CHAPTER II

FUNDAMENTAL CONSIDERATION

Theory of Thin, Isotropic Elastic Plates

The thin elastic plate bending theory is based on the assumptions [13] that plane sections remain plane during bending and that the deflection are small comparing with the thickness of the plate. The effect of shear forces on the deflection of plates are also disregarded.

A thin plate element of thickness h , in polar co-ordinates (ρ,θ) , as shown in Fig. 1a and 1b together with the positive state of stress resultants. These moments and shear forces all acts per unit length, while the slopes and deflections refer to the middle surface of the plate. The expressions of the stress resultants in terms of the deflection, w , may be written as [13]:

$$M_\rho = -D\left[\frac{\partial^2 w}{\partial \rho^2} + \nu\left(\frac{1}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right)\right], \qquad (1)$$

$$M_\theta = -D\left[\nu\frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\cdot\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right], \qquad (2)$$

$$M_{\theta\rho} = -D(1-\nu)\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial w}{\partial\theta}\right), \tag{3}$$

$$Q_{\rho} = -D\frac{\partial}{\partial\rho}(\nabla^2 w), \tag{4}$$

$$Q_{\theta} = -D\frac{1}{\rho}\frac{\partial}{\partial\theta}(\nabla^2 w), \tag{5}$$

where M_{ρ} = normal bending moment per unit arc length,

M_{θ} = transverse bending moment per unit length,

$M_{\rho\theta} = -M_{\theta\rho}$ = twisting moment per unit length,

Q_{ρ} = normal shear per unit arc length,

Q_{θ} = transverse shear per unit length,

$\nabla^2 = \partial^2/\partial\rho^2 + (1/\rho)\partial/\partial\rho + (1/\rho^2)\partial^2/\partial\theta^2$

= Laplacian opertor,

$D = Eh^3 / 12(1-\nu^2)$ = flexural rigidity.

In which, E is the modulus of elasticity, h the plate thickness and ν the Poisson's ratio.

Furthermore, the Kirchhoff's shears per unit length V_{ρ} and V_{θ}, may also be computed as:

$$V_{\rho} = Q_{\rho} + \frac{1}{\rho}\frac{\partial M_{\theta\rho}}{\partial\theta} \tag{6}$$

$$V_{\theta} = Q_{\theta} + \frac{\partial M_{\theta\rho}}{\partial\rho} \tag{7}$$

By considering equilibrium of the stress resultants of plate element, Fig. 1, it leads to the governing equation [13]:

$$\nabla^2\nabla^2 w(\rho,\theta) = q(\rho,\theta)/D. \quad (8)$$

In which, $q(\rho,\theta)$ denotes transverse load per unit area applied to the plate.

Betti's Reciprocal Theorem

The boundary integral equations may be conveniently formulated by using the Betti's reciprocal theorem based on energy considerations [14]. It stated that if a linearly elastic body subject to two separate force systems is in equilibrium and compatibility, the work that would be done by the first system of forces in acting through the corresponding displacements produced by the second system of forces is equal to the work that would be done by the second system of forces in acting through the corresponding displacements produced by the first system of forces.

Boundary Integral Equations Formulation

To solve a problem of plate bending, it would be perfect if one can seek a function that satisfies the governing equation, Eq. 8, and the prescribed boundary conditions of the plate. However, for plates of irregular plan forms or boundary conditions, it will extremely complicated, if possible at all, to obtain such solution functions. Alternatively, a boundary element technique, based on

Betti's reciprocal theorem, may be formulated as follows:

Consider an annular plate with a bounded domain A, as shown in Fig. 2, in which, a and b denote the outer and inner radius respectively. Two distinct systems of compatible deflections and equilibrating stresses, one real and one virtual (designated by asterisks), are considered acting separately on the same plate domain. The real one (Fig. 2a), which is loaded by transverse loads of intensity $q(\xi,\alpha)$, are prescribed by mixed boundary conditions of simple, clamped and free edges supports with N_c interior columns. Let the outer edge of the real plate be divided into K sections of different boundary conditions with the beginning angle γ^I ;$I = 1,2,3,\ldots,K$, while sections $K+1$ to L belong to the inner edge. Along each of these sections of the edge and at each interior column, the boundary conditions may be any one of the following:

For simple supports, $w = M_\rho = 0.$ (9a)

For clamped supports, $w = \partial w/\partial\rho = 0.$ (9b)

For free edges, $V_\rho = M_\rho = 0.$ (9c)

For interior column supports, $w = 0.$ (10)

In which, w, $\partial w/\partial\rho$ = deflection and normal slope, V_ρ = normal Kirchhoff's shear, and M_ρ = normal bending moment.

In the virtual system, Fig. 2b, a fundamental solution is taken as:

$$w^{*}(\rho,\theta;\xi,\alpha) \quad = \quad r^{2}\ln r \ / \ 8\pi D, \tag{11}$$

which is the deflection at a point (ξ,α) due to a unit singular load acting at a point (ρ,θ), where the distance between the two points are denoted by:

$$r \quad = \quad \{\rho^{2} + \xi^{2} - 2\rho\xi \cos(\alpha-\theta)\}^{1/2}. \tag{12}$$

This deflection function which satisfies the governing equation, Eq. 8, is compatible and the associated stress resultants obtained by appropriate differentiation, Eqs. 1 to 7, are in equilibrium.

Since the real system in Fig. 2a must also be in equilibrium and compatibility, and the material is assumed to be linear elastic, a virtual work equation of the two systems based on Betti's reciprocal theorem may be written:

$$\phi\, w(\rho,\theta) \quad = \quad \int_{A} q(\xi,\alpha)\; w^{*}(\rho,\theta;\xi,\alpha)\; dA(\xi,\alpha)$$

$$+ \sum_{I=1}^{L} \int_{\gamma^{I}}^{\gamma^{I+1}} \Big[\, V_{\rho}(\xi^{I},\alpha)\; w^{*}(\rho,\theta;\xi^{I},\alpha) - M_{\rho}(\xi^{I},\alpha)\; \frac{\partial w^{*}(\rho,\theta;\xi^{I},\alpha)}{\partial \xi}$$

$$- w(\xi^{I},\alpha)\; V_{\xi}^{*}(\rho,\theta;\xi^{I},\alpha) + \frac{\partial w(\xi^{I},\alpha)}{\partial \rho}\; M_{\xi}^{*}(\rho,\theta;\xi^{I},\alpha) \,\Big]\, \xi^{I} d\alpha$$

$$+\sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\ w^*(\rho,\theta;\xi_n,\alpha_n) \qquad ; b\leq\rho\leq a,\ 0\leq\theta<2\pi, \qquad (13)$$

where ξ^I = a ; $1 \leq I \leq K$, for the outer edge,

b ; $K+1 \leq I \leq L$, for the inner edge. (14)

R_c = column unknown reaction.

And ϕ = 1 when (ρ,θ) is inside the plate domain,

1/2 when (ρ,θ) is right on the boundary. (15)

The latter value of ϕ is due to the fact that when a unit load of the virtual system, Fig. 2b, acts right on a smooth boundary point, only a half of it is in the domain of the plate to produce virtual external work.

In Eq. 13 above, $\partial w^*/\partial\xi$, $V_\xi{}^*$ and $M_\xi{}^*$ denote normal slope, Kirchhoff's shear and normal bending moment corresponding to the deflection, w^*, in Eq. 11 of the virtual system.

