Chapter II

Theory

The array of data obtained from the caliper tool can be interpreted to identify locations of casing damage and scale precipitation. As mentioned before, an algorithm currently used to process the data is the vector sum, and we propose a new algorithm called the ellipse fit. This chapter discusses the theory involved in the two methods.

In general, the actual casing shape may take any form due to different forces acting upon the casing. In this study, we assume that the casing shape is either circular or oval.

2.1 The Vector Sum

In this method, each measurement of distance (or radius) from the sensor of the tool to the inner surface of the casing is considered as a vector. As being rotated to measure the radii, the tool takes 72 measurements at each depth. These 72 radii are equivalent to 72 vectors (vector 1, vector 2, vector 3, … , and vector 72). Each vector represents the distance from the sensor to the casing wall and the angle θ of such distance. It is inconvenient to illustrate 72 vectors in a figure, thus it is assumed that the tool measures only 5 points or radii as shown in Fig. 2.1.

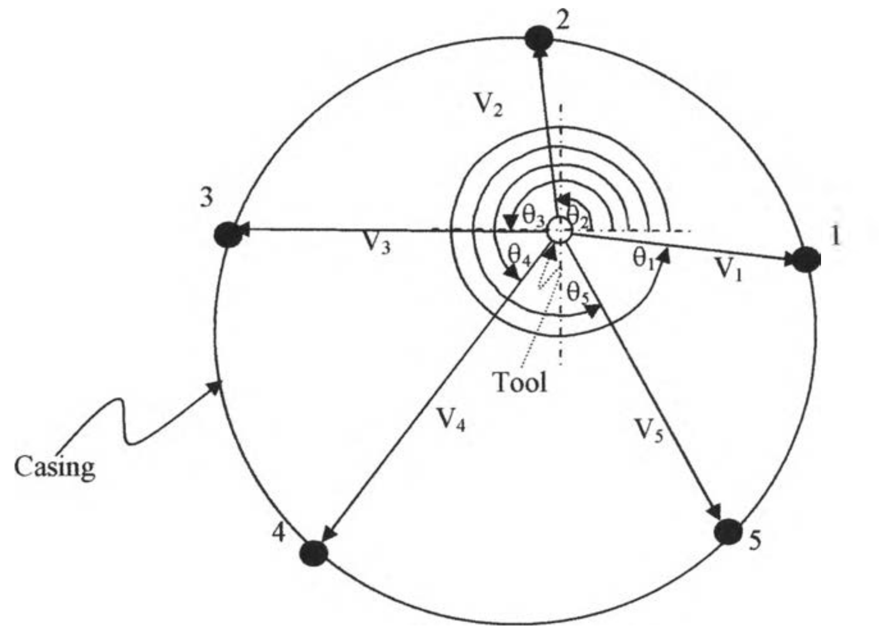

Figure 2.1: The original measurements of radii.

The principles of the vector sum are based on the fact that

1. The summation of all vectors originating from the center of a circle to its perimeter equates to zero, i.e., Σ vector = 0.

2. The casing is round; thus, the measured radii lie on a circle.

3. The tool measuring the radii may not be at the center of a circle (and the center of the circle is the center of the casing).

As the tool is not at the center of the casing, this effect causes an eccentricity. The vector sum plays a role to eliminate this eccentricity.

The method to find the reference point which is the center of the circle or the casing can be described as follows:

1. To determine the center of the circle, the tool is imaginarily moved to center of the circle as shown in Fig. 2.2. The tool center is moved such that the sum of all the vectors is zero. The radii and angles measured from the corrected center to the casing wall are changed to new values.

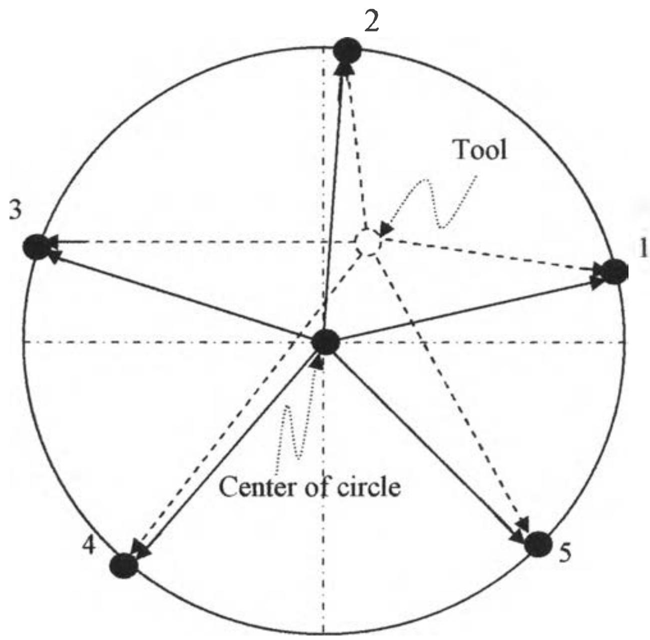

Figure 2.2: The corrected center after applying the vector sum method.

