|\int_X f\,d\mu\right| = \int_X |f|\,d\mu$, then there exists $\alpha \in \mathbb{H}$ such that $\alpha f = |f|$ a.e. on X.

<u>Proof of (a)</u> Put $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions $f_{i'}$, $i' \leq 4$. Let $E = \{x \in X /\ f_1(x) \geqslant 0\}$. Then $E \in \mathfrak{M}$, so $0 = \int_E f\,d\mu = \int_E f_1 d\mu + i\int_E f_2 d\mu + j\int_E f_3 d\mu + k\int_E f_4 d\mu$, hence $\int_E f_{i'} d\mu = 0$ for all $i' \leq 4$. But $0 = \int_E f_1 d\mu = \int_E f_1^+ d\mu$ (since $f_1 = f_1^+$ on E). By Theorem 4.48, $f_1^+ = 0$ a.e. on E. Since $f_1 < 0$ on E^c, $f_1^+ = 0$ on E^c. Then $f_1^+ = 0$ a.e. on X. Similarly, $f_1^- = f_2^+ = f_2^- = f_3^+ = f_3^- = f_4^+ = f_4^- = 0$ a.e. on X. Hence $f = 0$ a.e. on X.

<u>Proof of (b)</u> Let $h = \int_X f\,d\mu$, so $h \in \mathbb{H}$. Let

$$\alpha = \begin{cases} 1 & \text{if } h = 0, \\ \frac{|h|}{h} & \text{if } h \neq 0. \end{cases}$$

Then $|\alpha| = 1$ and $\alpha h = |h|$. Let $\alpha f = g_1 + ig_2 + jg_3 + kg_4$ for some real measurable functions g_i, $i \leq 4$. Hence

$$\int_X g_1\,d\mu + i\int_X g_2\,d\mu + j\int_X g_3\,d\mu + k\int_X g_4\,d\mu = \int_X \alpha f\,d\mu = \alpha\int_X f\,d\mu$$

$= \left|\int_X f\,d\mu\right| = \int_X |f|\,d\mu \geqslant 0$. It follows that $\int_X (|f| - g_1)\,d\mu$ $= 0$. Since $|f| = |\alpha f| \geqslant g_1$, $|f| - g_1 \geqslant 0$. By Theorem 4.48, $|f| = g_1$ a.e. on X. Then $|\alpha f| = g_1$ a.e. on X. Hence $\alpha f = g_1$ a.e. on X. Hence $\alpha f = |f|$ a.e. on X. #

4.75 <u>Theorem</u> Let μ be a finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Suppose $f \in L^1(\mu)$, S is a closed set in $\mathbb{H}$, and the averages

$$A_E(f) = \frac{1}{\mu(E)}\int_E f\,d\mu$$

lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

<u>Proof</u>[9] Let $\alpha \in S^c$. Since S is closed, there exists $r > 0$ such that $\overline{B(\alpha;r)} \subseteq S^c$. Let $E = f^{-1}(\overline{B(\alpha;r)})$, so $E \in \mathfrak{M}$. Suppose $\mu(E) > 0$. Then

$$\begin{aligned}
|A_E(f) - \alpha| &= \left|\frac{1}{\mu(E)}\int_E f\,d\mu - \frac{1}{\mu(E)}\int_E \alpha\,d\mu\right| \\
&= \frac{1}{\mu(E)}\left|\int_E f\,d\mu - \int_E \alpha\,d\mu\right| \\
&= \frac{1}{\mu(E)}\left|\int_E (f-\alpha)\,d\mu\right| \quad \text{(by Theorem 4.70 and } \mu(X) < \infty) \\
(*) \qquad &\leq \frac{1}{\mu(E)}\int_E |f-\alpha|\,d\mu \quad \text{(by Theorem 4.71).}
\end{aligned}$$

For each $x \in E$, $|f(x)-\alpha| \leq r$, i.e., $0 \leq |f-\alpha| \leq r$ on E. Then $\int_E |f-\alpha| d\mu \leq \int_E r d\mu = r\mu(E)$. From (*), we have $|A_E(f)-\alpha| \leq r$ which is impossible since $A_E(f) \in S$. Hence $\mu(E) = 0$.

This proves that every closed ball $\bar{B} \subseteq S^c$, $\mu(f^{-1}(\bar{B})) = 0$. By Theorem 1.30, S^c is a countable union of closed balls contained in S^c. We have that $\mu(f^{-1}(S^c)) = 0$ and $f(X \setminus f^{-1}(S^c)) \subseteq S$. #

4.76 <u>Theorem</u> Let X be a locally compact, σ-compact Hausdorff space, and let Λ be a positive left(right) linear functional on $C_c(X)$ over $\mathbb{H}$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique σ-finite positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that

(a) $\Lambda f = \int_X f d\mu$ for every $f \in C_c(X)$, and which has the following additional properties:

(b) $\mu(K) < \infty$ for every compact set $K \subseteq X$.

(c) For every $E \in \mathfrak{M}$, we have

$$\mu(E) = \inf\{\mu(V) / E \subseteq V, V \text{ open}\}$$

(d) The relation

$$\mu(E) = \sup\{\mu(K) / K \subseteq E, K \text{ compact}\}$$

holds for every open set E, and for every $E \in \mathfrak{M}$ with $\mu(E) < \infty$.

(e) If $E \in \mathfrak{M}$, $A \subseteq E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$.

<u>Proof</u> <u>Uniqueness of μ</u> Let μ_1 and μ_2 be σ-finite positive measures on $\mathfrak{M}$ for which the theorem holds. First

we shall show that $\mu_1(K) = \mu_2(K)$ for every compact set K of X. Let K be a compact set of X and let $\varepsilon > 0$ be given. By (b) and (c), there exists an open set $V \supseteq K$ such that

$$\mu_2(V) < \mu_2(K) + \varepsilon .$$

By Urysohn's Lemma, there exists $f \in C_c(X)$ such that

$$K \prec f \prec V.$$

Then $\chi_K \leq f \leq \chi_V$. Hence

$$\mu_1(K) = \int_X \chi_K d\mu_1 \leq \int_X f d\mu_1 = \Lambda f = \int_X f d\mu_2$$

$$\leq \int_X \chi_V d\mu_2 = \mu_2(V) < \mu_2(K) + \varepsilon .$$

Since $\varepsilon > 0$ is an arbitrary, $\mu_1(K) \leq \mu_2(K)$. By interchanging the roles of μ_1 and μ_2, we obtain $\mu_2(K) \leq \mu_1(K)$. Hence $\mu_1(K) = \mu_2(K)$. It follows by (d) that $\mu_1(E) = \mu_2(E)$ for every open set E of X. Hence by (c), $\mu_1(E) = \mu_2(E)$ for every $E \in \mathfrak{M}$.

Construction of μ and $\mathfrak{M}$

For every open set V in X, define

(1) $\mu(V) = \sup\{\Lambda f /\ f \in C_c(X) \text{ and } f \prec V\}$.

From (1), if V_1, V_2 are open sets such that $V_1 \subseteq V_2$, then $\mu(V_1) \leq \mu(V_2)$. Hence, if $E \subseteq X$ is open,

$$\mu(E) = \inf\{\mu(V) /\ E \subseteq V,\ V \text{ open}\}.$$

For $E \subseteq X$, define

(2) $\mu(E) = \inf\{\mu(V) /\ E \subseteq V,\ V \text{ open}\}$.

Let $\mathfrak{M}_F$ = the class of all $E \subseteq X$ such that

$$\mu(E) < \infty$$

and

(3) $\mu(E) = \sup\{\mu(K)/\ K \subseteq E,\ K \text{ compact}\}$.

Let $\mathcal{M}$ be the class of all $E \subseteq X$ such that $E \cap K \in \mathcal{M}_F$ for every compact set K of X. Observe that

(i) If $A, B \subseteq X$ are such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(ii) If $E \subseteq X$ and $\mu(E) = 0$, then $E \in \mathcal{M}_F$ and $E \in \mathcal{M}$.

Thus (e) holds. By the definition of μ, (c) holds. If $f \leq g$ in $C_c(X)$, then $\Lambda g = \Lambda f + \Lambda(g-f) \geq \Lambda f$ since $\Lambda(g-f) \geq 0$.

<u>Claim</u> I If $E_1, E_2, \ldots$ are arbitrary subsets of X, then

(4) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

To prove this, let V_1, V_2 be open sets in X. Let $g \in C_c(X)$ be such that $g \prec V_1 \cup V_2$. By Theorem 1.41, there exist h_1, $h_2 \in C_c(X)$ such that $h_i \prec V_i$, $i = 1,2$, and $h_1(x) + h_2(x) = 1$ for all $x \in$ support g. Hence $gh_i \prec V_i$, $i = 1,2$ (since $\{x \in X/\ g(x)h_i(x) \neq 0\} \subseteq \{x \in X/\ h_i(x) \neq 0\}$) and $g = gh_1 + gh_2$. Thus $\Lambda g = \Lambda(gh_1) + \Lambda(gh_2) \leq \mu(V_1) + \mu(V_2)$ (from (1)). This proves that $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$. By induction, if $V_1, \ldots, V_n$ are open in X, then

$$\mu(V_1 \cup V_2 \cup \ldots \cup V_n) \leq \mu(V_1) + \ldots + \mu(V_n).$$

If $\mu(E_i) = \infty$ for some i, (4) holds clearly. Suppose $\mu(E_i) < \infty$ for all i. Let $\varepsilon > 0$ be given. By (2), for each i there exists an open set V_i such that $V_i \supseteq E_i$ and

$$\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{2^i}.$$

Put $V = \bigcup_{i=1}^{\infty} V_i$. Let $f \in C_c(X)$ be such that $f \prec V$. Since support $f \subseteq V = \bigcup_{i=1}^{\infty} V_i$, there exists $n \in \mathbb{N}$ such that support $f \subseteq V_1 \cup \ldots \cup V_n$. Hence

$$\Lambda f \leq \mu(V_1 \cup \ldots \cup V_n) \leq \mu(V_1) + \ldots + \mu(V_n)$$

$$< \sum_{i=1}^{n}\left(\mu(E_i)+\frac{\varepsilon}{2^i}\right) < \sum_{i=1}^{\infty}\mu(E_i)+\varepsilon .$$

Hence

$$\mu(V) \leq \sum_{i=1}^{\infty}\mu(E_i)+\varepsilon .$$

But $\bigcup_{i=1}^{\infty}E_i \subseteq V$, so $\mu(\bigcup_{i=1}^{\infty}E_i) \leq \mu(V)$. Hence $\mu(\bigcup_{i=1}^{\infty}E_i) \leq \sum_{i=1}^{\infty}\mu(E_i)+\varepsilon$. This holds for all $\varepsilon > 0$, thus $\mu(\bigcup_{i=1}^{\infty}E_i) \leq \sum_{i=1}^{\infty}\mu(E_i)$, so we have claim I.

