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SPECTRA OF UNITARY CAYLEY GRAPHS OF MATRICES OVER FINITE COMMUTATIVE RINGS

Mr. Jitsupat Rattanakangwanwong

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By	Mr. Jitsupat Rattanakangwanwong
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.... Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Professor Patanee Udomkavanich, Ph.D.)

.... Thesis Advisor

(Professor Yotsanan Meemark, Ph.D.)

.... Examiner

(Associate Professor Tuangrat Chaichana, Ph.D)

.... External Examiner

(Associate Professor Utsanee Leerawat, Ph.D.)

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สำหรับริงจำกัด R ที่มีเอกลักษณ์ กราฟเคย์เลย์ยูนิแทรีของ R, C_R, คือกราฟที่มีเซตของจุดยอดเป็น R และสำหรับทุก $x, y \in R$ x เชื่อมกับ y ก็ต่อเมื่อ $x-y$ เป็นยูนิตใน R ในวิทยานิพนธ์นี้ เราหาค่าเฉพาะบางค่าของ $\mathrm{C}_{\mathrm{M}_n(F)}$ เมื่อ F เป็นฟีลด์จำกัดโดยใช้คาแรกเตอร์การบวก และนำค่าเฉพาะเหล่านี้มาวิเคราะห์ความปกติอย่างเข้ม ไฮเพอร์เอเนอร์จีติกกราฟ และรามานุจันกราฟ ต่อมาเราขยายผลเหล่านี้ไปสู่ $\mathrm{C}_{\mathrm{M}_n(R)}$ เมื่อ R เป็นริงเฉพาะที่จำกัด เราบอกลักษณะของริงเฉพาะที่ R และจำนวนนับ $n \geq 2$ ทั้งหมดที่ทำให้ $\mathrm{C}_{\mathrm{M}_n(R)}$ เป็นกราฟปกติอย่างเข้มและเป็นกราฟรามานุจัน เราแสดงต่อว่ากราฟเคย์เลย์ยูนิแทรีของผลคูณของเมตริกซ์ริงมีสมบัติไฮเพอร์เอเนอร์จีติก สุดท้ายเราพิสูจน์ว่ากราฟเคย์เลย์ยูนิแทรีของของผลคูณของเมตริกซ์ริงไม่มีสมบัติปกติอย่างเข้มและรามานุจัน

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For a finite ring R with identity, the *unitary Cayley graph of* R, C_R, is the graph with vertex set R and for each $x, y \in R$, x and y are adjacent if and only if $x - y$ is a unit of R. In this thesis, we determine some eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$, where F is a finite field, by using the additive characters and use these eigenvalues to analyze strong regularity, hyperenergetic graphs and Ramanujan graphs. Next, we extend the results to $\mathrm{C}_{\mathrm{M}_n(R)}$, where R is a local ring. We characterize all local rings R and $n \geq 2$ such that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular and Ramanujan and also show that the graph is hyperenergetic. Moreover, we show that the unitary Cayley graph of product of matrix rings is hyperenergetic. Finally, we prove that the unitary Cayley graph of product of matrix rings is neither a strongly regular graph nor a Ramanujan graph.

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CONTENTS

CHAPTER I
PRELIMINARIES

In this chapter, we give some definitions, notation and results which will be used for this thesis. Throughout, all rings have identity $1 \neq 0$.

1.1 Basic Knowledge in Algebra

Here, we recall some definitions and elementary theorems in group and ring theories that are referred in this thesis. The quoted of more advanced results are cited with references.

Definition 1.1. An ideal of M of a ring R is **maximal** if $M \neq R$ and for every ideal J of R,

$$M \subseteq J \subseteq R \Rightarrow J = M \text{ or } J = R.$$

Theorem 1.2. *Let R be a commutative ring and M an ideal of R. Then M is a maximal ideal of R if and only if R/M is a field.*

Definition 1.3. A **local ring** is a commutative ring which has a unique maximal ideal.

Theorem 1.4. *Let R be a local ring with a unique maximal ideal M. If u is a unit in R and $m \in M$, then $u + m$ is a unit in R*

Let R be a ring and $n \in \mathbb{N}$. Let $R^\times$ denote the group of units of R. Let $\mathrm{M}_n(R)$ denote the ring of $n \times n$ matrices over R. The group of all invertible matrices over R is denoted by $\mathrm{GL}_n(R)$. we write I_n for the $n \times n$ identity matrix and $\mathbf{0}_{n\times n}$ for the $n \times n$ zero matrix.

Theorem 1.5. *Let R be a ring, I an ideal of R and $n \in \mathbb{N}$. Then $\mathrm{M}_n(R)/\mathrm{M}_n(I) \cong \mathrm{M}_n(R/I)$.*

Let F be the field of q elements.

Proposition 1.6. *The number of invertible matrices in* $\mathrm{M}_n(F)$ *is* $(q^n-1)(q^n-q)\dots(q^n-q^{n-1})$.

Theorem 1.7. *[13] The number of* $n\times n$ *matrices of rank* k *over a field* F *is*

$$\frac{[(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})]^2}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

Definition 1.8. A matrix in $\mathrm{M}_n(F)$ is a **linear derangement** if it is invertible and does not fix any nonzero vector.

Theorem 1.9. *[13] Let* e_n *be the number of linear derangements in* $\mathrm{M}_n(F)$ *and define* $e_0=1$. *Then* e_n *satisfies the recursion*

$$e_n=e_{n-1}(q^n-1)q^{n-1}+(-1)^nq^{\frac{n(n-1)}{2}}.$$

1.2 Basic Knowledge in Graph Theory

We give some terminologies and quote results from graph theory in this section.

Definition 1.10. Let G be a graph. A **clique** is a subgraph that is a complete graph and *clique number* of G is the size of largest clique in G, denoted by $\omega(G)$. A set I of vertices of G is called an **independent set** if no distinct vertices of I are adjacent. The **independence number** of G is the size of a maximal independent set, denoted by $\alpha(G)$. The **chromatic number of** G is the least number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. We write $\chi(G)$ for the chromatic number of G. The **edge chromatic number of** G is the least nuber of colors needed to color edges of G so that no two edges having a common vertex share the same color. We write $\chi'(G)$ for the edge chromatic number

Definition 1.11. If every vertex of a graph G is adjacent to k vertices, then G is a k-regular graph. We say that a k-regular graph G is **edge regular** if there exists a parameter λ such that for any two adjacent vertices, there are exactly λ vertices adjacent to both of them. If an edge regular graph with parameters k, λ also satisfies an additional property that for any two non-adjacent vertices, there are exactly μ vertices adjacent to both of them, then it is called a **strongly regular graph** with parameters k, λ, μ.

Definition 1.12. The **adjacency matrix** of a simple graph G with vertex set $\{v_1, \ldots, v_n\}$ is the $n \times n$ symmetric A_G in which entry a_{jk} is the number of edges (0 or 1) in G with endpoints $\{v_j, v_k\}$ for all $j, k \in \{1, 2, \ldots, n\}$.

Definition 1.13. An **eigenvalue** of a graph G is an eigenvalue of the adjacency matrix of a graph G. The **spectrum** of a graph G is the list of its eigenvalues together with their multiplicities. If $\lambda_1, \ldots, \lambda_r$ are eigenvalues of a graph G with multiplicities $m_1, \ldots, m_r$, respectively, we write $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \ldots & \lambda_r \\ m_1 & \ldots & m_r \end{pmatrix}$ to describe the spectrum of G

Theorem 1.14. *[2] If G is a connected regular graph which is not a complete graph, then G is strongly regular if and only if G has exactly three distinct eigenvalues.*

Definition 1.15. The **complete graph** K_n is the graph with n vertices such that every are adjacent. Moreover, the complete graph with vertex set X with a loop on each vertex is written as $\mathring{X}$.

Theorem 1.16. *Let X be a set of n vertices. Then*

$$\operatorname{Spec}\left(\mathring{X}\right) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}.$$

Theorem 1.17. *If G is a connected $k-$regular graph, then k is an eigenvalue of G with multiplicity* 1.