2. From the measurement at position 1 (shown in Fig. 2.3), the summation of vectors can be expressed as

$$V_{C1}+V_{A1}=V_{B1}$$

or

$$V_{C1}+V_{A1}-V_{B1}=0 \qquad (2.1)$$

where

V_{B1} is vector of coordinate x-y from the original center to position 1.

V_{A1} is vector of coordinate x-y from the corrected center to position 1.

V_{C1} is a correcting vector.

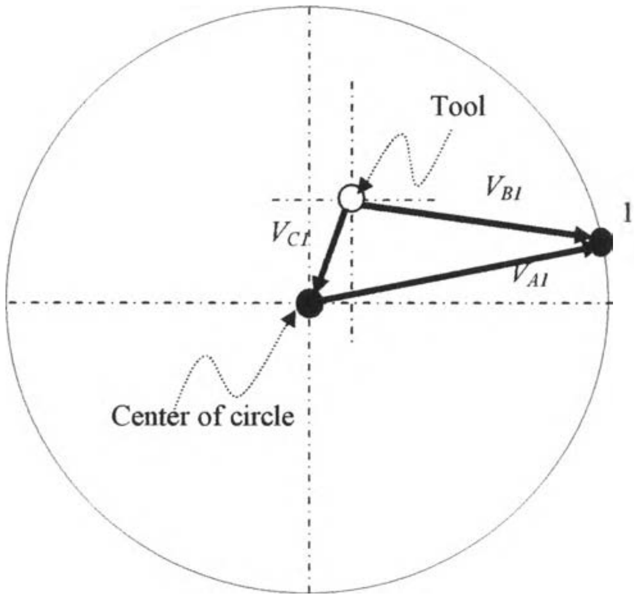

Figure 2.3: The three vectors, V_{B1}, V_{A1} and V_{C1} at position 1.

3. For a measurement at location i,

$$V_{Ci} + V_{Ai} = V_{Bi}$$

or

$$V_{Ci} + V_{Ai} - V_{Bi} = 0 \qquad (2.2)$$

The summation of all vectors for every position from position 1 to position 72 is shown in Fig. 2.4.

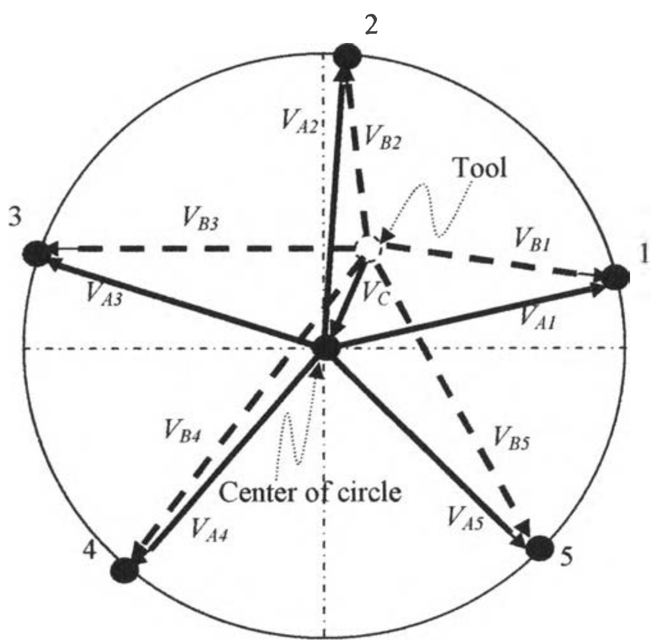

Figure 2.4: The three vectors, V_{Bi}, V_{Ai} and V_{Ci} at each position.

All vectors are collected and equated to zero.

$$\sum_{i=1}^{72} V_{Ci} + \sum_{i=1}^{72} V_{Ai} - \sum_{i=1}^{72} V_{Bi} = 0 \qquad (2.3)$$

where

$\sum_{i=1}^{72} V_{Ci}$ is the summation of the correcting vectors from positions 1 to 72.

$\sum_{i=1}^{72} V_{Ai}$ is the summation of the corrected vectors from positions 1 to 72.

$\sum_{i=1}^{72} V_{Bi}$ is the summation of the original vectors from positions 1 to 72.