<u>Claim II</u> $\mathfrak{M}_F$ contains every compact set. Hence (b) holds. To prove this , let K be a compact set. Let $f \in C_c(X)$ be such that $K \prec f$ (use Urysohn's Lemma). Let $V = \{x \in X/\ f(x) > \frac{1}{2}\}$. Then $K \subseteq V$ and $g \leq 2f$ for all $g \in C_c(X)$ such that $g \prec V$. Hence

$\mu(K) \leq \mu(V) = \sup\{\Lambda g/\ g \in C_c(X)$ such that $g \prec V\}$

$\leq \Lambda 2f = 2\Lambda f < \infty$ (since $\Lambda f \in \mathbb{H}$).

Clearly, K satisfies (3). So $K \in \mathfrak{M}_F$ and we have claim II.

<u>Claim III</u> Every open set satisfies (3). Hence $\mathfrak{M}_F$ contains every open set V with $\mu(V) < \infty$. To prove this, let V be an open set. Let $\alpha \in \mathbb{R}$ be such that $\alpha < \mu(V)$. There exists an $f \in C_c(X)$ such that $f \prec V$ with $\alpha < \Lambda f$ (since α is not an upper bounded of $\{\Lambda f/\ f \in C_c(X)$ and $f \prec V$). If W is any open set containing support f, then $f \prec W$, hence $\Lambda f \leq \mu(W)$. Thus by (2),

$$\Lambda f \leq \mu(\text{support } f).$$

Then $\mu(\text{support } f) > \alpha$. This shows that

$$\sup\{\mu(K)/\ K \subseteq V,\ K \text{ compact}\} > \alpha$$

for all $\alpha \in (-\infty, \mu(V))$. Hence

$$\sup\{\mu(K)/\ K \subseteq V,\ K \text{ compact}\} \geq \mu(V).$$

But $\sup\{\mu(K) / K \subseteq V,\ K \text{ compact}\} \leq \mu(V)$. Thus V satisfies (3) and we have claim III.

<u>Claim IV</u> Suppose $E_1, E_2, \ldots$ are pairwise disjoint members of $\mathfrak{M}_F$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

If $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) < \infty$, then $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{M}_F$. To prove this, let K_1 and K_2 be compact subsets of X such that $K_1 \cap K_2 = \phi$. Let $\varepsilon > 0$ be given. Since $K_1 \subseteq K_2^c$ and K_2^c is open (since a compact subset of a Hausdorff space is closed), by Theorem 1.36 there exists an open set V_1 such that

$$K_1 \subseteq V_1 \subseteq \overline{V}_1 \subseteq K_2^c .$$

Since $K_2 \subseteq \overline{V}_1^c$ and $\overline{V}_1^c$ is open, there exists an open set V_2 such that

$$K_2 \subseteq V_2 \subseteq \overline{V}_2 \subseteq \overline{V}_1^c \quad (\subseteq V_1^c) .$$

Then $V_1 \cap V_2 = \phi$. By claim II, $\mu(K_1 \cup K_2) < \infty$. From (2), there exists an open set W such that $W \supseteq K_1 \cup K_2$ and

$$\mu(W) < \mu(K_1 \cup K_2) + \varepsilon .$$

Since $W \cap V_i$ is open, $i = 1,2$, from (1) there exist $f_i \in C_c(X)$ such that $f_i \prec W \cap V_i$ and $\Lambda f_i > \mu(W \cap V_i) - \varepsilon$, $i = 1,2$. Since $K_i \subseteq W \cap V_i$ and $f_1 + f_2 \prec W$, we obtain

$$\mu(K_1) + \mu(K_2) \leq \mu(W \cap V_1) + \mu(W \cap V_2) < \Lambda f_1 + \Lambda f_2 + 2\varepsilon$$

$$= \Lambda(f_1 + f_2) + 2\varepsilon \leq \mu(W) + 2\varepsilon < \mu(K_1 \cup K_2) + 3\varepsilon .$$

This holds for all $\varepsilon > 0$, so

$$\mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2).$$

By claim I, $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$. Hence

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2).$$

By induction, if $K_1,\dots,K_n$ are pairwise disjoint compact sets, then

$$\mu(K_1\cup\dots\cup K_n) = \mu(K_1)+\dots+\mu(K_n).$$

If $\mu(\bigcup_{i=1}^{\infty}E_i) = \infty$, then from claim I, we are done. Assume $\mu(\bigcup_{i=1}^{\infty}E_i)<\infty$. Let $\varepsilon>0$ be given. Since $E_i\in\mathcal{M}_F$, there exists a compact set $H_i\subseteq E_i$ such that

$$\mu(H_i) > \mu(E_i)-\frac{\varepsilon}{2^i} \qquad (i = 1,2,\dots).$$

For each n, put $K_n = \bigcup_{i=1}^{n}H_i$. Then K_n is compact for all n. Hence

$$\mu(\bigcup_{i=1}^{\infty}E_i) \geq \mu(K_n) = \sum_{i=1}^{n}\mu(H_i) > \sum_{i=1}^{n}\mu(E_i)-\varepsilon .$$

This holds for all $n\in\mathbb{N}$ for all $\varepsilon>0$, hence

$$\mu(\bigcup_{i=1}^{\infty}E_i) \geq \sum_{i=1}^{\infty}\mu(E_i).$$

Also, $\mu(\bigcup_{i=1}^{\infty}E_i) \leq \sum_{i=1}^{\infty}\mu(E_i)$ by claim I, we thus have that

$$\mu(\bigcup_{i=1}^{\infty}E_i) = \sum_{i=1}^{\infty}\mu(E_i)$$

Assume that $\mu(\bigcup_{i=1}^{\infty}E_i)<\infty$. Let $\varepsilon>0$ be given. Since $\lim_n \sum_{i=1}^{n}\mu(E_i) = \sum_{i=1}^{\infty}\mu(E_i)<\infty$, there exists $N\in\mathbb{N}$ such that

$$\mu(\bigcup_{i=1}^{\infty}E_i) \leq \sum_{i=1}^{N}\mu(E_i)+\varepsilon .$$

Thus

$$\mu(\bigcup_{i=1}^{\infty}E_i) < \sum_{i=1}^{N}\mu(H_i)+2\varepsilon = \mu(K_N)+2\varepsilon ,$$

K_N is compact and $K_N\subseteq\bigcup_{i=1}^{\infty}E_i$. Hence

$$\begin{aligned}\mu(\bigcup_{i=1}^{\infty}E_i) &\leq \sup\{\mu(K)+2\varepsilon/\ K\subseteq\bigcup_{i=1}^{\infty}E_i,\ K \text{ compact}\}\\ &= \sup\{\mu(K)/\ K\subseteq\bigcup_{i=1}^{\infty}E_i,\ K \text{ compact}\}+2\varepsilon .\end{aligned}$$

This is true for all $\varepsilon>0$, so

$$\begin{aligned}\mu(\bigcup_{i=1}^{\infty}E_i) &\leq \sup\{\mu(K)/\ K\subseteq\bigcup_{i=1}^{\infty}E_i,\ K \text{ compact}\}\\ &\leq \mu(\bigcup_{i=1}^{\infty}E_i) \qquad \text{(from (i))}.\end{aligned}$$

Then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sup\{\mu(K) / K \subseteq \bigcup_{i=1}^{\infty} E_i,\ K \text{ compact}\}$, so $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{M}_F$ and we have claim IV.

<u>Claim V</u>. If $E \in \mathfrak{M}_F$ and $\varepsilon > 0$, then there exist a compact set K and an open set V such that $K \subseteq E \subseteq V$ and $\mu(V \setminus K) < \varepsilon$. To prove this, let $E \in \mathfrak{M}_F$ and $\varepsilon > 0$. From (2), there exists an open set V such that $E \subseteq V$ and

$$\mu(V) - \frac{\varepsilon}{2} < \mu(E).$$

From (3), there exists a compact set K such that $K \subseteq E$ and

$$\mu(E) < \mu(K) + \frac{\varepsilon}{2}.$$

By claim II, $\mu(K) < \infty$, so

$$\mu(V) < \mu(K) + \varepsilon < \infty.$$

Then $V \setminus K$ is open and $\mu(V \setminus K) \leq \mu(V) < \infty$. By claim III, $V \setminus K \in \mathfrak{M}_F$. Hence claim IV implies that

$$\mu(K) + \mu(V \setminus K) = \mu(V),$$

and thus

$$\mu(V \setminus K) < \varepsilon.$$

So we have claim V.

<u>Claim VI</u> If $A, B \in \mathfrak{M}_F$, then $A \setminus B$, $A \cup B$, $A \cap B \in \mathfrak{M}_F$. To prove this, let $A, B \in \mathfrak{M}_F$. Then $\mu(A \setminus B) \leq \mu(A) < \infty$. Let $\varepsilon > 0$ be given. Claim V shows that there exist compact sets K_1, K_2 and open sets V_1, V_2 such that

$$K_1 \subseteq A \subseteq V_1,\quad K_2 \subseteq B \subseteq V_2,$$

$$\mu(V_1 \setminus K_1) < \varepsilon \quad \text{and} \quad \mu(V_2 \setminus K_2) < \varepsilon.$$

Since $A \setminus B \subseteq V_1 \setminus K_2 \subseteq (V_1 \setminus K_1) \cup (K_1 \setminus V_2) \cup (V_2 \setminus K_2)$, by claim I, we have

$$\mu(A\setminus B) \leq 2\varepsilon + \mu(K_1\setminus V_2).$$

Because $K_1\setminus V_2$ is a compact subset of $A\setminus B$, $\mu(K_1\setminus V_2)<\infty$.

$$\mu(A\setminus B) \leq 2\varepsilon + \sup\{\mu(K)/\ K\subseteq A\setminus B,\ K \text{ compact}\}.$$

This is true for all $\varepsilon > 0$, so

$$\mu(A\setminus B) \leq \sup\{\mu(K)/\ K\subseteq A\setminus B,\ K \text{ compact}\}$$
$$\leq \mu(A\setminus B).$$

Thus $\mu(A\setminus B) = \sup\{\mu(K)/\ K\subseteq A\setminus B,\ K \text{ compact}\}$. This proves that $A\setminus B \in \mathfrak{M}_F$.