Definition 1.18. The **energy** of a graph G, $E(G)$, is the sum of absolute value of its eigenvalues. That is, if $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \ldots & \lambda_r \\ m_1 & \ldots & m_r \end{pmatrix}$, then

$$E(G) = m_1|\lambda_1| + \cdots + m_r|\lambda_r|.$$

A graph G on n vertices is said to be **hyperenergytic** if $E(G) > 2(n-1)$. A k-regular graph G is a **Ramanujan graph** if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ of G other than $\pm k$.

Definition 1.19. Let A be an $n \times n$ matrix. The **trace**, $\mathrm{tr}(A)$, of A is the sum of the diagonal entries of A.

Theorem 1.20. *Let G be a graph with e edges and A the adjacency matrix of G. If $\lambda_1, \ldots \lambda_n$ are eigenvalues of G, then*

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(A) = 0 \text{ and } \sum_{i=1}^{n} {\lambda_i}^2 = \mathrm{tr}(A^2) = 2e.$$

Definition 1.21. Let G and H be undirected graphs. The **product graph** $G \times H$ is the graph consisting a vertex set $V(G) \otimes V(H)$ and an edge set $\{\{(x_1, y_1), (x_2, y_2)\} :$ x_1 is adjacent to x_2 in G and y_1 is adjacent to y_2 in $H\}$.

Theorem 1.22. *Let $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots \mu_n$ be eigenvalues of graphs G and H, respectively. Then the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Definition 1.23. Let G and H be graphs. We say that G is **isomorphic to** H, denoted by $G \cong H$ is there is a bijection f from G onto H such that for any $x, y \in V(G)$, x is adjacent to y in G if and only if $f(x)$ and $f(y)$ is adjacent in H.

1.3 Additive character

To introduce our methology, we recall some results on characters of finite abelian groups.

Definition 1.24. Let G be a finite abelian group. A map $\chi : G \to (\mathbb{C} \setminus \{0\}, \cdot)$ is a **character** if χ is a group homomorphism.

Proposition 1.25. *[10] Let G be a finite abelian group. Then the set of all characters of G, denoted by $\widehat{G}$, forms an abelian group under pointwise multiplication* *where for any characters χ_1, χ_2 of G, we define*

$$\chi_1 \cdot \chi_2 : G \to (\mathbb{C} \setminus \{0\}, \cdot)$$

by $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g)$ *for all* $g \in G$.

Theorem 1.26. *[10] Let G_1, G_2 be finite abelian groups. Then there is a canonical isomorphism from $\widehat{G_1} \times \widehat{G_2}$ onto $\widehat{G_1 \times G_2}$ given by $(\chi_1, \chi_2) \to \chi_1\chi_2$ for all $\chi_1 \in \widehat{G_1}$ and $\chi_2 \in \widehat{G_2}$.*

Definition 1.27. Let G be a finite group and let $S \subset G$ be a subset. The **Cayley graph**, $\mathrm{Cay}(G, S)$, is a graph with vertex set G and for each $g, h \in G$, x is adjacent to y if and only if $gh^{-1} \in S$.

Theorem 1.28. *[14] Let G be a finite abelian group and let S be a subset of G such that $e \notin S$ and $s^{-1} \in S$ for all $s \in S$, called a **symmetric subset** of G. Then the eigenvalues of* Cay*(G,S) are given by*

$$\lambda = \sum_{s \in S} \chi(s)$$

as χ ranges over all characters of G.

Definition 1.29. Let F be a finite field extension of $\mathbb{Z}_p$ which has order p^r for some $r \in \mathbb{N}$ and a prime p. The **trace map** from F to $\mathbb{Z}_p$ is the $\mathbb{Z}_p$-linear map

$$\mathrm{Tr} : x \mapsto x + x^p + \cdots + x^{p^{r-1}}.$$

Note that $\mathrm{Tr}\big|_{\mathbb{Z}_p} = \mathrm{id}_{\mathbb{Z}_p}$

Theorem 1.30 (Hilbert's Theorem 90)**.** *The trace map is a surjective map.*

Theorem 1.31. *[10] Let F be a finite field extension of $\mathbb{Z}_p$. Each character of the group $(F, +)$ is given by*

$$\chi_a(x) = e^{\frac{2\pi i}{p}\,\mathrm{Tr}(ax)} \text{ for all } x \in F$$

where $a \in F$ is fixed.

1.4 Our objectives

We first define our main object.

Definition 1.32. Let R be a ring. The **unitary Cayley graph** of R, denoted by C_R, with $V(\mathrm{C}_R) = R$ and for each $x, y \in R$, x is adjacent to y if and only if $x - y \in R^\times$.

The unitary Cayley graphs have been widely studied by many authors (see, for example, [3, 9, 5, 1, 6]). As discovered in [1, 6], if R is a finite commutative ring, then R can be decomposed as a direct product of finite local rings $R_1, \ldots, R_s$ and C_R is the tensor product of the graphs $\mathrm{C}_{R_1}, \ldots, \mathrm{C}_{R_s}$. In addition, if R is a finite local ring with maximal ideal M, then C_R is a complete multi-partite graph whose partite sets are the cosets of M. Thus, the unitary Cayley graphs of finite commutative rings are well studied. Their spectral properties including the energies are also well known (see [6]).

For non-commutative rings, Kiani et al. [7] worked on unitary Cayley graphs of the ring $\mathrm{M}_{n_1}(F_1) \times \cdots \times \mathrm{M}_{n_k}(F_k)$ where $n_1, \ldots, n_k \in \mathbb{N}$ and $F_1, \ldots, F_k$ are finite fields. They obtained the clique number, the chromatic number and the independence number of the graph. They also studied the role between C_R and the structure of R. Later in [8], they proved that if F is a finite field, then $\mathrm{C}_{\mathrm{M}_n(F)}$ is an edge regular graph with $k = |\,\mathrm{GL}_n(F)|$ and $\lambda = |(I_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)| = e_n$. Kiani showed further that $\mathrm{C}_{\mathrm{M}_2(F)}$ is strongly regular with $\mu = \left| \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathrm{GL}_2(F) \right) \cap \mathrm{GL}_2(F) \right|$ but $\mathrm{C}_{\mathrm{M}_3(F)}$ is not strongly regular.

Note that $(\mathrm{M}_n(F), +) \cong (F, +) \times (F, +) \times \cdots \times (F, +)$ (n^2 copies). By theorem 1.26, we may identify a character of $\mathrm{M}_n(F)$ as $\chi_A = \prod_{1 \le i,j \le n} \chi_{a_{ij}}$ where $A = [a_{ij}]_{n \times n}$ is in $\mathrm{M}_n(F)$ and so it follows from Theorem 1.28 that the eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ are given by

$$\rho_A = \sum_{S \in \mathrm{GL}_n(F)} \chi_A(S)$$

as A ranges over all matrices in $\mathrm{M}_n(F)$.

The thesis is organized as follows. We focus on $\mathrm{C}_{\mathrm{M}_n(F)}$ and $\mathcal{L}(\mathrm{C}_{\mathrm{M}_n(F)})$ for all $n \geq 2$. In Chapter II, we use the above additive characters to find some eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ and show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$. Next, we show that the graph is hyperenergetic and characterize all fields F and $n \geq 2$ such that $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan. In Chapter III, we use the lifting theorem to extend the results on $\mathrm{C}_{\mathrm{M}_n(F)}$ to the results on $\mathrm{C}_{\mathrm{M}_n(R)}$, where R is a finite local ring. We show that if R is a local ring which is not a field, then the graph is neither strongly regular nor Ramanujan and prove that it is hyperenergetic. We end this chapter by proving that the unitary Cayley graph of product of matrix rings is also hyperenergetic.

