CHAPTER III

GROUP DIVISIBLE DESIGNS

WITH m or $n \equiv 0$ or $1 \pmod 3$, $m \neq 2$ and $n \neq 2$

3.1 Introduction

The main work in this thesis is to show the sufficiency part of the existence problem of our GDDs. In particular, we show the construction of group divisible designs that satisfy Table 1.1. In this chapter, GDDs with m or $n \equiv 0$ or $1 \pmod 3$ and $m, n \neq 2$ are of our interest while the GDDs with m and $n \equiv 2 \pmod 3$ and $m, n \neq 2$ will be considered in Chapter IV.

To construct a desired GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$, we show that there is a K_3-decomposition of the corresponding graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1' K_n$. Recall that

$$M_m = \{x_1, x_2, x_3, \ldots, x_m\}$$

and

$$N_n = \{y_1, y_2, y_3, \ldots, y_n\}$$

are disjoint sets of elements and the notations $\lambda_1 K_m(M_m)$ and $\lambda_1' K_n(N_n)$ stand for the complete multigraphs lying on the sets M_m and N_n, respectively.

The following observations are basic tools for our construction. Thus, we conclude them in Lemma 3.1 for future references.

Lemma 3.1. *Let m and n be positive integers and let $\lambda_1, \lambda_1', \lambda_2, \gamma_1, \gamma_1'$ and γ_2 be nonnegative integers.*

(i) *If there exist a* $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ *and a* $\mathsf{GDD}(m,n;\gamma_1,\gamma_1',\gamma_2)$*, then there exists a* $\mathsf{GDD}(m,n;\lambda_1+\gamma_1,\lambda_1'+\gamma_1',\lambda_2+\gamma_2)$.

(ii) *If there exists a* $\mathsf{TS}(m;\lambda_1-\lambda_2)$*, a* $\mathsf{TS}(n;\lambda_1'-\lambda_2)$ *and a* $\mathsf{TS}(m+n;\lambda_2)$*, then there exists a* $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$.

Proof. (i) Let $(M_m,N_n,\mathcal{B}_1)$ and $(M_m,N_n,\mathcal{B}_2)$ be a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ and a $\mathsf{GDD}(m,\, n;\gamma_1,\gamma_1',\gamma_2)$, respectively. Then, the 3-tuple $(M_m,N_n,\mathcal{B}_1\cup\mathcal{B}_2)$ is a desired GDD.

(ii) Let $(M_m,\varnothing,\mathcal{B}_1)$ be a $\mathsf{TS}(m;\lambda_1-\lambda_2)$, let $(\varnothing,N_n,\mathcal{B}_2)$ be a $\mathsf{TS}(n;\lambda_1'-\lambda_2)$ and let $(M_m,N_n,\mathcal{B}_3)$ be a $\mathsf{TS}(m+n;\lambda_2)$. Then, $(M_m,N_n,\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3)$ is a desired GDD.

□

In our construction, several techniques will be used. One of them is to find some small designs and combine them to get a larger one. By the existence of well-known triple systems from Theorem 2.3 with the observations in Lemma 3.1, we can construct some of our desired GDDs directly in Theorem 3.2. However, note that a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ exists if and only if the corresponding graph $\lambda_1K_m\vee_{\lambda_2}\lambda_1'K_n$ has a K_3-decomposition. This means that each edge in the graph must belong to exactly one triangle. Thus, when $m=n=1$ and $\lambda_2>0$, a $\mathsf{GDD}(1,1;\lambda_1,\lambda_1',\lambda_2)$ does not exist.

Theorem 3.2. *Let* m *and* n *be positive integers such that* $m\neq 2, n\neq 2$ *and* $mn\neq 1$*. Let* λ_1,λ_1' *and* λ_2 *be nonnegative integers such that* $\lambda_1\geq\lambda_2$ *and* $\lambda_1'\geq\lambda_2$*. Then, there exists a* $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ *with parameters* m,n,λ_1,λ_1' *and* λ_2 *satisfying one of the following:*

(i) $m\equiv 1$ *or* $3 \pmod 6$, $n\equiv 0$ *or* $4 \pmod 6$, $\lambda_1'\equiv\lambda_2 \pmod 2$ *and if* $m+n\equiv 5 \pmod 6$*, then* $\lambda_2\equiv 0$ *or* $3 \pmod 6$.

(ii) $m, n \equiv 1$ *or* $3 \pmod 6$, $\lambda_2 \equiv 0 \pmod 2$ *and*

if $m + n \equiv 2 \pmod 6$, *then* $\lambda_2 \equiv 0 \pmod 6$.

(iii) $m \equiv 1$ *or* $4 \pmod 6, n \equiv 5 \pmod 6, 3|(\lambda_1' + 2\lambda_2)$,

if $m \equiv 1 \pmod 6$, *then* $\lambda_2 \equiv 0 \pmod 2$ *and*

if $m \equiv 4 \pmod 6$, *then* $\lambda_1 \equiv \lambda_2 \pmod 2$.

(iv) $m \equiv 1 \pmod 6, n \equiv 2 \pmod 6, \lambda_1' \equiv \lambda_2 \pmod 2$ *and* $\lambda_1' + 2\lambda_2 \equiv 0 \pmod 3$.

Proof. First, since $\lambda_1, \lambda_1' \geq \lambda_2$, we have that $\lambda_1 - \lambda_2$ and $\lambda_1' - \lambda_2$ are nonnegative integers.

(i) Assume that $\lambda_1' \equiv \lambda_2 \pmod 2$. Then, $\lambda_1' - \lambda_2$ is even. Note that $m+n \equiv 1, 3$ or $5 \pmod 6$; moreover if $m + n \equiv 5 \pmod 6$, we also have that $\lambda_2 \equiv 0$ or $3 \pmod 6$. Thus, by Theorem 2.3, a $\mathsf{TS}(m + n; \lambda_2)$, a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ and a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$ exist. Hence, by Lemma 3.1, we obtain our desired GDD.

(ii) Note that m, n and λ_2 satisfy one of the cases: $m + n \equiv 0$ or $4 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 2$; or $m + n \equiv 2 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$. By Theorem 2.3, a $\mathsf{TS}(m + n; \lambda_2)$, a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ and a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$ exist. Applying Lemma 3.1, these triple systems form a desired GDD.