Then,

$$\sum_{i=1}^{72} V_{Ci} + \sum_{i=1}^{72} V_{Ai} = \sum_{i=1}^{72} V_{Bi} \qquad (2.4)$$

The summation of the corrected vectors is zero. Thus,

$$\sum_{i=1}^{72} V_{Ai} = 0 \qquad (2.5)$$

Substituting Eq. 2.5 into Eq. 2.4

$$\sum_{i=1}^{72} V_{Ci} + 0 = \sum_{i=1}^{72} V_{Bi} \qquad (2.6)$$

Since V_C is the same vector for every position

$$\sum_{i=1}^{72} V_{Ci} = 72\ V_{Ci} \qquad (2.7)$$

Substituting Eq. 2.7 into Eq. 2.6, thus

$$72V_C + 0 = \sum_{i=1}^{72} V_{Bi} \qquad (2.8)$$

$$72V_C = \sum_{i=1}^{72} V_{Bi} \qquad (2.9)$$

$$V_C = \frac{\sum_{i=1}^{72} V_{Bi}}{72} \qquad (2.10)$$

We can write Eq. 2.10 in term of x and y components as

$$V_{CX} = \frac{\sum_{i=1}^{72} V_{BXi}}{72} \qquad (2.11)$$

$$V_{CY} = \frac{\sum_{i=1}^{72} V_{BY\,i}}{72} \tag{2.12}$$

where

V_{CX} is the x component of the correcting vector.

V_{CY} is the y component the correcting vector.

$\sum_{i=1}^{72} V_{BX\,i}$ is the summation of the x component of the original vectors.

$\sum_{i=1}^{72} V_{BY\,i}$ is the summation of the y component of the original vectors.

And

$$\sum_{i=1}^{72} V_{BX\,i} = \sum_{i=1}^{72} R_i \cos\theta_i \tag{2.13}$$

$$\sum_{i=1}^{72} V_{BY\,i} = \sum_{i=1}^{72} R_i \sin\theta_i \tag{2.14}$$

where

R_i is the radius measurement at location i.

θ_i is the angle of radius measurement at location i.

Substituting Eq. 2.13 into Eq. 2.11 and substituting Eq. 2.14 into Eq. 2.12, we get

$$V_{CX} = \frac{\sum_{i=1}^{72} R_i \cos\theta_i}{72} \tag{2.15}$$

$$V_{CY} = \frac{\sum_{i=1}^{72} R_i \sin\theta_i}{72} \tag{2.16}$$

Then, the angle of the correcting vector V_C can be computed from

$$\tan\beta = \frac{V_{CY}}{V_{CX}} = \frac{\sum_{i=1}^{72} V_{BY\,i}}{\sum_{i=1}^{72} V_{BX\,i}} = \frac{\sum_{i=1}^{72} R_i \sin\theta_i}{\sum_{i=1}^{72} R_i \cos\theta_i} \tag{2.17}$$

where β is the angle of the correcting vector.

Finally, we can calculate the corrected radii by

$$R_{CXi} = R_i \cos\theta_i - V_{CX} \tag{2.18}$$

$$R_{CYi} = R_i \sin\theta_i - V_{CY} \tag{2.19}$$

$$R_{Ci} = \sqrt{(R_{CXi})^2 + (R_{CYi})^2} \tag{2.20}$$

where

R_{CY} is the corrected radii in y axis.

R_{CX} is the corrected radii in x axis.

R_C is the corrected radii.

To determine the corrected radii of the casing, Eq. 2.15 and Eq. 2.16 are first calculated. Then, Eq. 2.20 is used to compute the new radii. We can use the new radii to find out the location of the damage and scale precipitation as shown in Chapters 3 and 4.

2.2 Ellipse Fit

When data are missing or casing is elliptic, the vector sum algorithm is no longer valid (as discussed in details in Chapter 4). A new algorithm, which is the least squares fitting of ellipses, is proposed to correct these problems. The least squares fitting of ellipses is a mathematical method to find the best fitting ellipse for a given set of points by minimizing the sum of the squares of the offsets of the points from the curve.

For the least squares fitting of ellipses, a general conic equation of an implicit second order polynomial is expressed as:

$$F(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{2.21}$$

where

a, b, c, d, e, f are the coefficients of the ellipse.

(x, y) are coordinate system.

$F(x,y)$ is called the "algebraic distance" of the point (x,y) to the specified conic.