CHAPTER II

SPECTRAL PROPERTIES OF $\mathrm{C}_{\mathrm{M}_n(F)}$

2.1 Strong regularity of $\mathrm{C}_{\mathrm{M}_n(F)}$

Let F be the finite field with q elements and $n \geq 2$. Our main work is to show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$. We begin by determining some eigenvalues of the graph by considering three matrices in $\mathrm{M}_n(F)$, namely,

$$A_1 = \mathbf{0}_{n\times n}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, we have

$$\rho_{A_1} = |\,\mathrm{GL}_n(F)| = (q^n - 1)(q^n - q)\dots(q^n - q^{n-1}).$$

Note that

$$\rho_{A_2} = \sum_{m\in F} N_m e^{\frac{2\pi i}{p}\mathrm{Tr}(m)}$$

where N_m is the number of invertible matrices with m at the left-top corner for all $m \in F$. If an invertible matrix has the left-top corner being 0, then the other $n-1$ elements in the first column cannot be all zeros, so there are $q^{n-1} - 1$ choices for the first column. Thus,

$$N_0 = (q^{n-1} - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$$

because the second column must not be multiple of the first column, and the jth column must not be a linear combination of the previous $j-1$ columns for all

$j \in \{2, \dots, n\}$, so there are $q^n - q^{j-1}$ choices for jth column. Now, we have

$$|\operatorname{GL}_n(F)| - N_0 = (q^n - q^{n-1})(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$$

invertible matrices with the top−left corner being nonzero. Since $m \operatorname{GL}_n(F) = \operatorname{GL}_n(F)$ for all $m \neq 0$, $N_m = N_1$ for all $m \neq 0$, so we have

$$(q-1)N_1 = \sum_{m\neq 0} N_m = (q^n - q^{n-1})(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$$

so

$$N_1 = q^{n-1}(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

It follows that

$$\begin{aligned}
\rho_{A_2} &= N_0 e^{\frac{2\pi i}{p}\operatorname{Tr}(0)} + N_1 \sum_{m\neq 0} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} \\
&= (q^{n-1} - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}) + N_1 \sum_{m\neq 0} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} \\
&= -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}) + q^{n-1}(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}) \\
&\qquad + N_1 \sum_{m\neq 0} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} \\
&= -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}) + N_1 \sum_{m\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)}.
\end{aligned}$$

By Hilbert's theorem 90, we know that the trace map is surjective and $\operatorname{Tr}_{|\mathbb{Z}_p} = \operatorname{id}_{\mathbb{Z}_p}$, so we get

$$\sum_{m\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} = |\ker \operatorname{Tr}| \sum_{m\in\mathbb{Z}_p} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} = |\ker \operatorname{Tr}| \sum_{m\in\mathbb{Z}_p} e^{\frac{2\pi i}{p} m} = 0.$$

Here, the last sum is the sum of pth root of unity which equals to zero. Therefore,

$$\rho_{A_2} = -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

Finally, we determine ρ_{A_3}. Since

$$\rho_{A_3} = N(m_1, m_2, \ldots, m_{n+1}) \sum_{m_1, m_2, \ldots, m_{n+1} \in F} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_1 + m_2 + \cdots + m_n + m_{n+1})}$$

where $N(m_1, m_2, \ldots, m_{n+1})$ is the number of invertible matrices of the form

$$\begin{bmatrix} m_1 & m_{n+1} & \cdots & * \\ m_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ m_n & * & \cdots & * \end{bmatrix}$$

and $m_1, m_2, \ldots, m_{n+1} \in F$. For $m_1 = 0$, we can determine $N(0, m_2, \ldots, m_{n+1})$ according to m_{n+1} as follows. If $m_{n+1} \neq 0$, then the first column and the second column are linearly independent, so the second column can be arbitrarily chosen. If $m_{n+1} = 0$, then the second column must not be multiple of the first column and the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \ldots, n\}$. Thus, $N(0, m_2, \ldots, 0) = (q^{n-1})(q^n - q^2)\ldots(q^n - q^{n-1})$ and $N(0, m_2, \ldots, m_{n+1}) = (q^{n-1})(q^n - q^2)\ldots(q^n - q^{n-1})$ if $m_{n+1} \neq 0$. Now, assume that $m_1 \neq 0$. Then $N(m_1, m_2, \ldots, m_{n+1}) = N(1, m_2, \ldots, m_{n+1})$ for all $m_2, \ldots, m_{n+1} \in F$. To find $N(1, m_2, \ldots, m_{n+1})$, we note that the second column cannot be m_{n+1}-multiple of the first column and similarly the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \ldots, n\}$, so

$$N(1, m_2, \ldots, m_{n+1}) = (q^{n-1} - 1)(q^n - q^2)\ldots(q^n - q^{n-1}).$$

Now, we compute

$$\begin{aligned} \rho_{A_3} = {} & (q^{n-1} - q)(q^n - q^2)\ldots(q^n - q^{n-1})(q^n + 1) \sum{}' e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n)} \\ & + q^{n-1}(q^n - q^2)\ldots(q^n - q^{n-1}) \sum{}' \sum_{m_{n+1} \neq 0} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n + m_{n+1})} \\ & + (q^{n-1} - 1)(q^n - q^2)\ldots(q^n - q^{n-1}) \sum_{m_1 \neq 0} \sum{}' \sum_{m_{n+1} \in F} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_1 + m_2 + \ldots m_n + m_{n+1})} \end{aligned}$$

where $\sum'$ denotes the sum over $m_2, \ldots, m_n \in F$ such that $\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$ is the first column of an invertible matrix. Note that

$$\sum_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_1+m_2+\ldots m_n+m_{n+1})} = e^{\frac{2\pi i}{p}\operatorname{Tr}(m_1+m_2+\ldots m_n)} \sum_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_{n+1})}$$

Since $\sum_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_{n+1})} = 0$, the last sum is 0, so we can rewrite ρ_{A_3} as

$$\begin{aligned}\rho_{A_3} = q^{n-1}(q^n - q^2)\ldots(q^n - q^{n-1}) {\sum}' \sum_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\ldots m_n+m_{n+1})} \\ - q(q^n - q^2)\ldots(q^n - q^{n-1}) {\sum}' e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\ldots m_n)}\end{aligned}$$

The first sum is again zero because m_{n+1} varies over F. Now, since $m_1 = 0$, $m_2, \ldots, m_n$ cannot be all zeros and so

$$\begin{aligned}{\sum}' e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\ldots m_n)} &= \sum_{\{m_2,\ldots,m_n\}\neq\{0\}} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\ldots m_n)} \\ &= \sum_{m_2,\ldots,m_n\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\ldots m_n)} - 1 = -1.\end{aligned}$$

Hence, $\rho_{A_3} = q(q^n - q^2)\ldots(q^n - q^{n-1})$.

Let A and B be $n \times n$ matrices over F. Assume that $\operatorname{rank} A = \operatorname{rank} B$. Then there exist invertible matrices P and Q such that $A = PBQ$. Consider $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $P = [p_{ij}]_{n\times n}$ and $Q = [q_{ij}]_{n\times n}$. For $S = [s_{ij}]_{n\times n} \in \mathrm{GL}_n(F)$, we have

$$\chi_A(S) = e^{\frac{2\pi i}{p}\operatorname{Tr}\left(\sum_{1\leq i,j\leq n} a_{ij}s_{ij}\right)}$$

From

$$\sum_{1\le i,j\le n} a_{ij}s_{ij} = \sum_{1\le i,j\le n}\Big(\sum_{1\le k,l\le n} p_{il}b_{lk}q_{kj}\Big)s_{ij}$$
$$= \sum_{1\le i,j\le n}\sum_{1\le k,l\le n} b_{lk}(p_{il}s_{ij}q_{kj})$$
$$= \sum_{1\le k,l\le n} b_{lk}\sum_{1\le i,j\le n}(p_{il}s_{ij}q_{kj}).$$

and $\sum_{1\le i,j\le n} p_{il}s_{ij}q_{kj} = \left(P^tSQ^t\right)_{lk}$, it follows that $\chi_A(S) = \chi_B(P^tSQ^t)$. Since P and Q are invertible, $\mathrm{GL}_n(F) = P^t\,\mathrm{GL}_n(F)Q^t$, so

$$\sum_{S\in \mathrm{GL}_n(F)} \chi_A(S) = \sum_{S\in \mathrm{GL}_n(F)} \chi_B(S).$$

Hence, we have shown:

Theorem 2.1. *If A and B are $n\times n$ matrices over F of the same rank, then $\rho_A = \rho_B$.*