(iii) If $m \equiv 1 \pmod 6$, then $m + n \equiv 0 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 2$. By Theorem 2.3, there exist a $\mathsf{TS}(m + n; \lambda_2)$ and a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$. Note that $3|(\lambda_1' - \lambda_2)$. Thus, there exists a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$. Then, by Lemma 3.1, we obtain our GDD.

If $m \equiv 4 \pmod 6$, then $m + n \equiv 3 \pmod 6$. Thus, there exists a $\mathsf{TS}(m + n; \lambda_2)$ by Theorem 2.3. Note that $\lambda_1 - \lambda_2$ is even and $3|(\lambda_1' - \lambda_2)$. Thus, a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ and a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$ exist. These triple systems form a GDD by Lemma 3.1.

(iv) Since $\lambda_1' \equiv \lambda_2 \pmod 2$ and $\lambda_1' + 2\lambda_2 \equiv 0 \pmod 3$, $\lambda_1' - \lambda_2 \equiv 0 \pmod 6$. Then, by Theorem 2.3, there exist a $\mathsf{TS}(m + n; \lambda_2)$, a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ and a

$\mathsf{TS}(n; \lambda_1' - \lambda_2)$. Applying Lemma 3.1, we obtain a desired GDD.

□

Theorem 3.2 shows a construction of some cases in Table 1.1. To proceed our investigation of GDDs where m or $n \equiv 0$ or 1 (mod 3), we first note that a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ is equivalent to a $\mathsf{GDD}(n, m; \lambda_1', \lambda_1, \lambda_2)$. Thus. it suffices to consider only when $m \equiv 0$ or 1 (mod 3) and run the value of n. Hence, we separate our construction in this chapter into three sections, depending on the value of n.

3.2 m and $n \equiv 0$ or 1 (mod 3)

In this section, we consider the case that both m and $n \equiv 0$ or 1 (mod 3). Due to the construction in Theorem 3.2 (i) and (ii), it remains to construct a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1'. \lambda_2)$ when $(m, n) \in \{(\bar{0}, \bar{0}), (\bar{0}, \bar{4}), (\bar{4}, \bar{4})\}$. Again, by the observations in Lemma 3.1 together with the existing GDDs when $\lambda_1 = \lambda_1'$ in Lemma 2.13, we obtain our desired GDDs in the following theorem.

Theorem 3.3. *Let m and n be positive integers such that m and $n \equiv 0$ or 4* (mod 6). *Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. There exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ *provided that*

(i) *λ_1 and λ_1' are even and*

(ii) *if m, $n \equiv 4$* (mod 6), *then $\lambda_2 \equiv 0$* (mod 3).

Proof. When λ_2 is even, by Theorem 2.3, there exist a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ and a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$. Note that $m + n \equiv 0$ or 2 or 4 (mod 6); moreover if $m + n \equiv 2$ (mod 6), then $\lambda_2 \equiv 0$ (mod 6). Thus, by Theorem 2.3, a $\mathsf{TS}(m + n; \lambda_2)$ exists. Applying Lemma 3.1, we obtain a desired GDD.

Now, assume that λ_2 is odd, then $\lambda_2 - 1$ is even. When $m+n \equiv 0$ or $4 \pmod 6$, we obtain from the previous case that there exists a $\mathsf{GDD}(m,n;\lambda_1-2,\lambda_1'-2,\lambda_2-1)$. Together with a $\mathsf{GDD}(m,n;2,2,1)$ from Lemma 2.13, we obtain our desired GDD. When $m+n \equiv 2 \pmod 6$, we have that $\lambda_2 \equiv 3 \pmod 6$. From the previous case, there is a $\mathsf{GDD}(m,n;\lambda_1-4,\lambda_1'-4,\lambda_2-3)$. Together with a $\mathsf{GDD}(m,n;4,4,3)$ from Lemma 2.13 (iii), we obtain our desired GDD. □

3.3 $m \equiv 0$ or $1 \pmod 3$, $n \equiv 2 \pmod 6$ and $n \neq 2$

This section is to consider GDDs where $m \equiv 0$ or $1 \pmod 3, n \equiv 2 \pmod 6$ and $n \neq 2$. Note that $m \equiv 0, 1, 3$ or $4 \pmod 6$. By Theorem 3.2 (iii), it remains to construct the GDDs when $(m,n) \in \{(\bar{0},\bar{2}),(\bar{3},\bar{2}),(\bar{4},\bar{2})\}$.

First, we construct GDDs with $m \equiv 0 \pmod 6, n \equiv 2 \pmod 6$ and $n \neq 2$. The main construction is provided in Theorem 3.7, which requires the existence of some small GDDs in Lemmas 3.4 - 3.6.

Lemma 3.4. *There exists a* $\mathsf{GDD}(6,2;\lambda_1,6,\lambda_2)$ *where* $(\lambda_1,\lambda_2) \in \{(0,1),(2,1),(2,2),(4,3),(4,4)\}$.

Proof. Let $\mathcal{B} = \{\{x_i,y_1,y_2\} : i \in \{1,2,3,\ldots,6\}\}$. Then, $(M_6,N_2,\mathcal{B})$ forms a $\mathsf{GDD}(6,2;0,6,1)$. For $(\lambda_1,\lambda_2) = (2,1)$, the graph $2K_6(M_6)$ can be considered as a $\mathsf{TS}(6;2)$. Thus, by Lemma 3.1, a $\mathsf{GDD}(6,2;2,6,1)$ exists. For $(\lambda_1,\lambda_2) \in \{(2,2),(4,3),(4,4)\}$, we note that $2(\lambda_2-1) \equiv \lambda_1 \pmod 2$ and $2(\lambda_2-1) \leq 5\lambda_1$. By Theorem 2.12, we can decompose the graph $\lambda_1 K_6(M_6)$ into a collection of triangles $\mathcal{T}$ and $2(\lambda_2-1)$ 1-factors, say $F_{i,j}$ where $i \in \{1,2\}$, $j \in \{1,2,3,\ldots,\lambda_2-1\}$. Let $\mathcal{F}$ be a collection of triangles defined by

$$\mathcal{F} = \{y_i + F_{i,j} : i \in \{1,2\}, j \in \{1,2,3,\ldots,\lambda_2-1\}\}.$$

Hence, $\mathcal{B} \cup \mathcal{T} \cup \mathcal{F}$ is a K_3-decomposition of the graph $\lambda_1 K_6(M_6) \vee_{\lambda_2} 6K_2(N_2)$. Then, there exists a $\mathsf{GDD}(6, 2; \lambda_1, 6. \lambda_2)$ where $(\lambda_1, \lambda_2) \in \{(2, 2), (4, 3), (4, 4)\}$. □

Lemma 3.5. *There exists a* $\mathsf{GDD}(6, 8; 4, 6, 4)$.