Eq. 2.21 can be expressed in a vector form as

$$F_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{a} = 0 \tag{2.22}$$

where

$\boldsymbol{a} = [\, a, b, c, d, e, f]^{T}$

$\boldsymbol{x} = [x^2, xy, y^2, x, y, 1]$

In order to fit a conic, the algebraic distance over the set of n data points is minimized using the least squares approach:

$$\min \sum_{i=1}^{n} F(x_i, y_i)^2 = \min \sum_{i=1}^{n} (F_{\boldsymbol{a}}(\boldsymbol{x}_i))^2 \tag{2.23}$$

The elliptic coefficients a, b, c, d, e and f in Eq. 2.22 can be determined by the least squares method with an ellipse-specific constraint which Pilu *et.al.*[3, 4] introduced. This is an equality constraint, $4ac - b^2 = 1$. With this constraint, Eq. 2.23 can be rewritten as

$$\min E = \| \boldsymbol{Da} \|^2 \text{, subject to } \boldsymbol{a}^T\boldsymbol{Ca} = 1 \tag{2.24}$$

where

E is the sum of the squares of errors where each error is the difference between the coordinate (x_i, y_i) of the data point and the coordinate of the ellipse fit.

$\boldsymbol{D}$ is called "the design matrix" of the size n x 6 matrix and

$$\boldsymbol{D} = \begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n & 1 \end{bmatrix}$$

C is called "the constraint matrix" of the size 6 x 6 matrix and

$$C = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the definition of matrix norm, Eq. 2.24 is rearranged as

$$\| \boldsymbol{Da} \|^2 = \left(\sqrt{(ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f)^2} \right)^2 + \ldots \\ \left(\sqrt{(ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f)^2} \right)^2 + \ldots \\ \left(\sqrt{(ax_n^2 + bx_ny_n + cy_n^2 + dx_n + ey_n + f)^2} \right)^2 \tag{2.25}$$

The derivative of $\| \boldsymbol{Da} \|^2$ with respect to the coefficients a, b, c, d, e and f can be written as

$$\frac{\partial}{\partial a} \| \boldsymbol{Da} \|^2 = 2x_1^2 (F(x,y))_1 + \ldots + 2x_n^2 (F(x,y))_n \tag{2.26}$$

$$\frac{\partial}{\partial b} \| \boldsymbol{Da} \|^2 = 2x_1y_1 (F(x,y))_1 + \ldots + 2x_ny_n (F(x,y))_n \tag{2.27}$$

$$\frac{\partial}{\partial c} \| \boldsymbol{Da} \|^2 = 2y_1^2 (F(x,y))_1 + \ldots + 2y_n^2 (F(x,y))_n \tag{2.28}$$

$$\frac{\partial}{\partial d} \| \boldsymbol{Da} \|^2 = 2x_1 (F(x,y))_1 + \ldots + 2x_n (F(x,y))_n \tag{2.29}$$

$$\frac{\partial}{\partial e} \| \boldsymbol{Da} \|^2 = 2y_1 (F(x,y))_1 + \ldots + 2y_n (F(x,y))_n \tag{2.30}$$

$$\frac{\partial}{\partial f} \| \boldsymbol{Da} \|^2 = 2(F(x,y))_1 + \ldots + 2(F(x,y))_n \tag{2.31}$$

where

$$(F(x,y))_1 = ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f$$

$$(F(x,y))_n = ax_n^2 + bx_ny_n + cy_n^2 + dx_n + ey_n + f$$

Or presented in term of

$$\frac{\partial}{\partial \boldsymbol{a}} \| \boldsymbol{Da} \|^2 = 2\boldsymbol{D}^T \boldsymbol{Da} \tag{2.32}$$

where

$$\boldsymbol{Da} = \begin{bmatrix} ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f \\ \vdots \\ ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f \\ \vdots \\ ax_n^2 + bx_ny_n + cy_n^2 + dx_n + ey_n + f \end{bmatrix}$$

$$\boldsymbol{D}^T = \begin{bmatrix} x_1^2 & \cdots & x_i^2 & \cdots & x_n^2 \\ x_1y_1 & \cdots & x_iy_i & \cdots & x_ny_n \\ y_1^2 & \cdots & y_i^2 & \cdots & y_n^2 \\ x_1 & \cdots & x_i & \cdots & x_n \\ y_1 & \cdots & y_i & \cdots & y_n \\ 1 & \cdots & 1 & \cdots & 1 \end{bmatrix}$$

To determine the minimum, the partial derivative of the sum of squares of errors, E, with respect to the coefficients of ellipse, $\boldsymbol{a}$, is set to equal to zero as:

$$\frac{\partial}{\partial \boldsymbol{a}} E = 0 \tag{2.33}$$

Substituting Eq. 2.32 into Eq. 2.33

$$\frac{\partial}{\partial \boldsymbol{a}} \| \boldsymbol{Da} \|^2 = 2\boldsymbol{D}^T \boldsymbol{Da} = 0 \tag{2.34}$$