Since $\mathrm{C}_{\mathrm{M}_n(F)}$ is connected and $|\mathrm{GL}_n(F)|$-regular, ρ_{A_1} induced from the zero matrix has multiplicity 1. Observe that ρ_{A_2} and ρ_{A_3} are induced by matrices of rank 1 and 2, respectively. Since the set of characters are linearly independent, the multiplicities of them are the number of matrices of such rank. Suppose $n = 2$. The number of matrices of rank 1 is $\frac{(q^2-1)^2}{q-1} = (q-1)(q+1)^2$ and the number of matrices of rank 2 is $(q^2-1)(q^2-q)$. Then

$$\begin{aligned}
E(\mathrm{C}_{\mathrm{M}_2(F)}) &= (q^2-1)(q^2-q) + (q^2-q)(q-1)(q+1)^2 + q(q^2-1)(q^2-q)\\
&= (q^2-1)(q^2-q)[1 + (q+1) + q]\\
&= 2(q^2-1)(q^2-q)(q+1)\\
&= 2q(q^2-1)(q-1)(q+1)\\
&= 2q(q^2-1)^2.
\end{aligned}$$

We record this result in:

Theorem 2.2. $\operatorname{Spec} \mathrm{C}_{\mathrm{M}_2(F)} = \begin{pmatrix} (q^2-1)(q^2-q) & -(q^2-q) & q \\ 1 & (q-1)(q+1)^2 & (q^2-1)(q^2-q) \end{pmatrix}$ *and* $E(\mathrm{C}_{\mathrm{M}_2(F)}) = 2q(q^2-1)^2$.

If $n = 3$, then $\rho_{A_1} = (q^3-1)(q^3-q)(q^3-q^2)$, $\rho_{A_2} = -(q^3-q)(q^3-q^2)$ and $\rho_{A_3} = q(q^3-q^2)$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_3(F)}$ induced from matrices of rank 0, 1 and 2, respectively. Let λ be the eigenvalue induced from matrices of rank 3. Since the sum of all eigenvalues is zero, counting the number of matrices of each rank gives

$$\begin{aligned}(q^3-1)(q^3-q)(q^3-q^2) - (q^3-q)(q^3-q^2)\frac{(q^3-1)^2}{q-1} \\ + q(q^3-q^2)\frac{(q^3-1)^2(q^3-q)^2}{(q^2-1)(q^2-q)} + (q^3-1)(q^3-q)(q^3-q^2)\lambda = 0.\end{aligned}$$

Dividing by $(q^3-1)(q^3-q)(q^3-q^2)$ gives

$$1 - \frac{q^3-1}{q-1} + q\frac{(q^3-1)(q^3-q)}{(q^2-1)(q^2-q)} + \lambda = 0$$

Hence, we have

$$\begin{aligned}\lambda &= -1 + \frac{q^3-1}{q-1} - q\frac{(q^3-1)(q^3-q)}{(q^2-1)(q^2-q)} \\ &= -1 + (q^2+q+1) - q^2\frac{(q-1)(q^2+q+1)(q^2-1)}{(q^2-1)q(q-1)} \\ &= -1 + q^2 + q + 1 - q^3 - q^2 - q = -q^3\end{aligned}$$

This proves the following theorem.

Theorem 2.3. $\operatorname{Spec} \mathrm{C}_{\mathrm{M}_3(F)} = \begin{pmatrix} (q^3-1)(q^3-q)(q^3-q^2) & -(q^3-q)(q^3-q^2) & q(q^3-q^2) & -q^3 \\ 1 & (q^3-1)(q^2+q+1) & (q^3-1)(q^3-q)(q^2+q+1) & (q^3-1)(q^3-q)(q^3-q^2) \end{pmatrix}.$

Recall that a connected regular graph with exactly three distinct eigenvalues is strongly regular. So, we can conclude from Theorem 2.2 that $\mathrm{C}_{\mathrm{M}_2(F)}$ is strongly regular. Next, we assume that $n \geq 3$ and $C_{M_n(F)}$ is strongly regular. According to [4], $C_{M_n(F)}$ has only three eigenvalues. From our computation, they must be ρ_{A_1}, ρ_{A_2} and ρ_{A_3}. Suppose the multiplicities of ρ_{A_2} and ρ_{A_3} are m_2 and m_3, respectively. Since the sum of eigenvalues of $C_{M_n(F)}$ is 0, we have

$$(q^n-1)(q^n-q)\dots(q^n-q^{n-1})-(q^n-q)\dots(q^n-q^{n-1})m_2+q(q^n-q^2)\dots(q^n-q^{n-1})m_3 = 0.$$

Dividing by $(q^n - q^2)\dots(q^n - q^{n-1})$ gives

$$(q^n - 1)(q^n - q) - (q^n - q)m_2 + qm_3 = 0.$$

Note that $1 + m_2 + m_3 = q^{n^2}$, so $m_2 = q^{n^2} - m_3 - 1$. Putting m_2 in the previous equation gives $m_3 = q(q^{n-1} - 1)(q^{n^2-n} - 1)$. By theorem 1.20, the sum of square of eigenvalues of the adjacency matrix A is the trace of A^2 which is twice of the number of edges of the graph. Since the sum of degree of all vertices equals twice of the number of edge in the graph and our graph is $|GL_n(F)|$-regular, if E_n is the number of edges, then

$$2E_n = q^{n^2}(q^n - 1)\dots(q^n - q^{n-1}).$$

This yields another relation on m_2 and m_3 given by

$$\begin{aligned}((q^n - 1)(q^n - q)\dots(q^n - q^{n-1}))^2 + ((q^n - q)\dots(q^n - q^{n-1}))^2 m_2 \\ +(q(q^n - q^2)\dots(q^n - q^{n-1}))^2 m_3 = q^{n^2}(q^n - 1)\dots(q^n - q^{n-1}).\end{aligned}$$

Dividing by $(q^n - q^2)\dots(q^n - q^{n-1})$ and substituting $m_3 = q(q^{n-1} - 1)(q^{n^2-n} - 1)$ give

$$\begin{aligned}(q^n - 1)^2(q^n - q)^2(q^n - q^2)\dots(q^n - q^{n-1}) + q(q^n - q)^2(q^n - q^2)\dots(q^n - q^{n-1})m_2 \\ + q^3(q^n - q^2)\dots(q^n - q^{n-1})(q^{n-1} - 1)(q^{n^2-n} - 1) \\ = q^{n^2}(q^n - 1)(q^n - q)\end{aligned}$$

Since $q^{n^2-n}-1 = (q^{n-1})^n - 1$, the left hand side is divisible by $(q^{n-1}-1)^2$, so $(q^{n-1}-1)^2$ divides $q^{n^2}(q^n-1)(q^n-q)$. It follows that $q^{n-1}-1$ divides $q^{n^2+1}(q^n-1)$. Since q and q^n-1 are relatively prime, we have $q^{n-1}-1$ divides $q^n-1 = q^n-q+(q-1)$, so $q^{n-1}-1$ divides $q-1$ which is a contradiction because $n \geq 3$. Therefore, we have our desired result.

Theorem 2.4. *The graph* $\mathrm{C}_{\mathrm{M}_n(F)}$ *is strongly regular if and only if* $n = 2$.