Since $A\cup B = (A\setminus B)\cup B$, $(A\setminus B)\cap B = \emptyset$ and $A\setminus B$, $B\in\mathfrak{M}_F$, it follows by claim IV that

$$\mu(A\cup B) = \mu(A\setminus B)+\mu(B).$$

But $\mu(A\setminus B)<\infty$ and $\mu(B)<\infty$, so $\mu(A\cup B)<\infty$. By claim IV $A\cup B = (A\setminus B)\cup B\in\mathfrak{M}_F$.

Since $A, A\setminus B\in\mathfrak{M}_F$, we have that $A\setminus(A\setminus B)\in\mathfrak{M}_F$. But $A\cap B = A\setminus(A\setminus B)$, so $A\cap B\in\mathfrak{M}_F$ and we have claim VI.

<u>Claim VII</u> $\mathfrak{M}$ is a σ-algebra in X which contains all Borel sets. To prove this, let C be closed. If K is a compact set, then $C\cap K$ is compact, so $C\cap K\in\mathfrak{M}_F$ by claim II. Thus $C\in\mathfrak{M}$. This shows that $\mathfrak{M}$ contains all closed sets.

Let $A\in\mathfrak{M}$. If K is a compact set, then $K, A\cap K\in\mathfrak{M}_F$ and so $A^c\cap K = K\setminus A\cap K$ which belongs to $\mathfrak{M}_F$ by claim VI. Hence $A^c\in\mathfrak{M}$. This proves that if $A\in\mathfrak{M}$, then $A^c\in\mathfrak{M}$.

Because $\emptyset$ is closed, $\emptyset^c = X\in\mathfrak{M}$. If A is an open set, then $A^c\in\mathfrak{M}$, so $A = (A^c)^c\in\mathfrak{M}$. Hence $\mathfrak{M}$ contains all open sets.

Next, suppose $A_1, A_2, \ldots \in \mathfrak{M}$. Let K be a compact set. Put

$$B_1 = A_1 \cap K \ , \ B_2 = (A_2 \cap K) \sim B_1 \ ,$$

$$B_n = (A_n \cap K) \sim (B_1 \cup B_2 \cup \ldots \cup B_{n-1}), \ n = 2,3,4,\ldots \ .$$

By the definition of $\mathfrak{M}$, $A_n \cap K \in \mathfrak{M}_F$ for all n. By claim VI, $B_n \in \mathfrak{M}_F$ for all n and $B_i \cap B_j = \emptyset$ if $i \neq j$. Since $\bigcup_{i=1}^{\infty} B_i \subseteq K$, $\mu(\bigcup_{i=1}^{\infty} B_i) \leq \mu(K) < \infty$. By claim IV, $\bigcup_{i=1}^{\infty} B_i \in \mathfrak{M}_F$. Since $(\bigcup_{i=1}^{\infty} A_i) \cap K = \bigcup_{i=1}^{\infty} (A_i \cap K) = \bigcup_{i=1}^{\infty} B_i$, $(\bigcup_{i=1}^{\infty} A_i) \cap K \in \mathfrak{M}_F$, hence $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$, hence $\mathfrak{M}$ is a σ-algebra containing all open sets, so $\mathfrak{M}$ contains all Borel sets. Hence we have claim VII.

<u>Claim VIII</u> $\mathfrak{M}_F = \{E \in \mathfrak{M} / \mu(E) < \infty\}$. Hence (d) holds. To prove this, let $E \in \mathfrak{M}_F$. Claim II and claim VI imply that $E \cap K \in \mathfrak{M}_F$ for all compact set K, so $E \in \mathfrak{M}$. Also, by the definition of $\mathfrak{M}_F$, $\mu(E) < \infty$, so $\mathfrak{M}_F \subseteq \{E \in \mathfrak{M} / \mu(E) < \infty\}$

Suppose $E \in \mathfrak{M}$ is such that $\mu(E) < \infty$. Let $\varepsilon > 0$ be given. Then there exists an open set $V \supseteq E$ such that $\mu(V) < \infty$ by (2). By claim III, $V \in \mathfrak{M}_F$. By claim V, there exists a compact set K such that $K \subseteq V$ with $\mu(V \sim K) < \varepsilon$. Since $E \cap K \in \mathfrak{M}_F$, there is a compact set $H \subseteq E \cap K$ such that

$$\mu(E \cap K) < \mu(H) + \varepsilon \ .$$

Since $E \subseteq (E \cap K) \cup (V \sim K)$, $\mu(E) \leq \mu(E \cap K) + \mu(V \sim K)$, hence $\mu(E) < \mu(H) + 2\varepsilon$. Thus

$$\mu(E) \leq \sup\{\mu(A) / A \subseteq E, \text{ A compact}\} + 2\varepsilon \ .$$

This implies that

$$\mu(E) = \sup\{\mu(A) / A \subseteq E, \text{ A compact}\} \ .$$

Hence $E \in \mathcal{M}_F$, so $\{E \in \mathcal{M} / \mu(E) < \infty\} \subseteq \mathcal{M}_F$ and we have claim VIII

Claim IX μ is a σ-finite positive measure on $\mathcal{M}$. The countable additivity of μ on $\mathcal{M}$ follows from claim IV and claim VIII. Then μ is a positive measure on $\mathcal{M}$. Since X is σ-compact and by (b) implies that μ is a σ-finite positive measure on $\mathcal{M}$. So we have claim IX.

Claim X For each $f \in C_c(X)$, $\Lambda f = \int_X f d\mu$. This proves (a). To prove this, first, let $f \in C_c(X)$ be a real-valued function. Since $f(x) \subseteq f(\text{support } f) \cup \{0\}$ and support f is compact and f is continuous, $f(\text{support } f) \cup \{0\}$ is compact in $\mathbb{R}$, so there exist $a, b \in \mathbb{R}$, $b > 0$ such that $f(x) \subseteq [a,b]$. Let $\varepsilon > 0$ be given. Let $y_0, \ldots, y_n \in \mathbb{R}$ be such that

$$y_0 < a < y_1 < y_2 < \cdots < y_n = b$$

and $y_i - y_{i-1} < \varepsilon$ for all $i = 1,2,\ldots,n$. For each $i \in \{1,2,\ldots,n\}$, let $E_i = \{x \in X / y_{i-1} < f(x) \le y_i\} \cap$ support f, i.e., $E_i = f^{-1}(y_{i-1}, y_i] \cap$ support f. Since f is continuous, f is a Borel measurable function. Hence $f^{-1}(y_{i-1}, y_i]$ is a Borel set. Hence the set E_i are disjoint Borel sets, since f is a function. $\bigcup_{i=1}^{n} E_i$ = support f and for each i,

$$\mu(E_i) \le \mu(\text{support } f) < \infty \quad \text{(by claim II)}.$$

From (2), for each $i \in \{1,2,\ldots,n\}$ there is an open set $V_i' \supseteq E_i$ such that

$$\mu(V_i') < \mu(E_i) + \frac{\varepsilon}{n} .$$

Let $V_i = V_i' \cap f^{-1}(y_{i-1}, y_i + \varepsilon)$ for all i. Then for each $i \in \{1,2,\ldots,n\}$, $V_i \supseteq E_i$,

$$\mu(V_i) \leq \mu(V_i') < \mu(E_i) + \frac{\varepsilon}{n}$$

and

$$f(x) < y_i + \varepsilon$$

for all $x \in V_i$. Since support $f \subseteq \bigcup_{i=1}^{n} V_i$, by Theorem 1.41, we have there exist $h_i \in C_c(X)$ such that

$$h_i \prec V_i \quad (i = 1,2,\ldots,n)$$

and

$$h_1+h_2+h_3+\cdots+h_n = 1$$

on support f. Hence

$$f = h_1 f+h_2 f+\ldots+h_n f.$$

Since for each $i \in \{1,2,\ldots,n\}$, $h_i f \leq (y_i+\varepsilon)h_i$ on X and since $y_i - \varepsilon < f(x)$ on E_i (since $y_i - \varepsilon < y_{i-1} < f(x)$), we have

$$\begin{aligned}
\Lambda f &= \sum_{i=1}^{n} \Lambda(h_i f) \\
&\leq \sum_{i=1}^{n} (y_i+\varepsilon)\Lambda h_i \\
&\leq \sum_{i=1}^{n} (y_i+\varepsilon)\mu(V_i) \qquad \text{(from (1))} \\
&\leq \sum_{i=1}^{n} (y_i+\varepsilon)\mu(E_i) + \sum_{i=1}^{n} (y_i+\varepsilon)\frac{\varepsilon}{n} \\
&< \sum_{i=1}^{n} (y_i-\varepsilon)\mu(E_i) + 2\varepsilon\sum_{i=1}^{n}\mu(E_i) + \sum_{i=1}^{n}(b+\varepsilon)\frac{\varepsilon}{n} \\
&= \sum_{i=1}^{n} (y_i-\varepsilon)\mu(E_i) + 2\varepsilon\mu(\text{support } f) + (b+\varepsilon)\varepsilon \\
&\leq \sum_{i=1}^{n} \int_{E_i} f\,d\mu + \varepsilon\left[2\mu(\text{support } f) + (b+\varepsilon)\right] \\
&= \int_{\text{support } f} f\,d\mu + \varepsilon\left[2\mu(\text{support } f) + (b+\varepsilon)\right] \\
&= \int_X \chi_{\text{support } f} f\,d\mu + \varepsilon\left[2\mu(\text{support } f) + (b+\varepsilon)\right] \\
&= \int_X f\,d\mu + \varepsilon\left[2\mu(\text{support } f) + (b+\varepsilon)\right]
\end{aligned}$$

(because $\chi_{\text{support } f} f = f$ on X).

Since this is true for all $\varepsilon > 0$,

$$\Lambda f \leq \int_X f d\mu .$$

Hence $-\Lambda f = \Lambda(-f) \leq \int_X (-f) d\mu = -\int_X f d\mu$, so $\Lambda f \geq \int_X f d\mu$.

Hence $\Lambda f = \int_X f d\mu$.