Proof. Let $B = \{y_1, y_2\} \subseteq N_8$. By Lemma 3.4, there exists a $\mathsf{GDD}(6, 2; 0, 6, 1)$ on the vertex set $M_6 \cup B$, namely $(M_6, B, \mathcal{B})$. Since $\lambda_1(m - 1) = 20$, by Theorem 2.12, we can decompose the graph $4K_6(M_6)$ into a collection of triangles $\mathcal{T}_1$ and 18 1-factors, say $F_{i,j}$ and $F_{p,q}$ where $i, q \in \{1, 2\}$, $j \in \{1, 2, 3\}$ and $p \in \{3, 4, 5, \dots, 8\}$. Let $\mathcal{F}$ be a collection of triangles defined by

$$\mathcal{F} = \{y_i + F_{i,j}, y_p + F_{p,q} : i, q \in \{1, 2\}, j \in \{1, 2, 3\}, p \in \{3, 4, 5, \dots, 8\}\}.$$

Since $\lambda_1'(n - 3) = 30$, we can apply Theorem 2.12 again to decompose the graph $6K_6(N_8 \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and 24 1-factors, say $H_{i,j}$ and $H_{p,q}$ where $i \in \{1, 2, 3, \dots, 6\}$, $q \in \{7, 8, 9, \dots, 12\}$ and $j, p \in \{1, 2\}$. Let $\mathcal{H}$ be a collection of triangles defined by

$$\mathcal{H} = \{x_i + H_{i,j}, y_l + H_{p,q} : i \in \{1, 2, 3, \dots, 6\}, q \in \{7, 8, 9, \dots, 12\}, j, p \in \{1, 2\}\}.$$

Thus, $(M_6, N_8, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F} \cup \mathcal{H})$ is a $\mathsf{GDD}(6, 8; 4, 6, 4)$. □

The GDDs from Lemma 3.4 are useful to construct some larger GDDs in the following lemma.

Lemma 3.6. *Let m and n be positive integers such that* $m \equiv 0 \pmod 6), n \equiv 2 \pmod 6$ *and* $n \neq 2$. *There exists a* $\mathsf{GDD}(m, n; \lambda_1, 6. \lambda_2)$ *where* $(\lambda_1, \lambda_2) \in \{(2, 1), (2, 2), (4, 3), (4, 4)\}$.

Proof. We write $m = 6h + 6$ and $n = 6k + 8$ for nonnegative integers h, k. Let $A = \{x_1, x_2, x_3, \dots, x_6\}, B = \{y_1, y_2\}$ and let $(\lambda_1, \lambda_2) \in \{(2, 1), (2, 2), (4, 3), (4, 4)\}$. We separate our construction in three cases.

Case (i) $h > k$. Since λ_1 and n are even, $m > n$ and $\lambda_1 \geq \lambda_2$, by Theorem 2.12, we can decompose the graph $\lambda_1 K_m(M_m)$ into a collection of triangles $\mathcal{T}$ and $\lambda_2 n$ 1-factors, say $F_{i,j}$ where $i \in \{1, 2, 3, \ldots, n\}$, $j \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}$ be a collection of triangles defined by

$$\mathcal{F} = \{y_i + F_{i,j} : i \in \{1, 2, 3, \ldots, n\}, j \in \{1, 2, 3, \ldots, \lambda_2\}\}.$$

Besides, the graph $6K_n(N_n)$ can be considered as a $\mathsf{TS}(n; 6)$, namely $(N_n, \mathcal{B})$. Hence, $(M_m, N_n, \mathcal{T} \cup \mathcal{F} \cup \mathcal{B})$ is a $\mathsf{GDD}(m, n; \lambda_1, 6, \lambda_2)$.

Case (ii) $1 \neq h \leq k$. We first note that the construction of a $\mathsf{GDD}(6, 8; 4, 6, 4)$ is done in Lemma 3.5. By Lemma 3.4, there exists a $\mathsf{GDD}(6, 2; \lambda_1, 6. \lambda_2)$ on a vertex set $A \cup B$, namely $(A, B, \mathcal{B})$. Since $h \neq 1$ and $\lambda_1 \geq \lambda_2$, by Theorem 2.12, we can decompose the graph $\lambda_1 K_{6h}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $6\lambda_1 + 2\lambda_2$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 6\}$, $j \in \{1, 2, 3, \ldots. \lambda_1\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j} : i \in \{1, 2, 3, \ldots, 6\}, j \in \{1, 2, 3, \ldots, \lambda_1\}\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3. \ldots, \lambda_2\}\}.$$

Since $\lambda_2 < 6$ and $h \leq k$, we can apply Theorem 2.12 again to decompose the graph $6K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $\lambda_2 m + 12$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3. \ldots, \lambda_2\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, 6\}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be collections of triangles defined by

$$\mathcal{H}_1 = \{x_i + H_{i,j} : i \in \{1, 2, 3, \ldots, m\}, j \in \{1, 2, 3, \ldots, \lambda_2\}\}$$

and

$$\mathcal{H}_2 = \{y_p + H'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3, \ldots, 6\}\}.$$

Then, $(M_m, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_2)$ is a desired GDD.