From the above theorem, we learn that $\mathrm{C}_{\mathrm{M}_n(F)}$ is not strongly regular for $n \geq 3$. Since it is edge regular with $\lambda = e_n$, there are more than one value of the number of common neighborhoods of non-adjacent vertices in $\mathrm{C}_{\mathrm{M}_n(F)}$. If $A, B \in \mathrm{M}_n(F)$ and $\operatorname{rank}(A-B) = r$ for some $0 < r \leq n$, then there exist invertible matrices P, Q such that

$$P(A-B)Q = \begin{bmatrix} I_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{bmatrix}$$

For $A \in \mathrm{M}_n(F)$, let $N(A)$ be the set of neighbors of A. According to Kiani (Lemma 2.1 of [8]), we have

$$|N(A) \cap N(B)| = \left| \left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathrm{GL}_n(F) \right) \cap \mathrm{GL}_n(F) \right|$$

for all $A, B \in \mathrm{M}_n(F)$ with $A \neq B$. It gives the number of common neighbors of any pair of two vertices A and B in $\mathrm{M}_n(F)$. For $1 \leq r \leq n$, we define

$$d(n,r) = \left| \left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathrm{GL}_n(F) \right) \cap \mathrm{GL}_n(F) \right|.$$

Since two matrices A and B are adjacent if and only if $\operatorname{rank}(A-B) = n$, $d(n,n) = e_n$ mentioned in chapter1. Observe that $d(n,r)$ is the number of invertible matrices A such that $A - \begin{bmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{bmatrix}$ is also invertible. Now, let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis of F^n. Consider the set $\mathcal{X}$ of vectors given by

$$\mathcal{X} = \left\{ A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \in \mathrm{GL}_n(F) : \vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\} \right\}.$$

Note that if $A \in \mathcal{X}$, then A is invertible but $A - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ is not invertible. We proceed to compute $d(n,1)$. Since $d(n,1) = |\mathrm{GL}_n(F)| - |\mathcal{X}|$, we shall determine the cardinality of $\mathcal{X}$. Let $A = [a_{ij}]_{n\times n}$ be in $\mathcal{X}$. Then $\operatorname{rank} A = n$ and $\operatorname{rank}\left(A - \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = n-1$. It follows that $\vec{a}_1 \notin \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$ but $\vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$. This forces that $\vec{e}_1 \notin \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$. Also, $\{\vec{a}_2, \dots, \vec{a}_n\}$ must be linearly independent. Thus, there are $(q^n - q)\dots(q^n - q^{n-1})$ choices for $\{\vec{a}_2, \dots, \vec{a}_n\}$. As for $\vec{a}_1$, it suffices to count under a condition $\vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$ because if $\vec{a}_1 \in \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$, then $\vec{e}_1 \in \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$, which is absurd, so there are q^{n-1} choices for $\vec{a}_1$. Hence,

$$|\mathcal{X}| = q^{n-1}(q^n - q)\dots(q^n - q^{n-1}).$$

Then

Theorem 2.5. $d(n,1) = |\mathrm{GL}_n(F)| - |\mathcal{X}| = (q^n - q^{n-1} - 1)(q^n - q)\dots(q^n - q^{n-1})$.

Remark 2.6. For $r \geq 2$, we can find a lower bound for $d(n,r)$. Consider a matrix of the form $Y = \begin{bmatrix} A & \mathbf{0} \\ B & C \end{bmatrix}$ where A, B and C are $r \times r$, $(n-r) \times r$ and $(n-r) \times (n-r)$ matrices, respectively. It is easy to see that $\det Y = \det A \det C$, and $\det\left(X - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \det(A - I_r)\det C$. If we choose A to be a derangement matrix and C is an invertible matrix, then Y and $Y - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ are invertible. Since there are e_r choices for A , $q^{r(n-r)}$ choices for B, and $(q^{n-r} - 1)\dots(q^{n-r} - q^{n-r-1})$

choices for C, we have

$$d(n,r) \geq e_r q^{r(n-r)}(q^{n-r}-1)\dots(q^{n-r}-q^{n-r-1}) = e_r(q^n-q^r)\dots(q^n-q^{n-1}).$$

2.2 Hyperenegetic graphs and Ramanujan graphs

Let F be the finite field with q elements. In this section, without explicitly computing the spectrum of the graph, we show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic for all $n \geq 2$ and characterize all n and q such that $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan.

Since $q^3-1=(q-1)(q^2+q+1) > q^2+q$, we get $q(q^2-1)=q^3-q > q^2+1$, so $E(\mathrm{C}_{\mathrm{M}_2(F)}) = 2q(q^2-1)^2 > 2(q^4-1)$. Then $\mathrm{C}_{\mathrm{M}_2(F)}$ is hyperenergetic. Next, we assume that $n \geq 3$. Recall that $\rho_{A_3} = q(q^n-q^2)\dots(q^n-q^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_n(F)}$ with multiplicities at least $\dfrac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)}$. It follows that

$$E(\mathrm{C}_{\mathrm{M}_n(F)}) > q(q^n-q^2)\dots(q^n-q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)}.$$

Thus, to show that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic, it suffices to prove

$$q(q^n-q^2)\dots(q^n-q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} > 2(q^{n^2}-1).$$

Since $|\operatorname{GL}_n(F)| = (q^n-1)(q^n-q)\dots(q^n-q^{n-1})$, the above inequality is equivalent to

$$|\operatorname{GL}_n(F)| > \frac{2(q^2-1)(q^2-q)(q^{n^2}-1)}{q(q^n-1)(q^n-q)}.$$

We shall use induction on $n \geq 3$ to show that this inequality holds and conclude that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic. If $n=3$, then the right-hand side becomes

$$\begin{aligned}\frac{2(q^2-1)(q^2-q)(q^9-1)}{q(q^3-1)(q^3-q)} &= \frac{2(q-1)(q+1)q(q-1)(q^3-1)(q^6+q^3+1)}{q(q^3-1)q(q-1)(q+1)} \\ &= \frac{2(q-1)}{q}(q^6+q^3+1)\end{aligned}$$

and $|\operatorname{GL}_3(F)| = (q^3-1)(q^3-q)(q^3-q^2) = (q-1)(q^2+q+1)(q-1)(q^2+q)(q^2)(q-1) = (q-1)^3(q^6+2q^5+2q^4+q^3) > (q-1)^3(q^6+q^3+1)$. Since $q \geq 2$, we have $q(q-1)^2 \geq 2$. Then $(q-1)^3 \geq \dfrac{2(q-1)}{q}$ and the inequality is valid for $n = 3$. Now, let $n \geq 4$ and assume that

$$\begin{aligned}
|\operatorname{GL}_{n-1}(F)| &> \frac{2(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^{n-1}-1)(q^{n-1}-q)} \\
&= \frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^{n-1}-q)} \\
&\geq \frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^n-1)}
\end{aligned}$$

where the last inequality comes from $q^n - 1 - (q^{n-1} - q) = (q^{n-1}+1)(q-1) \geq 0$. Since $|\operatorname{GL}_n(F)| = (q^n-1)(q^n-q)\ldots(q^n-q^{n-1}) = q^{n-1}(q^n-1)|\operatorname{GL}_{n-1}(F)|$, it follows from the previous inequality that

$$|\operatorname{GL}_n(F)| > q^{n-1}(q^n-1)\frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^n-1)}$$

and so it remains to show that $q^n(q^n-1)(q^{(n-1)^2}-1) \geq q^{n^2}-1$. Rewrite

$$\begin{aligned}
q^n(q^n-1)(q^{(n-1)^2}-1) - q^{n^2} + 1 &= q^n(q^{n^2-n+1} - q^{n^2-2n+1} - q^n + 1) - q^{n^2} + 1 \\
&= q^{n^2+1} - q^{n^2-n+1} - q^{n^2} - q^{2n} + q^n + 1 \\
&= q^{n^2-n+1}\left(q^{n-1}(q-1)-1\right) - q^{2n} + q^n + 1.
\end{aligned}$$

Since $n \geq 4$ and $q \geq 2$,

$$q^{n^2-n+1}\left(q^{n-1}(q-1)-1\right) - q^{2n} \geq q^{n^2-n+1} - q^{2n} = q^{2n}(q^{n^2-3n+1}-1) \geq 0.$$

This completes the proof of the next theorem.

Theorem 2.7. $\mathrm{C}_{\mathrm{M}_n(F)}$ *is hyperenergetic for all* $n \geq 2$.