Finally, let $f \in C_c(X)$. Then $f = f_1 + if_2 + jf_3 + kf_4$, $f_i : X \longrightarrow \mathbb{R}$, $i \leq 4$, so f_1, f_2, f_3 and f_4 are continuous. Since support $f_i \subseteq$ support f, $i \leq 4$, so support f_i is compact, $i \leq 4$. Hence $f_i \in C_c(X)$ and are all real-valued for all $i \leq 4$. By the previous proof, $\Lambda f_i = \int_X f_i d\mu$, $i \leq 4$. Thus

$$\begin{aligned}\Lambda f &= \Lambda(f_1 + if_2 + jf_3 + kf_4) = \Lambda f_1 + i\Lambda f_2 + j\Lambda f_3 + k\Lambda f_4 \\ &= \int_X f_1 d\mu + i\int_X f_2 d\mu + j\int_X f_3 d\mu + k\int_X f_4 d\mu \\ &= \int_X f d\mu .\end{aligned}$$

So we have claim X . #

4.77 <u>Definition</u> Let X be a locally compact, σ-compact Hausdorff space and $\mathcal{B}$ = the σ-algebra of all Borel sets. An arbitrary measure μ on $\mathcal{B}$ is called a <u>Borel measure</u> on X. If μ is σ-finite positive, a Borel set $E \subseteq X$ is called <u>outer regular</u> if

$$\mu(E) = \inf\{\mu(V) / \ E \subseteq V, \ V \text{ open}\}$$

and it is called <u>inner regular</u> if

$$\mu(E) = \sup\{\mu(K) / \ K \subseteq E, \ K \text{ compact}\} .$$

If every Borel set in X is both outer and inner regular, μ is called <u>regular</u>. That is, μ is regular if and only if for all $E \in \mathcal{B}$,

$$\mu(E) = \inf\{\mu(V) / E \subseteq V, V \text{ open}\}$$

$$= \sup\{\mu(K) / K \subseteq E, K \text{ compact}\} .$$

4.78 <u>Theorem</u> Let X be a locally compact, σ-compact Hausdorff space. Let $\mathcal{M}$ and μ be as described in the statement of Theorem 4.76. If $E \in \mathcal{M}$ and $\varepsilon > 0$, then there exists a closed set F and an open set V such that $F \subseteq E \subseteq V$ and $\mu(V \setminus F) < \varepsilon$.

<u>Proof</u>[9] Let $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is a compact subset of X. Let $E \in \mathcal{M}$ and let $\varepsilon > 0$ be given. Then $\mu(K_n \cap E) < \infty$ for all n. Thus for each n there exists an open set $V_n \supseteq K_n \cap E$ such that

$$\mu(V_n \setminus K_n \cap E) = \mu(V_n) - \mu(K_n \cap E) < \frac{\varepsilon}{2^{n+1}} .$$

Let $V = \bigcup_{i=1}^{\infty} V_i$. Then V is open, $E \subseteq V$ and $V \setminus E \subseteq \bigcup_{n=1}^{\infty}(V_n \setminus K_n \cap E)$, thus $\mu(V \setminus E) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus K_n \cap E) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}$.

Similarly, there exists an open set $W \supseteq E^c$ such that $\mu(W \setminus E^c) < \frac{\varepsilon}{2}$. Let $F = W^c$, then F is closed and $F \subseteq E \subseteq V$. Since $E \setminus F = W \setminus E^c$ and $V \setminus F \subseteq (V \setminus E) \cup (E \setminus F)$, we have

$$\mu(V \setminus F) \leq \mu(V \setminus E) + \mu(E \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \#$$

4.79 <u>Lusin's Theorem</u> Let X be a locally compact, σ-compact Hausdorff space. Let $\mathcal{M}$ and μ be as described in the statement of Theorem 4.76. Suppose $f: X \longrightarrow \mathbb{H}$ is a measurable function and there is a set $A \in \mathcal{M}$ such that $\mu(A) < \infty$ and

$f(x) = 0$ for all $x \notin A$. Then for all $\varepsilon > 0$ there is $g \in C_c(X)$ such that

$$\mu(\{x \in X / f(x) \neq g(x)\}) < \varepsilon .$$

Furthermore, we may arrange it so that

$$\sup_{x \in X} \{|g(x)|\} \leq \sup_{x \in X} \{|f(x)|\} .$$

<u>Proof</u> <u>Step I</u> $0 \leq f < 1$ and A is compact.

For each $n \in \mathbb{N}$, let s_n be defined as in the proof of Theorem 3.15, i.e.,

$$E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) , \quad 1 \leq i \leq n2^n ,$$

$$F_n = f^{-1}[n, \infty] ,$$

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n \chi_{F_n}$$

Put

$$t_1 = s_1, \quad t_n = s_n - s_{n-1}, \quad n = 2, 3, \ldots .$$

To show that for all $n \in \mathbb{N}$, $2^n t_n$ is the characteristic function of a set $T_n \subseteq A$. Since $0 \leq f < 1$, $E_{n,k} = \emptyset$ if $2^n + 1 \leq k \leq n2^n$, and $F_n = \emptyset$ for all $n \in \mathbb{N}$. Then $s_n = \sum_{i=1}^{2^n} \frac{i-1}{2^n} \chi_{E_{n,i}}$ for all $n \in \mathbb{N}$. Thus $s_n - s_{n-1} = \sum_{i=1}^{2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} - \sum_{i=1}^{2^{n-1}} \frac{i-1}{2^{n-1}} \chi_{E_{n-1,i}}$

Let $T_n = \bigcup_{i=1}^{2^{n-1}} E_{n,2i}$, for all $n \in \mathbb{N}$. We have

$$2^n t_n(x) = 2^n (s_n - s_{n-1})(x) = \begin{cases} 0 & \text{if } x \notin T_n, \\ 1 & \text{if } x \in T_n, \end{cases}$$

for all $n \in \mathbb{N}$. Since $E_{n,2i}$ is a measurable set for all $n \in \mathbb{N}$, for all $i = 1, 2, \ldots, 2^{n-1}$, T_n is a measurable set, hence $2^n t_n$ is a measurable function. That is $2^n t_n$ is the characteristic function of T_n. Next, we will show that for all $n \in \mathbb{N}$, $T_n \subseteq A$. To prove this, let $n \in \mathbb{N}$ and let $x \in T_n$. So $x \in E_{n,2i} = f^{-1}([\frac{2i-1}{2^n}, \frac{2i}{2^n}))$ for some $i \in \{1,2,\ldots,2^{n-1}\}$. Then $f(x) \in [\frac{2i-1}{2^n}, \frac{2i}{2^n})$. Since $\frac{2i-1}{2^n} > 0$, $f(x) \neq 0$. Then $x \in A$, so $T_n \subseteq A$.

By Theorem 3.15, $\lim_{n\to\infty} s_n(x) = f(x)$ for all $x \in X$. Since for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} t_i = s_n$, we have that $f(x) = \lim_{n\to\infty} s_n(x) = \sum_{n=1}^{\infty} t_n$ for all $x \in X$.

Let V be an open set, $A \subseteq V$ and $\overline{V}$ compact. Such a V exists. To prove this, for all $p \in A$, choose a nbhd U_p containing p such that $\overline{U}_p$ is compact. Then $A \subseteq \bigcup_{p \in A} U_p$. Since A is compact, there exists a finite subcover, say $U_{p_1}, U_{p_2}, \ldots, U_{p_{n_o}}$ such that

$$A \subseteq \bigcup_{i=1}^{n_o} U_{p_i} \subseteq \overline{\bigcup_{i=1}^{n_o} U_{p_i}} = \bigcup_{i=1}^{n_o} \overline{U}_{p_i}$$

which is compact. Now, $T_n \subseteq V$. Since $\overline{V}$ is compact, we apply Theorem 4.78, given $\varepsilon > 0$ for all $n \in \mathbb{N}$ there exists a compact set K_n and an open set V_n such that $K_n \subseteq T_n \subseteq V_n \subseteq V$ and $\mu(V_n \setminus K_n) < \frac{\varepsilon}{2^n}$. By Urysohn's Lemma, for all $n \in \mathbb{N}$ there exist $h_n \in C_c(x)$ such that

$$K_n \prec h_n \prec V_n .$$

Define $g(x) = \sum_{n=1}^{\infty} \frac{h_n(x)}{2^n}$ for all $x \in X$. Since $0 \leq \frac{h_n(x)}{2^n} \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$ for all $x \in X$, the series converges uniformly on X. By Theorem 1.32, g is continuous on X.

Since $\{x \in X/\ g(x) \neq 0\} \subseteq \bigcup_{n=1}^{\infty}\{x \in X/\ h_n(x) \neq 0\}$, $\overline{\{x \in X/\ h_n(x) \neq 0\}} \subseteq V \subseteq \overline{V}$ for all $n \in \mathbb{N}$ and $\overline{V}$ compact, we have support g is compact, hence $g \in C_c(X)$. Since for all $n \in \mathbb{N}$, $K_n \subseteq T_n \subseteq V_n$ and $K_n \prec h_n \prec V_n$, for all $n \in \mathbb{N}$, $h_n(x) = 2^n t_n(x)$ except in $V_n \setminus K_n$, so for all $n \in \mathbb{N}$, $\frac{h_n(x)}{2^n} = t_n(x)$ except in $V_n \setminus K_n$. Thus $g(x) = f(x)$ except in $\bigcup_{n=1}^{\infty}(V_n \setminus K_n)$ and $\mu(\bigcup_{n=1}^{\infty}(V_n \setminus K_n)) < \varepsilon$. Hence $\mu(\{x \in X/\ f(x) \neq g(x)\}) < \varepsilon$, since $\{x \in X/\ g(x) \neq f(x)\} \subseteq \bigcup_{n=1}^{\infty}(V_n \setminus K_n)$.

<u>Step II</u> A is compact and f is bounded.

<u>Case 1</u> Let f be real. Then there exists $M > 0$ be such that $|f| < M$, so $-1 < \frac{f}{M} < 1$, hence $0 \leq \frac{f^+}{M} < 1$ and $0 \leq \frac{f^-}{M} < 1$. By Step I, there exist $g_1, g_2 \in C_c(X)$ such that g_1 and g_2 are real, $\mu(\{x \in X/\ \frac{f^+}{M}(x) \neq g_1(x)\}) < \frac{\varepsilon}{2}$ and $\mu(\{x \in X/\ \frac{f^-}{M}(x) \neq g_2(x)\}) < \frac{\varepsilon}{2}$. Since $f(x) = f^+(x) - f^-(x)$ for all $x \in X$ and if $f(x) = f^+(x) - f^-(x) \neq Mg_1(x) - Mg_2(x)$, then $f^+(x) \neq Mg_1(x)$ or $f^-(x) \neq Mg_2(x)$ hence $\{x \in X/\ f(x) \neq M(g_1 - g_2)(x)\} \subseteq \{x \in X/\ f^+(x) \neq Mg_1(x)\} \cup \{x \in X/\ f^-(x) \neq Mg_2(x)\}$. Hence $\mu(\{x \in X/\ f(x) \neq M(g_1 - g_2)\}) < \varepsilon$ and $M(g_1 - g_2) \in C_c(X)$.