Case (iii) $1 = h \leq k$. Then, $m = 12$. By Lemma 3.4, there exists a GDD$(6, 2; 0, 6, 1)$ on $A \cup B$, namely $(A, B, \mathcal{B})$. Since $\lambda_1 \geq \lambda_2$, by Theorem 2.12, we can decompose the graph $\lambda_1 K_6(A)$ into a collection of triangles $\mathcal{T}_1$ and $2(\lambda_2 - 1) + \frac{\lambda_1}{2}(6)$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{7, 8, 9, \ldots, 12\}$, $j \in \{1, 2, 3, \ldots, \frac{\lambda_1}{2}\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2 - 1\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \left\{x_i + F_{i,j} : i \in \{7, 8, 9, \ldots, 12\}, j \in \left\{1, 2, 3, \ldots, \frac{\lambda_1}{2}\right\}\right\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3, \ldots, \lambda_2 - 1\}\}.$$

Again, by Theorem 2.12, we can decompose the graph $\lambda_1 K_6(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_2$ and $\frac{\lambda_1}{2}(6) + 2\lambda_2$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 6\}$, $j \in \{1, 2, 3, \ldots, \frac{\lambda_1}{2}\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be collections of triangles defined by

$$\mathcal{H}_1 = \left\{x_i + H_{i,j} : i \in \{1, 2, 3, \ldots, 6\}, j \in \left\{1, 2, 3, \ldots, \frac{\lambda_1}{2}\right\}\right\}$$

and

$$\mathcal{H}_2 = \{y_p + H'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3, \ldots, \lambda_2\}\}.$$

Lastly, since $h \leq k$ and $\lambda_2 < 6$, we can apply Theorem 2.12 again to decompose the graph $6K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_3$ and $\lambda_2 m + 12$ 1-factors, say $G_{i,j}$ and $G'_{p,q}$ where $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, \lambda_2\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, 6\}$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be collections of triangles defined by

$$\mathcal{G}_1 = \{x_i + G_{i,j} : i \in \{1, 2, 3, \ldots, m\}, j \in \{1, 2, 3, \ldots, \lambda_2\}\}$$

and

$$\mathcal{G}_2 = \{y_p + G'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3, \ldots, 6\}\}.$$

Hence, $(M_{12}, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{G}_1 \cup \mathcal{G}_2)$ is a desired GDD. $\square$

The following theorem completes the proof of the existence of GDDs where $m \equiv 0 \pmod 6, n \equiv 2 \pmod 6$ and $n \neq 2$.

Theorem 3.7. *Let m and n be positive integers such that $m \equiv 0 \pmod 6, n \equiv 2 \pmod 6$ and $n \neq 2$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If $\lambda_1 \equiv 0 \pmod 2$ and $\lambda_1' \equiv 0 \pmod 6$, then there exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. The construction is done as usual by applying Theorem 2.3 and Lemma 3.1. First, write $\lambda_2 \equiv a \pmod 6$ where $a \in \{1, 2, 3, \ldots, 6\}$. If a is even, by Theorem 2.3, there exist a $\mathsf{TS}(m+n; (\lambda_2 - a))$, a $\mathsf{TS}(m; (\lambda_1 - a) - (\lambda_2 - a))$ and a $\mathsf{TS}(n; (\lambda_1' - 6) - (\lambda_2 - a))$. It follows from Lemma 3.1 that there exists a $\mathsf{GDD}(m, n; \lambda_1 - a, \lambda_1' - 6. \lambda_2 - a)$. Similarly, if a is odd, then there exists a $\mathsf{GDD}(m, n; \lambda_1 - (a+1), \lambda_1' - 6, \lambda_2 - a)$. Together with a $\mathsf{GDD}(m, n; a, 6, a)$ when $a \in \{2, 4, 6\}$, and a $\mathsf{GDD}(m, n; a+1, 6, a)$ when $a \in \{1, 3, 5\}$ from Lemmas 2.13 (iv), 3.5 and 3.6, we have our desired GDD. $\square$

Now, we consider GDDs with $m \equiv 3 \pmod 6$, $n \equiv 2 \pmod 6$ and $n \neq 2$. We first construct a $\mathsf{GDD}(m, n; 1, 3, 1)$ in Lemma 3.8, using a graph decomposition, then utilize this GDD to construct a GDD for any values of λ_1, λ_1' and λ_2 in Theorem 3.9.

Lemma 3.8. *Let m and n be positive integers such that $m \equiv 3 \pmod 6$, $n \equiv 2 \pmod 6$ and $n \neq 2$. There exists a* $\mathsf{GDD}(m, n; 1, 3, 1)$.

Proof. We write $m = 6h + 3$ and $n = 6k + 8$ for nonnegative integers h and k. Let $A = \{x_1, x_2, x_3\}$ and $B = \{y_1, y_2\}$. First, let $\mathcal{B} = \{\{x_1, y_1, y_2\}, \{x_2, y_1, y_2\}, \{x_3, y_1, y_2\}, \{x_1, x_2, x_3\}\}$.

Case (i) $h \leq k+1$. By Theorem 2.12, we can decompose the graph $3K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $6h+9$ 1-factors, say $F_{i,j}$ and F_p where $i \in \{1,2\}, j \in \{1,2,3\}$ and $p \in \{1,2,3,\ldots,6h+3\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j}, x_p + F_p : i \in \{1,2\}, j \in \{1,2,3\}\}, p \in \{1,2,3,\ldots,6h+3\}.$$

If $h=0$, then the construction is done here and $(M_3, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{F}_1)$ is our desired GDD. Now, assume that $h > 1$. Since $h \leq k+1$, we can apply Theorem 2.12 again to decompose the graph $K_{6h}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_2$ and five 1-factors, say $H_{1,1}, H_{2,1}, H_{1,2}, H_{2.2}$ and $H_{3.2}$. Let $\mathcal{F}_2$ be the collection of triangles defined by

$$\mathcal{F}_2 = \{y_i + H_{i,1}, x_j + H_{j,2} : i \in \{1,2\}, j \in \{1,2,3\}\}.$$

Then, $(M_m, N_n, \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{B})$ is a desired GDD.

Case (ii) $h > k+1$. Since $k \geq 0$ and $h \geq k+2$, by Theorem 2.12, we can decompose the graph $3K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and nine 1-factors, say $F_{i,j}$ and F_p where $i \in \{1,2\}$ and $j, p \in \{1,2,3\}$; and decompose the graph $K_{6h}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_2$ and $6k+11$ 1-factors, say H_i and H_j where $i \in \{1,2,3,\ldots,6h+8\}$ and $j \in \{1,2,3\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j}, x_p + F_p : i \in \{1,2\}, j, p \in \{1,2,3\}\}$$

and

$$\mathcal{F}_2 = \{y_i + H_i, x_j + H_j : i \in \{1,2,3,\ldots,6h+8\}, j \in \{1,2,3\}\}.$$