Recall that a k-regular graph is Ramanujan if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ other than $\pm k$. Since eigenvalues of a graph are real numbers, this inequality is equivalent to $\lambda^2 - 4(k-1) \le 0$. We know that $\mathrm{C}_{\mathrm{M}_n(F)}$ is regular with parameter $k = (q^n - 1)(q^n - q)\dots(q^n - q^{n-1})$. If $n = 2$, then its eigenvalues are $q, -(q^2 - q)$ and $(q^2 - 1)(q^2 - q)$. Since $q \ge 2$, we have $q^2 - q \ge 2$, so

$$q^2 + 4 \le 4q^2 \quad \text{and} \quad (q^2 - q)^2 + 4 \le 4(q^2 - q).$$

The first inequality gives $q^2 + 4 \le 4q(q+1)(q-1)^2$ which is equivalent to $q^2 - 4(q^2-1)(q^2-q) + 4 \le 0$ and the second inequality directly proves $(q^2 - q)^2 < 4(q^2 - 1)(q^2 - q) - 4$. Thus, $\mathrm{C}_{\mathrm{M}_2(F)}$ is Ramanujan. Now suppose that $n \ge 3$ and $\mathrm{C}_{\mathrm{M}_n(F)}$ is a Ramanujan graph. From the computation in the previous section, $\rho_{A_2} = -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_n(F)}$, so

$$0 \ge \rho_{A_2}^2 - 4(q^n-1)(q^n-q)\dots(q^n-q^{n-1}) + 4 = \rho_{A_2}^2 - 4(q^n-1)|\rho_{A_2}| + 4 = (\rho_{A_2}+2)^2 - 4q^n|\rho_{A_2}|.$$

It follows that $4q^n|\rho_{A_2}| \ge (\rho_{A_3} + 2)^2 > |\rho_{A_2}^2|$, so $4q^n > \rho_{A_2}$. For $n = 3$, we have $4q^3 > (q^3 - q)(q^3 - q^2)$, so $4 > (q^2 - 1)(q - 1)$ which implies that $q = 2$ and for $n \ge 4$, we have $n + 2 \le \dfrac{(n-1)n}{2}$ and so

$$4q^n > |\rho_{A_2}| = q^{\frac{(n-1)n}{2}}(q^{n-1} - 1)(q^{n-2} - 1)\dots(q-1) > q^{\frac{(n-1)n}{2}}$$

which leads to a contradiction for all $q \ge 2$. Finally, if $n = 3$ and $q = 2$, by Theorem 2.3, we have $-(2^3 - 2)(2^3 - 2^2) = -24, 2(2^3 - 2^2) = 8$ and $-2^3 = -8$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ and $4((2^3 - 1)(2^3 - 2)(2^3 - 2^2) - 1) = 668$ is greater than 24^2 and 8^2. Hence, $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ is also Ramanujan.

We record this result in the following theorem.

Theorem 2.8. *The graph* $\mathrm{C}_{\mathrm{M}_n(F)}$ *is Ramanujan if and only if* $n = 2$ *or* ($n = 3$ *and* $F = \mathbb{Z}_2$).

CHAPTER III
THE UNITARY CAYLEY GRAPH OF PRODUCT OF MATRIX RINGS

In this chapter, we study the unitary Cayley graph of product of matrix rings. We introduce the lifting theorem in the first section. In the second section, we use the lifting theorem to extend the results from finite fields to finite local rings. Finally, we study the unitary Cayley graph of product of matrix rings. We determine the clique number, the chromatic number and the independence number of the graph, and show that the graph is hyperenergetic.

3.1 Lifting theorem

Let R be a local ring with unique maximal ideal M and residue field $\Bbbk$. Recall that $R/M \cong \Bbbk$ results in $\mathrm{M}_n(R)/\mathrm{M}_n(M) \cong \mathrm{M}_n(R/M) \cong \mathrm{M}_n(\Bbbk)$. Then elements in R can be partitioned into cosets of M and can be viewed as lifting from elements of $\Bbbk$. Suppose $|M| = m$ and $|\Bbbk| = q$. We fix $A_1, \ldots, A_{q^{n^2}}$ to be coset representatives of $\mathrm{M}_n(M)$ in $\mathrm{M}_n(R)$.

Lemma 3.1. *Let $A \in \mathrm{M}_n(R)$ and $X \in \mathrm{M}_n(M)$. Then*

$$\det(A + X) = (\det A) + m' \text{ for some } m' \in M.$$

In particular, A is invertible if and only if $A + X$ is invertible.

Proof. Write $A = [a_{ij}]_{n\times n}$ and $X = [m_{ij}]_{n\times n}$. Then

$$\begin{aligned}\det(A+X) &= \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma)(a_{1\sigma(1)} + m_{1\sigma(1)}) \ldots (a_{n\sigma(n)} + m_{n\sigma(n)}) \\ &= \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma)(a_{1\sigma(1)} \ldots a_{n\sigma(n)}) + m' = (\det A) + m'\end{aligned}$$

for some $m' \in M$. Moreover, $\det A$ is a unit if and only if $\det(A+X) = \det A + m'$ is a unit by Theorem 1.4. $\square$

The above lemma directly implies the following theorem.

Theorem 3.2. *1. For $A, B \in \mathrm{M}_n(R)$, A and B are adjacent in $\mathrm{C}_{\mathrm{M}_n(R)}$ if and only if $A + \mathrm{M}_n(M)$ and $B + \mathrm{M}_n(M)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$.*

2. The set $\mathrm{M}_n(R)/\mathrm{M}_n(M) = \{A_1 + \mathrm{M}_n(M), \ldots, A_{q^{n^2}} + \mathrm{M}_n(M)\}$ is a partition of the vertex set of $\mathrm{C}_{\mathrm{M}_n(R)}$ such that

(a) for each $i \in \{1, \ldots, q^{n^2}\}$, any two distinct vertices in $A_i + \mathrm{M}_n(M)$ are nonadjacent vertices, and

(b) for $i, j \in \{1, \ldots, q^{n^2}\}$, A_i and A_j are adjacent in $\mathrm{C}_{\mathrm{M}_n(R)}$ if and only if $A_i + \mathrm{M}_n(M)$ and $A_j + \mathrm{M}_n(M)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$.

3. Let $\mathring{\mathrm{M}}_n(M)$ be the complete graph of $|\mathrm{M}_n(M)|$ vertices with a loop on every vertex. Define $f : \mathrm{M}_n(\Bbbk) \times \mathrm{M}_n(M) \to \mathrm{M}_n(R)$ by $f(A_i + \mathrm{M}_n(M), X) = A_i + X$ for all $i \in \{1, \ldots, q^{n^2}\}$ and $X \in \mathrm{M}_n(M)$. Then f is an isomorphism from the graph $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ onto the graph $\mathrm{C}_{\mathrm{M}_n(R)}$.

Proof. The above discussion implies (1) and (2). For (3), we first show that f is an injection. Let $i, j \in \{1, \ldots, q^{n^2}\}$ and $X, Y \in \mathrm{M}_n(M)$ such that $A_i + X = A_j + Y$. Then $A_i - A_j = Y - X \in \mathrm{M}_n(M)$. This forces that $A_i + \mathrm{M}_n(M) = A_j + \mathrm{M}_n(M)$ in $\mathrm{M}_n(\Bbbk)$, so $i = j$ and $X = Y$. Since $|\mathrm{M}_n(\Bbbk) \times \mathrm{M}_n(M)| = |\mathrm{M}_n(R)|$, f is a bijection. Finally, for $i, j \in \{1, \ldots, q^{n^2}\}$ and $X, Y \in \mathrm{M}_n(M)$, we have $(A_i + \mathrm{M}_n(M), X)$ and $(A_j + \mathrm{M}_n(M), Y)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ if and only if $A_i + \mathrm{M}_n(M)$ and $A_j + \mathrm{M}_n(M)$ are adjacent if and only if A_i and A_j are adjacent by (2). Hence, f is a graph isomorphism. $\square$

3.2 Unitary Cayley graph of product of matrix rings

First, we assume that R is a finite local ring which is not a field with unique maximal ideal M and residue field $\Bbbk$. Let $|M| = m$ and $|\Bbbk| = q$. Since the adjacency matrix of $\dot{\mathrm{M}}_n(M)$ is the all-ones matrix of size m^{n^2}, we have $\mathrm{Spec}\Big(\mathring{\mathrm{M}}_n(M)\Big) = \begin{pmatrix} m^{n^2} & 0 \\ 1 & m^{n^2}-1 \end{pmatrix}$ and $(q^n-1)(q^n-q)\dots(q^n-q^{n-1}), -(q^n-q)\dots(q^n-q^{n-1})$ and $q(q^n-q^2)\dots(q^n-q^{n-1})$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$. Since the eigenvalues of $G \otimes H$ are $\lambda_i\mu_j$ where λ_i's and μ_j's are eigenvalues of G and H, respectively, we can conclude from the isomorphism in Theorem 3.2 (3) that $0, m^{n^2}(q^n-1)(q^n-q)\dots(q^n-q^{n-1}), -m^{n^2}(q^n-q)\dots(q^n-q^{n-1})$ and $m^{n^2}q(q^n-q^2)\dots(q^n-q^{n-1})$ are distinct eigenvalues of $\mathrm{C}_{\mathrm{M}_n(R)}$. Then we have shown the following theorem.