<u>Case 2</u> Let f be quaternion. Then $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions $f_{i'}$, $i' \leq 4$. Since f is bounded, $f_{i'}$ is bounded for all $i' \leq 4$. Then there exist $g_{i'} \in C_c(X)$ such that $g_{i'}$ is real for all $i' \leq 4$ and $\mu(\{x \in X/\ f_{i'}(x) \neq g_{i'}(x)\}) < \frac{\varepsilon}{4}$ for all $i' \leq 4$. Let $g = g_1 + ig_2 + jg_3 + kg_4$. Then $g \in C_c(X)$.

Since $\{x \in X/ \ f(x) \neq g(x)\} = \bigcup_{i=1}^{4} \{x \in X/ \ f_i'(x) \neq g_i'(x)\}$, $\mu(\{x \in X/ \ f(x) \neq g(x)\}) < \varepsilon$.

Step III f is bounded. Since $\mu(A) = \sup\{\mu(K)/ \ K \subseteq A,$ K compact$\}$, there exists a compact set $K \subseteq A$ such that $\mu(A \setminus K) < \frac{\varepsilon}{2}$. Thus $f\chi_K$ is measurable and bounded. Also, $f\chi_K = 0$ on K^c. By Step II, there exists $g \in C_c(X)$ such that $\mu(\{x \in X/ \ f\chi_K(x) \neq g(x)\}) < \frac{\varepsilon}{2}$. If $x \notin A$, $f(x) = 0 = f\chi_K(x)$ and if $x \in K$, then $f\chi_K(x) = f(x)$. Thus

$$\{x \in X/ \ f(x) \neq g(x)\} \subseteq \{x \in X/ \ f\chi_K(x) \neq g(x)\} \cup (A \setminus K),$$

so $\mu(\{x \in X/ \ f(x) \neq g(x)\}) < \varepsilon$.

Step IV For each $n \in \mathbb{N}$, let $B_n = \{x \in X/ \ |f(x)| > n\}$. Then $A \supseteq B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} B_n = \emptyset$ since f is quaternion. Hence $\lim_{n \to \infty} \mu(B_n) = \mu(\bigcap_{n=1}^{\infty} B_n) = \mu(\emptyset) = 0$. So there exists $n_o \in \mathbb{N}$ such that $\mu(B_{n_o}) < \frac{\varepsilon}{2}$. Now $|f| \leq n_o$ on $B_{n_o}^c$, so $|f\chi_{B_{n_o}^c}| \leq n_o$ on X. Morever, $\mu(A \setminus B_{n_o}) < \infty$ and $f\chi_{B_{n_o}^c} = 0$ on $A^c \cup B_{n_o} = (A \setminus B_{n_o})^c$. By Step III, there exists $g \in C_c(X)$ such that

$$\mu(\{x \in X/ \ f\chi_{B_{n_o}^c}(x) \neq g(x)\}) < \frac{\varepsilon}{2} .$$

Since $\{x \in X/ \ f(x) \neq g(x)\} \subseteq \{x \in X/ \ f\chi_{B_{n_o}^c}(x) \neq g(x)\} \cup B_{n_o}$, $\mu(\{x \in X/ \ f(x) \neq g(x)\}) < \varepsilon$.

Next, to show that we can arrange g so that $\sup_{x \in X}\{|g(x)|\} \leq \sup_{x \in X}\{|f(x)|\}$. To prove this, let $R = \sup_{x \in X}\{|f(x)|\}$. If $R = \infty$, so we are done. If $R = 0$, then $f \equiv 0$ and choose

$g \equiv 0$ and we are done. Assume $0 < R < \infty$. Let $\varphi : \mathbb{H} \to \mathbb{H}$ be defined by

$$\varphi(h) = \begin{cases} h & \text{if } |h| \leq R, \\ \dfrac{Rh}{|h|} & \text{if } |h| > R. \end{cases}$$

Then φ is continuous on $\mathbb{H}$. Let $g_1 = \varphi \circ g$, g_1 is continuous and support $g_1 \subseteq$ support g, so $g_1 \in C_c(X)$. And

$$g_1(x) = \varphi(g(x)) = \begin{cases} g(x) & \text{if } |g(x)| \leq R, \\ \dfrac{Rg(x)}{g(x)} & \text{if } |g(x)| > R. \end{cases}$$

Since $\{x \in X / \ f(x) \neq g_1(x)\} \subseteq \{x \in X / \ f(x) \neq g(x)\}$,

$\mu(\ x \in X / \ f(x) \neq g_1(x)\}) < \varepsilon$ and $\sup\limits_{x \in X} \{|g_1(x)|\} \leq R = \sup\limits_{x \in X} \{|f(x)|\}$. #

4.80 <u>Theorem</u> (Jensen's Inequality) Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω such that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$ and $a, b \in \mathbb{R}$ such that $a < f(x) < b$ for all $x \in \Omega$ and if φ is convex in (a,b), then

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \varphi \circ f d\mu .$$

<u>Proof</u>[9] Let $t = \int_\Omega f d\mu$. Then $a < t < b$. Let $\beta = \sup\left\{ \dfrac{\varphi(t) - \varphi(s)}{t-s} / \ a < s < t \right\}$. Then $\beta \leq \dfrac{\varphi(u) - \varphi(t)}{u-t}$ for all $u \in (t,b)$. Hence $\varphi(u) \geq \varphi(t) + \beta(u-t)$ for all $u \in (a,b)$. ($a < u < t \Rightarrow \beta \geq \dfrac{\varphi(t) - \varphi(u)}{t-u} \Rightarrow \beta(t-u) \geq \varphi(t) - \varphi(u)$).

Hence $\varphi(f(x)) \geq \varphi(t) + \beta(f(x) - t)$ for all $x \in \Omega$. Since φ is convex on (a,b), by Theorem 1.45, φ is continuous on (a,b). Hence $\varphi \circ f$ is measurable. Hence

$$\begin{aligned} \int_\Omega \varphi \circ f d\mu &\geq \int_\Omega \varphi(t) d\mu + \beta \int_\Omega (f(x) - t) d\mu \\ &= \varphi(t)\mu(\Omega) + \beta\left(\int_\Omega f d\mu - t\mu(\Omega)\right) \\ &= \varphi\left(\int_\Omega f d\mu\right) \qquad (\mu(\Omega) = 1). \ \# \end{aligned}$$

<u>Remark:</u> Suppose φ in Theorem 4.80 is $\varphi(x) = e^x$. Then

$$(*) \qquad \exp\left\{\int_\Omega f d\mu\right\} = e^{\int_\Omega f d\mu} \leq \int_\Omega e^f d\mu .$$

Assume $\Omega = \{p_1, p_2, \ldots, p_n\}$ is such that $p_i \neq p_j$ if $i \neq j$. Let $\mathcal{M} = \mathcal{P}(\Omega)$ and let $\mu(\{p_i\}) = \frac{1}{n}$ for all $i = 1,2,\ldots,n$, and $f(p_i) = x_i \in \mathbb{R}$ for all $i = 1,2,\ldots,n$. Then from (*),

$$\exp\left\{\frac{1}{n}(x_1+x_2+\ldots+x_n)\right\} \leq \frac{1}{n}(e^{x_1}+e^{x_2}+\ldots+e^{x_n}).$$

It then follows that for all $n \in \mathbb{N}$ for all $y_1, \ldots y_n \in (0,\infty)$

$$(y_1 y_2 \ldots y_n)^{\frac{1}{n}} \leq \frac{1}{n}(y_1+y_2+\ldots+y_n)$$

(because for each $y_i \in (0,\infty)$ there exists $x_i \in \mathbb{R}$ such that $y_i = e^{x_i}$)

If we take $\mu(\{p_i\}) = \alpha_i > 0$ for all $i = 1,2,\ldots,n$ $\sum_{i=1}^n \alpha_i = 1$, then we obtain

$$y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} \leq \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n$$

for all $y_1, \ldots, y_n \in (0,\infty)$.

4.81 <u>Definition</u> If p and q are positive real numbers such that $p+q = pq$ or equivalently $\frac{1}{p} + \frac{1}{q} = 1$, then we call p and q a pair of <u>conjugate</u> <u>exponents</u>.

Observe that if p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then $1 < p < \infty$ and $1 < q < \infty$, and $p \to 1$ implies that $q \to \infty$. Consequently, 1 and ∞ are also regarded as a pair of conjugate exponents.

4.82 <u>Theorem</u> Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ in X. Let p and q be conjugate exponents,

$1 < p < \infty$. Let $f, g : X \longrightarrow [0, \infty]$ be measurable functions. Then

(1) $$\int_X fg\,d\mu \leq \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} \left\{\int_X g^q d\mu\right\}^{\frac{1}{q}}$$

and

(2) $$\left\{\int_X (f+g)^p d\mu\right\}^{\frac{1}{p}} \leq \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} + \left\{\int_X g^p d\mu\right\}^{\frac{1}{p}}.$$

Note: (1) is called the Hölder's inequality; (2) is called the Minkowski's inequality. If $p = q = 2$, then (1) is called as the Schwarz's inequality.

Proof [9] Let $A = \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}}$ and $B = \left\{\int_X g^q d\mu\right\}^{\frac{1}{q}}$. If $A = 0$, $f^p = 0$ a.e. on X by Theorem 4.48, so $f = 0$ a.e. on X and hence $fg = 0$ a.e. on X and thus (1) holds by Theorem 4.45. Similarly, $B = 0$ implies (1) holds. If ($A > 0$ and $B = \infty$) or ($A = \infty$ and $B > 0$), (1) clearly holds.

Assume $0 < A < \infty$ and $0 < B < \infty$. Put

$$F = \frac{f}{A}, \quad G = \frac{g}{B}.$$

Then $\int_X F^p d\mu = \int_X G^q d\mu = 1$. Let $x \in X$ be such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$. Then there exist $s, t \in \mathbb{R}$ such that $F(x) = e^{\frac{s}{p}}$ and $G(x) = e^{\frac{t}{q}}$. Since $\frac{1}{p} + \frac{1}{q} = 1$, the convexity of the exponential function implies that $e^{\frac{s}{p} + \frac{t}{q}} \leq \frac{e^s}{p} + \frac{e^t}{q}$. Then $F(x)G(x) \leq \frac{(F(x))^p}{p} + \frac{(G(x))^q}{q}$. Hence for all $x \in X$

$$F(x)G(x) \leq \frac{(F(x))^p}{p} + \frac{(G(x))^q}{q}.$$

Thus $\int_X FG\,d\mu \leq \frac{1}{p} + \frac{1}{q} = 1$. Hence

$$\int_X fg\,d\mu \leq AB = \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} \left\{\int_X g^q d\mu\right\}^{\frac{1}{q}}.$$

Thus (1) holds.

