Chapter 3
The Quantum-Classical Boundary

3.1 The Classical Limit for a Heavy Mass

In 1961. Aharonov and Bohm [26] considered the time-energy uncertainty relation. They discussed the nature of time in quantum mechanics and cleaned up many misconceptions on the time-energy uncertainty relation by using the variables determining the time of measurement. called the quantum-mechanical side of the cut which now we call the quantum-classical boundary. Aharonov and Bohm introduced these variables into the wave function, so that they are in this way led to a many-body SE. It implies that an additional observing apparatus on the classical side of the cut. with the aid of the many-body system under discussion can be observed. The probabilities for the result of such observations are determined by the wave function. which takes the form

$$\Psi = \Psi(x, y, z, t) \tag{3.1}$$

where z represents the apparatus variable on the quantum-mechanical side of the cut (which includes those describing the time of measurement). x represents the coordinate of the observed particle. and y that of the test particle. Aharonov and Bohm stated that:

1) The time of measurement was determined by an interaction between the test particle and the observed particle which was assumed to last for some interval Δt.

2) If there is a time-dependent interaction between apparatus and observed system which last for an interval Δt. then the SE will have to have a corresponding

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potential, which represents this interaction. The form of this potential will depend on where we place "the cut", z.

3) If the apparatus determining the time of interaction is taken to be on the classical side, then the potential will be a certain well defined function of time, which is nonzero only in the specified interval of length Δt. We may write this potential as

$$V(x,y,z) \rightarrow V(x,y,z(t)) = V(x,y,t). \tag{3.2}$$

4) If, on the other hand, the variable determining the time of interaction are placed on the quantum mechanical side of the cut then we cannot regard the potential as a well-defined function of time. Instead, we must write $V = V(x,y,z)$.

5) If the particles determining (or the apparatus) the time of interaction are heavy enough, then they will move in an essentially classical way, very nearly following a definite orbit, $z = z(t)$.

To the extent that this happens, we obtain, as a good approximation,

$$V(x,y,z) \approx V(x,y,z(t)). \tag{3.3}$$

To treat this problem mathematically, Aharonov and Bohm started with the SE for the whole system,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,y,z,t) = [\mathbf{H}_0+\mathbf{H}_y+\mathbf{H}_A + V(x,y,z)]\,\Psi(x,y,z,t) \tag{3.4}$$

where $\mathbf{H}_o$ represents the Hamiltonian of the observed particle, $\mathbf{H}_y$ that of the test particle, $\mathbf{H}_A$ that of the time determining variable, z (or the apparatus) and $V(x,y,z)$ represents the interaction potential.

They simplify this problem by letting the time determining variable be represented by a heavy free particle mass M, for which we have

$$\mathbf{H}_A = \frac{\mathbf{P}^2}{2M} \tag{3.5}$$

and suppose that the initial state of the time-determining variable can be represented by a wave packet narrow enough in z space, so that $\Delta t = \Delta z/|\dot{z}|$ can be made as small as necessary. This procedure is similar to those developed by Armstrong in 1957 [39]. The wave packet is

$$\Phi_0(z,t) = \sum_{P_z} C_{P_Z} \exp\left\{\frac{i}{\hbar}\left[zP_z - \frac{P_z^2}{2M}t\right]\right\} \tag{3.6}$$

where P_z is the momentum of the apparatus system. Because M is very large, the wave packet will spread very slowly, and to a good approximation. The wave packet becomes

$$\Phi_0(z,t) = \Phi(z - v_z t)\exp\left\{\frac{i}{\hbar}\left[z\bar{P}_z - \frac{(\bar{P}_z)^2}{2M}t\right]\right\} \tag{3.7}$$

where $v_z = \frac{\bar{P}_z}{M}$ is the mean velocity. $\bar{P}_z$ is the mean momentum and $\Phi(z - v_z t)$ is just a form factor for the wave packet which is, in general, a fairly regular function which varies slowly in comparison with the wavelength of the apparatus system.

$$\bar{\lambda} = h/\bar{P}_z. \tag{3.8}$$

If the interaction, $V(x, y, z)$ is neglected, a solution for the whole problem will be

$$\Psi(x, y, z, t) = \Phi_0(z, t)\psi_0(x, y, t) \tag{3.9}$$

where $\psi_0(x, y, t)$ is a solution of the equation

$$i\hbar\frac{\partial}{\partial t}\psi_0(x, y, t) = (\mathbf{H}_0 + \mathbf{H}_y)\,\psi_0(x, y, t). \tag{3.10}$$

When this interaction is taken into account, the general solution will, take the form

$$\Psi(x, y, z, t) = \sum_n C_n \Phi_n(z, t)\psi_n(x, y, t) \tag{3.11}$$

where C_n is the coefficients of the expansion. The sum is taken over the respective eigenfunctions, $\Phi_n(z, t)$ and $\psi_n(x, y, t)$ of $\mathbf{H}_A$ and $(\mathbf{H}_0 + \mathbf{H}_y)$ respectively.

