CHAPTER I

PRELIMINARIES

In this chapter I shall give some notation and some background on semirings. To make the exposition smoother, most definitions and results proved by other authers used in this thesis will be combined with the original material in Chapter II, and, in a few cases, Chapter III.

1.1. Notation

My general notational conventions are as follows:

$\mathbb{Z} =$ the set of all integers;

$\mathbb{Z}^+ =$ the set of all positive integers;

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$;

$\mathbb{Q}_0^+ =$ the set of all nonnegative rational numbers;

$\bar{n} = \{1, 2, \ldots, n\}$, where $n \in \mathbb{Z}^+$;

$(A_i)_{i \in I} =$ the family of the sets A_i, where i runs over the index set I;

$f|_A =$ the restriction of the function f to the set A;

$1_A =$ the identity map on a set A;

$1_{A,B} =$ the inclusion map from a set A into a superset B of A; and

$o(a) =$ order of an element a of a group.

1.2. Semirings

I will not say much about semirings, since they appear in this work only because the semimodules which are the primary objects of interest must be defined with respect to some semiring. For a good account of the basics of semirings, the reader is advised to consult *Golan*'s book *[1]*. Moreover, many of the constructions involving semirings are similar to ones involving rings, which may be found, for example, in *Ribenboim*'s book *[2]*.

A **semigroup** $(A, *)$ consists of a nonempty set A on which an associative operation $*$ is defined. If A is a semigroup in which there exists an element e satisfying $a * e = a = e * a$ for all $a \in A$ then A is called a **monoid** having **identity** element e. This element can easily be seen to be unique, and is usually denoted by 1_A, or just 1, if the monoid A is clear. A semigroup $(A, *)$ is **commutative** iff $a * a' = a' * a$ for all $a, a' \in A$, and is **cancellative** iff $a * b = a * c$ implies that $b = c$ for all $a, b, c \in A$ and $b * a = c * a$ implies that $b = c$ for all $a, b, c \in A$.

A **semiring** is a nonempty set S on which operations of addition $(+)$ and multiplication $(\cdot)$ have been defined such that the following conditions are satisfied:

(i) $(S, +)$ is a commutative monoid with identity element 0;

(ii) $(S, \cdot)$ is a monoid with identity element 1;

(iii) Multiplication distributes over addition from either side; and

(iv) $0 \cdot s = 0 = s \cdot 0$ for all s in S.

In other words, a semiring is just like a ring, except that inverses are not required to exist in the additive structure. (As a rule, I will write xy instead of $x \cdot y$

whenever x and y are elements of a semiring with the multiplication $\cdot$). Note that if $1 = 0$ then $s = s1 = s0 = 0$ for each element s of S and so $S = \{0\}$. In order to avoid this trivial case, I will assume that all semirings under consideration satisfy the additional condition

(v) $1 \neq 0$.

Note that 0 is clearly the only element of S satisfying (iv); indeed, if z is an element of S satisfying $zs = z = sz$ for all s in S then $0 = 0z = z$. Also note that conditions (iv) and (v) insure that the operations of addition and multiplication are not the same.

Example. Some simple examples of semirings are $(\mathbb{Z}_0^+, +, \cdot)$ and $(\mathbb{Q}_0^+, +, \cdot)$.

Example. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}_0^+ \right\}$ is easily seen to be a noncommutative semiring.

Example. Consider the set $S = \mathbb{Z}_0^+ \cup \{\infty\}$, where ∞ is a new object not in $\mathbb{Z}_0^+$. Extend the usual operations of $+$ and $\cdot$ on $\mathbb{Z}_0^+$ to S by defining $a + \infty = \infty + a = \infty$ for all $a \in S$, $a \cdot \infty = \infty \cdot a = \infty$ for all $a \in S \setminus \{0\}$, and $0 \cdot \infty = \infty \cdot 0 = 0$. It is tedious but straightforward to show that with these definitions $(S, +, \cdot)$ is a semiring.

If S and S' are semirings then a function $f \colon S \to S'$ is a **morphism of semirings** iff:

(i) $f(0_S) = 0_{S'}$;

(ii) $f(1_S) = 1_{S'}$; and

(iii) $f(s+s') = f(s) + f(s')$ and $f(ss') = f(s)f(s')$ for all s, s' in S.

A morphism of semirings which is both injective and surjective is called an **isomorphism**. If there exists an isomorphism between semirings S and S' I will write $S \cong S'$.