To prove (2), we write $(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}$. From (1)

$$\int_X f(f+g)^{p-1} d\mu \leq \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} \left\{\int_X (f+g)^{q(p-1)} d\mu\right\}^{\frac{1}{q}},$$

$$\int_X g(f+g)^{p-1} d\mu \leq \left\{\int_X g^p d\mu\right\}^{\frac{1}{p}} \left\{\int_X (f+g)^{q(p-1)} d\mu\right\}^{\frac{1}{q}}$$

Then

(3) $$\int_X (f+g)^p d\mu \leq \left\{\int_X (f+g)^{q(p-1)} d\mu\right\}^{\frac{1}{q}} \left(\left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} + \left\{\int_X g^p d\mu\right\}^{\frac{1}{p}}\right)$$

If $\left\{\int_X (f+g)^p d\mu\right\}^{\frac{1}{p}} = 0$ or $\left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} = \infty$ or $\left\{\int_X g^p d\mu\right\}^{\frac{1}{p}} = \infty$, then (2) holds. Assume $\left\{\int_X (f+g)^p d\mu\right\}^{\frac{1}{p}} > 0$, $\left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} < \infty$ and $\left\{\int_X g^p d\mu\right\}^{\frac{1}{p}} < \infty$. Since the function x^p is convex in $(0,\infty)$, it follows that

$$\left(\frac{f+g}{2}\right)^p \leq \tfrac{1}{2}(f^p + g^p).$$

Then $\frac{1}{2^p}\int_X (f+g)^p d\mu \leq \frac{1}{2}\left(\int_X f^p d\mu + \int_X g^p d\mu\right) < \infty$. Thus

$$0 < \left\{\int_X (f+g)^p d\mu\right\}^{-\frac{1}{q}} < \infty,$$

so by (3) and $q(p-1) = p$, we have

$$\left\{\int_X (f+g)^p d\mu\right\}^{1-\frac{1}{q}} \leq \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} + \left\{\int_X g^p d\mu\right\}^{\frac{1}{p}}.$$

This proves (2). #

4.83 Definition Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. If $1 \leq p < \infty$ and $f: X \longrightarrow \mathcal{H}$ is measurable, define

$$\|f\|_p = \left\{\int_X |f|^p d|\mu|\right\}^{\frac{1}{p}}.$$

and let

$$L^p(\mu) = \{f:X \longrightarrow \mathbb{H} \,/\, f \text{ is measurable and } \|f\|_p < \infty\},$$

we call $\|f\|_p$ the <u>L^p-norm</u> of f.

4.84 <u>Definition</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. Suppose $g:X \longrightarrow [0,\infty]$ is measurable. Let $S = \{\alpha \in [0,\infty) \,/\, |\mu|(g^{-1}(\alpha,\infty]) = 0\}$. Note that if $a \in S$, then $b \in S$ for all $b \geq a$. Put

$$\beta = \begin{cases} \inf S & \text{if } S \neq \emptyset, \\ \infty & \text{if } S = \emptyset. \end{cases}$$

Claim that if $\beta = \inf S$, then $\beta \in S$. To prove this, since $g^{-1}(\beta,\infty] = \bigcup_{n=1}^{\infty} g^{-1}(\beta+\frac{1}{n},\infty]$ and $|\mu|(g^{-1}(\beta+\frac{1}{n},\infty]) = 0$ for all $n \in \mathbb{N}$, we have $|\mu|(g^{-1}(\beta,\infty]) = 0$, thus $\beta \in S$. So we have the claim. We call β the <u>essential</u> <u>supremum</u> of g.

If $f:X \longrightarrow \mathbb{H}$ is measurable, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, i.e.,

$$\|f\|_\infty = \begin{cases} \infty \text{ if } |\mu|(|f|^{-1}(\alpha,\infty]) \neq 0 \text{ for all } \alpha \in [0,\infty), \\ \inf\{\alpha \in [0,\infty) \,/\, |\mu|(|f|^{-1}(\alpha,\infty]) = 0\}; \end{cases}$$

and $L^\infty(\mu) = \{f:X \longrightarrow \mathbb{H} \,/\, f \text{ is measurable and } \|f\|_\infty < \infty\}$.

Observe that if $f:X \longrightarrow \mathbb{H}$ is measurable, then $|f(x)| \leq \lambda$ for almost all x iff $\lambda \geq \|f\|_\infty$, hence $|f(x)| \leq \|f\|_\infty$ for almost all x.

4.85 <u>Theorem</u> Let μ be an arbtrary measure on a σ-algebra $\mathfrak{M}$ in X. If p and q are conjugate exponents, $1 \leq p \leq \infty$, and

$f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q .$$

<u>Proof[9]</u><u>Case $1 < p < \infty$</u> Then the theorem follows by a applying Theorem 4.82 to the function $|f|$ and $|g|$.

<u>Case $p = \infty$</u> Then $q = 1$. We have $|f(x)| \leq \|f\|_\infty$ for almost all x, so $|f(x)g(x)| = |f(x)||g(x)| \leq \|f\|_\infty |g(x)|$ for almost all x. Then there exists $A \in \mathfrak{M}$ such that $|\mu|(A) = 0$ and $|f(x)g(x)| \leq \|f\|_\infty |g(x)|$ for all $x \in A^c$. Hence $\int_{A^c} |fg| d|\mu| \leq$ $\int_{A^c} \|f\|_\infty |g| d|\mu|$. Since $X = A \cup A^c$ and $|\mu|(A) = 0$, $\int_X |fg| d|\mu| \leq \int_X \|f\|_\infty |g| d|\mu| = \|f\|_\infty \int_X |g| d|\mu|$. Hence $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_\infty \|g\|_1$.

<u>Case $p = 1$</u> Then $q = \infty$. Similar to case $p = \infty$, we have $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$. #

4.86 <u>Theorem</u> Let μ be an arbtrary measure on a σ-algebra $\mathfrak{M}$ in X. Suppose $1 \leq p \leq \infty$, $f \in L^p(\mu)$ and $g \in L^p(\mu)$. Then $f+g \in L^p(\mu)$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

<u>Proof[9]</u><u>Case $1 < p < \infty$</u> Then the theorem follows from Minkowski's inequality and $|f+g|^p \leq (|f|+|g|)^p$.

<u>Case $p = 1$</u> Then $\|f\|_1 = \int_X |f| d|\mu| < \infty$ and $\|g\|_1 = \int_X |g| d|\mu| < \infty$, so $\|f+g\|_1 = \int_X |f+g| d|\mu| \leq \int_X (|f|+|g|) d|\mu| =$ $\int_X |f| d|\mu| + \int_X |g| d|\mu| = \|f\|_1 + \|g\|_1 < \infty$.

Case $p = \infty$ Since $|f+g| \le |f| + |g|$, $|f(x)+g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$ for almost all x. Hence $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty < \infty$. #

4.87 Theorem Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X and $1 \le p \le \infty$. Assume $f \in L^p(\mu)$ and $\alpha \in \mathbb{H}$. Then $\alpha f \in L^p(\mu)$ and $\|\alpha f\|_p = |\alpha| \|f\|_p$.

Proof[9] Case $1 \le p < \infty$ Then $\|\alpha f\|_p = \left\{\int_X |\alpha f|^p d\mu\right\}^{\frac{1}{p}}$ $= |\alpha| \left\{\int_X |f|^p d\mu\right\}^{\frac{1}{p}} = |\alpha| \|f\|_p < \infty$.

Case $p = \infty$ Since $|f(x)| \le \|f\|_\infty$ for almost all x, $|\alpha f(x)| \le |\alpha| \|f\|_\infty$ for almost all x. Thus

(*) $$\|\alpha f\|_\infty \le |\alpha| \|f\|_\infty .$$

If $\alpha = 0$, it is clear that $\|\alpha f\|_\infty = 0 = |\alpha| \|f\|_\infty$. Assume $\alpha \ne 0$. From (*) $\|f\|_\infty = \|\frac{1}{\alpha}. \alpha f\|_\infty \le \frac{1}{|\alpha|}\|\alpha f\|_\infty$, so $|\alpha| \|f\|_\infty \le \|\alpha f\|_\infty$. Hence $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$. #

Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. Let $1 \le p \le \infty$. If $f, g, h \in L^p(\mu)$, then $\|f-h\|_p = \|(f-g)+(g-h)\|_p \le \|f-g\|_p + \|g-h\|_p$ (by Theorem 4.86). Define

$$d : L^p(\mu) \times L^p(\mu) \longrightarrow \mathbb{R} \text{ by}$$
$$d(f,g) = \|f-g\|_p .$$

Then for all $f,g,h \in L^p(\mu)$, we have $0 \le d(f,g) < \infty$, $d(f,f) = 0$, $d(f,g) = d(g,f)$ (since $\|f-g\|_p = |(-1)| \|g-f\|_p$) and $d(f,h) \le d(f,g) + d(g,h)$.

Next, to show that $d(f,g) = 0$ iff $f = g$ a.e. $[\mu]$. First, assume $d(f,g) = 0$. If $1 \le p < \infty$, then

$\|f-g\|_p = \{\int_X |f-g|^p d|\mu|\}^{\frac{1}{p}} = 0$, so $f = g$ a.e. $[|\mu|]$. If $p = \infty$ then $\|f-g\|_\infty = 0$, so $|f(x)-g(x)| = 0$ for almost all x, with respect to $|\mu|$. Then $f = g$ a.e. $[|\mu|]$. This proves that if $d(f,g) = 0$, then $f = g$ a.e. $[|\mu|]$.

Finally, assume $f = g$ a.e. $[|\mu|]$. Then $|f-g| = 0$ a.e. $[|\mu|]$. If $1 \leq p < \infty$, then $|f-g|^p = 0$ a.e. $[|\mu|]$, so $\int_X |f-g|^p d|\mu| = 0$ which implies that $\|f-g\|_p = 0$. If $p = \infty$, $|\mu|(|f-g|^{-1}(0,\infty]) = |\mu|(\{x \in X/ \ |f-g|(x) \neq 0\}) = 0$. Hence $\|f-g\|_\infty = 0$. This proves that if $f = g$ a.e. $[|\mu|]$, then $d(f,g) = 0$.