Applying the Lagrange multiplier, λ, to the constraint $\boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} = 1$, we have

$$\lambda\left(\boldsymbol{a}^T \boldsymbol{C}\boldsymbol{a} - 1\right) = 0 \tag{2.35}$$

But $\boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} = 1$ is equivalent to $4ac - b^2 = 1$. Then,

$$\lambda\left(4ac - b^2 - 1\right) = 0 \tag{2.36}$$

Differentiating Eq. 2.36 with respect to the coefficients a, b, and c, we obtain

$$\frac{\partial}{\partial a}\left[\lambda\left(4ac - b^2 - 1\right)\right] = 4\lambda c \tag{2.37}$$

$$\frac{\partial}{\partial b}\left[\lambda\left(4ac - b^2 - 1\right)\right] = -2\lambda b \tag{2.38}$$

$$\frac{\partial}{\partial c}\left[\lambda\left(4ac - b^2 - 1\right)\right] = 4\lambda a \tag{2.39}$$

The derivatives can be written as:

$$\frac{\partial}{\partial \boldsymbol{a}}\left[\lambda\left(\boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} - 1\right)\right] = 2\lambda\begin{bmatrix} 2c \\ -b \\ 2a \end{bmatrix} = 2\lambda\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 2\lambda\boldsymbol{C}\boldsymbol{a} \tag{2.40}$$

Substituting Eq. 2.34 and Eq. 2.40 into Eq. 2.24, we have

$$2\,\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{a} - 2\,\lambda\boldsymbol{C}\boldsymbol{a} = 0 \tag{2.41}$$

$$\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{a} = \lambda\boldsymbol{C}\boldsymbol{a} \tag{2.42}$$

$$\boldsymbol{S}\boldsymbol{a} = \lambda\boldsymbol{C}\boldsymbol{a} \tag{2.43}$$

where

$\boldsymbol{S} = \boldsymbol{D}^T\boldsymbol{D}$, called "the scatter matrix" of the size 6 x 6 and

$$
\boldsymbol{S} = \begin{bmatrix}
S_{x^4} & S_{x^3y} & S_{x^2y^2} & S_{x^3} & S_{x^2y} & S_{x^2} \\
S_{x^3y} & S_{x^2y^2} & S_{xy^3} & S_{x^2y} & S_{xy^2} & S_{xy} \\
S_{x^2y^2} & S_{xy^3} & S_{y^4} & S_{xy^2} & S_{y^3} & S_{y^2} \\
S_{x^3} & S_{x^2y} & S_{xy^2} & S_{x^2} & S_{xy} & S_{x} \\
S_{x^2y} & S_{xy^2} & S_{y^3} & S_{xy} & S_{y^2} & S_{y} \\
S_{x^2} & S_{xy} & S_{y^2} & S_{x} & S_{y} & S_{1}
\end{bmatrix}
$$

and

$$
S_{x^a y^b} = \sum_{i=1}^{n} x_i^a y_i^b
$$

Eq. 2.43 is solved by a generalized eigensystem. Six pairs of eigenvalues and eigenvectors are obtained. The eigenvector a_k corresponding to the smallest positive eigenvaule is selected. Then, the solutions of the minimization problem of Eq. 2.24, which are the coefficients a, b, c, d, e and f of ellipse, are calculated. After the coefficients are obtained, the ellipse can be drawn.

2.2.1 The coefficients *a*, *b*, *c*, *d*, *e* and *f* of ellipse

Since Eq. 2.43 is a generalized eigenvalue problem, there are many methods to solve for the eigenvalues and eigenvectors. Here is an example of how to solve the eigensystem using Cholesky decomposition.

From Eq. 2.43,

$$
\boldsymbol{Sa} = \lambda \boldsymbol{Ca} \tag{2.43}
$$

Using Cholesky decomposition, the matrix $\boldsymbol{S}$ can be rewritten as

$$
\boldsymbol{S} = \boldsymbol{L}\,\boldsymbol{L}^{\mathrm{T}} \tag{2.44}
$$

where

$\boldsymbol{L}$ is a lower triangular matrix.

$\boldsymbol{L}^{\mathrm{T}}$ is a transpose of the lower triangular matrix.

