CHAPTER II

PRELIMINARIES

To proceed with the investigation of our GDDs, we recall some known designs here which will be used in our constructions.

2.1 Triple Systems

First, we describe the well-known designs called *triple systems*. A *triple system*, denoted by $\mathsf{TS}(n;\lambda)$, is an ordered pair $(S,\mathcal{T})$ where S is an n-set and $\mathcal{T}$ is a collection of 3-subsets of S (called *blocks* or *triples*) such that each pair of distinct elements of S occurs together in exactly λ blocks. Note that a $\mathsf{TS}(n;\lambda)$ can be considered as a group divisible design with only one group.

Example 2.1. Let $S_1 = \{1,2,3,\ldots,7\}$ and $\mathcal{T}_1 = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{1,5,6\},\{2,6,7\},\{1,3,7\}\}$. Then, $(S_1,\mathcal{T}_1)$ is a $\mathsf{TS}(7;1)$.

Example 2.2. Let $S_2 = \{1,2,3,4\}$ and $\mathcal{T}_2 = \{\{1,2,3\},\{2,3,4\},\{1,3,4\},\{1,2,4\}\}$. Then, $(S_2,\mathcal{T}_2)$ is a $\mathsf{TS}(4;2)$.

The following theorem guarantees the existence of such $\mathsf{TS}(n;\lambda)$, see details in [4].

Theorem 2.3. [4] *Let $n \geq 3$. Then, a $\mathsf{TS}(n;\lambda)$ exists if and only if λ and n are in one of the following cases:*

(i) $\lambda \equiv 0 \pmod 6$,

(ii) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and* $n \equiv 1$ *or* $3 \pmod 6$,

(iii) $\lambda \equiv 2$ *or* 4 (mod 6) *and* $n \equiv 0$ *or* 1 (mod 3) *and*

(iv) $\lambda \equiv 3$ (mod 6) *and* n *is odd.*

2.2 Factorizations

A variety of the techniques that we use to construct GDDs are related to k-factors.

Definition 2.4. A *k-factor* of a graph G is a spanning k-regular subgraph of G.

In particular, a 1-factor is a perfect matching. A 2-factor is a union of cycles which span all vertices of the graph. Besides, if every cycle in a 2-factor is K_3, then such 2-factor is called a *$\triangle$-factor.*

Theorems 2.5 and 2.7 are classical results of decompositions of complete graphs into 1-factors and 2-factors, respectively.

Theorem 2.5. [10] *For all even integer n, the complete graph K_n can be decomposed into $(n-1)$ 1-factors.*

Example 2.6. A complete graph K_4 can be decomposed into three 1-factors.

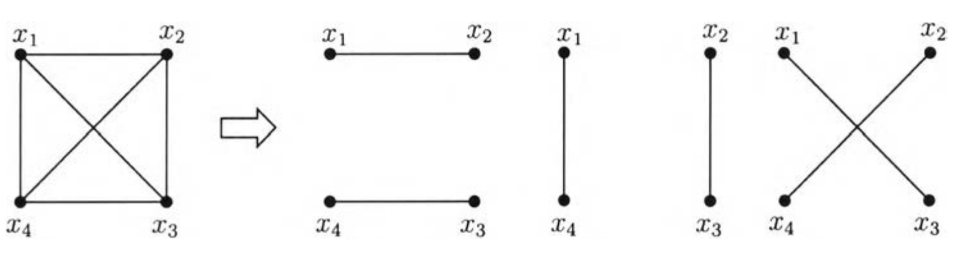

Figure 2.1: Three different 1-factors of K_4

Theorem 2.7. [10] *For all odd integer n, the complete graph K_n can be decomposed into $\frac{n-1}{2}$ 2-factors.*

Example 2.8. A complete graph K_5 can be decomposed into two 2-factors.