Hence, $(M_m, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD. $\square$

Theorem 3.9. *Let m and n be positive integers such that $m \equiv 3 \pmod 6$, $n \equiv 2 \pmod 6$ and $n \neq 2$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$*

and $\lambda_1' \geq \lambda_2$. *If* $\lambda_1' \equiv 0 \pmod 3$ *and* $\lambda_1' \equiv \lambda_2 \pmod 2$, *then there exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. The construction is done as usual by applying Theorem 2.3 and Lemma 3.1. We assume that $\lambda_2 \equiv b \pmod 3$ where $b \in \{0, 1, 2\}$. Then, there exist a $\mathsf{TS}(m + n; \lambda_2 - b)$ and a $\mathsf{TS}(m; (\lambda_1 - b) - (\lambda_2 - b))$. Note that $b \leq \lambda_2 \leq \lambda_1', \lambda_1' \equiv 0 \pmod 3$ and $\lambda_1' \equiv \lambda_2 \pmod 2$. Then, $\lambda_1' \geq 3b$. Hence, $(\lambda_1' - 3b) - (\lambda_2 - b) \equiv 0 \pmod 6$ and there is a $\mathsf{TS}(n; (\lambda_1' - 3b) - (\lambda_2 - b))$. By Lemma 3.1, there is a $\mathsf{GDD}(m, n; \lambda_1 - b, \lambda_1' - 3b, \lambda_2 - b)$. Together with b copies of $\mathsf{GDD}(m, n; 1, 3, 1)$ from Lemma 3.8, we obtain a desired GDD. □

The last case in this section is to construct GDDs with $m \equiv 4 \pmod 6$, $n \equiv 2 \pmod 6$ and $n \neq 2$. The main construction is shown in Theorem 3.12.

Lemma 3.10. *There exists a* $\mathsf{GDD}(4, 2; 2, 4, 1)$.

Proof. Let
$$\mathcal{B} = \{\{x_1, y_1, y_2\}, \{x_2, y_1, y_2\}, \{x_3, y_1, y_2\}, \{x_4, y_1, y_2\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}\}.$$
Then, $(M_4, N_2, \mathcal{B})$ is a $\mathsf{GDD}(4, 2; 2, 4, 1)$. □

Next, we show the existence of a $\mathsf{GDD}(m, n; 2, 4, 1)$, which will be used in our main construction.

Lemma 3.11. *Let* m *and* n *be positive integers such that* $m \equiv 4 \pmod 6$, $n \equiv 2 \pmod 6$ *and* $n \neq 2$. *There exists a* $\mathsf{GDD}(m, n; 2, 4, 1)$.

Proof. We write $m = 6h + 4$ and $n = 6k + 8$ for nonnegative integers h, k and let $A = \{x_1, x_2, x_3, x_4\}$ and $B = \{y_1, y_2\}$.

Case (i) $h \leq k + 1$. By Lemma 3.10, there is a $\mathsf{GDD}(4, 2; 2, 4, 1)$ on the vertex set $A \cup B$, namely $(A, B, \mathcal{B})$. Since $h \leq k+1$, by Theorem 2.12, we can decompose the graph $4K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $6h + 12$ 1-factors, say $F_{i,j}$ and F_p where $i \in \{1,2\}, j \in \{1,2,3,4\}$ and $p \in \{1,2,3,\ldots,6h+4\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j} : i \in \{1,2\}, j \in \{1,2,3,4\}\}$$

and

$$\mathcal{F}_2 = \{x_p + F_p : p \in \{1,2,3,\ldots,6h+4\}\}.$$

If $h = 0$, then our proof is done here and $(M_4, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ yields a desired GDD. Let $h \neq 0$. By Theorem 2.12, the graph $2K_{m-4}(M_m \setminus A)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and ten 1-factors, say H_i and $H_{p,q}$ where $i \in \{1,2\}, p \in \{1,2,3,4\}$ and $q \in \{1,2\}$. Let $\mathcal{F}_3$ and $\mathcal{F}_4$ be collections of triangles defined by

$$\mathcal{F}_3 = \{y_i + H_i : i \in \{1,2\}\}$$

and

$$\mathcal{F}_4 = \{x_p + H_{p,q} : p \in \{1,2,3,4\}, q \in \{1,2\}\}.$$

Thus, $(M_m, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4)$ is a desired GDD.

Case (ii) $h > k+1$. From Lemma 3.10, there is a GDD$(4,2;2,4,1)$ on the vertex set $A \cup B$, namely $(A, B, \mathcal{B})$. By Theorem 2.12, we can decompose the graph $4K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and 12 1-factors, say $F_{i,j}$ and F_p where $i \in \{1,2\}$ and $j, p \in \{1,2,3,4\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j}, x_p + F_p : i \in \{1,2\}, j, p \in \{1,2,3,4\}\}.$$

Since $h > k+1$, by Theorem 2.12, the graph $2K_{m-4}(M_m \setminus A)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and $6k+12$ 1-factors, say H_i and $H_{p,q}$ where $i \in \{1,2,3,\ldots,6k+8\}, p \in \{1,2,3,4\}$ and $q \in \{1,2\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{y_i + H_i : i \in \{1,2,3,\ldots,6k+8\}\}$$

and

$$\mathcal{F}_3 = \{x_p + H_{p,q} : p \in \{1,2,3,4\}, q \in \{1,2\}\}.$$

Therefore, $(M_m, N_n, \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD. □

Now, we conclude this case in the following theorem.

Theorem 3.12. *Let m and n be positive integers such that $m \equiv 4 \pmod 6$, $n \equiv 2 \pmod 6$ and $n \neq 2$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If both λ_1 and λ_1' are even and $\lambda_1' + 2\lambda_2 \equiv 0 \pmod 3$, then there exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. First, we note that λ_1' is even and $\lambda_1' - \lambda_2 \equiv 0 \pmod 3$. That is, $\lambda_1' - \lambda_2 \equiv 0$ or $3 \pmod 6$. Hence, if λ_2 is even, then $\lambda_1' - \lambda_2 \equiv 0 \pmod 6$; and if λ_2 is odd, then $\lambda_1' - \lambda_2 \equiv 3 \pmod 6$.