Define the relation $\sim$ on $L^p(\mu)$ by

$$f \sim g \iff d(f,g) = 0.$$

Thus $f \sim g \iff f = g$ a.e. $[|\mu|]$. So $\sim$ is an equivalence relation on $L^p(\mu)$ which partitions $L^p(\mu)$ into equivalence classes.

4.88 Theorem Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X and $1 \leq p \leq \infty$. If $f,g: X \longrightarrow \mathbb{H}$ are measurable such that $f = g$ a.e. $[|\mu|]$, then $\|f\|_p = \|g\|_p$.

Proof Case $1 \leq p < \infty$ Then $|f| = |g|$ a.e. $[|\mu|]$, hence $|f|^p = |g|^p$ a.e. $[|\mu|]$. Thus $\int_X |f|^p d|\mu| = \int_X |g|^p d|\mu|$ which implies that $\|f\|_p = \|g\|_p$.

Case $p = \infty$ Since $|f| = |g|$ a.e. $[|\mu|]$, there exists $E \in \mathfrak{M}$ such that $|\mu|(E) = 0$ and $|f| = |g|$ on E^c. Then for $\alpha \in [0,\infty)$,

$|\mu|(|f|^{-1}(\alpha,\infty]) = |\mu|((|f|^{-1}(\alpha,\infty])\cap E^c) =$
$|\mu|((|g|^{-1}(\alpha,\infty])\cap E^c = |\mu|(|g|^{-1}(\alpha,\infty])$. Thus

$$\|f\|_\infty = \|g\|_\infty \quad . \#$$

Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X and $1 \leq p \leq \infty$. If F and G are two equivalence classes, choose $f \in F$, $g \in G$, define

$$d(F,G) = d(f,g) = \|f-g\|_p.$$

If $f \sim f_1$ and $g \sim g_1$, then $d(f,g) = d(f_1,g_1)$ (Because $f = f_1$ a.e. $[|\mu|]$ and $g = g_1$ a.e. $[|\mu|]$, then $f-g = f_1-g_1$ a.e.$[|\mu|]$ which implies that $\|f-g\|_p = \|f_1-g_1\|_p$). Hence d is well-defined.

Now, the set of all equivalence classes is a metric space with metric d.

When $L^p(\mu)$ is regarded as a metric space, then the space which really under consideration is therefore not a space whose elements are functions, but a space whose elements are equivalence classes of functions.

4.89 <u>Definition</u> Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X and $(f_n)_{n\in\mathbb{N}}$ a sequence in $L^p(\mu)$, $1 \leq p \leq \infty$. If $f \in L^p(\mu)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0,$$

then we say that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^p(\mu)$ or $(f_n)_{n\in\mathbb{N}}$ is <u>L^p-convergent</u> to f.

Observe that if $(f_n)_{n\in\mathbb{N}}$ converges to f and g in

$L^p(\mu)$, then $f = g$ a.e. $[|\mu|]$.

Proof $\lim_{n\to\infty} \|f_n - f\|_p = 0 = \lim_{n\to\infty} \|f_n - g\|_p$. Let $\varepsilon > 0$. There exists $n_o \in \mathbb{N}$ such that $\|f_{n_o} - f\|_p < \frac{\varepsilon}{2}$ and $\|f_{n_o} - g\|_p < \frac{\varepsilon}{2}$. Then $0 \leq \|f-g\|_p \leq \|f - f_{n_o}\|_p + \|f_{n_o} - g\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|f-g\|_p = 0$, i.e., $f = g$ a.e. $[|\mu|]$. #

If for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \varepsilon$ for all $m, n \geq N$, then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mu)$.

If every Cauchy sequence in $L^p(\mu)$ converges to an element in $L^p(\mu)$, then $L^p(\mu)$ is said to be a complete metric space.

4.90 Theorem Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X. Then $L^p(\mu)$ is a complete metric space for $1 \leq p \leq \infty$

Proof[9]Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\mu)$.

Case $1 \leq p < \infty$ Claim that there is a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $\|f_{n_{i+1}} - f_{n_i}\|_p < \frac{1}{2^i}$, $i = 1,2,3,\ldots$. There exists $n_1 \in \mathbb{N}$ such that $\|f_n - f_{n_1}\|_p < \frac{1}{2}$ for all $n \geq n_1$ and there exists $n_2' \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \frac{1}{2^2}$ for all $n, m \geq n_2'$. Choose $n_2 = n_2' + n_1$. Then $\|f_n - f_{n_2}\|_p < \frac{1}{2^2}$ for all $n \geq n_2$ and $n_2 > n_1$. There exists $n_3' \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \frac{1}{2^3}$ for all $n, m \geq n_3'$. Choose $n_3 = n_3' + n_2$. Then

$\|f_n - f_{n_3}\|_p < \frac{1}{2^3}$ for all $n \geqslant n_3$ and $n_3 > n_2$. By this process, for all $i \geqslant 2$ there exists $n_i \in \mathbb{N}$ such that $\|f_n - f_{n_i}\|_p < \frac{1}{2^i}$ for all $n \geqslant n_i$ and $n_i > n_{i-1}$. Fix $i \geqslant 2$. There exists $n_{i+1} \in \mathbb{N}$ such that $\|f_n - f_{n_{i+1}}\|_p < \frac{1}{2^{i+1}}$ for all $n \geqslant n_{i+1}$ and $n_{i+1} > n_i$. So we have $\|f_{n_{i+1}} - f_{n_i}\|_p < \frac{1}{2^i}$. Hence we have the claim. Put

$$g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

and for each $k \in \mathbb{N}$, let

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|.$$

By Minkowski's inequality, we have that for all $k \in \mathbb{N}$

$$\|g_k\|_p = \left\{\int_X |g_k|^p d|\mu|\right\}^{\frac{1}{p}} = \left\{\int_X \left(\sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|\right)^p d|\mu|\right\}^{\frac{1}{p}}$$

$$\leqslant \sum_{i=1}^{k} \left\{\int_X |f_{n_{i+1}} - f_{n_i}|^p d|\mu|\right\}^{\frac{1}{p}} = \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p$$

$$< \sum_{i=1}^{k} \frac{1}{2^i} < 1 .$$

Now $\|g\|_p^p = \int_X |g|^p d|\mu| = \int_X g^p d|\mu| = \int_X (\lim_{k\to\infty} g_k)^p d|\mu|$

$$= \int_X (\lim_{k\to\infty} g_k^p) d|\mu| = \int_X (\liminf_{k\to\infty} g_k^p) d|\mu|$$

$$\leqslant \liminf_{k\to\infty} \int_X g_k^p d|\mu| \quad \text{(by Fatou's Lemma)}$$

$$= \liminf_{k\to\infty} \|g_k\|_p^p \leqslant 1 .$$

Hence $\|g\|_p \leqslant 1$. Then we see that $g(x) < \infty$ a.e. $[|\mu|]$ (Because if $|\mu|(\{x \in X / g(x) = \infty\}) \neq 0$, then $|\mu|(\{x \in X / |g(x)|^p = \infty\}) \neq 0$ which implies that $\int_X |g|^p d|\mu| = \infty$, so $\|g\|_p = \infty$, a contradiction).

Hence the series

$$(1) \qquad f_{n_1}(x) + \sum_{i=1}^{\infty}(f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges absolutely to a finite value for almost all $x \in X$. Then there exists $E \in \mathcal{M}$ such that the series (1) converges on E^c and $|\mu|(E) = 0$. Let $f: X \longrightarrow \mathbb{H}$ be defined by

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{i=1}^{\infty}(f_{n_{i+1}}(x) - f_{n_i}(x)) & \text{if } x \in E^c, \\ 0 & \text{if } x \in E. \end{cases}$$

Since $f_{n_1} + \sum_{i=1}^{k}(f_{n_{i+1}} - f_{n_i}) = f_{n_k}$, it follows that $f(x) = \lim_{i \to \infty} f_{n_i}(x)$ for all $x \in E^c$. Then $f(x) = \lim_{i \to \infty} f_{n_i}(x)$ a.e. $[|\mu|]$.

Claim that $f \in L^p(\mu)$ and $\lim_{n \to \infty} \|f_n - f\|_p = 0$. To prove this, let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \varepsilon$ for all $m, n > N$. Fix $m > N$, by Fautou's Lemma,

$$\begin{aligned} \int_X |f - f_m|^p d|\mu| &= \int_X \left|\lim_{i \to \infty} f_{n_i} - f_m\right|^p d|\mu| \\ &= \int_X (\lim_{i \to \infty} |f_{n_i} - f_m|)^p d|\mu| \\ &= \int_X \lim_{i \to \infty} |f_{n_i} - f_m|^p d|\mu| \\ &= \int_X \liminf_{i \to \infty} |f_{n_i} - f_m|^p d|\mu| \\ &\leq \liminf_{i \to \infty} \int_X |f_{n_i} - f_m|^p d|\mu| \leq \varepsilon^p. \end{aligned}$$

Hence $\|f - f_m\|_p^p \leq \varepsilon^p$, so $\|f - f_m\|_p \leq \varepsilon < \infty$. Then $f - f_m \in L^p(\mu)$. Since $f_m \in L^p(\mu)$, $f \in L^p(\mu)$. Since $\|f - f_m\|_p \leq \varepsilon$, this shows that $\|f - f_m\|_p \leq \varepsilon$ for all $m \geq N$. So we have the claim.