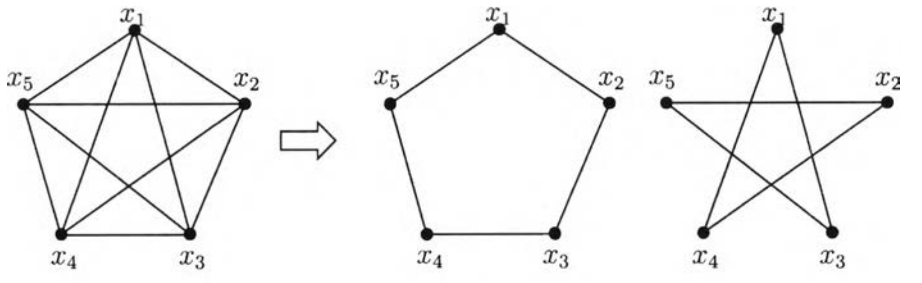

Figure 2.2: Two different 2-factors of K_5

2.3 Packings

A *packing with triangles* of a complete graph K_n is a triple $(S, \mathcal{T}, \mathcal{L})$ where S is the vertex set of K_n; $\mathcal{T}$ is a collection of edge-disjoint complete subgraph K_3 of K_n; and $\mathcal{L}$ is the collection of edges in K_n not belonging to any K_3 in $\mathcal{T}$. The collection of edges $\mathcal{L}$ is called the *leave*. If $|\mathcal{L}|$ is as small as possible, then $(S, \mathcal{T}, \mathcal{L})$ is called a *maximum packing* of order n.

We can consider maximum packings as a generalization of triple systems with $\lambda = 1$. According to Theorem 2.3, we have that the complete graph K_n can be decomposed into triangles if and only if $n \equiv 1$ or $3 \pmod 6$. However, if we decompose the complete graph K_n for any value of n into triangles, the following result about maximum packings shows what remains as the leave.

Theorem 2.9. [4] *Let n be a positive integer. If $(S, \mathcal{T}, \mathcal{L})$ is a maximum packing of order n, then the leave is*

(i) *a 1-factor, if $n \equiv 0$ or $2 \pmod 6$,*

(ii) *a 4-cycle, which is a cycle on 4 vertices, if $n \equiv 5 \pmod 6$,*

(iii) *a tripole, which is a spanning graph with each vertex having odd degree and containing $\frac{n+2}{2}$ edges, if $n \equiv 4 \pmod 6$ and*

(iv) *the empty set, if $n \equiv 1$ or $3 \pmod 6$.*

2.4 Uniformly Resolvable Designs

A *uniformly resolvable design* $\mathsf{URD}(n,k)$ is a decomposition of the complete graph K_n into t 1-factors and $k-t$ $\triangle$-factors where $t = 2k-n+1$. The following theorem provides the existence of a $\mathsf{URD}(n,k)$.

Theorem 2.10. [7] *Let n and k be positive integers such that $n \equiv 0 \pmod 6$ and $\frac{n}{2}+1 \le k \le n-1$. Then, there exists a $\mathsf{URD}(n,k)$.*

Note that when $n \equiv 0 \pmod 6$, every vertex in the complete graph K_n has odd degree. Besides, any triangle contributes two to the value of vertex degree. Thus, the number of 1-factors in a $\mathsf{URD}(n,k)$ is always odd. Therefore, by Theorem 2.10, the complete graph K_n can be decomposed into $\frac{n-1-t}{2}$ $\triangle$-factors and t 1-factors, when t is odd and $3 \le t \le n-1$. In other words, K_n can be decomposed into t 1-factors and a collection of triangles when $t \ge 3$ is odd. Furthermore, by Theorem 2.9, the leave of a maximum packing with triangles when $n \equiv 0 \pmod 6$ is a 1-factor. This means that K_n can be decomposed into one 1-factor and a collection of triangles. We conclude this result in the following theorem.

Theorem 2.11. *Let $n \equiv 0 \pmod 6$. The complete graph K_n can be decomposed into a collection of triangles and a collection of t 1-factors, where t is odd and $1 \le t \le n-1$.*

We extend the above result to a decomposition of any complete multigraph λK_n when $n \equiv 0 \pmod 6$ and $\lambda \ge 1$ in Theorem 2.12.

Theorem 2.12. *Let $n \equiv 0 \pmod 6$ and λ be a positive integer. Then, there is a decomposition of the graph λK_n into a collection of triangles and a collection of k 1-factors, where $k \equiv \lambda \pmod 2$ and $1 \le k \le \lambda(n-1)$.*

Proof. When $1 \le k \le \lambda$, by Theorem 2.11, we decompose k copies of K_n into triangles and k 1-factors. Since $\lambda - k$ is even, the remaining $(\lambda-k)K_n$ can be considered as a $\mathsf{TS}(n; \lambda - k)$. Now, assume that $\lambda \le k \le \lambda(n-1)$. Since $k \equiv \lambda \pmod 2$ and $1 \le k \le \lambda(n-1)$, we can write $k = \lambda a + 2b$ for nonnegative integers a and b such that a is odd, $1 \le a \le n-1$ and $0 \le b < \lambda$. Again, by Theorem 2.11, we decompose each of b copies of K_n into triangles and $a+2$ 1-factors and decompose each of the remaining $(\lambda - b)K_n$ into triangles and a 1-factors. □