Theorem 3.3. *If R is a local ring which is not a field and $n \geq 2$, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is not strongly regular.*

However, it turns out that the graph $\mathrm{C}_{\mathrm{M}_n(R)}$ is hyperenergetic.

Theorem 3.4. *If R is a local ring, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is hyperenergetic for all $n \geq 2$.*

Proof. Let $\Bbbk$ be the residue field of R and assume that $|\Bbbk| = q$. Recall that $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$ is hyperenergetic and $\mathrm{C}_{\mathrm{M}_n(R)}$ has $-m^{n^2}q(q^n-q^2)\dots(q^n-q^{n-1})$ as an eigenvalue with multiplicities at least $\dfrac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)}$. The proof of Theorem 2.7 tells us that

$$q(q^n-q^2)\dots(q^n-q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} > 2(q^{n^2}-1).$$

Note that the left-hand side is a multiple of q. It follows that

$$q(q^n-q^2)\dots(q^n-q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} \geq 2q^{n^2}$$

Multiplying by m^{n^2} both sides gives

$$m^{n^2}q(q^n-q^2)\dots(q^n-q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} \geq 2(mq)^{n^2} > 2((mq)^{n^2}-1)$$

which completes the proof. □

Theorem 3.5. *If R is a local ring which is not a field, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is not Ramanujan for all $n \geq 2$*

Proof. For simplicity, let $k = |\operatorname{GL}_n(\Bbbk)|$. We first handle case $n \geq 3$ and $q \geq 3$. Then $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$ is not Ramanujan by Theorem 2.8. From the proof of Theorem 2.8, we have $(q^n - q)\dots(q^n - q^{n-1}) \geq 2\sqrt{k-1}$. Thus,

$$m^{n^2}(q^n - q)\dots(q^n - q^{n-1}) \geq 2m^{n^2}\sqrt{k-1},$$

so we must show that $m^{n^2}\sqrt{k-1} > \sqrt{m^{n^2}k - 1}$. Rewrite

$$m^{2n^2}(k-1) - (m^{n^2}k - 1) = (m^{n^2} - 1)(m^{n^2}k - m^{n^2} - 1).$$

Since R is not a field, we have $m \geq 2$, so $(m^{n^2} - 1)(m^{n^2}k - m^{n^2} - 1) > 0$ and the desired inequality follows. Next, we assume that $n = 3$ and $q = 2$. Then $-m^9(2^3 - 2)(2^3 - 2^2) = -24m^9$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_3(R)}$. Moreover, $k = m^9(2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 168m^9$. We have $576m^{18} - 4(168m^9 - 1) = m^9(576m^9 - 672) + 4$. Since $m \geq 2$, we get $24m^9 > 2\sqrt{168m^9 - 1}$. Finally, if $n = 2$, then $-m^4(q^2 - q)$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_2(R)}$ and $k = m^4(q^2 - 1)(q^2 - q)$, so

$$\begin{aligned} m^8(q^2 - q)^2 - 4(m^4(q^2 - 1)(q^2 - q) - 1) &= m^8(q^2 - q)^2 - 4m^4(q^2 - 1)(q^2 - q) + 4 \\ &\geq m^8(q^2 - q)^2 - 4m^4(q^2 - q)^2 + 4 \\ &= (m^8 - 4m^4)(q^2 - q)^2 + 4 > 0 \end{aligned}$$

because $m \geq 2$. Hence, $\mathrm{C}_{\mathrm{M}_2(R)}$ is not Ramanujan. □

Let $R_1, \dots, R_s$ be finite local rings with maximal ideals $M_1, \dots, M_s$ and residue fields $\Bbbk_1, \dots, \Bbbk_s$, respectively. Let $\mathcal{R} = \mathrm{M}_{n_1}(R_1) \times \dots \times \mathrm{M}_{n_s}(R_s)$ where $n_1, \dots n_s \in \mathbb{N}$. By Theorem 3.8 of [7], we have

$$\chi(\mathrm{C}_{\mathcal{R}}) = \omega(\mathrm{C}_{\mathcal{R}}) = \omega(\mathrm{C}_{\mathrm{M}_{n_1}(\Bbbk_1)\times\dots\times\mathrm{M}_{n_k}(\Bbbk_k)}) = \min_{1\leq i\leq s}\{|\Bbbk_i|^{n_i}\}$$

Finally, we compute $\alpha(\mathrm{C}_{\mathcal{R}})$. Theorem 3.2 (3) gives

$$\mathrm{C}_{\mathcal{R}} \cong \left(\mathrm{C}_{\mathrm{M}_{n_1}(\Bbbk_1)} \otimes \dots \otimes \mathrm{C}_{\mathrm{M}_{n_s}(\Bbbk_s)}\right) \otimes \left(\mathring{\mathrm{M}}_{n_1}(M_1) \otimes \dots \otimes \mathring{\mathrm{M}}_{n_s}(M_s)\right).$$

Since the second product is a complete graph with a loop on each vertex, we can see that

$$\alpha(\mathrm{C}_{\mathcal{R}}) = \alpha(\mathrm{C}_{\mathrm{M}_{n_1}(\mathbb{k}_1)} \otimes \cdots \otimes \mathrm{C}_{\mathrm{M}_{n_s}(\mathbb{k}_s)}) \prod_{i=1}^{s} |\mathrm{M}_{n_i}(M_i)|$$

$$= \frac{\prod_{i=1}^{s} |\mathrm{M}_{n_i}(\mathbb{k}_i)|}{\min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}} \prod_{i=1}^{s} |\mathrm{M}_{n_i}(M_i)| = \frac{|\mathcal{R}|}{\min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}}.$$

Thus, we prove:

Theorem 3.6. $\omega(\mathrm{C}_{\mathcal{R}}) = \chi(\mathrm{C}_{\mathcal{R}}) = \min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}$ *and* $\alpha(\mathrm{C}_{\mathcal{R}}) = \dfrac{|\mathcal{R}|}{\min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}}$.

For each $1 \le i \le s$, let $|M_i| = m_i$ and $|\mathbb{k}_i| = q_i$. Recall that $\rho_i = -{m_i}^{{n_i}^2} q_i({q_i}^{n_i} - {q_i}^2) \dots ({q_i}^{n_i} - {q_i}^{n_i-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ with multiplicities at least t_i where $t_i = \dfrac{({q_i}^{n_i}-1)^2({q_i}^{n_i}-q_i)^2}{({q_i}^2-1)({q_i}^2-q_i)}$ for all i. Hence, $\prod_{i=1}^{s} \rho_i$ is an eigenvalue of $\mathrm{C}_{\mathcal{R}}$ with multiplicities at least $\prod_{i=1}^{s} t_i$. By Theorem 3.4, we have $\rho_i t_i > 2(|\mathrm{M}_{n_i}(R_i)| - 1)$ for all $1 \le i \le s$. Note that the left-hand side is a multiple of q_i. We can conclude that $\rho_i t_i \ge 2|R_i|^{{n_i}^2}$. It follows that

$$\prod_{i=1}^{s}\rho_i \prod_{i=1}^{s} t_i = \prod_{i=1}^{s}\rho_i t_i \ge \prod_{i=1}^{s} 2|\mathrm{M}_{n_i}(R_i)| = 2^s \prod_{i=1}^{s} |\mathrm{M}_{n_i}(R_i)| > 2\left(\prod_{i=1}^{s} |\mathrm{M}_{n_i}(R_i)| - 1\right).$$

This shows that:

Theorem 3.7. *The graph* $\mathrm{C}_{\mathcal{R}}$ *is hyperenergetic. In particular, if* R *is a finite commutative ring, then* $\mathrm{C}_{\mathrm{M}_n(R)}$ *is hypergeometric for all* $n \ge 2$.