If λ_2 is even, by Theorem 2.3, a $\mathsf{TS}(m+n; \lambda_2)$, a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$ and a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ exist. We obtain a desired GDD by Lemma 3.1. Assume that λ_2 is odd. Thus, by Theorem 2.3, there exist a $\mathsf{TS}(m+n; \lambda_2 - 1)$, a $\mathsf{TS}(m; (\lambda_1 - 2) - (\lambda_2 - 1))$ and a $\mathsf{TS}(n; (\lambda_1' - 4) - (\lambda_2 - 1))$. Again, we can apply Lemma 3.1 to construct a $\mathsf{GDD}(m, n; \lambda_1 - 2, \lambda_1' - 4, \lambda_2 - 1)$. Together with a $\mathsf{GDD}(m, n; 2, 4, 1)$ from Lemma 3.11, we obtain our desired GDD. □

3.4 $m \equiv 0$ or $1 \pmod 3$ and $n \equiv 5 \pmod 6$

Our last section in this chapter is to consider GDDs with $m \equiv 0$ or $1 \pmod 3 \equiv 0, 1, 3$ or $4 \pmod 6$ and $n \equiv 5 \pmod 6$. From Theorem 3.2 (iv), it remains to construct a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ where $m \equiv 0$ or $3 \pmod 6$ and $n \equiv 5 \pmod 6$. The main construction is concluded in Theorem 3.18. Lemmas 3.13 - 3.15 are for small GDDs as these GDDs are too small to be constructed by the general construction in Lemma 3.16. Therefore, we construct each of them individually by slightly different techniques.

Lemma 3.13. *There exists a* GDD$(6, 11; 2, 3, 2)$.

Proof. In this case, we let $B = \{y_1, y_2, y_3\}$. First, we can apply Theorem 2.12 to decompose the graph $2K_6(M_6)$ into a collection of triangles $\mathcal{T}$ and six 1-factors, say $F_{i,j}$ where $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j} : i \in \{1, 2, 3\}, j \in \{1, 2\}\}.$$

Let $\mathcal{B} = \{\{y_1, y_2, y_3\}, \{y_1, y_2, y_3\}, \{y_1, y_2, y_3\}\}$ be a collection of triangles on B. Now, by Theorem 2.5, the graph $3K_8(N_{11} \setminus B)$ can be decomposed into 21 1-factors, say $H_{i,j}$ and $H_{p,q}$ where $i \in \{1, 2, 3, \ldots, 6\}$, $j \in \{1, 2\}$ and $p, q \in \{1, 2, 3\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_{i,j}, y_p + H_{p,q} : i \in \{1, 2, 3, \ldots, 6\}, j \in \{1, 2\}, p, q \in \{1, 2, 3\}\}.$$

Therefore, $(M_6, N_{11}, \mathcal{T} \cup \mathcal{B} \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is the desired GDD. □

Lemma 3.14. *There exists a* GDD$(12, 17; 2, 3, 2)$.

Proof. First, we let $B = \{y_1, y_2, y_3, y_4, y_5\}$. By Theorem 2.5, the graph $2K_{12}(M_{12})$ can be decomposed into 22 1-factors, say F_i and $F_{j,2}$ where $i \in \{1, 2, 3, \ldots, 17\}$ and $j \in \{1, 2, 3, 4, 5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_i, y_j + F_{j,2} : i \in \{1, 2, 3, \ldots, 17\}, j \in \{1, 2, 3, 4, 5\}.$$

We consider the graph $3K_5(B)$ as a TS$(5; 3)$, namely $(B, \mathcal{B})$. By Theorem 2.12, the graph $3K_{12}(N_{17} \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}$ and 27 1-factors, say H_i and $H_{p,q}$ where $i \in \{1, 2, 3, \ldots, 12\}$, $p \in \{1, 2, 3, 4, 5\}$ and $q \in \{1, 2, 3\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_i, y_p + H_{p,q} : i \in \{1, 2, 3, \ldots, 12\}, p \in \{1, 2, 3, 4, 5\}, q \in \{1, 2, 3\}\}.$$

Then, $(M_{12}, N_{17}, \mathcal{B} \cup \mathcal{T} \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is the GDD. □

Lemma 3.15. *There exists a* GDD$(6, 11; 1, 3, 1)$.

Proof. Let $B = \{y_1, y_2, y_3, y_4, y_5\}$. By Theorem 2.5, the graph $K_6(M_6)$ can be decomposed into five 1-factors, say F_1, F_2, F_3, F_4 and F_5. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_i : i \in \{1, 2, 3, 4, 5\}\}.$$

By Theorem 2.12, the graph $3K_6(N_{11} \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}$ and eleven 1-factors, say $H_{i,1}$ and $H_{j,2}$ where $i \in \{1, 2, 3, \ldots, 6\}$ and $j \in \{1, 2, 3, 4, 5\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + T_{i,1}, y_j + T_{j,2} : i \in \{1, 2, 3, \ldots, 6\}, j \in \{1, 2, 3, 4, 5\}\}.$$

By Theorem 2.7, the graph $3K_5(B)$ can be decomposed into six 2-factors, say $C_6, C_7, C_8, \ldots, C_{11}$. Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{y_j + C_j : j \in \{6, 7, 8, \ldots, 11\}\}.$$

Hence, $(M_6, N_{11}, \mathcal{T} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{C})$ is a GDD$(6, 11; 1, 3, 1)$. □

Now, we establish a construction of GDD$(m, n; 1, 3, 1)$ and GDD$(m, n; 2, 3, 2)$ for all $m \equiv 0 \pmod 6$ and $n \equiv 5 \pmod 6$ in Lemma 3.16. These GDDs will be used to construct GDDs for all possible λ_1, λ_1' and λ_2.

Lemma 3.16. *Let m and n be positive integers such that $m \equiv 0 \pmod 6$ and $n \equiv 5 \pmod 6$. There exist a* GDD$(m, n; 1, 3, 1)$ *and a* GDD$(m, n; 2, 3, 2)$.

Proof. We consider a construction in the following two cases.

Case (i) $m > n$. For each $\lambda \in \{1, 2\}$, by Theorem 2.12, the graph $\lambda K_m(M_m)$ can be decomposed into a collection of triangles $\mathcal{T}_1$ and λn 1-factors, say $F_{i,j}$ when

$i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{1, 2, 3, \ldots, \lambda\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j} : i \in \{1, 2, 3, \ldots, n\}, j \in \{1, 2, 3, \ldots, \lambda\}\}.$$

By Theorem 2.3, the graph $3K_n(N_n)$ can be considered as a $\mathsf{TS}(n; 3)$, namely $(N_n, \mathcal{B})$. Hence, $(M_m, N_n, \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{B})$ is a $\mathsf{GDD}(m, n; \lambda, 3, \lambda)$.

