CHAPTER I

PRELIMINARIES

Let $\boldsymbol{N}$, $\boldsymbol{Z}$ and $\boldsymbol{R}$ denote the set of all positive integers, the set of all integers and the set of all real numbers. For $n \in \boldsymbol{N}$, let the notation $(\boldsymbol{Z}_n, \cdot)$ denote the multiplicative semigroup of integers modulo n and for $x \in \boldsymbol{Z}$, let $\bar{x} \in \boldsymbol{Z}_n$ be the equivalence class modulo n containing x.

Let S be a semigroup. If S has a zero 0 and $xy = 0$ for all $x, y \in S$, we call S a *zero semigroup*. S is said to be *regular* if for every $a \in S$, $a = axa$ for some $x \in S$. A nonempty subset A of S is called a *left* [right] *ideal* of S if $SA \subseteq A$ $[AS \subseteq A]$. We call S a *left* [right] *simple semigroup* if S is the only left [right] ideal of S. If S has a zero 0, $S^2 \neq \{0\}$ and S and $\{0\}$ are the only left [right] ideals of S, then S is said to be *left* [right] *0-simple*. A nonempty subset Q of S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. We call a nonempty subset B of S a *bi-ideal* of S if $BS^1B \subseteq B$ where $S^1 = S$ if S has an identity and S^1 is the semigroup S with an identity adjoined if S does not have an identity.

The following statements hold clearly.

(1) If S is commutative, then a nonempty subset Q of S is a quasi-ideal of S if and only if $SQ \subseteq Q$.

(2) If S is commutative, then a nonempty subset B of S is a bi-ideal of S if and only if $S^1B^2 \subseteq B$.

(3) If S has a zero 0, then every quasi-ideal and every bi-ideal of S contains 0.

(4) If S is commutative, S has a zero 0, B is a nonempty subset of S containing 0 and $B^2 = \{0\}$, then B is a bi-ideal of S.

(5) If S has an identity and Q is a quasi-ideal of S containing a unit of S, then $Q = S$.

(6) If S has an identity and B is a bi-ideal of S containing a unit of S, then $B = S$.

Quasi-ideals are a generalization of left ideals and right ideals and it is given in [11] that every quasi-ideal of a semigroup is a bi-ideal. However, a bi-ideal of a semigroup need not be a quasi-ideal. Let ***BQ*** denote the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. The following three theorems show some kinds of semigroups belonging to ***BQ***.

Theorem 1.1 ([6], S. Lajos). *Every regular semigroup belongs to the class* ***BQ***.

Theorem 1.2 ([5], K.M. Kapp). *Every left simple semigroup and every right simple semigroup belongs to* ***BQ***.

Theorem 1.3 ([5], K.M. Kapp). *Every left 0-simple semigroup and every right 0-simple semigroup belongs to* ***BQ***.

The class ***BQ*** does not contain only these kinds of semigroups. A zero semigroup containing more than one element is an obvious example.

Let S be a semigroup. The intersection of any set of quasi-ideals of S is either empty or a quasi-ideal of S. This is also true for bi-ideals. For $\phi \neq A \subseteq S$, the *quasi-ideal of S generated by A* is the intersection of all quasi-ideals of S containing A which is denoted by $(A)_q$. The *bi-ideal of S generated by $A \subseteq S$* with $A \neq \phi$ is defined similarly and it is denoted $(A)_b$. For $x_1, x_2, \ldots, x_n \in S$, let $(x_1, x_2, \ldots, x_n)_q$ and $(x_1, x_2, \ldots, x_n)_b$ denote $(\{x_1, x_2, \ldots, x_n\})_q$ and $(\{x_1, x_2, \ldots, x_n\})_b$, respectively. The following results are known.

Theorem 1.4 ([2], page 85). *For a nonempty subset A of S, $(A)_q = S^1A \cap AS^1$.*

Theorem 1.5 ([2], page 84). *For a nonempty subset A of S, $(A)_b = AS^1A \cup A$.*

Then for $x \in S$, $(x)_q = S^1x \cap xS^1$ and $(x)_b = xS^1x \cup \{x\}$.

A characterization of semigroups in ***BQ*** was give by J. Calais in [1] as follows:

Theorem 1.6 ([1], J.Calais). *Let S be a semigroup. Then* $S \in \boldsymbol{BQ}$ *if and only if* $(x, y)_q = (x, y)_b$ *for all* $x, y \in S$.

The characterization given in Theorem 1.6 is not practical to use to determine whether a given semigroup belongs to ***BQ***.

The following theorem was introduced in [8] and the detail of a proof was given by S. Ritkeao in [9]. This theorem shows that there are exactly 15 types of multiplicative interval semigroups on ***R***.

Theorem 1.7 ([8], K.R. Pearson). *A subset S of **R** is a multiplicative interval semigroup on **R** if and only if S is one of the following types :*

(1) $\boldsymbol{R}$, (2) $\{0\}$, (3) $\{1\}$, (4) $(0, \infty)$, (5) $[0, \infty)$,

(6) (a, ∞) where $a \geq 1$,

(7) $[a, \infty)$ where $a \geq 1$,

(8) $(0, b)$ where $0 < b \leq 1$,

(9) $(0, b]$ where $0 < b \leq 1$,

(10) $[0, b)$ where $0 < b \leq 1$,

(11) $[0, b]$ where $0 < b \leq 1$,

(12) (a, b) where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(13) $(a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(14) $[a, b)$ where $-1 < a < 0 < a^2 < b \leq 1$,

(15) $[a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$.

There are exactly 6 types of additive interval semigroups. This was proved by K. Palasri in [7].

Theorem 1.8 ([7], K. Palasri). *A subset S of **R** is an additive interval semigroup on **R** if and only if S is one of the following types:*

(1) $\{0\}$, (2) $\boldsymbol{R}$,

(3)	(a, ∞) *where* $a \geq 0$,	*(4)*	$[a, \infty)$ *where* $a \geq 0$,
(5)	$(-\infty, b)$ *where* $b \leq 0$,	*(6)*	$(-\infty, b]$ *where* $b \leq 0$.

For $n \in N$, n is said to be *square-free* if n is not divisible by the square of any integer greater than 1. G. Ehrlich has proved the following result in [4].

Theorem 1.9 ([4], G. Ehrlich). *For any positive integer n, the semigroup* $(\boldsymbol{Z_n}, \cdot)$ *is regular if and only if n is square-free.*

Let X be a set. Let

P_X = the partial transformation semigroup on X,

T_X = the full transformation semigroup on X,

I_X = the one-to-one partial transformation semigroup on X (the symmetric inverse semigroup on X),

M_X = the semigroup of one-to-one transformations of X,

E_X = the semigroup of onto transformations of X and

G_X = the symmetric group on X .

P_X , T_X , I_X and G_X are standard transformation semigroups and all of them are regular for any cardinality of X. By Theorem 1.1, P_X , T_X , I_X , $G_X \in BQ$ for every cardinality of X. M_X and E_X are also standard transformation semigroups and each of them is the symmetric group on X if X is finite. Then M_X , $E_X \in BQ$ if X is finite.

Let I be an interval on $\boldsymbol{R}$ with $|I| > 1$. Let C_I and D_I denote the semigroup of continuous functions and the semigroup of differentiable functions of I into itself under usual topology on I. Then C_I and D_I are subsemigroups of T_I.