Remark 3.8. The later statement comes from the fact that every finite commutative ring is isomorphic to a direct product of finite local rings. Indeed, we can use this fact and Theorem 3.6 to compute the clique number, chromatic number and independence number for the unitary Cayley graph of a matrix ring over a finite commutative ring.

Moreover, if $s \ge 2$, then we can show that $\mathrm{C}_{\mathcal{R}}$ is neither a strongly regular graph nor a Ramanujan graph.

Theorem 3.9. *If $s \geq 2$, then $\mathrm{C}_{\mathcal{R}}$ is not strongly regular.*

Proof. If there exists $1 \leq i \leq s$ such that the graph $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ is not strongly regular, then $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ has more than three distinct eigenvalues which implies that $\mathrm{C}_{\mathcal{R}}$ has more than three distinct eigenvalues, so it is not stongly regular.

Assume that $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ is strongly regular for all $i \in \{1, 2, \ldots, s\}$. By Theorems 2.4 and 3.5, we have $n_i = 2$ and $R_i \cong \mathbb{k}_i$ for all $i \in \{1, 2, \ldots, s\}$. Thus, $\rho_1 = \prod_{i=1}^{s}({q_i}^2 - 1)({q_i}^2 - q_i)$, $\rho_2 = (-1)^s \prod_{i=1}^{s}({q_i}^2 - q_i)$ and $\rho_3 = \prod_{i=1}^{s} q_i$ are eigenvalues of $\mathrm{C}_{\mathcal{R}}$. If there exists $i \in \{1, 2, \ldots, s\}$ such that $q_i > 2$ say $i = 1$, then ρ_1, ρ_2 and ρ_3 are three distinct eigenvalues of $\mathrm{C}_{\mathcal{R}}$. Let $\rho = -({q_1}^2 - q_1) \prod_{i=2}^{s} q_i$. It is clear that $\rho \neq \rho_1$. Since ${q_1}^2 - q_1 > q_1$, we can conclude that $\rho \neq \rho_3$. Next, we assume $\rho = \rho_2$, so $-q_2 \ldots q_s = (-1)^{s-1} \prod_{i=2}^{s}({q_i}^2 - q_i)$. This forces that s is even and $q_2 = \cdots = q_s = 2$. Now, $\mathcal{R} \cong \mathrm{M}_2(\mathbb{k}_1) \times (\mathrm{M}_2(\mathbb{Z}_2))^{s-1}$ where s is even, and $\rho_1 = ({q_1}^2 - 1)({q_1}^2 - q_1)2^{s-1}$, $\rho_2 = (-1)^s({q_1}^2 - q_1)2^{s-1}$ and $\rho_3 = 2^{s-1}q_1$. Recall that -2 is an eigenvalue of $\mathrm{C}_{\mathrm{M}_2(\mathbb{Z}_2)}$. Let $\mu = -q_1 2^{s-1}$. Then $\mu \neq \rho_1$ and $\mu \neq \rho_3$. Also, ${q_1}^2 - q_1 > q_1$ implies $\rho \neq \rho_2$. Hence, $\mathrm{C}_{\mathcal{R}}$ has more than three distinct eigenvalues, so it is not strongly regular.

Finally, we assume that $q_i = 2$ for all $i \in \{1, 2, \ldots, s\}$. If $s \geq 3$, then $6^s, 2^s, 6 \cdot 2^{s-1}$ and $2 \cdot 6^{s-1}$ are 4 distinct eigenvalues of $\mathrm{C}_{\mathcal{R}}$. If $s = 2$, then $6, 2$ and -2 are eigenvalues of $\mathrm{C}_{\mathrm{M}_2(\mathbb{Z}_2)}$, so we have $36, 4, 12, -12$ are 4 distinct eigenvalues of $\mathrm{C}_{\mathrm{M}_2(\mathbb{Z}_2) \times \mathrm{M}_2(\mathbb{Z}_2)}$. □

Theorem 3.10. *If $s \geq 2$, then $\mathrm{C}_{\mathcal{R}}$ is not Ramanujan.*

Proof. Let $r_i = \mathrm{GL}_{n_i}(R_i)$ for all $i \in \{1, 2, \ldots, s\}$. If there exist $1 \leq i \leq s$ such that the graph $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ is Ramanujan, then $\rho = -{m_i}^{{n_i}^2}({q_i}^{n_i} - q) \ldots ({q_i}^{n_i} - {q_i}^{n_i} - 1)$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ other than $\pm r_i$ such that $|\rho| > 2\sqrt{r_i - 1}$. We may assume $i = s$. Then $|r_1 \ldots r_{s-1}\rho| > 2r_1 \ldots r_{s-1}\sqrt{r_s - 1}$. Let $m = r_1 \ldots r_{s-1} \geq 2$. We have $4m^2(r_s - 1) - 4(mr_s - 1) = 4mr_s(m - 1) > 0$, so $|r_1 \ldots r_{s-1}\rho| > 2\sqrt{r_1 \ldots r_s - 1}$. Hencec, $\mathrm{C}_{\mathcal{R}}$ is not Ramanujan.

Next, suppose that $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ is Ramanujan for all $i \in \{1, \ldots, s\}$. Then for any

i, we have

$$(n_i = 2 \text{ and } R = \Bbbk_i \text{ is a field}) \text{ or } (n_i = 3 \text{ and } R_i = \mathbb{Z}_2).$$

We may assume $n_1 = \ldots n_t = 2$ and $n_{t+1} = \ldots = n_s = 3$ where $t \geq 0$. We have $R_i \cong \Bbbk_i$ is a field for all $1 \leq i \leq t$ and $R_i = \mathbb{Z}_2$ for all $t+1 \leq i \leq s$. Recall that $-(2^3-2)(2^3-2^2) = -24$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ and $|\operatorname{GL}_3(\mathbb{Z}_2)| = 168$. Suppose that $s > t$. Let $\lambda = r_1 \ldots r_t(-24)^{s-t}$ and assume that

$$|\lambda| \leq 2\sqrt{r_1 \ldots r_t(168)^{s-t} - 1} < 2\sqrt{r_1 \ldots r_t(168)^{s-t}}.$$

If $t > 0$, then it follows that $6 \leq r_1 \ldots r_t < 4\left(\frac{168}{576}\right)^{s-t} \leq \frac{672}{576} < 2$ which is absurd. If $t = 0$, then $576^s < 4(168)^s$ which implies that $1 < 4\left(\frac{168}{576}\right)^2$ which is absurd again. Hence, we have $s = t$ and $\mathcal{R} \cong \mathrm{M}_2(\Bbbk_1) \times \cdots \times \mathrm{M}_2(\Bbbk_s)$. Let $\mu = -({q_1}^2 - q_1)r_2 \ldots r_s$. Suppose $|\mu| \leq 2\sqrt{r_1 \ldots r_s - 1} < 2\sqrt{r_1 \ldots r_s}$. Since $r_1 = ({q_1}^2-1)({q_1}^2-q_1)$, we can conclude that $r_2 \ldots r_s < 4\left(\frac{{q_1}^2-1}{{q_1}^2-q_1}\right)$. Moreover, we get $\frac{{q_1}^2-1}{{q_1}^2-q_1} \leq \frac{3}{2}$ because $q_1 \geq 2$. It follows that $6 \leq r_2 \ldots r_s < 4 \cdot \frac{3}{2} < 6$ which is a contradiction. □

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VITA

Name	:	Mr. Jitsupat Rattnakangwanwong
Date of Birth	:	22 August 1995
Place of Birth	:	Samutprakan, Thailand
Education	:	B.Sc. (Mathematics), Chulalongkorn University, 2016
Scholarship	:	H.M. the King Bhumibhol Adulyadej's 72nd Birthday Anniversary Scholarship, Graduate School, Chulalongkorn University