Furthermore, in any section on the boundary, two of the four unknown integrand functions, w, $\partial w/\partial\rho$, V_ρ and M_ρ will be prescribed by one of the boundary conditions in Eq. 9. Thus, in each section of the plate boundary, there remains only a pair of unknown functions. Therefore, altogether there are 2L unknown integrand functions plus N_c unknown values of column reactions, R_c. These unknowns can be determined by approaching the unit load in the virtual system to each and every section of the boundary and to each point of the column location, and consider the deflection function, as derived in Eq. 13 together with its normal slope as follow:

At each section on the boundary,

$$\frac{1}{2} w(\rho^J,\theta) = \int_A q(\xi,\alpha)\, w^*(\rho^J,\theta;\xi,\alpha)\, dA(\xi,\alpha)$$

$$+ \sum_{I=1}^{L} \int_{\gamma^I}^{\gamma^{I+1}} \Big[V_\rho(\xi^I,\alpha)\, w^*(\rho^J,\theta;\xi^I,\alpha) - M_\rho(\xi^I,\alpha)\, \frac{\partial w^*(\rho^J,\theta;\xi^I,\alpha)}{\partial \xi}$$

$$- w(\xi^I,\alpha)\, V_\xi^*(\rho^J,\theta;\xi^I,\alpha) + \frac{\partial w(\xi^I,\alpha)}{\partial \rho}\, M_\xi^*(\rho^J,\theta;\xi^I,\alpha) \Big] \xi^I d\alpha$$

$$+ \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\, w^*(\rho^J,\theta;\xi_n,\alpha_n) \quad ; \gamma^J \leq \theta < \gamma^{J+1},\ J=1,2,3,\ldots,L, \quad (16)$$

$$\frac{1}{2} \frac{\partial w(\rho^J,\theta)}{\partial \rho} = \int_A q(\xi,\alpha)\, \frac{\partial w^*(\rho^J,\theta;\xi,\alpha)}{\partial \rho}\, dA(\xi,\alpha)$$

$$+ \sum_{I=1}^{L} \int_{\gamma^I}^{\gamma^{I+1}} \Big[V_\rho(\xi^I,\alpha)\, \frac{\partial w^*(\rho^J,\theta;\xi^I,\alpha)}{\partial \rho} - M_\rho(\xi^I,\alpha)\, \frac{\partial^2 w^*(\rho^J,\theta;\xi^I,\alpha)}{\partial \rho \partial \xi}$$

$$- w(\xi^I,\alpha)\, \frac{\partial V_\xi^*(\rho^J,\theta;\xi^I,\alpha)}{\partial \rho} + \frac{\partial w(\xi^I,\alpha)}{\partial \rho}\, \frac{\partial M_\xi^*(\rho^J,\theta;\xi^I,\alpha)}{\partial \rho} \Big] \xi^I d\alpha$$

$$+ \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\, \frac{\partial w^*(\rho^J,\theta;\xi_n,\alpha_n)}{\partial \rho} \quad ; \gamma^J \leq \theta < \gamma^{J+1},\ J=1,2,3,\ldots,L. \quad (17)$$

In which, $\rho^{J} = a$; for $1 \le J \le K$,

b ; for $K+1 \le J \le L$. (18)

At each column location,

$$0 = \int_A q(\xi,\alpha)\, w^*(\rho_m,\theta_m;\xi,\alpha)\, dA(\xi,\alpha)$$

$$+ \sum_{I=1}^{L} \int_{\gamma^I}^{\gamma^{I+1}} \Big[V_\rho(\xi^I,\alpha)\, w^*(\rho_m,\theta_m;\xi^I,\alpha) - M_\rho(\xi^I,\alpha)\, \frac{\partial w^*(\rho_m,\theta_m;\xi^I,\alpha)}{\partial \xi}$$

$$- w(\xi^I,\alpha)\, V_\xi^*(\rho_m,\theta_m;\xi^I,\alpha) + \frac{\partial w(\xi^I,\alpha)}{\partial \rho}\, M_\xi^*(\rho_m,\theta_m;\xi^I,\alpha) \Big]\, \xi^I d\alpha$$

$$+ \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\, w^*(\rho_m,\theta_m;\xi_n,\alpha_n) \qquad ;\ m=1,2,3,\ldots,N_c. \qquad (19)$$

The integral equations which are formulated in Eqs. 16, 17 and 19 are sufficient to be solved for the 2L unknown integrand functions and the N_c unknown values of column reactions, R_c , by the familiar numerical technique as to be elaborated in the next chapter.