6) If the mass M, of the time determining particle is great enough, so that the potential $V(x,y,z)$ does not significant variation in the wave-length, $\lambda = \hbar/\bar{P}_z$, then, as is well known, the adiabatic approximation will be applied. In this case, one can obtain a simple solution, consisting of a single product, even when interaction is taken into account. Aharonov and Bohm obtain the solution in the form

$$\Psi(x,y,z,t) = \Phi_0(z,t)\psi(x,y,z,t). \tag{3.12}$$

When this function is substituted into the SE, Eq.(3.4), the result is

$$i\hbar\frac{\partial}{\partial t}\psi(x,y,z,t) = \left(\mathbf{H}_0+\mathbf{H}_y + V(x,y,z) - \frac{\hbar^2}{M}\frac{\partial}{\partial z}\ln\Phi_0\frac{\partial}{\partial z} - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z,t). \tag{3.13}$$

If M is large and if the potential dose not vary very rapidly as a function of z, the last term on the right-hand side of Eq.(3.13) in the above equation can be neglected, if $V(x,y,z)$ varies very rapidly, then $\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2}\psi_0(x,y,z,t)$ will not be negligible, even when M is large. Moreover,

$$\frac{\partial}{\partial z}\ln\Phi_0 = \frac{i}{\hbar}\left[\bar{P}_z + \hbar\frac{\partial}{\partial z}\ln\Phi(z-v_zt)\right]. \tag{3.14}$$

Because $\Phi(z-v_zt)$ does not vary significantly in a wave-length, this term also can be neglected in the above equation, and we obtains

$$i\hbar\frac{\partial}{\partial t}\psi(x,y,z,t) = \left(\mathbf{H}_0+\mathbf{H}_y + V(x,y,z) - i\hbar v_z\frac{\partial}{\partial z}\right)\psi(x,y,z,t). \tag{3.15}$$

Aharonov and Bohm then make the substitution, $z - v_zt = u$ and

$$\psi'(x,y,u,t) = \psi(x,y,z,t) = \psi(x,y,u+v_zt,t). \tag{3.16}$$

With the relation

$$\frac{\partial\psi'}{\partial t} = \frac{\partial\psi}{\partial t} + v_z\frac{\partial\psi}{\partial z}, \tag{3.17}$$

we have

$$i\hbar\frac{\partial}{\partial t}\psi'(x,y,u,t) = [\mathbf{H}_0+\mathbf{H}_y + V(x,y,u+v_zt)]\,\psi(x,y,u+vt,t). \tag{3.18}$$

Note that this equation does not contain derivatives of u, so that u can be given a definite value in it.

The complete wave function is, of course, obtained by multiplying $\psi'(x, y, u, t)$ by $\Phi(z - v_z t) = \Phi(u)$. Now, this was assumed to be a narrow packet centering at $u = 0$, such that the spread of u can be neglected. As a result, we can write $u = 0$ in the above equation. The result is

$$i\hbar\frac{\partial}{\partial t}\psi'(x, y, u = 0, t) = [\mathbf{H}_0 + \mathbf{H}_y + V(x, y, v_z t)]\,\psi(x, y, t). \tag{3.19}$$

In this way, we have obtained the SE for x, y, with the appropriate time-dependent potential $V(x, y, v_z, t)$, the relationship between the time parameter t and the time determining variable z being, in this case, $t = z/v_z$.

Above is a discussion as the same what Mandelstamm and Tamm had done in 1945 [40] who had formulated for the justification of the time-energy uncertainty relationship. Mandelstamm and Tamm considered an arbitrary operator $\mathbf{A}$, which is a function of the time (e.g., the location of the needle on a clock dial or the position of a free particle in motion) and which can therefore be used to indicate time. If $\Delta A = \sqrt{\langle(\mathbf{A} - \langle\mathbf{A}\rangle)^2\rangle}$ is the uncertainty in $\mathbf{A}$, then the uncertainty in time is

$$\Delta t = \frac{\Delta A}{\left|\left\langle\dot{\mathbf{A}}\right\rangle\right|}, \tag{3.20}$$

provided that $\dot{\mathbf{A}}$ does not change significantly during the time period, Δt, and that $\Delta\dot{A}/\left|\left\langle\dot{\mathbf{A}}\right\rangle\right|$ is negligible. From the relation

$$\Delta A\Delta E \geq |\langle\mathbf{A}, \mathbf{H}\rangle| = \hbar\left|\left\langle\dot{\mathbf{A}}\right\rangle\right|, \tag{3.21}$$

where $\mathbf{H}$ represents the Hamiltonian of the isolated system and $\Delta E = \sqrt{\langle(\mathbf{H} - \langle\mathbf{H}\rangle)^2\rangle}$ is the uncertainty in energy of the system. We obtain the time-energy uncertainty relation

$$\frac{\Delta A}{\left|\left\langle\dot{\mathbf{A}}\right\rangle\right|}\Delta E = \Delta t\Delta E \geq \hbar. \tag{3.22}$$