Substituting Eq. 2.44 into Eq. 2.43, we obtain

$$\boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{a} = \lambda \boldsymbol{C}\boldsymbol{a} \tag{2.45}$$

Multiplying $\boldsymbol{L}^{-1}$ both side of Eq. 2.45, we get

$$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{a} = \lambda\, \boldsymbol{L}^{-1}\boldsymbol{C}\boldsymbol{a} \tag{2.46}$$

Multiplying ($(\boldsymbol{L}^{-1})^{\mathrm{T}}\, \boldsymbol{L}^{\mathrm{T}}$) on the right side of Eq. 2.46, it becomes

$$(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{a}) = \lambda\, \boldsymbol{L}^{-1}\boldsymbol{C}\,(\, (\boldsymbol{L}^{-1})^{\mathrm{T}}\, \boldsymbol{L}^{\mathrm{T}}\,)\boldsymbol{a} \tag{2.47}$$

Rearranging Eq. 2.47, we have

$$(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{a}) = \lambda\, (\boldsymbol{L}^{-1}\boldsymbol{C}\,(\boldsymbol{L}^{-1})^{\mathrm{T}}\,)\,(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{a}) \tag{2.48}$$

or

$$\boldsymbol{V} = \lambda\, \boldsymbol{E}\, \boldsymbol{V} \tag{2.49}$$

where

$\boldsymbol{V} = (\boldsymbol{L}^{\mathrm{T}}\boldsymbol{a})$

$\boldsymbol{E} = (\boldsymbol{L}^{-1}\boldsymbol{C}\,(\boldsymbol{L}^{-1})^{\mathrm{T}}\,)$

For actual implementation, the method to evaluate the coefficients *a*, *b*, *c*, *d*, *e* and *f* of ellipse is summarized below:

1. Calculate the matrix $\boldsymbol{D}$ by transforming the radius measurements to (x, y) coordinates.
2. Calculate the scatter matrix $\boldsymbol{S}$.
3. Calculate the lower triangular matrix $\boldsymbol{L}$.
4. Calculate the inverse matrix $\boldsymbol{L}^{-1}$.
5. Calculate the matrix $\boldsymbol{E}$.
6. Calculate the eigenvalues (λ) and eigenvectors ($\boldsymbol{V}$) by Jacobi transformation of the matrix $\boldsymbol{E}$.

7. Calculate the unnormalized eigenvectors (***a***) by

$$\boldsymbol{a} = (\boldsymbol{L}^{-1})^{T} \boldsymbol{V} \tag{2.50}$$

8. Calculate the unit eigenvectors which have length of 1. To obtain unit eigenvectors, each vector is divided by the square root of the sum of the squares of their components.
9. Select the positive eigenvalue. Then, the eigenvector corresponding to the eigenvalue is obtained.

2.2.2 Ellipticity

The change in ellipticity of the casing can be used to indicate the change in casing shape as a result of rock stress and strain. To calculate the semimajor and semiminor axes of an ellipse, we must convert the general conic equation to the standard form of ellipse equation. The standard ellipse equation is expressed as

$$(x''/A)^2 + (y''/B)^2 = 1 \tag{2.51}$$

where

A is the semimajor axis.

B is the semiminor axis.

If we have an ellipse such as the one shown in Fig. 2.5, there is a need to (1) rotate the x and y axes such that these axes become horizontal and vertical, respectively, and (2) shift the horizontal and vertical axes such that the center of the ellipse is at the origin.

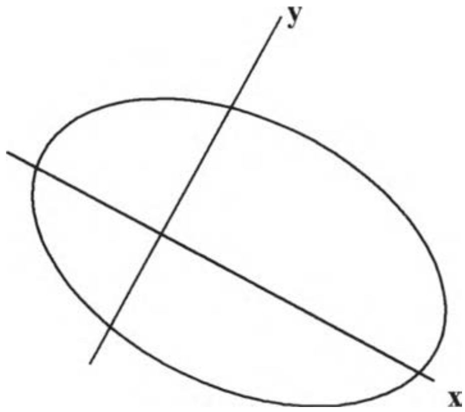

Figure 2.5: An ellipse of which orientation is not aligned with horizontal and vertical axes.

Since the x-y axes of Eq. 2.21 takes an angle θ with the x′-y′ axes, we must rotate the x-y axes to x′-y′ axes as shown in Fig. 2.6. Then, we need to shift the x′-y′ axes to the x″-y″ axes in order to move the center of the ellipse to the origin as shown in Fig. 2.7. The semimajor and semiminor axes of the ellipse lie on the x″-y″ axes of the rectangular coordinate system.