<u>Case $p = \infty$</u> For each $k \in \mathbb{N}$, let

$$A_k = \{x \in X / \ |f_k(x)| > \|f_k\|_\infty\}$$

and for $m, n \in \mathbb{N}$, let

$$B_{m,n} = \{x \in X / \ |f_n(x)-f_m(x)| > \|f_n-f_m\|_\infty\}.$$

Then $|\mu|(A_k) = 0 = |\mu|(B_{m,n})$ for all $k,m,n \in \mathbb{N}$. Let

$$E = (\bigcup_{k=1}^{\infty} A_k) \cup (\bigcup_{m,n \in \mathbb{N}} B_{m,n}) .$$

Then $E \in \mathfrak{M}$ and $|\mu|(E) = 0$, and also for all $k,m,n \in \mathbb{N}$,

$$|f_k(x)| \leq \|f_k\|_\infty$$

for all $x \in E^c$ and

(2) $$|f_m(x)-f_n(x)| \leq \|f_m-f_n\|_\infty$$

for all $x \in E^c$. Since $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and by (2), we have for all $x \in E^c$, $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}$. Thus there exists $f: E^c \longrightarrow \mathbb{H}$ such that $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for all $x \in E^c$. Hence f is measurable on E^c and $\lim\limits_{n \to \infty} |f_n(x)-f(x)| = 0$ for all $x \in E^c$. Let $\varepsilon > 0$ be given, from (2) we get that there exists $N \in \mathbb{N}$ such that $|f_m(x)-f_n(x)| < \frac{\varepsilon}{2}$ for all $m,n \geqslant N$ and for all $x \in E^c$. Let $m > N$ be fixed. Then $\lim\limits_{n \to \infty} |f_m(x)-f_n(x)| = |f_m(x)-f(x)| < \varepsilon$ for all $x \in E^c$, so $|f(x)| < \varepsilon + |f_m(x)| \leq \varepsilon + \|f_m\|_\infty < \infty$ for all $x \in E^c$. Thus $f_n \longrightarrow f$ uniformly on E^c and f is bounded on E^c. Define $\bar{f}: X \longrightarrow \mathbb{H}$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in E^c, \\ 0 & \text{if } x \in E. \end{cases}$$

Then $\bar{f}$ is measurable and $\bar{f}$ is bounded, so $\bar{f} \in L^\infty(\mu)$. Next, we shall show that $f_n \longrightarrow \bar{f}$ in $L^\infty(\mu)$. To prove this, let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $|f_n(x)-\bar{f}(x)| < \frac{\varepsilon}{2}$ for all $x \in E^c$ for all $n \geqslant N$, so $|\mu|(|f_n-\bar{f}|^{-1}(\frac{\varepsilon}{2}, \infty]) = |\mu|((|f_n-\bar{f}|^{-1}(\frac{\varepsilon}{2},\infty]) \cap E^c) = |\mu|(\emptyset) = 0$. Then $\|f_n-\bar{f}\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon$ for all $n \geqslant N$. This prove that $f_n \longrightarrow \bar{f}$ on $L^\infty(\mu)$. #

The proof of Theorem 4.90 contains the following result:

4.91 Theorem Let μ be an arbitrary measure on a σ-algebra $\mathfrak{M}$ in X and $1 \leq p \leq \infty$. If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mu)$, with limit f, then $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges pointwise almost everywhere to f(x).

4.92 Theorem Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $\mathcal{S}$ be the class of all quaternion simple functions s on X such that

$$\mu(\{x\in X/\ s(x) \neq 0\}) < \infty.$$

If $1 \leq p < \infty$, then $\mathcal{S}$ is dense in $L^p(\mu)$.

Proof For $s \in \mathcal{S}$, we have

$$\begin{aligned}\|s\|_p &= \Big\{\int_X |s|^p d\mu\Big\}^{\frac{1}{p}} \\ &= \Big\{\int_{\{x/\ s(x)=0\}} |s|^p d\mu + \int_{\{x/\ s(x)\neq 0\}} |s|^p d\mu\Big\}^{\frac{1}{p}} \\ &= \Big\{\int_{\{x/\ s(x)\neq 0\}} |s|^p d\mu\Big\}^{\frac{1}{p}} < \infty,\end{aligned}$$

because $|s|$ is bounded and $\mu(\{x/\ s(x) \neq 0\}) < \infty$. Then $\mathcal{S} \subseteq L^p(\mu)$. Let $f \in L^p(\mu)$.

Case 1 $f \geq 0$. By Theorem 3.15, there exists a simple measurable functions s_n $(n\in\mathbb{N})$ on X such that

$$0 \leq s_1 \leq s_2 \leq s_3 \leq \ldots \leq f$$

and

$$\lim_{n\to\infty} s_n(x) = f(x)$$

for all $x \in X$. Since $s_n \leq f$, $s_n \in L^p(\mu)$. Claim that $s_n \in \mathcal{S}$

for all $n \in \mathbb{N}$. To prove this, let $n \in \mathbb{N}$. If $s_n \equiv 0$, then $s_n \in \mathcal{S}$. Assume $s_n \not\equiv 0$. Let $A = \{x \in X / \ s_n(x) \neq 0\}$. By the definition of simple measurable function, let

$$s_n = \sum_{i=1}^{m} \alpha_i \chi_{A_i} + 0\,\chi_{A^c}$$

where $\alpha_1, \ldots, \alpha_n$ are positive distinct values of s_n, $\alpha_i \neq 0$ and $A_i = s_n^{-1}(\alpha_i)$, $i = 1,2,\ldots,m$. Suppose $\mu(A) = \infty$. so there is $j \in \{1,2,\ldots,m\}$ such that $\mu(A_j) = \infty$. Then

$$\|s_n\|_p^p = \int_X |s_n|^p d\mu = \int_A |s_n|^p d\mu$$

$$= \int_A \Big(\sum_{i=1}^{m} \alpha_i \chi_{A_i}\Big)^p d\mu \geq \int_A \alpha_j^p \chi_{A_j} d\mu$$

$$= \alpha_j^p \mu(A_j) = \infty,$$

which contradicts to $s_n \in L^p(\mu)$. Hence $\mu(A) < \infty$, so $s_n \in \mathcal{S}$ and we have the claim. Since $|f - s_n|^p \leq f^p$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(f(x) - s_n(x))^p = 0$ for all $x \in X$ and $f^p \in L^1(\mu)$, by Lebesgue's Dominated Convergence Theorem, we see that

$$\lim_{n\to\infty} \|f - s_n\|_p^p = \lim_{n\to\infty} \int_X |f - s_n|^p d\mu = \lim_{n\to\infty} \int_X (f - s_n)^p d\mu$$

$$= \int_X \lim_{n\to\infty} (f - s_n)^p d\mu = 0.$$

Hence $\lim_{n\to\infty} \|f - s_n\|_p = 0$.

<u>Case 2</u> f is real. Then $f = f^+ - f^-$, so there exist sequences $(s_n)_{n\in\mathbb{N}}$ and $(s_n')_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\lim_{n\to\infty} \|f^+ - s_n\|_p = 0$ and $\lim_{n\to\infty} \|f^- - s_n'\|_p = 0$. For each $n \in \mathbb{N}$,

$$0 \leq \|f - (s_n - s_n')\|_p \leq \|f^+ - s_n\|_p + \|f^- - s_n'\|_p,$$

and so

$$\lim_{n\to\infty} \|f - (s_n - s_n')\|_p = 0 \text{ and } s_n - s_n' \in \mathcal{S} \text{ for all } n \in \mathbb{N}.$$

<u>Case 3</u> f is quaternion. Then $f = f_1+if_2+jf_3+kf_4$ for some real measurable functions f_i, $i \le 4$. By Case 2, there exist sequences $(s_n^{(1)})_{n\in\mathbb{N}}$, $(s_n^{(2)})_{n\in\mathbb{N}}$, $(s_n^{(3)})_{n\in\mathbb{N}}$ and $(s_n^{(4)})_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that

$$\lim_{n\to\infty}\|f_1-s_n^{(1)}\|_p = \lim_{n\to\infty}\|f_2-s_n^{(2)}\|_p = \lim_{n\to\infty}\|f_3-s_n^{(3)}\|_p$$

$$= \lim_{n\to\infty}\|f_4-s_n^{(4)}\|_p = 0.$$

Then for all $n\in\mathbb{N}$, $s_n^{(1)}+is_n^{(2)}+js_n^{(3)}+ks_n^{(4)} \in \mathcal{S}$. For all $n\in\mathbb{N}$,

$$0 \le \|f-(s_n^{(1)}+is_n^{(2)}+js_n^{(3)}+ks_n^{(4)})\|_p$$

$$\le \|f_1-s_n^{(1)}\|_p + \|f_2-s_n^{(2)}\|_p + \|f_3-s_n^{(3)}\|_p + \|f_4-s_n^{(4)}\|_p .$$

Then $\lim_{n\to\infty}\|f-(s_n^{(1)}+is_n^{(2)}+js_n^{(3)}+ks_n^{(4)})\|_p = 0$. #

<u>Note</u>: If $\mu(K)<\infty$ for all compact sets K of X, then $C_c(X)\subseteq L^p(\mu)$, $1\le p<\infty$.

<u>Proof</u> Let $f\in C_c(X)$. Then support f is compact, so there exists $M>0$ such that $|f|<M$, hence $|f|^p < M^p < \infty$. Thus $\int_X |f|^p d\mu = \int_{\text{support } f} |f|^p d\mu \le M^p \mu(\text{support } f) < \infty$. Hence $f\in L^p(\mu)$. #

4.93 <u>Theorem</u> Let X be a locally compact, σ-compact Hausdorff space and let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X with the properties stated in Theorem 4.76. Then for $1\le p<\infty$, $C_c(X)$ is dense in $L^p(\mu)$.

<u>Proof</u>[9] Let $\mathcal{S} = \{s : X\to\mathbb{H}$ is simple measurable/ $\mu(\{x\in X/ s(x)\ne 0\})<\infty\}$. By Theorem 4.92, $\mathcal{S}$ is dense in $L^p(\mu)$.

Let $f \in L^p(\mu)$ and $\varepsilon > 0$. Then there exists $s \in \mathcal{S}$ such that $\|f-s\|_p < \frac{\varepsilon}{2}$. Let $A = \{x \in X / \ s(x) \neq 0\}$. Then $\mu(A) < \infty$ and $s = 0$ on A^c. By Lusin's Theorem, there exists $g \in C_c(X)$ such that

$$\mu(\{x \in X / \ s(x) \neq g(x)\}) < \left(\frac{\varepsilon}{4(\left|\sup\limits_{x \in X}|s(x)|\right| + 1)}\right)^p$$

and

$$\sup_{x \in X}|g(x)| \leq \sup_{x \in X}|s(x)| .$$

Then

$$\|g-s\|_p = \left\{\int_X |g-s|^p d\mu\right\}^{\frac{1}{p}}$$

$$= \left\{\int_{\{x/ \ s(x) \neq g(x)\}} |g-s|^p d\mu\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_{\{x/ \ s(x) \neq g(x)\}} (|g|+|s|)^p d\mu\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_{\{x/ \ s(x) \neq g(x)\}} (2\sup_{x \in X}|s(x)|)^p d\mu\right\}^{\frac{1}{p}}$$

$$< (2\sup_{x \in X}|s(x)|) \frac{\varepsilon}{4(\sup\limits_{x \in X}|s(x)| + 1)} < \frac{\varepsilon}{2}$$

Hence

$$\|f-g\|_p \leq \|f-s\|_p + \|g-s\|_p \quad . \#$$