Case (ii) $m < n$. We note that a $\mathsf{GDD}(6, 11; 2, 3, 2)$, a $\mathsf{GDD}(12, 17; 2, 3, 2)$ and a $\mathsf{GDD}(6, 11; 1, 3, 1)$ are done in Lemmas 3.13 - 3.15. Here, we construct other GDDs, which are the cases when $n \geq 23$ or $(m, n, \lambda) \in \{(6, 17, 1), (12, 17, 1), (6, 17, 2)\}$. We let $B = \{y_1, y_2, y_3, y_4, y_5\}$. First, we note that $3(n-6) \geq \lambda m + 15$ if and only if $3(n-11) \geq \lambda m$. Thus, except the cases when $(m, n, \lambda) \in \{(6, 11, 2), (12, 17, 2), (6, 11, 1)\}$, we can apply Theorem 2.12 to decompose the graph $3K_{n-5}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $\lambda m + 15$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, \lambda\}$, $p \in \{1, 2, 3, 4, 5\}$ and $q \in \{1, 2, 3\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + H_{i,j} : i \in \{1, 2, 3, \ldots, m\}, j \in \{1, 2, 3, \ldots, \lambda\}\}$$

and

$$\mathcal{F}_2 = \{y_p + H'_{p,q} : p \in \{1, 2, 3, 4, 5\}, q \in \{1, 2, 3\}\}.$$

Since $m \neq 0$, by Theorem 2.12, the graph $\lambda K_m(M_m)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and 5λ 1-factors, say $F_{i,j}$ where $i \in \{1, 2, 3, 4, 5\}$ and $j \in \{1, 2, 3, \ldots, \lambda\}$. Let $\mathcal{F}_3$ be a collection of triangles defined by

$$\mathcal{F}_3 = \{y_i + F_{i,j} : i \in \{1, 2, 3, 4, 5\}, j \in \{1, 2, 3, \ldots, \lambda\}\}.$$

By Theorem 2.3, the graph $3K_5(B)$ can be considered as a $\mathsf{TS}(5; 3)$, namely $(B, \mathcal{B})$. Therefore, $(M_m, N_n, \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{B})$ is a desired GDD. □

Next, we show a construction of GDD$(m, n; 2, 3, 2)$ where $m \equiv 3 \pmod 6$ and $n \equiv 5 \pmod 6$. The construction technique used in Lemma 3.17 relies on the existence of a maximum packing in Theorem 2.9. This technique is powerful and establishes the desired GDDs in a short solution.

Lemma 3.17. *Let m and n be positive integers such that $m \equiv 3 \pmod 6$ and $n \equiv 5 \pmod 6$. There exists a* GDD$(m, n; 2, 3, 2)$.

Proof. We write $m = 6h + 3$ and $n = 6k + 5$ for nonnegative integers h and k. We give the construction depending on the values of h and k in the following four cases.

Case (i) $h \geq k$. Let $B = \{y_1, y_2, y_3\}$. We note that the graph $2K_m(M_m) \vee_2 3K_3(B)$ can be considered as a graph $2K_{6h+6}(M_m \cup B)$ together with a triangle $T_0 = \{y_1, y_2, y_3\}$. Since $h \geq k$, by Theorem 2.12, the graph $2K_{6h+6}(M_m \cup B)$ can be decomposed into a collection of triangles $\mathcal{T}_1$ and $2(6k) + 4$ 1-factors, say $F_{i,j}$ where $i \in \{1, 2, 3, \ldots, 6k + 2\}$ and $j \in \{1, 2\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j} : i \in \{1, 2, 3, \ldots, 6k + 2\}, j \in \{1, 2\}\}.$$

By Theorem 2.9, each copy of K_{6k+2} in $3K_{6k+2}(N_n \setminus B)$ can be considered as a maximum packing of order $6k + 2$, having a 1-factor as the leave. Thus, there are a total of three 1-factors, denoted by F_1, F_2 and F_3. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{y_r + F_r : r \in \{1, 2, 3\}\}.$$

Then, $(M_m, N_n, \{T_0\} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{T}_1)$ is a desired GDD.

Case (ii) $2 \leq h < k$. Let $A = \{x_1, x_2, x_3\}$ and $B = \{y_1, y_2, y_3, y_4, y_5\}$. From the previous case, there exists a GDD$(3, 5; 2, 3, 2)$ on the vertex set $A \cup B$, say $(A, B, \mathcal{B})$. Since $h \geq 2$, by Theorem 2.12, we can decompose the graph $2K_{6h}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and 16 1-factors, say $F_{i,j}$ and $F_{p,q}$ where $i \in \{1,2,3\}$, $j \in \{1,2\}$, $p \in \{1,2,3,4,5\}$ and $q \in \{3,4\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F_{p,q} : i \in \{1,2,3\}, j \in \{1,2\}, p \in \{1,2,3,4,5\}, q \in \{3,4\}\}.$$

Since $h < k$, by Theorem 2.12, the graph $3K_{6k}(N_n \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and $2(6h+3)+15$ 1-factors, say $H_{i,j}$ and $H_{p,q}$ where $i \in \{1,2,3,\dots,6h+3\}$, $j \in \{1,2\}$, $p \in \{1,2,3,4,5\}$ and $q \in \{3,4,5\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_{i,j} : i \in \{1,2,3,\dots,6h+3\}, j \in \{1,2\}\}$$

and

$$\mathcal{F}_3 = \{y_p + H_{p,q} : p \in \{1,2,3,4,5\}, q \in \{3,4,5\}\}.$$

Then, $(M_m, N_n, \mathcal{B} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{T}_1 \cup \mathcal{T}_2)$ is a desired GDD.