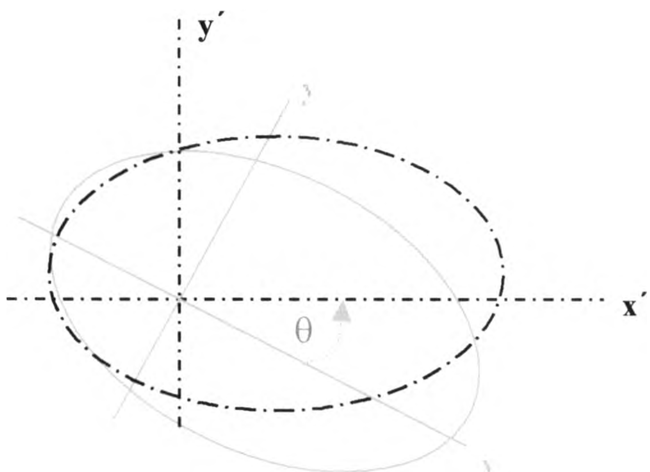

Figure 2.6: Simplification of equation by rotation of axes.

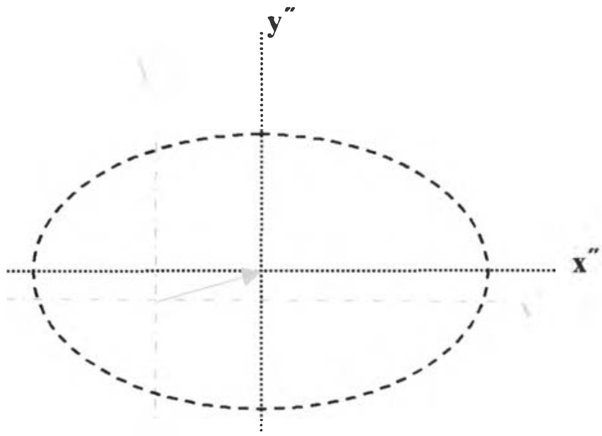

Figure 2.7: Simplification of equation by translation of axes.

The method to transform the general conic equation to the standard form of ellipse equation is described in details as follows :

(1) Rotation from x-y axes to x′-y′ axes by an angle θ
From Eq. 2.21, the general conic equation of an implicit second order polynomial is expressed as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (2.21)$$

The elliptic equation in the (x′-y′) coordinate can be expressed as

$$a' x'^2 + c' y'^2 + d' x' + e' y' + f' = 0 \quad (2.52)$$

The ellipse coefficients in the (x′-y′) coordinate can be expressed as

$$a' = a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta \quad (2.53)$$

$$b' = b\cos 2\theta + (c - a)\sin 2\theta \quad (2.54)$$

$$c' = a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta \quad (2.55)$$

$$d' = d\cos\theta + e\sin\theta \quad (2.56)$$

$$e' = e\cos\theta - d\sin\theta \quad (2.57)$$

$$f' = f \quad (2.58)$$

where

θ	is angle from x axis to x′ axis
a, b, c, d, e, f	are the coefficients of ellipse in coordinate (x, y)
a', b', c', d', e', f'	are the coefficients of ellipse in coordinate (x′, y′)

Since we rotate from x-y axes to x′-y′ axes by an angle θ , b' in Eq. 2.54 is equal to zero

$$b' = b\cos 2\theta + (c - a)\sin 2\theta = 0$$

or

$$(a - c) / b = \cot 2\theta \qquad (2.59)$$

From Eq. 2.59, we can draw a right triangle of angle 2θ as shown in Fig. 2.8. From the law of Pythagoras, hypotenuse of the right angle 2θ is equal to $((a-c)^2 + b^2)^{1/2}$. Thus, the $\cos 2\theta$ and $\sin 2\theta$ from the right triangle of angle 2θ are calculated as

$$\cos 2\theta = (a - c) / ((a-c)^2 + b^2)^{1/2} \qquad (2.60)$$

$$\sin 2\theta = b / ((a-c)^2 + b^2)^{1/2} \qquad (2.61)$$

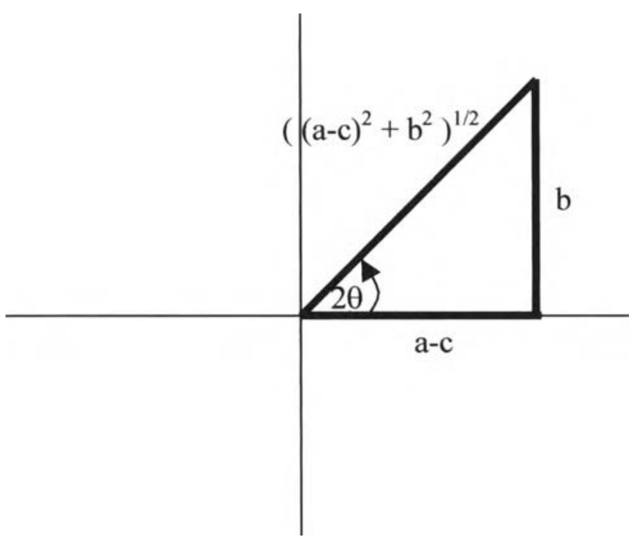

Figure 2.8: A right triangle of 2θ.