Our work in Chapter III and Chapter IV is to construct the GDDs that satisfy the necessary conditions in Theorem 1.2. However, we first note that if $\lambda_1 = \lambda_1'$, a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ can be considered as an original GDD with two associate classes, namely $\mathsf{GDD}(v = m+n, 2, 3; \lambda_1, \lambda_2)$, in which the existence problem had been already done [5, 6, 8, 9]. We state some results in the following lemma, which will be ingredients for our construction.

Lemma 2.13. [5, 6] *Let h and k be nonnegative integers. All of the following* GDD*s exist:*

(i) *a* $\mathsf{GDD}(6h+6, 6k+6; 2, 2, 1)$,

(ii) *a* $\mathsf{GDD}(6h+6, 6k+4; 2, 2, 1)$,

(iii) *a* $\mathsf{GDD}(6h+4, 6k+4; 4, 4, 3)$,

(iv) *a* $\mathsf{GDD}(6h+6, 6k+2; 6, 6, 5)$, *where $k \neq 0$ and*

(v) *a* $\mathsf{GDD}(6h+2, 6k+2; 4, 4, 1)$, *where $h \neq 0$ and $k \neq 0$.*

The following notations will be used throughout this thesis.

(1) When we say that $\mathcal{B}$ is a *collection* of blocks of a v-set, $\mathcal{B}$ may contain repeated blocks. Thus, the union in our constructions is referred to the union of multisets.

(2) Let m be a positive integer. We write $m = \bar{a}$ for $m \equiv a \pmod 6$ where $a \in \{0, 1, 2, \dots, 5\}$.

(3) Let $e = \{u, v\}$ be an edge in a graph G. We use $a + e$ for the triple $\{a, u, v\}$. If X is a set of edges of a graph G, then $a + X$ stands for $|X|$ triples (or,

equivalently, triangles) in $\{a+e : e \in X\}$. For a subgraph H of G, we also write $a+H$ instead of $a+E(H)$.

Example 2.14. Let G be a graph with the vertex set $V(G)=\{a, x_1, x_2, x_3, x_4, x_5\}$ and let C be a 2-factor of G with 4 vertices x_1, x_2, x_3 and x_4. Thus, C is a 4-cycle, say $C=(x_1x_2x_3x_4)$. Then, $a+C$ stands for 4 triangles $\{a, x_1, x_2\}$, $\{a, x_2, x_3\}$, $\{a, x_3, x_4\}$ and $\{a, x_4, x_1\}$, illustrated in Figure 2.3.

Example 2.15. Let $X=\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_1, x_2\}\}$ be a set of edges in a graph G. Then, $a+X$ stands for 4 triangles $\{a, x_1, y_1\}, \{a, x_2, y_2\}, \{a, x_3, y_3\}$ and $\{a,\ x_1, x_2\}$, illustrated in Figure 2.4.

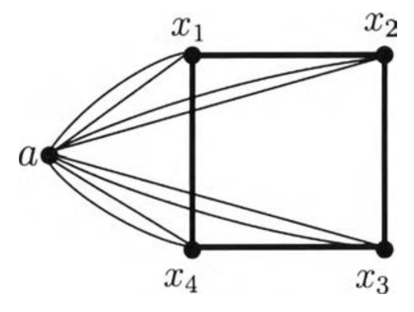

Figure 2.3: The graph obtained from the union of all triples in $a+C$

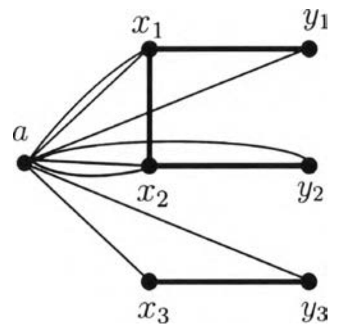

Figure 2.4: The graph obtained from the union of all triples in $a+X$

Remark 2.16. Note that if H is a spanning subgraph of degree k, then for any vertex x in the graph H, there exist k triples in $a+H$ that contain both vertices

a and x. In particular, if H is a 1-factor (or 2-factor), then a and x occur together once (or twice, respectively) in $a + H$, for each vertex x in H.