Case (iii) $1 = h < k$. In this case, we let $B_1 = \{y_1, y_2, y_3\}$ and $B_2 = \{y_4, y_5\}$. Note that the graph $2K_9(M_9) \vee_2 3K_3(B_1)$ can be considered as a graph $2K_{12}$ together with a triangle $T_0 = \{y_1, y_2, y_3\}$. By Theorem 2.12, we can decompose the graph $2K_{12}$ into a collection of triangles $\mathcal{T}_1$ and four 1-factors, say $F_{4,1}, F_{4,2}, F_{5,1}$ and $F_{5,2}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_p + F_{p,q} : p \in \{4,5\}, q \in \{1,2\}\}.$$

Let $\mathcal{B} = \{\{y_1, y_4, y_5\}, \{y_2, y_4, y_5\}, \{y_3, y_4, y_5\}\}$. Now, since $k \geq 2$, we can apply Theorem 2.12 to decompose the graph $3K_{6k}(N_n \setminus (B_1 \cup B_2))$ into a collection of triangles $\mathcal{T}_2$ and 33 1-factors, say $H_{i,j}$ and $H_{p,q}$ where $i \in \{1,2,3,\dots,9\}$, $j \in \{1,2\}$, $p \in \{1,2,3,4,5\}$ and $q \in \{3,4,5\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_{i,j} : i \in \{1,2,3,\dots,9\}, j \in \{1,2\}\}$$

and

$$\mathcal{F}_3 = \{y_p + H_{p,q} : p \in \{1,2,3,4,5\}, q \in \{3,4,5\}\}.$$

Then, $(M_9, N_n, \{T_0\} \cup \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD.

Case (iv) $0 = h < k$. For $k = 1$, we let $N_{11} = \{y_1\} \cup \{y_2, y_3, y_4, \ldots, y_{11}\}$. Let $\mathcal{B}_1 = \{\{x_1, x_2, x_3\}, \{x_1, x_2, y_1\}, \{x_2, x_3, y_1\}, \{x_1, x_3, y_1\}\}$. By Theorem 2.5, the graph $K_{10}(N_{11} \setminus \{y_1\})$ can be decomposed into nine 1-factors, say $F_{i,j}$ and F_p where $i \in \{1,2,3\}$, $j \in \{1,2\}$ and $p \in \{1,2,3\}$. Let $\mathcal{F}$ be a collection of triangles defined by

$$\mathcal{F} = \{x_i + F_{i,j}, y_1 + F_p : i \in \{1,2,3\}, j \in \{1,2\}, p \in \{1,2,3\}\}.$$

Note that the graph $2K_{10}(N_{11} \setminus \{y_1\})$ can be considered as a $\mathsf{TS}(10; 2)$, namely $(N_{11} \setminus \{y_1\}, \mathcal{B}_2)$. Hence, $(M_3, N_{11}, \mathcal{B}_1 \cup \mathcal{F} \cup \mathcal{B}_2)$ is a desired GDD. Now, we assume that $k \geq 2$. Let $B = \{y_1, y_2, y_3, y_4, y_5\}$. From Case (i), there exists a $\mathsf{GDD}(3, 5; 2, 3, 2)$ on the vertex set $M_3 \cup B$, namely $(M_3, B, \mathcal{B}_3)$. Since $k \geq 2$, by Theorem 2.12, the graph $3K_{6k}(N_n \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}_1$ and 21 1-factors, say $F_{i,j}$ and $F_{p,q}$ where $i \in \{1,2,3\}$, $j \in \{1,2\}$, $p \in \{1,2,3,4,5\}$ and $q \in \{3,4,5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F_{p,q} : i \in \{1,2,3\}, j \in \{1,2\}, p \in \{1,2,3,4,5\}, q \in \{3,4,5\}\}.$$

Therefore, $(M_3, N_n, \mathcal{B}_3 \cup \mathcal{T}_1 \cup \mathcal{F}_1)$ is a desired GDD. $\square$

Finally, we conclude the construction for any possible values of λ_1, λ_1' and λ_2 in the following theorem.

Theorem 3.18. *Let m and n be positive integers such that $m \equiv 0$ or $3 \pmod 6$ and $n \equiv 5 \pmod 6$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If $\lambda_1' \equiv 0$ or $3 \pmod 6$ and m, λ_1, λ_2 satisfy:*

(i) *if $m \equiv 0 \pmod 6$, then $\lambda_1 \equiv \lambda_2 \pmod 2$ and*

(ii) *if* $m \equiv 3 \pmod 6$, *then* $\lambda_2 \equiv 0 \pmod 2$,

then there exists a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. We separate the construction depending on the value of λ_2 in the following three cases. All triple systems in this proof exist by Theorem 2.3.

Case (i) $\lambda_2 \equiv 0$ or $3 \pmod 6$. Then, there exists a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$. If $m \equiv 0 \pmod 6$, then a $\mathsf{TS}(m+n; \lambda_2)$ and a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ exist. If $m \equiv 3 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 6$. Hence, there exist a $\mathsf{TS}(m+n; \lambda_2)$ and a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$. By Lemma 3.1, we have our desired GDD.

Case (ii) $\lambda_2 \equiv 1$ or $4 \pmod 6$. When $m \equiv 0 \pmod 6$, a $\mathsf{TS}(m+n; \lambda_2 - 1)$ and a $\mathsf{TS}(m; (\lambda_1 - 1) - (\lambda_2 - 1))$ exist. Since $\lambda_1 \geq \lambda_2$, we have that $\lambda_1' \geq 3$ and there is a $\mathsf{TS}(n; (\lambda_1' - 3) - (\lambda_2 - 1))$. By Lemma 3.1, there is a $\mathsf{GDD}(m, n; \lambda_1 - 1, \lambda_1' - 3, \lambda_2 - 1)$. Together with a $\mathsf{GDD}(m, n; 1, 3, 1)$ from Lemma 3.16, we have a desired GDD. When $m \equiv 3 \pmod 6$, $\lambda_2 \equiv 4 \pmod 6$. Then, there exist a $\mathsf{TS}(m+n; \lambda_2 - 4)$ and a $\mathsf{TS}(m; (\lambda_1 - 4) - (\lambda_2 - 4))$. Since $\lambda_1' \geq \lambda_2 \geq 4$ and $\lambda_1' \equiv 0$ or $3 \pmod 6$, there is a $\mathsf{TS}(n; (\lambda_1' - 6) - (\lambda_2 - 4))$. By Lemma 3.1, a $\mathsf{GDD}(m, n; \lambda_1 - 4, \lambda_1' - 6, \lambda_2 - 4)$ exists. Together with two copies of $\mathsf{GDD}(m, n; 2, 3, 2)$ from Lemma 3.17, we have a desired GDD.

Case (iii) $\lambda_2 \equiv 2$ or $5 \pmod 6$. Similar to Case (ii), a $\mathsf{GDD}(m, n; \lambda_1 - 2, \lambda_1' - 3, \lambda_2 - 2)$ exists by Lemma 3.1. Then, together with a $\mathsf{GDD}(m, n; 2, 3, 2)$ from Lemmas 3.16 and 3.17, we obtain the desired GDD. $\square$