Then, the $cos\theta$ and $sin\theta$ are

$$cos\theta = ((1 + cos\ 2\theta) / 2)^{1/2} \tag{2.62}$$

$$sin\theta = ((1 - cos\ 2\theta) / 2)^{1/2} \tag{2.63}$$

Knowing $sin\ \theta$, $cos\ \theta$, $sin2\theta$ and $cos2\theta$ in term of the coefficients a, b, c, d, e, and f in the x-y coordinate, the coefficients a', b', c', d', e', and f' in the x′-y′ coordinate expressed in Eqs. 2.53 - 2.58 can be computed.

(2) Translation from x′-y′ axes to x″-y″ axes

The elliptic equation in the x″-y″ coordinate can be expressed as

$$a''x''^2 + c''y''^2 + f'' = 0 \tag{2.64}$$

The ellipse coefficients in the (x″-y″) coordinate can be computed as

$$a'' = a' \tag{2.65}$$

$$b'' = b' = 0 \tag{2.66}$$

$$c'' = c' \tag{2.67}$$

$$d'' = 2a'h + b'k + d' \tag{2.68}$$

$$e'' = b'h + 2c'k + e' \tag{2.69}$$

$$f'' = a'h^2 + b'hk + c'k^2 + d'h + e'k + f' \tag{2.70}$$

where

a'', b'', c'', d'', e'', f'' are the coefficients of ellipse in (x″, y″) coordinate system.

h is horizontal shift of x′ axis.

k is vertical shift of y′ axis.

Since we translate from x′-y′ axes to x″-y″ axes, d'' in Eq. 2.68 and e'' in Eq. 2.69 are equal to zero as

$$d'' = 2a'h + b'k + d' = 0 \tag{2.71}$$

$$e'' = b'h + 2c'k + e' = 0 \tag{2.72}$$

We can calculate *h* and *k* by solving Eq. 2.71 and 2.72 as

$$h = (2c'd' - b'e') / (b'^2 - 4a'c') \tag{2.73}$$

$$k = (2a'e' - b'd') / (b'^2 - 4a'c') \tag{2.74}$$

Eq. 2.64 can be rearranged as

$$(x'' / (-f'' / a''))^2 + (y'' / (-f'' / c''))^2 = 1 \tag{2.75}$$

Eq. 2.75 is the standard form of the ellipse equation that is obtained by rotation and translation of axes of the general conic equation. The simplified equation can be used to find the semimajor axis (A) and the semiminor axis (B) of the ellipse as shown in Fig. 2.9. The ratio of A/B is calculated to determine the ellipticity of the ellipse.

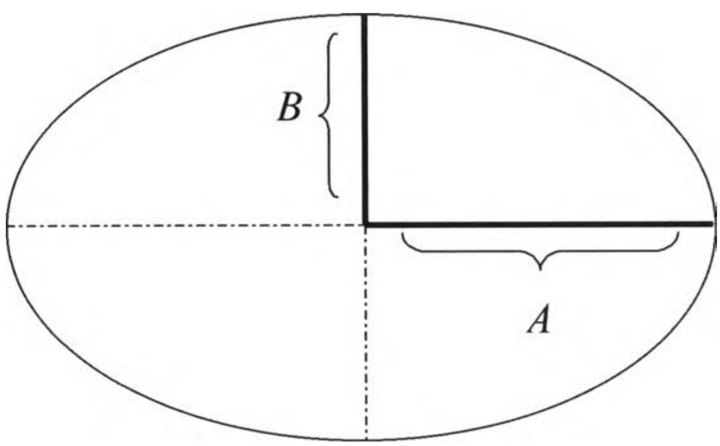

Figure 2.9: The semimajor axis (A) and the semiminor axis (B) of the ellipse.

Comparing Eq. 2.75 with Eq. 2.51, we obtain

$$A = -f'' / a'' \tag{2.76}$$

$$B = -f'' / c'' \tag{2.77}$$

Then,

$$A/B = (-f'' / a'') / (-f'' / c'') \tag{2.78}$$

$$A/B = c'' / a'' \tag{2.79}$$

By fitting the ellipse to the measurements, the coefficients of the ellipse are calculated. Then, the semimajor and semiminor axes are calculated. The ratio of the semimajor axis to semiminor axis tells us the ellipticity of the casing.